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ON THE COHOMOLOGY OF CALOGERO-MOSER SPACES

by

CÉDRIC BONNAFÉ & PENG SHAN

Abstract. — We compute the equivariant cohomology of smooth Calogero-Moser spaces and some associated symplectic resolutions of symplectic quotient singularities.

1. Notation and main results

All along this note, we will abbreviate $\otimes_{\mathbb{C}}$ as \otimes . By an algebraic variety, we mean a reduced scheme of finite type over \mathbb{C} .

1.A. Reflection group. — Let V be a \mathbb{C} -vector space of finite dimension n and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}$$

and we assume that

$$W = \langle \mathrm{Ref}(W) \rangle.$$

We set $\varepsilon : W \rightarrow \mathbb{C}^{\times}$, $w \mapsto \det(w)$.

If $s \in \mathrm{Ref}(W)$, we denote by α_s^{\vee} and α_s two elements of V and V^* respectively such that $V^s = \mathrm{Ker}(\alpha_s)$ and $V^{*s} = \mathrm{Ker}(\alpha_s^{\vee})$, where α_s^{\vee} is viewed as a linear form on V^* .

If $w \in W$, we set

$$\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$$

and we define a filtration $\mathcal{F}_{\bullet}(\mathbb{C}W)$ of the group algebra of W as follows: let

$$\mathcal{F}_i(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \leq i} \mathbb{C}w.$$

Then

$$\mathbb{C}\mathrm{Id}_V = \mathcal{F}_0(\mathbb{C}W) \subset \mathcal{F}_1(\mathbb{C}W) \subset \cdots \subset \mathcal{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathcal{F}_{n+1}(\mathbb{C}W) = \cdots$$

is a filtration of $\mathbb{C}W$. For any subalgebra A of $\mathbb{C}W$, we set $\mathcal{F}_i(A) = A \cap \mathcal{F}_i(\mathbb{C}W)$, so that

$$\mathbb{C}\mathrm{Id}_V = \mathcal{C}\mathcal{F}_0(A) \subset \mathcal{F}_1(A) \subset \cdots \subset \mathcal{F}_n(A) = A = \mathcal{F}_{n+1}(A) = \cdots$$

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is also a filtration of A . Let \hbar be a formal variable, and write

$$\begin{aligned} \text{Rees}_{\mathcal{F}}^{\bullet}(A) &= \bigoplus_{i \geq 0} \hbar^i \mathcal{F}_i(A) \subset \mathbb{C}[\hbar] \otimes A \quad (\text{the Rees algebra}), \\ \text{gr}_{\mathcal{F}}^{\bullet}(A) &= \bigoplus_{i \geq 0} \mathcal{F}_i(A) / \mathcal{F}_{i-1}(A). \end{aligned}$$

Recall that $\text{gr}_{\mathcal{F}}^{\bullet}(A) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(A) / \hbar \text{Rees}_{\mathcal{F}}^{\bullet}(A)$.

1.B. Rational Cherednik algebra at $t = 0$. — All along this note, we fix a function $c : \text{Ref}(W) \rightarrow \mathbb{C}$ which is invariant under conjugacy. We define the rational Cherednik algebra \mathbf{H}_c to be the quotient of the algebra $\text{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\text{T}(V \oplus V^*)$ with the group W) by the relations

$$(\mathcal{H}_c) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \rightarrow \mathbf{H}_c$ and $\mathbb{C}[V^*] \rightarrow \mathbf{H}_c$. Recall [5, Theorem 1.3] that we have an isomorphism of \mathbb{C} -vector spaces

$$(1.1) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_c$$

induced by multiplication (this is the so-called *PBW-decomposition*).

We denote by \mathbf{Z}_c the center of \mathbf{H}_c : it is well-known [5] that \mathbf{Z}_c is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and that it is a free \mathbf{P} -module of rank $|W|$. We denote by \mathcal{Z}_c the algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_c]$ is \mathbf{Z}_c : this is the *Calogero-Moser space* associated with the datum (V, W, c) .

Using the PBW-decomposition, we define a \mathbb{C} -linear map $\Omega_{\mathbf{H}}^c : \mathbf{H}_c \rightarrow \mathbb{C}W$ by

$$\Omega_{\mathbf{H}}^c(fwg) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W -equivariant for the action on both sides by conjugacy, so it induces a well-defined \mathbb{C} -linear map

$$\Omega^c : \mathbf{Z}_c \rightarrow Z(\mathbb{C}W).$$

Recall from [3, Corollary 4.2.11] that Ω^c is a morphism of algebras, and that

$$(1.2) \quad \mathcal{Z}_c \text{ is smooth if and only if } \Omega^c \text{ is surjective.}$$

The “only if” part is essentially due to Gordon [7, Corollary 5.8] (see also [3, Proposition 9.6.6 and (16.1.2)]) while the “if” part follows from the work of Bellamy, Schedler and Thiel [14, Corollary 1.4].

1.C. Grading. — The algebra $\text{T}(V \oplus V^*) \rtimes W$ can be \mathbb{Z} -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a \mathbb{Z} -grading on \mathbf{H}_c since the defining relations (\mathcal{H}_c) are homogeneous. Since the center of a graded algebra is always graded, it follows that \mathbf{Z}_c also inherits a

\mathbb{Z} -grading. By definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of \mathbf{Z}_c . So the Calogero-Moser space \mathcal{Z}_c inherits a regular \mathbb{C}^\times -action.

1.D. Main results. — For a complex algebraic variety \mathcal{X} (equipped with its classical topology), we denote by $H^i(\mathcal{X})$ its i -th singular cohomology group with coefficients in \mathbb{C} . If \mathcal{X} carries a regular action of a torus \mathbf{T} , we denote by $H_{\mathbf{T}}^i(\mathcal{X})$ its i -th \mathbf{T} -equivariant cohomology group (still with coefficients in \mathbb{C}). Note that $H^{2\bullet}(\mathcal{X}) = \bigoplus_{i \geq 0} H^{2i}(\mathcal{X})$ is a graded \mathbb{C} -algebra and $H_{\mathbf{T}}^{2\bullet}(\mathcal{X}) = \bigoplus_{i \geq 0} H_{\mathbf{T}}^{2i}(\mathcal{X})$ is a graded $H_{\mathbf{T}}^{2\bullet}(\mathbf{pt})$ -algebra. The following result [5, Theorem 1.8] describes the algebra structure on the cohomology of \mathcal{Z}_c (with coefficients in \mathbb{C}):

Theorem 1.3 (Etingof-Ginzburg). — *Assume that \mathcal{Z}_c is smooth. Then:*

- (a) $H^{2i+1}(\mathcal{Z}_c) = 0$ for all i .
- (b) *There is an isomorphism of graded \mathbb{C} -algebras $H^{2\bullet}(\mathcal{Z}_c) \simeq \mathfrak{gr}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$.*

In this note, we prove an equivariant version of this statement (we identify $R = H_{\mathbb{C}^\times}^*(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ in the usual way):

Theorem A. — *Assume that \mathcal{Z}_c is smooth. Then:*

- (a) $H_{\mathbb{C}^\times}^{2i+1}(\mathcal{Z}_c) = 0$ for all i .
- (b) *There is an isomorphism of graded R -algebras $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{Z}_c) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$.*

Note that Theorem A(a) just follows from statement (a) of Etingof-Ginzburg Theorem by Proposition 2.4(a) below. As a partial consequence of Theorem A, we also obtain the following application to the equivariant cohomology of some symplectic resolutions.

Theorem B. — *Assume that the symplectic quotient singularity $(V \times V^*)/W$ admits a symplectic resolution $\pi : \mathcal{X} \rightarrow (V \times V^*)/W$. Recall that the \mathbb{C}^\times -action on $(V \times V^*)/W$ lifts (uniquely) to \mathcal{X} (see [12, Theorem 1.3(ii)]). Then:*

- (a) $H_{\mathbb{C}^\times}^{2i+1}(\mathcal{X}) = 0$ for all i .
- (b) *There is an isomorphism of graded R -algebras $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$.*

Note that for $W = \mathfrak{S}_n$ acting on \mathbb{C}^n , Theorem B describes the equivariant cohomology of the Hilbert scheme of n points in \mathbb{C}^2 : this was already proved (by different methods) by Vasserot [15]. In [6, Conjecture 1.3], Ginzburg-Kaledin proposed a conjecture for the equivariant cohomology of a symplectic resolution of a symplectic quotient singularity E/G , where E is a finite dimensional symplectic vector space and G is a finite subgroup of $\mathbf{Sp}(E)$. However, their conjecture cannot hold as stated, because they considered the \mathbb{C}^\times -action by dilatation, which is contractible. Theorem B shows that the correct equivariant cohomological realization of the Rees algebra is provided by the symplectic \mathbb{C}^\times -action, which exists only when G stabilizes a Lagrangian subspace of E .

Remark 1.4. — Recall from the works of Etingof-Ginzburg [5], Ginzburg-Kaledin [6], Gordon [7] and Bellamy [1] that the existence of a symplectic resolution of $(V \times V^*)/W$ is equivalent to the existence of a parameter c such that \mathcal{Z}_c is smooth, and that it can only occur if all the irreducible components of W are of type $G(d, 1, n)$ (for some $d, n \geq 1$) or G_4 in Shephard-Todd classification. ■

1.E. Conjectures. — In [3, Chapter 16, Conjectures COH and ECOH], Rouquier and the first author proposed the following conjecture which aims to generalize the above Etingof-Ginzburg Theorem 1.3 into two directions: it includes singular Calogero-Moser spaces and it extends to equivariant cohomology.

Conjecture 1.5. — *With the above notation, we have:*

- (1) $H^{2i+1}(\mathcal{Z}_c) = 0$ for all i .
- (2) We have an isomorphism of graded \mathbb{C} -algebras $H^{2\bullet}(\mathcal{Z}_c) \simeq \text{gr}_{\mathcal{F}}(\text{Im}(\Omega^c))$.
- (3) We have an isomorphism of graded R -algebras $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{Z}_c) \simeq \text{Rees}_{\mathcal{F}}(\text{Im}(\Omega^c))$.

By (1.2), when \mathcal{Z}_c is smooth, the image of Ω^c coincide with the center of $\mathbb{C}W$. So Theorem A proves this conjecture for smooth \mathcal{Z}_c .

We will also prove in Example 3.6 the following result:

Proposition 1.6. — *If $\dim_{\mathbb{C}}(V) = 1$, then Conjecture 1.5 holds.*

1.F. Structure of the paper. — The paper is organized as follows. The proof of Theorem A relies on classical theorems about restriction to fixed points. In the next Section 2, we first recall basic properties on equivariant cohomology and equivariant K-theory and restriction to fixed points. Section 3 explains how these general principles can be applied to Calogero-Moser spaces. Theorem A will be proved in Section 4. The cyclic group case (Proposition 1.6) will be handled in Example 3.6. The proof of Theorem B will be given in Section 5.

2. Equivariant cohomology, K-theory and fixed points

2.A. Equivariant cohomology. — Let \mathcal{X} be a complex algebraic variety equipped with a regular action of a torus \mathbf{T} . Recall that the *equivariant cohomology* of \mathcal{X} is defined by

$$H_{\mathbf{T}}^{\bullet}(\mathcal{X}) = H^{\bullet}(E_{\mathbf{T}} \times_{B_{\mathbf{T}}} X),$$

where $E_{\mathbf{T}} \rightarrow B_{\mathbf{T}}$ is a universal \mathbf{T} -bundle. The pullback of the structural morphism $x : \mathcal{X} \rightarrow \mathbf{pt}$ yields a ring homomorphism $H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \rightarrow H_{\mathbf{T}}^{\bullet}(\mathcal{X})$, which makes $H_{\mathbf{T}}^{\bullet}(\mathcal{X})$ a graded $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -algebra.

Denote by $X(\mathbf{T})$ the character lattice of \mathbf{T} . For each $\chi \in X(\mathbf{T})$, denote by \mathbb{C}_{χ} the one dimensional \mathbf{T} -module of character χ , the first Chern class c_{χ} of the line bundle $E_{\mathbf{T}} \times_{\mathbf{T}} \mathbb{C}_{\chi}$ on $B_{\mathbf{T}}$ is an element in $H^2(B_{\mathbf{T}})$. Identify the vector space $\mathbb{C} \otimes_{\mathbb{Z}} X(\mathbf{T})$ with the dual \mathfrak{t}^* of the Lie algebra \mathfrak{t} of \mathbf{T} via $\chi \mapsto d\chi$. Then the assignment $\chi \mapsto c_{\chi}$ yields an isomorphism of graded \mathbb{C} -algebras $S(\mathfrak{t}^*) = H_{\mathbf{T}}^{2\bullet}(\mathbf{pt})$.

2.B. Equivariant K-theory. — We denote by $K_{\mathbf{T}}(\mathcal{X})$ the Grothendieck ring of the category of \mathbf{T} -equivariant vector bundles on \mathcal{X} . Note that when \mathcal{X} is a point, a \mathbf{T} -equivariant vector bundles is the same as a finite dimensional \mathbf{T} -module. We have a canonical isomorphism $K_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})]$ which sends the class of a \mathbf{T} -module M to

$$(2.1) \quad \dim^{\mathbf{T}}(M) = \sum_{\chi \in X(\mathbf{T})} \dim_{\mathbb{C}}(M_{\chi})\chi,$$

where M_{χ} is the χ -weight space in M .

Let $\hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$ be the completion of $H_{\mathbf{T}}^{2\bullet}(\mathcal{X})$ with respect to the ideal $\bigoplus_{i>0} H_{\mathbf{T}}^{2i}(\mathcal{X})$. The equivariant Chern character provides a ring homomorphism

$$\mathrm{ch}_{\mathcal{X}} : K_{\mathbf{T}}(\mathcal{X}) \longrightarrow \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$$

with the following properties. First, when \mathcal{X} is a point \mathbf{pt} , we have

$$(2.2) \quad \begin{array}{ccc} \mathrm{ch}_{\mathbf{pt}} : K_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})] & \longrightarrow & \hat{H}_{\mathbf{T}}^{2\bullet}(\mathbf{pt}) = \hat{S}(\mathfrak{t}^*) \\ \chi & \longmapsto & \exp(d\chi). \end{array}$$

Next the Chern character commutes with pullback. More precisely, if \mathcal{Y} is another variety with a regular action of the same torus \mathbf{T} and if $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ is a \mathbf{T} -equivariant morphism, then the following diagram commutes

$$(2.3) \quad \begin{array}{ccc} K_{\mathbf{T}}(\mathcal{Y}) & \xrightarrow{\varphi^*} & K_{\mathbf{T}}(\mathcal{X}) \\ \mathrm{ch}_{\mathcal{Y}} \downarrow & & \downarrow \mathrm{ch}_{\mathcal{X}} \\ \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{Y}) & \xrightarrow{\hat{\varphi}^*} & \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X}). \end{array}$$

Here φ^* denotes both the pullback map in K-theory or in equivariant cohomology, and $\hat{\varphi}^*$ is the map induced after completion by the pullback map in equivariant cohomology. In particular, by applying the above diagram to $\mathcal{Y} = \mathbf{pt}$ and the structural morphism $\mathcal{X} \rightarrow \mathbf{pt}$, we may view $\mathrm{ch}_{\mathcal{X}}$ as a morphism of algebras over $K_{\mathbf{T}}(\mathbf{pt})$, with the $K_{\mathbf{T}}(\mathbf{pt})$ -algebra structure on $\hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$ provided by the embedding (2.2).

2.C. Fixed points. — We denote by $\mathcal{X}^{\mathbf{T}}$ the (reduced) variety consisting of fixed points of \mathbf{T} in \mathcal{X} . Let $i_{\mathcal{X}} : \mathcal{X}^{\mathbf{T}} \hookrightarrow \mathcal{X}$ be the natural closed immersion. Since \mathbf{T} acts trivially on $\mathcal{X}^{\mathbf{T}}$, we have $H_{\mathbf{T}}^{\bullet}(\mathcal{X}^{\mathbf{T}}) = H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes H^{\bullet}(\mathcal{X}^{\mathbf{T}})$ as $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -algebras. Recall that the \mathbf{T} -action on \mathcal{X} is called *equivariantly formal* if the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(B_{\mathbf{T}}; H^q(X)) \implies H_{\mathbf{T}}^{p+q}(X)$$

for the fibration $E_{\mathbf{T}} \times_{\mathbf{T}} X \rightarrow B_{\mathbf{T}}$ degenerates at E_2 . We have the following standard result on equivariant cohomology (see for instance [10, Proposition 2.1]):

Proposition 2.4. — Assume that $H^{2i+1}(\mathcal{X}) = 0$ for all i . Then \mathcal{X} is equivariantly formal, and:

- (a) There is an isomorphism of graded $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -modules $H_{\mathbf{T}}^{\bullet}(\mathcal{X}) \simeq H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes H^{\bullet}(\mathcal{X})$. In particular $H_{\mathbf{T}}^{2i+1}(\mathcal{X}) = 0$ for all i .
- (b) The pullback map $i_{\mathcal{X}}^* : H_{\mathbf{T}}^{\bullet}(\mathcal{X}) \rightarrow H_{\mathbf{T}}^{\bullet}(\mathcal{X}^{\mathbf{T}})$ is injective.
- (c) Let \mathfrak{m} be the unique graded maximal ideal of $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$. Then we have an isomorphism of algebras $H^{\bullet}(\mathcal{X}) \simeq H_{\mathbf{T}}^{\bullet}(\mathcal{X}) / \mathfrak{m}H_{\mathbf{T}}^{\bullet}(\mathcal{X})$.

In particular, this shows that Conjectures 1.5(1) and (3) imply Conjecture 1.5(2).

Example 2.5 (Blowing-up). — Let \mathcal{Y} be an affine variety with a \mathbf{T} -action and let \mathcal{C} be a \mathbf{T} -stable closed subvariety (not necessarily reduced) of \mathcal{Y} . Let \mathcal{X} be the blowing-up of \mathcal{Y} along \mathcal{C} . Write $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ for the natural morphism, and we equip \mathcal{X} with the unique \mathbf{T} -action such that π is equivariant. *We assume that $\mathcal{X}^{\mathbf{T}}$ is finite.* We write

$$H_{\mathbf{T}}^{2\bullet}(\mathcal{X}^{\mathbf{T}}) = \bigoplus_{x \in \mathcal{X}^{\mathbf{T}}} S(t^*)\mathbf{e}_x,$$

where $\mathbf{e}_x \in H_{\mathbf{T}}^0(\mathcal{X}^{\mathbf{T}})$ is the primitive idempotent associated with x (i.e., the fundamental class of x).

Then $\mathcal{D} = \pi^*(\mathcal{C})$ is a \mathbf{T} -stable effective Cartier divisor, and we denote by $[\mathcal{D}]$ the class in $K_{\mathbf{T}}(\mathcal{X})$ of its associated line bundle (which is \mathbf{T} -equivariant). We denote by $\text{ch}_{\mathcal{X}}^1([\mathcal{D}]) \in H_{\mathbf{T}}^2(\mathcal{X})$ its first \mathbf{T} -equivariant Chern class. We want to compute $i_{\mathcal{X}}^*(\text{ch}_{\mathbf{T}}^1(\mathcal{D}))$.

First, let I be the ideal of $\mathbb{C}[\mathcal{Y}]$ associated with \mathcal{C} . As it is \mathbf{T} -stable, we can find a family of \mathbf{T} -homogeneous generators $(a_i)_{1 \leq i \leq k}$ of I . We denote by $\lambda_i \in X(\mathbf{T})$ the \mathbf{T} -weight of a_i . The choice of this family of generators induces a \mathbf{T} -equivariant closed immersion $\mathcal{X} \hookrightarrow \mathcal{Y} \times \mathbb{P}^{k-1}(\mathbb{C})$. We denote by \mathcal{X}_i the affine chart corresponding to “ $a_i \neq 0$ ”. If $x \in \mathcal{X}^{\mathbf{T}}$, we denote by $\mathbf{i}(x) \in \{1, 2, \dots, k\}$ an element such that $x \in \mathcal{X}_{\mathbf{i}(x)}$. Then

$$(2.6) \quad i_{\mathcal{X}}^*(\text{ch}_{\mathbf{T}}^1(\mathcal{D})) = -\hbar \sum_{x \in \mathcal{D}^{\mathbf{T}}} (d\lambda_{\mathbf{i}(x)})\mathbf{e}_x.$$

Indeed, we just need to compute the local equation of \mathcal{D} around $x \in \mathcal{X}^{\mathbf{T}}$, and this can be done in $\mathcal{X}_{\mathbf{i}(x)}$. But $\mathcal{D} \cap \mathcal{X}_i$ is principal for all i , defined by

$$\mathcal{D} \cap \mathcal{X}_i = \{(y, \xi) \in \mathcal{Y} \times \mathbb{P}^{k-1}(\mathbb{C}) \mid (y, \xi) \in \mathcal{X}_i \text{ and } a_i(y) = 0\}.$$

So $\mathcal{D} \cap \mathcal{X}_i$ is defined by a \mathbf{T} -homogeneous equation of degree $\lambda_i \in X(\mathbf{T})$, and so (2.6) follows. ■

3. Localization and Calogero-Moser spaces

In this section, we apply the previous discussions to $\mathcal{X} = \mathcal{X}_c$ and $\mathbf{T} = \mathbb{C}^\times$. Denote by $q : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ the identity map. Then $X(\mathbb{C}^\times) = q^{\mathbb{Z}}$, and we have

$$K_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}], \quad H_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[\hbar],$$

with $\hbar = c_q$, following the notation of Section 2.A. So $\hat{H}_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[[\hbar]]$ and the Chern map in this case is given by

$$\text{ch}_{\mathbf{pt}} : \mathbb{Z}[q, q^{-1}] \hookrightarrow \mathbb{C}[[\hbar]], \quad q \mapsto \exp(\hbar).$$

Note also that a finite \mathbb{C}^\times -module is nothing but a finite dimensional \mathbb{Z} -graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that \mathbb{C}^\times acts on M_i by the character q^i . The identification $K_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}]$ sends the class of M to its *graded dimension* (or *Hilbert series*):

$$\dim^{\text{gr}}(M) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}}(M_i)q^i \in \mathbb{N}[q, q^{-1}].$$

3.A. Fixed points. — For $\chi \in \text{Irr}(W)$, we denote by $\omega_\chi : Z(CW) \rightarrow \mathbb{C}$ the associated morphism of algebras, i.e. $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar through which z acts on the irreducible CW -module with the character χ . We denote by e_χ^W the unique primitive idempotent of $Z(CW)$ such that $\omega_\chi(e_\chi^W) = 1$. If \mathcal{E} is a subset of $\text{Irr}(W)$, then we set $e_\mathcal{E}^W = \sum_{\chi \in \mathcal{E}} e_\chi^W$.

Now, consider the algebra homomorphism

$$\Omega_\chi^c = \omega_\chi \circ \Omega^c : \mathbf{Z}_c \longrightarrow \mathbb{C}.$$

Its kernel is a maximal ideal of \mathbf{Z}_c , denote by z_χ the corresponding closed point in \mathcal{Z}_c . It follows from [3, Lemma 10.2.3 and (14.2.2)] that $z_\chi \in \mathcal{Z}_c^{\text{C}^\times}$ and that the map

$$(3.1) \quad \begin{array}{ccc} z : \text{Irr}(W) & \longrightarrow & \mathcal{Z}_c^{\text{C}^\times} \\ \chi & \longmapsto & z_\chi \end{array}$$

is surjective. The fibers of this map are called the *Calogero-Moser c -families*. They were first considered by Gordon [7] and Gordon-Martino [9]: see also for instance [3, §9.2]. Let $\text{CM}_c(W)$ be the set of Calogero-Moser c -families. For $\mathcal{E} \in \text{CM}_c(W)$, we denote by $z_\mathcal{E} \in \mathcal{Z}_c^{\text{C}^\times}$ its image under the map z . On the other hand, by [3, (16.1.2)] we have

$$(3.2) \quad \text{Im}(\Omega^c) = \bigoplus_{\mathcal{E} \in \text{CM}_c(W)} \mathbb{C} e_\mathcal{E}^W.$$

Hence we get an isomorphism of \mathbb{C} -algebras

$$H^{2\bullet}(\mathcal{Z}_c^{\text{C}^\times}) \simeq \text{Im}(\Omega^c), \quad e_{z_\mathcal{E}} \mapsto e_\mathcal{E}^W,$$

which extends to an isomorphism of $\mathbb{C}[\hbar]$ -algebras $H^{2\bullet}(\mathcal{Z}_c^{\text{C}^\times}) \simeq \mathbb{C}[\hbar] \otimes \text{Im}(\Omega^c)$. For simplification, we set $i_c = i_{\mathcal{Z}_c} : \mathcal{Z}_c^{\text{C}^\times} \hookrightarrow \mathcal{Z}$ and, under the above identification, we view the pullback map i_c^* as a morphism of algebras

$$i_c^* : H_T^{2\bullet}(\mathcal{Z}_c) \longrightarrow \mathbb{C}[\hbar] \otimes \text{Im}(\Omega^c).$$

So, by Proposition 2.4, Conjecture 1.5 is implied by the following one:

Conjecture 3.3. — *With the above notation, we have:*

- (1) $H^{2i+1}(\mathcal{Z}_c) = 0$ for all i .
- (2) $\text{Im}(i_c^*) = \text{Rees}_{\mathcal{F}}(\text{Im}(\Omega^c))$.

Remark 3.4. — Set

$$\mathcal{F}_i^{\text{H}}(\text{Im}(\Omega^c)) = \{z \in \text{Im}(\Omega^c) \mid \hbar^i z \in \text{Im}(i_c^*)\}.$$

Then, by construction and Proposition 2.4, $\mathcal{F}_\bullet^{\text{H}}(\text{Im}(\Omega^c))$ is the filtration of $\text{Im}(\Omega^c)$ satisfying

$$H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{Z}_c) \simeq \text{Im}(i_c^*) = \text{Rees}_{\mathcal{F}^{\text{H}}}(\text{Im}(\Omega^c))$$

and

$$H^{2\bullet}(\mathcal{Z}_c) \simeq \text{gr}_{\mathcal{F}^{\text{H}}}(\text{Im}(\Omega^c)).$$

So showing Conjecture 3.3 is then equivalent to showing that the filtrations $\mathcal{F}_\bullet(\text{Im}(\Omega^c))$ and $\mathcal{F}_\bullet^{\text{H}}(\text{Im}(\Omega^c))$ coincide.

Note also that, since \mathcal{Z}_c is an affine variety of dimension $2n$, we have $H^i(\mathcal{Z}_c) = 0$ for $i > 2n$, so this shows that

$$(b) \quad \mathcal{F}_n(\text{Im}(\Omega^c)) = \text{Im}(\Omega^c) = \mathcal{F}_n^{\text{H}}(\text{Im}(\Omega^c)).$$

On the other hand, since \mathcal{X}_c is connected, we have

$$(\sharp) \quad \mathcal{F}_0(\mathrm{Im}(\Omega^c)) = \mathcal{F}_0^{\mathrm{H}}(\mathrm{Im}(\Omega^c)) = \mathbb{C}.$$

These two particular cases will be used below. ■

The following result, based on the previous remark, can be viewed as a reduction step for the proof of Conjecture 3.3:

Proposition 3.5. — Assume that $H^{2i+1}(\mathcal{X}_c) = 0$ and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all i . Then Conjecture 3.3 holds if and only if $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) \subset \mathrm{Im}(i_c^*)$.

Proof. — Assume that $H^{2i+1}(\mathcal{X}_c) = 0$ and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all i . We keep the notation of Remark 3.4. It then follows from this remark and the hypothesis that

$$\dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c))) = \dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim(\mathcal{F}_i^{\mathrm{H}}(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}^{\mathrm{H}}(\mathrm{Im}(\Omega^c)))$$

for all i . So, by induction, we get that $\dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))) = \dim(\mathcal{F}_i^{\mathrm{H}}(\mathrm{Im}(\Omega^c)))$ for all i (by the equality (\sharp) of Remark 3.4). This shows that $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) = \mathrm{Im}(i_c^*)$ if and only if $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) \subset \mathrm{Im}(i_c^*)$, as desired. □

Example 3.6. — Assume in this example, and only in this example, that $\dim_{\mathbb{C}}(V) = 1$. It is proved in [3, Theorem 18.5.8] that in this case $H^{2i+1}(\mathcal{X}_c) = 0$ and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all i . Since \mathcal{X}_c is affine of dimension 2, we have $H^i(\mathcal{X}_c) = 0$ if $i \notin \{0, 2\}$. So it follows from the equalities (b) and (\sharp) that Conjecture 3.3 holds in this case (this proves Proposition 1.6). ■

3.B. Chern map. — If \mathcal{E} is a Calogero-Moser family, we denote by $\mathfrak{m}_{\mathcal{E}} \subset \mathbf{Z}_c$ the ideal of functions vanishing at $z_{\mathcal{E}} \in \mathcal{X}_c^{\mathrm{C}^\times}$. We also set

$$\mathrm{Im}(\Omega^c)_{\mathbb{Z}} = \bigoplus_{\mathcal{E} \in \mathrm{CM}_c(W)} \mathbb{Z}e_{\mathcal{E}}^W.$$

We make the natural identification $\mathrm{K}_{\mathbb{C}^\times}(\mathcal{X}_c^{\mathrm{C}^\times}) = \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\Omega^c)_{\mathbb{Z}}$. Through these identifications, the Chern map $\mathrm{ch}_{\mathcal{X}_c^{\mathrm{C}^\times}}$ just becomes the natural inclusion $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\Omega^c)_{\mathbb{Z}} \hookrightarrow \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\Omega^c)$. Moreover, if P is a \mathbb{Z} -graded finitely generated projective \mathbf{Z}_c -module, then the commutativity of the diagram (2.3) just says that

$$(3.7) \quad i_c^*(\mathrm{ch}_{\mathcal{X}_c}([P])) = \sum_{\mathcal{E} \in \mathrm{CM}_c(W)} \dim^{\mathrm{gr}}(P/\mathfrak{m}_{\mathcal{E}}P) e_{\mathcal{E}}^W \subset \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\Omega^c).$$

4. Proof of Theorem A

Hypothesis and notation. *We assume in this section, and only in this section, that \mathcal{X}_c is smooth.*

If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and $N = \bigoplus_{i \in \mathbb{Z}} N_i$ are two finite-dimensional graded $\mathbb{C}W$ -modules, we set

$$\langle M, N \rangle_W^{\text{gr}} = \sum_{i, j \in \mathbb{Z}} \langle M_i, N_j \rangle_W q^{i+j},$$

where $\langle E, F \rangle_W = \dim \text{Hom}_{\mathbb{C}W}(E, F)$ for any finite dimensional $\mathbb{C}W$ -modules E and F . We extend this notation to the case where M or N is a graded virtual character (i.e. an element of $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z} \text{Irr}(W)$). Finally, we denote by $\mathbb{C}[V^*]^{\text{co}(W)}$ the *coinvariant algebra*, i.e. the quotient of the algebra $\mathbb{C}[V^*]$ by the ideal generated by the elements $f \in \mathbb{C}[V^*]^W$ such that $f(0) = 0$. Then $\mathbb{C}[V^*]^{\text{co}(W)}$ is a graded $\mathbb{C}W$ -module, which is isomorphic to the regular representation $\mathbb{C}W$ when one forgets the grading (we define similarly $\mathbb{C}[V]^{\text{co}(W)}$).

4.A. Localization in K-theory. — Recall from (1.2) that the smoothness of \mathcal{X}_c implies that $\text{Im}(\Omega^c) = Z(\mathbb{C}W)$, so that $z : \text{Irr}(W) \rightarrow \mathcal{X}_c^{\text{C}^\times}$ is bijective. Let $e = e_1^W = (1/|W|) \sum_{w \in W} w$. The smoothness of \mathcal{X}_c also implies that the functor

$$\begin{aligned} \mathbf{H}_c\text{-mod} &\longrightarrow \mathbf{Z}_c\text{-mod} \\ M &\longmapsto eM = e\mathbf{H}_c \otimes_{\mathbf{H}_c} M \end{aligned}$$

is an equivalence of categories [5, Theorem 1.7]. If E is a finite dimensional \mathbb{Z} -graded $\mathbb{C}W$ -module, the \mathbf{H}_c -module $\mathbf{H}_c \otimes_{\mathbb{C}W} E$ is finitely generated, \mathbb{Z} -graded and projective. Therefore, $e\mathbf{H}_c \otimes_{\mathbb{C}W} E$ is a finitely generated graded projective \mathbf{Z}_c -module, which can be viewed as a \mathbb{C}^\times -equivariant vector bundle on \mathcal{X}_c . For simplification, we set

$$\text{ch}_c(E) = i_c^*(\text{ch}_{\mathcal{X}_c}(e\mathbf{H}_c \otimes_{\mathbb{C}W} E)) \in \mathbb{C}[[\hbar]] \otimes \text{Im}(\Omega^c).$$

Proposition 4.1. — *Assume that \mathcal{X}_c is smooth. Let E be a finite dimensional graded $\mathbb{C}W$ -module. Then*

$$\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} q^{b_\chi} \frac{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \otimes E \rangle_W^{\text{gr}}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} e_\chi^W,$$

where b_χ is the valuation of $\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_G^{\text{gr}}$.

Proof. — As the formula is additive, we may, and we will, assume that E is an irreducible $\mathbb{C}W$ -module, concentrated in degree 0, and we denote by ψ its character.

Now, let $\chi \in \text{Irr}(W)$: we denote by \mathfrak{m}_χ the maximal ideal of \mathbf{Z}_c corresponding to the fixed point z_χ . We set $\mathfrak{p} = \mathfrak{m}_\chi \cap \mathbf{P}$: it does not depend on χ (it is the maximal ideal of $\mathbf{P} = \mathbb{C}[V/W \times V^*/W]$ of functions which vanishes at $(0, 0)$; see for instance [3, (14.2.2)]). By (3.7),

$$\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} \dim^{\text{gr}}((e\mathbf{H}_c \otimes_{\mathbb{C}W} E)/\mathfrak{m}_\chi(e\mathbf{H}_c \otimes_{\mathbb{C}W} E)) e_\chi^W.$$

Now, let $\bar{\mathbf{Z}}_c = \mathbf{Z}_c/\mathfrak{p}\mathbf{Z}_c$ and $\bar{\mathbf{H}}_c = \mathbf{H}_c/\mathfrak{p}\mathbf{H}_c$. We set $\bar{\mathfrak{m}}_\chi = \mathfrak{m}_\chi/\mathfrak{p}\mathbf{Z}_c$. Then $\bar{\mathbf{H}}_c$ is a finite dimensional \mathbb{C} -algebra (called the *restricted rational Cherednik algebra*) and again the bimodule

$e\bar{\mathbf{H}}_c$ induces a Morita equivalence between $\bar{\mathbf{H}}_c$ and $\bar{\mathbf{Z}}_c$. This implies that $e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi\bar{\mathbf{H}}_c = (\bar{\mathbf{Z}}_c/\bar{\mathfrak{m}}_\chi) \otimes_{\bar{\mathbf{Z}}_c} e\bar{\mathbf{H}}_c$ is a simple right $\bar{\mathbf{H}}_c$ -module (that will be denoted by $\mathcal{L}_c(\chi)$).

Since $\mathbb{C}W$ is semisimple, the $\mathbb{C}W$ -module E is flat, and so

$$\begin{aligned} (e\mathbf{H}_c \otimes_{\mathbb{C}W} E)/\mathfrak{m}_\chi(e\mathbf{H}_c \otimes_{\mathbb{C}W} E) &\simeq (e\mathbf{H}_c/\mathfrak{m}_\chi(e\mathbf{H}_c)) \otimes_{\mathbb{C}W} E \\ &\simeq (e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi(e\bar{\mathbf{H}}_c)) \otimes_{\mathbb{C}W} E. \\ &\simeq \mathcal{L}_c(\chi) \otimes_{\mathbb{C}W} E. \end{aligned}$$

But the graded dimension of $\mathcal{L}_c(\chi) \otimes_{\mathbb{C}W} E$ is known whenever \mathcal{Z}_c is smooth and is given by the expected formula (see [1, Lemma 3.3 and its proof]). \square

Let $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W)$ denote the Grothendieck group of the category of finite dimensional graded $\mathbb{C}W$ -modules. If E is a finite dimensional graded $\mathbb{C}W$ -module, we denote by $[E]$ its class in $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W)$. We still denote by $\text{ch}_c : \mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W) \rightarrow \mathbb{C}[[\hbar]] \otimes \mathbb{Z}(\mathbb{C}W)$ the map defined by

$$\text{ch}_c([E]) = \text{ch}_c(E).$$

Now, let W' be a parabolic subgroup of W and set $V' = V^{W'}$ and $r = \text{codim}_{\mathbb{C}}(V')$. We identify the dual V'^* of V' with $V^{*W'}$ and note that

$$(4.2) \quad V = V' \oplus (V'^*)^\perp.$$

We denote by $\wedge(V'^*)^\perp$ the element of $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W')$ defined by

$$\wedge(V'^*)^\perp = \sum_{i \geq 0} (-1)^i [\wedge^i (V'^*)^\perp].$$

Recall also that there exists n algebraically independent homogeneous polynomials f_1, \dots, f_n in $\mathbb{C}[V]^W$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$, and we denote by d_i the degree of f_i .

Corollary 4.3. — *Assume that \mathcal{Z}_c is smooth. Let E' be a finite dimensional graded $\mathbb{C}W'$ -module. Then*

$$\text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \frac{(1-q^{d_1}) \cdots (1-q^{d_n})}{(1-q)^{n-r}} \sum_{\chi \in \text{Irr}(W)} q^{b_\chi} \frac{\langle \chi, \text{Ind}_{W'}^W(E') \rangle_W^{\text{gr}}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} e_\chi^W.$$

Proof. — The group W' acts trivially on V' so it acts trivially on $\wedge^i V'$ for all i . Therefore,

$$(1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \text{ch}_c(\text{ind}_{W'}^W(\wedge V' \otimes \wedge(V'^*)^\perp \otimes [E'])).$$

But $\wedge V' \otimes \wedge(V'^*)^\perp = \text{Res}_{W'}^W(\wedge V)$ by (4.2), so, by Frobenius formula,

$$(1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \text{ch}_c(\wedge V \otimes \text{Ind}_{W'}^W(E')).$$

So it follows from Proposition 4.1 that

$$\begin{aligned} (1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) \\ = \sum_{\chi \in \text{Irr}(W)} \frac{q^{b_\chi}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} \langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \otimes (\wedge V) \otimes \text{Ind}_{W'}^W(E') \rangle_W^{\text{gr}} e_\chi^W. \end{aligned}$$

But, if $w \in W$, the Molien's formula implies that the graded trace of w on $\mathbb{C}[V^*]^{\text{co}(W)}$ is equal to

$$\frac{(1-q^{d_1})\cdots(1-q^{d_n})}{\det(1-wq)},$$

while its graded trace on $\wedge V$ is equal to $\det(1-wq)$. So the class of $\mathbb{C}[V^*]^{\text{co}(W)} \otimes \wedge V$ is equal to $(1-q^{d_1})\cdots(1-q^{d_n})$ times the class of the trivial module, and the corollary follows. \square

Proof of Theorem A. — Assume that \mathcal{Z}_c is smooth. This implies in particular that Conjecture 1.5(1) and (2) hold (Etingof-Ginzburg Theorem 1.3), and so the hypotheses of Proposition 3.5 are satisfied. Note also that $\text{Im}(\Omega^c) = Z(\mathbb{C}W)$ by (1.2). It is then sufficient to prove that $\hbar^i \mathcal{F}_i(Z(\mathbb{C}W)) \subset \text{Im}(i_c^*)$ for all i .

Let us introduce some notation. If G is a finite group and H is a subgroup, we define a \mathbb{C} -linear map $\text{Tr}_H^G : Z(\mathbb{C}H) \rightarrow Z(\mathbb{C}G)$ by

$$\text{Tr}_H^G(z) = \sum_{g \in [G/H]} \mathfrak{s}_z = \frac{1}{|H|} \sum_{g \in G} \mathfrak{s}_z$$

(here, $[G/H]$ denotes a set of representatives of elements of G/H and $\mathfrak{s}_z = gzg^{-1}$). It is easily checked that

$$(4.4) \quad \text{Tr}_H^G(e_\eta^H) = \frac{\eta(1)}{|H|} \sum_{\gamma \in \text{Irr}(G)} \frac{|G|}{\gamma(1)} \langle \gamma, \text{Ind}_H^G(\eta) \rangle_G e_\gamma^G$$

for all $\eta \in \text{Irr}(H)$. Also, if $h \in H$ and $\Sigma_H(h) \in Z(\mathbb{C}H)$ denotes the sum of the conjugates of h in H , then

$$(4.5) \quad \text{Tr}_H^G(\Sigma_H(h)) = \frac{|C_G(h)|}{|C_H(h)|} \Sigma_G(h).$$

Let $\mathcal{P}_r(W)$ denote the set of parabolic subgroups W' of W such that $\text{codim}_{\mathbb{C}}(V^{W'}) = r$. It follows from (4.5) that

$$(4.6) \quad \mathcal{F}_r(Z(\mathbb{C}W)) = \sum_{W' \in \mathcal{P}_r(W)} \text{Tr}_{W'}^W(Z(\mathbb{C}W')).$$

Therefore, by Proposition 3.5, it is sufficient to prove that

$$(\star) \quad \hbar^r \text{Tr}_{W'}^W(e_{\chi'}^{W'}) \in \text{Im}(i_c^*) \text{ for all } W' \in \mathcal{P}_r(W) \text{ and all } \chi' \in \text{Irr}(W').$$

But it turns out that the coefficient of \hbar^r in

$$\frac{\chi'(1)}{|W'|} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi'))$$

is equal, according to Corollary 4.3, to

$$\frac{\chi'(1)}{|W'|} \sum_{\chi \in \text{Irr}(W)} \frac{|W| \langle \chi, \text{Ind}_{W'}^W \chi' \rangle_W}{\chi(1)} e_\chi^W$$

(since $d_1 \cdots d_n = |W|$). Note that by definition $\frac{\chi'(1)}{|W'|} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi'))$ belongs to $\mathbb{C}[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \text{Im}(i_c^*)$, so each of its homogeneous components belong to $\text{Im}(i_c^*)$. By (4.4), this proves that (\star) holds, and the proof of Theorem A is complete. \square

5. Proof of Theorem B

Hypothesis and notation. *We assume in this section, and only in this section, that the symplectic quotient singularity $\mathcal{X}_0 = (V \times V^*)/W$ admits a symplectic resolution $\mathcal{X} \rightarrow \mathcal{X}_0$.*

Recall [12, Theorem 1.3(ii)] that the \mathbb{C}^\times -action on \mathcal{X}_0 lifts uniquely to \mathcal{X} . As it is explained in Remark 1.4, the existence of a symplectic resolution of \mathcal{X}_0 implies that all the irreducible components of W are of type $G(d, 1, n)$ or G_4 . Since the proof of Theorem B can be easily reduced to the irreducible case, we will separate the proof in two cases.

Proof of Theorem B for $W = G(d, 1, n)$. — Assume here that $W = G(d, 1, n)$. Let \mathbf{S}^1 be the group of complex numbers of modulus 1. In this case, it follows from [8] that \mathcal{X} is diffeomorphic to some smooth \mathcal{X}_c , and that the diffeomorphism might be chosen to be \mathbf{S}^1 -equivariant. As the \mathbf{S}^1 -equivariant cohomology is canonically isomorphic to the \mathbb{C}^\times -equivariant cohomology, this proves Theorem B in this case. \square

Proof of Theorem B for $W = G_4$. — Assume here that $W = G_4$. It is possible (probable?) that again \mathcal{X} is \mathbf{S}^1 -diffeomorphic to some smooth \mathcal{X}_c , but we are unable to prove it (it is only known that they are diffeomorphic). So we will prove Theorem B in this case by brute force computations.

We fix a primitive third root of unity ζ and we assume that $V = \mathbb{C}^2$ and that $W = \langle s, t \rangle$, where

$$s = \begin{pmatrix} \zeta & 0 \\ \zeta^2 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\zeta^2 \\ 0 & \zeta \end{pmatrix}.$$

By the work of Bellamy [2], there are only two symplectic resolutions of $\mathcal{X}_0 = (V \times V^*)/W$. They have both been constructed by Lehn and Sorger [13]: one can be obtained from the other by exchanging the role of V and V^* , so we will only prove Theorem B for one of them. Let us describe it.

Let $H = V^s$ and let \mathcal{H} denote the image of $H \times V^*$ in \mathcal{X}_0 , with its reduced structure of closed subvariety. We denote by $\beta : \mathcal{Y} \rightarrow \mathcal{X}_0$ the blowing-up of \mathcal{X}_0 along \mathcal{H} and we denote by $\alpha : \mathcal{X} \rightarrow \mathcal{Y}$ the blowing-up of \mathcal{Y} along its reduced singular locus \mathcal{S} . Then [13]

$$\pi = \beta \circ \alpha : \mathcal{X} \longrightarrow \mathcal{X}_0$$

is a symplectic resolution.

We now give more details, which all can be found in [13, §1]. First, $\mathbb{C}[\mathcal{X}_0]$ is generated by 8 homogeneous elements $(z_i)_{1 \leq i \leq 8}$ whose degrees are given by the following table:

(5.1)

z	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
$\deg(z)$	0	4	-4	2	-2	-6	6	0

The defining ideal of \mathcal{H} in $\mathbb{C}[\mathcal{X}_0]$ is generated by 6 homogeneous elements $(b_j)_{1 \leq j \leq 6}$ whose degrees are given by the following table:

(5.2)

b	b_1	b_2	b_3	b_4	b_5	b_6
$\deg(b)$	2	6	0	12	8	4

This defines a \mathbb{C}^\times -equivariant closed immersion $\mathcal{Y} \hookrightarrow \mathcal{X}_0 \times \mathbf{P}^5(\mathbb{C})$. We denote by \mathcal{Y}_i the affine chart defined by “ $b_i \neq 0$ ”. The equations of the zero fiber $\beta^{-1}(0)$ given in [13, §1] show that $\mathcal{Y}^{\mathbb{C}^\times} = \{p_2, p_3, p_4, p_6\}$, where p_i is the unique element of $\mathcal{Y}_i^{\mathbb{C}^\times}$. We use the notation of Example 2.5. By (2.6) and (5.2), we have

$$(\clubsuit) \quad i_{\mathcal{Y}}^*(\mathrm{ch}_{\mathcal{Y}}^1([\beta^* \mathcal{H}])) = -\hbar(6\mathbf{e}_{p_2} + 12\mathbf{e}_{p_4} + 4\mathbf{e}_{p_6}).$$

Now, \mathcal{S} is contained in $\mathcal{Y}_2 \cup \mathcal{Y}_3$, so α is an isomorphism in a neighborhood of p_4 and p_6 . So let $q_4 = \alpha^{-1}(p_4)$ and $q_6 = \alpha^{-1}(p_6)$. These are elements of $\mathcal{X}^{\mathbb{C}^\times}$.

On the other hand, \mathcal{Y}_2 is a transversal A_1 -singularity, so the defining ideal of $\mathcal{S} \cap \mathcal{Y}_2$ in $\mathbb{C}[\mathcal{Y}_2]$ is generated by three homogeneous elements a_+ , a_0 and a_- (of degree 6, 0 and -6 , by [13, §1]), and it is easily checked that $\alpha^{-1}(p_2)^{\mathbb{C}^\times} = \{q_2^+, q_2^-\}$ where q_2^\pm is the unique \mathbb{C}^\times -fixed element in the affine chart defined by “ $a_\pm \neq 0$ ”.

Also, \mathcal{Y}_3 is isomorphic to $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$, where \mathfrak{h} is the diagonal Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ and \mathfrak{S}_3 is the symmetric group on 3 letters, viewed as the Weyl group of $\mathfrak{sl}_3(\mathbb{C})$. Let $\mathbf{Hilb}_3(\mathbb{C}^2)$ denote the Hilbert scheme of 3 points in \mathbb{C}^2 , and let $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ denote the (reduced) closed subscheme defined as the Hilbert scheme of three points in \mathbb{C}^2 whose sum is equal to $(0, 0)$. By [11, Proposition 2.6],

$$(5.3) \quad \mathcal{X}_3 \simeq \mathbf{Hilb}_3^0(\mathbb{C}^2).$$

It just might be noticed that the isomorphism $\mathcal{Y}_3 \simeq (\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$ becomes \mathbb{C}^\times -equivariant if one “doubles the degrees” in $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$, that is, if \mathbb{C}^\times acts on \mathfrak{h} (respectively \mathfrak{h}^*) with weight 2 (respectively -2). Recall that \mathbb{C}^\times -fixed points in $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ are parametrized by partitions of 3: we denote by q_3^+ , q_3^0 and q_3^- the fixed points in \mathcal{X}_3 corresponding respectively to the partitions (3), (2, 1) and (1, 1, 1) of 3, so that

$$(5.4) \quad \mathcal{X}_3^{\mathbb{C}^\times} = \{q_3^+, q_3^0, q_3^-\}.$$

Finally, we have

$$(5.5) \quad \mathcal{X}^{\mathbb{C}^\times} = \{q_2^+, q_2^-, q_3^+, q_3^0, q_3^-, q_4, q_6\}.$$

Also, $\pi^*(\mathcal{H}) = \alpha^*(\beta^*(\mathcal{H}))$ is an effective Cartier divisor of \mathcal{X} , and it follows from (\clubsuit) and the commutativity of the diagram 2.3 that

$$(\diamond) \quad i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\pi^* \mathcal{H}])) = -\hbar(6\mathbf{e}_{q_2^+} + 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_4} + 4\mathbf{e}_{q_6}).$$

We now wish to compute $i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}]))$. As the singular locus \mathcal{S} is contained in $\mathcal{Y}_2 \cup \mathcal{Y}_3$, and contains p_2 and p_3 , there exists n_2^+ , n_2^- , n_3^+ , n_3^0 and n_3^- in \mathbb{Z} such that

$$i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}])) = -\hbar(n_2^+ \mathbf{e}_{q_2^+} + n_2^- \mathbf{e}_{q_2^-} + n_3^+ \mathbf{e}_{q_3^+} + n_3^0 \mathbf{e}_{q_3^0} + n_3^- \mathbf{e}_{q_3^-}).$$

Since a_+ and a_- have degree 6 and -6 , it follows from (2.6) that $n_2^+ = 6$ and $n_2^- = -6$.

As we can exchange the roles of \mathfrak{h} and \mathfrak{h}^* in the description of \mathcal{Y}_3 and its singular locus (because $\mathfrak{h} \simeq \mathfrak{h}^*$ as an \mathfrak{S}_3 -module), this shows that $n_3^- = -n_3^+$ and $n_3^0 = -n_3^0$. So $n_3^0 = 0$ and it remains to compute n_3^+ . So let U_+ be the open subset of $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ consisting of ideals J of codimension 3 of $\mathbb{C}[x, y]$ such that the classes of 1, x and x^2 form a basis of $\mathbb{C}[x, y]/J$. Then we have an isomorphism $J_+ : \mathbb{C}^4 \xrightarrow{\sim} U_+$ given by

$$(a, b, c, d) \longmapsto J_+(a, b, c, d) = \langle x^3 + ax + b, y - cx^2 - dx - \frac{2}{3}ac \rangle.$$

The form of the generators of the ideal $J_+(a, b, c, d)$ is here to ensure that $J_+(a, b, c, d) \in \mathbf{Hilb}_3^0(\mathbb{C}^2)$. The fixed point q_3^+ is the unique one in U_+ . Through this identification, the action of \mathbb{C}^\times on \mathbb{C}^4 is given by

$$\xi \cdot (a, b, c, d) = (\xi^4 a, \xi^6 b, \xi^{-6} c, \xi^{-4} d)$$

(remember that we must “double the degrees” of the usual action). The equation of $\alpha^*(\mathcal{S})$ on this affine chart $\simeq \mathbb{C}^4$ can then be computed explicitly and is given by $4a^3 + 27b^2 = 0$, so is of degree 12. This shows that $n_3^+ = 12$. Finally,

$$(\heartsuit) \quad i_{\mathcal{X}}^*(\text{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}])) = -\hbar(6\mathbf{e}_{q_2^+} - 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_3^+} - 12\mathbf{e}_{q_3^-}).$$

Let us now conclude. First, recall from [6, Theorem 1.2] that

$$(5.6) \quad \begin{cases} H^{2i+1}(\mathcal{X}) = 0 & \text{for all } i, \\ H^{2\bullet}(\mathcal{X}) \simeq \text{gr}_{\mathcal{F}}(Z(\mathbb{C}W)). \end{cases}$$

By Proposition 2.4, we get

$$(\spadesuit) \quad \dim_{\mathbb{C}} H_{\mathbb{C}^\times}^{2i}(\mathcal{X}) = \begin{cases} 1 & \text{if } i = 0, \\ 3 & \text{if } i = 1, \\ 7 & \text{if } i \geq 2. \end{cases}$$

Now, W has seven irreducible characters $1, \varepsilon, \varepsilon^2, \chi, \chi\varepsilon, \chi\varepsilon^2$ and θ , where χ is the unique irreducible character of degree 2 with rational values and θ is the unique one of degree 3. We denote by

$$\Psi : H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}^{\mathbb{C}^\times}) \xrightarrow{\sim} \mathbb{C}[[\hbar]] \otimes Z(\mathbb{C}W)$$

the isomorphism of $\mathbb{C}[[\hbar]]$ -algebras such that

$$\Psi(\mathbf{e}_{q_4}) = e_1^W, \quad \Psi(\mathbf{e}_{q_6}) = e_\theta^W, \quad \Psi(\mathbf{e}_{q_2^+}) = e_{\chi\varepsilon}^W, \quad \Psi(\mathbf{e}_{q_2^-}) = e_{\chi\varepsilon^2}^W,$$

$$\Psi(\mathbf{e}_{q_3^+}) = e_{\varepsilon^2}^W, \quad \Psi(\mathbf{e}_{q_3^-}) = e_\chi^W \quad \text{and} \quad \Psi(\mathbf{e}_{q_3^-}) = e_\varepsilon^W.$$

By Proposition 2.4, $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X})$ is isomorphic to its image by $\Psi \circ i_{\mathcal{X}}^*$ in $\mathbb{C}[[\hbar]] \otimes Z(\mathbb{C}W)$. But, by (\diamond) and (\heartsuit) , and after investigation of the character table of W , this image contains

$$\hbar(6\mathbf{e}_{\chi\varepsilon}^W - 6\mathbf{e}_{\chi\varepsilon^2}^W + 12\mathbf{e}_1^W + 4\mathbf{e}_\theta^W) = \hbar(4 + \Sigma_W(s) + \Sigma_W(s^2))$$

$$\text{and} \quad \hbar(12\mathbf{e}_{\varepsilon^2}^W - 12\mathbf{e}_\varepsilon^W + 6\mathbf{e}_{\chi\varepsilon} - 6\mathbf{e}_{\chi\varepsilon^2}) = \hbar((1 + 2\zeta)\Sigma_W(s) + (1 + 2\zeta^2)\Sigma_W(s^2)).$$

So it contains $\hbar\Sigma_W(s)$ and $\hbar\Sigma_W(s^2)$. Also, by (\spadesuit) and Proposition 2.4, it also contains $\hbar^2 Z(\mathbb{C}W)$, so

$$\text{Rees}_{\mathcal{F}}^\bullet(Z(\mathbb{C}W)) \subset \text{Im}(\Psi \circ i_{\mathcal{X}}^*).$$

Using again (\spadesuit) and Proposition 2.4, a comparison of dimensions yields that

$$\text{Rees}_{\mathcal{F}}^\bullet(Z(\mathbb{C}W)) = \text{Im}(\Psi \circ i_{\mathcal{X}}^*),$$

and the proof is complete. \square

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