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AN ASYMPOTIC CELL CATEGORY FOR CYCLIC GROUPS

by

Cédric Bonnafé & Raphaël Rouquier

In **[Ma]**, Malle has associated to any spetsial imprimitive complex reflection group *W* a set of *unipotent characters* (which is in natural bijection with the set of *unipotent characters* of the associated finite reductive group whenever *W* is a Weyl group). In this generalization of Lusztig's work, he also obtained a partition of this set into *families* and, to each family, he associated a \mathbb{Z} -fusion datum: by a \mathbb{Z} -fusion datum, we mean that we have all the axioms of a classical fusion datum (which will be called a \mathbb{Z}_+ -*fusion datum* in the present paper) except that the structure constants of the associated fusion ring might be negative.

It is a classical problem to find a categorification of a \mathbb{Z}_+ -fusion datum by a tensor category with suitable extra-structures (pivot, twist). The aim of this paper is to provide an *ad hoc* categorification of the \mathbb{Z} -fusion datum associated with the non-trivial family of the cyclic complex reflection group of order *d*: it is provided by the stable category of the Drinfeld double of the *Taft algebra* of dimension d^2 . It must be said that we have no theoretical explanation for this fact.

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1. The Drinfeld double of the Taft algebra

From now on, \otimes will denote the tensor product $\otimes_{\mathbb{C}}$. We fix a natural number $d \ge 2$ as well as a primitive *d*-th root of unity $\zeta \in \mathbb{C}^{\times}$. We denote by $\mu_d = \langle \zeta \rangle$ the group of *d*-th roots of unity. If $i \in \mathbb{Z}/d\mathbb{Z}$ and α is an element of a group such that $\alpha^d = 1$, we will denote by α^i the element $\alpha^{i'}$, where i' is any representative of *i*.

If $n \ge 1$ is a natural number and $\xi \in \mathbb{C}$, we set

$$(n)_{\xi} = 1 + \xi + \dots + \xi^{n-1}$$

 $(n)!_{\xi} = \prod_{i=1}^{n} (i)_{\xi}.$

and

We also set
$$(0)!_{\xi} = 1$$
.

1.A. The Taft algebra. — We denote by *B* the \mathbb{C} -algebra admitting the following presentation:

• Generators: K, E.

• Relations:
$$\begin{cases} K = 1, \\ E^d = 0, \\ KE = \zeta E K. \end{cases}$$

It follows from [Ka, Proposition IX.6.1] that:

(Δ) There exists a unique morphism of algebras Δ : $B \rightarrow B \otimes B$ such that

$$\Delta(K) = K \otimes K$$
 and $\Delta(E) = (1 \otimes E) + (E \otimes K).$

(ε) There exists a unique morphism of algebras $\varepsilon : B \to \mathbb{C}$ such that

$$\varepsilon(K) = 1$$
 and $\varepsilon(E) = 0$.

(S) There exists a unique anti-automorphism S of B such that

$$S(K) = K^{-1}$$
 and $S(E) = -EK^{-1}$.

With Δ as a coproduct, ε as a counit and *S* as an (invertible) antipode, *B* becomes a Hopf algebra, called the *Taft algebra* [**EGNO**, Example 5.5.6]. It is easily checked that

(1.1)
$$B = \bigoplus_{i,j=0}^{d-1} \mathbb{C}K^i E^j = \bigoplus_{i,j=0}^{d-1} \mathbb{C}E^i K^j.$$

1.B. Dual algebra. — Let *K*^{*} and *E*^{*} denote the elements of *B*^{*} such that

$$K^*(E^iK^j) = \delta_{i,0}\zeta^j$$
 and $E^*(E^iK^j) = \delta_{i,1}$.

Recall that *B*^{*} is a Hopf algebra [**Ka**, Proposition III.3.3] and it follows from [**Ka**, Lemma IX.6.3] that

(1.2)
$$(E^{*i}K^{*j})(E^{i'}K^{j'}) = \delta_{i,i'}(i)!_{\zeta}\zeta^{j(i+j')}.$$

We deduce easily that $(E^{*i}K^{*j})_{0 \le i,j \le d-1}$ is a \mathbb{C} -basis of B^* .

We will give explicit formulas for the coproduct, the counit and the antipode in the next subsection. We will in fact use the Hopf algebra $(B^*)^{cop}$, which is the Hopf algebra whose underlying space is B^* , whose product is the same as in B^* and whose coproduct is opposite to the one in B^* .

1.C. Drinfeld double. — We denote by D(B) the *Drinfeld quantum double* of *B*, as defined for instance in [**Ka**, Definition IX.4.1] or [**EGNO**, Definition 7.14.1]. Recall that D(B) contains *B* and $(B^*)^{cop}$ as Hopf subalgebras and that the multiplication induces an isomorphism of vector spaces $(B^*)^{cop} \otimes B \xrightarrow{\sim} D(B)$. A presentation of D(B), with generators *E*, E^* , *K*, K^* is given for instance in [**Ka**, Proposition IX.6.4]. We shall slightly modify it by setting

 $z = K^{*-1}K$ and $F = \zeta E^*K^{*-1}$.

Then [Ka, Proposition IX.6.4] can we rewritten as follows:

Proposition 1.3. — The \mathbb{C} -algebra D(B) admits the following presentation:

• Generators: E, F, K, z;
• Relations:
$$\begin{cases}
K^{d} = z^{d} = 1, \\
E^{d} = F^{d} = 0, \\
[z, E] = [z, F] = [z, K] = 0, \\
KE = \zeta E K, \\
KF = \zeta^{-1}F K, \\
[E, F] = K - zK^{-1}.
\end{cases}$$

The next corollary follows from an easy induction argument:

Corollary 1.4. — If $i \ge 1$, then $[E, F^{i}] = (i)_{\zeta} F^{i-1}(\zeta^{1-i}K - zK^{-1})$ and $[F, E^{i}] = (i)_{\zeta} E^{i-1}(\zeta^{1-i}zK^{-1} - K).$ The algebra D(B) is endowed with a structure of Hopf algebra, where the comultiplication, the counit and the antipode are still denoted by Δ , ε and S respectively (as they extend the corresponding objects for B). We have [**Ka**, Proposition IX.6.2]:

(1.5)
$$\begin{cases} \Delta(K) = K \otimes K, \\ \Delta(z) = z \otimes z, \\ \Delta(E) = (1 \otimes E) + (E \otimes K), \\ \Delta(F) = (F \otimes 1) + (zK^{-1} \otimes F), \end{cases} \begin{cases} S(K) = K^{-1}, \\ S(z) = z^{-1}, \\ S(E) = -EK^{-1}, \\ S(F) = -\zeta^{-1}FKz^{-1}, \end{cases}$$

(1.6)
$$\varepsilon(K) = \varepsilon(z) = 1 \quad \text{and} \quad \varepsilon(E) = \varepsilon(F) = 0. \end{cases}$$

1.D. Morphisms to \mathbb{C} . — If $\xi \in \mu_d$, we denote by $\varepsilon_{\xi} : D(B) \to \mathbb{C}$ the unique morphism of algebras such that

$$\varepsilon_{\xi}(K) = \xi, \quad \varepsilon_{\xi}(z) = \xi^2 \text{ and } \varepsilon_{\xi}(E) = \varepsilon_{\xi}(F) = 0.$$

It is easily checked that the ε_{ξ} 's are the only morphisms of algebras $D(B) \rightarrow \mathbb{C}$. Note that $\varepsilon_1 = \varepsilon$ is the counit.

1.E. Group-like elements. — It follows from (1.5) that *K* and *z* are group-like, so that $K^i z^j$ is group-like for all $i, j \in \mathbb{Z}$. The converse also holds (and is certainly already well-known):

Lemma 1.7. — If $g \in D(B)$ is group-like, then there exist $i, j \in \mathbb{Z}$ such that $g = K^i z^j$.

Proof. — Let $g \in D(B)$ be a group-like element. Let us write

$$g = \sum_{i,j,k,l=0}^{a-1} \alpha_{i,j,k,l} K^i z^j E^k F^l.$$

We denote by (k_0, l_0) the biggest pair (for the lexicographic order) such that there exist $i, j \in \{0, 1, ..., d-1\}$ such that $a_{i,j,k_0,l_0} \neq 0$. Then the coefficient of $K^i z^j E^{k_0} F^{l_0} \otimes K^i z^j E^{k_0} F^{l_0}$ in $g \otimes g$ is equal to a_{i,j,k_0,l_0}^2 , so it is different from 0.

But, if we compute the coefficient of $K^i z^j E^{k_0} F^{l_0} \otimes K^i z^j E^{k_0} F^{l_0}$ in

$$g \otimes g = \Delta(g) = \sum_{i,j,k,l=0}^{d-1} \alpha_{i,j,k,l} \Delta(K)^i \Delta(z)^j \Delta(E)^k \Delta(F)^l$$

using the formulas (1.5), we see that it is equal to 0 if $(k_0, l_0) \neq (0, 0)$. Therefore $(k_0, l_0) = (0, 0)$, and so g belongs to the linear span of the family $(K^i z^j)_{i,j \in \mathbb{Z}}$.

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Now the result follows from the linear independence of group-like elements. $\hfill \Box$

1.F. Braiding. — For $0 \le i, j \le d-1$, we set

$$\beta_{i,j} = \frac{E^{*i}}{d \cdot (i)!_{\zeta}} \sum_{k=0}^{d-1} \zeta^{-k(i+j)} K^{*k}.$$

It follows from (1.2) that $(\beta_{i,j})_{0 \le i,j \le d-1}$ is a dual basis to $(E^i K^j)_{0 \le i,j \le d-1}$. We then set

$$R = \sum_{i,j=0}^{d-1} E^i K^j \otimes \beta_{i,j} \in D(B) \otimes D(B).$$

Then *R* is a universal *R*-matrix for D(B) which endows D(B) with a structure of braided Hopf algebra [**Ka**, Theorem IX.4.4]). Using our generators *E*, *F*, *K*, *z*, we have:

(1.8)
$$R = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_{\zeta}} E^i K^j \otimes z^{-k} F^i K^k.$$

1.G. Twist. — Let us define

$$\begin{aligned} \tau : \quad D(B) \otimes D(B) & \longrightarrow \quad D(B) \otimes D(B) \\ a \otimes b & \longmapsto \quad b \otimes a. \end{aligned}$$

Following [Ka, §VIII.4], we set

$$u = \sum_{i,j=0}^{d-1} S(\beta_{ij}) E^i K^j \in D(B).$$

Recall that u is called the *Drinfeld element* of D(B). It satisfies several properties (see for instance [Ka, Proposition VIII.4.5]). For instance, u is invertible and we will recall only three equalities:

(1.9)
$$\varepsilon(u) = 1, \quad \Delta(u) = (\tau(R)R)^{-1}(u \otimes u) \text{ and } S^2(b) = u b u^{-1}$$

for all $b \in D(B)$. A straightforward computation shows that

(1.10)
$$S^2(b) = K b K^{-1}$$

for all $b \in D(B)$. We now set

 $\theta = K^{-1}u.$

Then it follows from (1.9) and (1.10) that:

Proposition 1.11. — The element θ is central and invertible in D(B) and satisfies $\varepsilon(\theta) = 1$ and $\Delta(\theta) = (\tau(R)R)^{-1}(\theta \otimes \theta).$

Let us give a formula for θ :

(1.12)
$$\theta = \frac{1}{d} \sum_{i,j,k=0}^{d-1} (-1)^{i} \frac{\zeta^{(i-k)(i+j)-i}}{(i)!_{\zeta}} z^{k-i} F^{i} E^{i} K^{i+j-k-1}$$

Corollary 1.13. — $S(\theta) = z \theta$.

Proof. — Let $g = S(\theta)\theta^{-1}$. Since $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ and $(S \otimes S)(R) = R$ (see for instance [**Ka**, Theorems III.3.4 and VIII.2.4], it follows from Proposition 1.11 that *g* is central and group-like. Hence, by Lemma 1.7, there exists $l \in \mathbb{Z}$ such that $S(\theta) = \theta z^l$. So, by (1.12), we have

(
$$\sharp$$
) $S(\theta)E^{d-1} = \theta z^{l}E^{d-1} = \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}}^{d-1} \zeta^{-jk} z^{k+l} K^{j-k-1} E^{d-1}.$

Let us now compute $S(\theta)E^{d-1}$ by using directly (1.12). We get

$$S(\theta)E^{d-1} = E^{d-1}S(\theta) = \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-jk} E^{d-1} z^{-k} K^{1+k-j}$$
$$= \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-jk} \zeta^{1+k-j} z^{-k} K^{1+k-j} E^{d-1}$$
$$= \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{(1-j)(1+k)} z^{-k} K^{1+k-j} E^{d-1}.$$

So, if we set j' = 1 - j and k' = -1 - k, we get

$$S(\theta)E^{d-1} = \frac{1}{d} \sum_{j',k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-j'k'} z^{k'+1} K^{j'-k'-1} E^{d-1}.$$

Comparing with (\sharp), we get that $z^{l} = z$.

2. D(B)-modules

Most of the result of this section are due to Chen **[Ch1]** or Edrmann, Green, Snashall and Taillefer **[EGST1]**, **[EGST2]**. By a D(B)-module, we mean a finite dimensional left D(B)-module. We denote by D(B)-mod the category of (finite dimensional left) D(B)-modules. If $\alpha_1, \ldots, \alpha_{l-1} \in \mathbb{C}$, we set

$$J_{l}^{+}(\alpha_{1},...,\alpha_{l-1}) = \begin{pmatrix} 0 & \alpha_{1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & 0 & \alpha_{l-1} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$
$$J_{l}^{-}(\alpha_{1},...,\alpha_{l-1}) = {}^{t}J_{l}^{+}(\alpha_{1},...,\alpha_{l-1}).$$

and

If *M* is a *D*(*B*)-module and $b \in D(B)$, we denote by $b|_M$ the endomorphism of *M* induced by *b*. For instance, $E|_M$ and $F|_M$ are nilpotent and $K|_M$ and $z|_M$ are semisimple.

2.A. Simple modules. — If $1 \le l \le d$ and $p \in \mathbb{Z}/d\mathbb{Z}$, we denote by $M_{l,p}$ the D(B)-module with \mathbb{C} -basis $\mathscr{M}^{(l,p)} = (e_i^{(l,p)})_{1 \le i \le l}$ and such that the action of z, K, E and F in the basis $\mathscr{M}^{(l,p)}$ are given by the following matrices:

$$\begin{split} z|_{M_{l,p}} &= \zeta^{2p+l-1} \operatorname{Id}_{M_{l,p}}, \\ K|_{M_{l,p}} &= \zeta^p \operatorname{diag}(\zeta^{l-1}, \zeta^{l-2}, \dots, \zeta, 1), \\ E|_{M_{l,p}} &= \zeta^p J_l^+((1)_{\zeta}(\zeta^{l-1}-1), (2)_{\zeta}(\zeta^{l-2}-1), \dots, (l-1)_{\zeta}(\zeta-1)), \\ F|_{M_{l,p}} &= J_l^-(1, \dots, 1). \end{split}$$

It is readily checked from the relations given in Proposition 1.3 that this defines a D(B)-module of dimension l. The next result is proved in [Ch1, Theorem 2.5].

Theorem 2.1 (Chen). — The map

$$\begin{array}{cccc} \{1,2,\ldots,d\}\times \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \operatorname{Irr}(D(B))\\ & & (l,p) & \longmapsto & M_{l,p} \end{array}$$

is bijective.

2.B. Blocks. — We denote by $\Lambda(d)$ the set $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Then $\Lambda(d)$ is in canonical bijection with the set $\{1, 2, ..., d\} \times \mathbb{Z}/d\mathbb{Z}$, which parametrizes the simple D(B)-modules. So if $\lambda \in \Lambda(d)$, we will denote by M_{λ} the corresponding simple D(B)-module. We also set $(\mathbb{Z}/d\mathbb{Z})^{\#} = (\mathbb{Z}/d\mathbb{Z}) \setminus \{0\}$ and $\Lambda^{\#}(d) = (\mathbb{Z}/d\mathbb{Z})^{\#} \times \mathbb{Z}/d\mathbb{Z}$. Finally, let $\Lambda^{0}(d) = \{0\} \times \mathbb{Z}/d\mathbb{Z}$ be the complement of $\Lambda^{\#}(d)$ in $\Lambda(d)$. Let

$$\begin{array}{cccc} \iota : & \Lambda(d) & \longrightarrow & \Lambda(d) \\ & (l,p) & \longmapsto & (-l,p+l). \end{array}$$

Then $\iota^2 = \text{Id}_{\Lambda(d)}$ and $\Lambda^0(d)$ is the set of fixed points of ι . If \mathscr{L} is a ι -stable subset of $\Lambda(d)$, we denote by $[\mathscr{L}/\iota]$ a set of representatives of ι -orbits in \mathscr{L} . The next result is proved in [**EGST1**, Theorem 2.26].

Theorem 2.2 (Erdmann-Green-Snashall-Taillefer). — Let λ , $\lambda' \in \Lambda(d)$. Then M_{λ} and $M_{\lambda'}$ belong to the same block of D(B) if and only if λ and λ' are in the same *ι*-orbit.

We have constructed in §1.G a central element, namely θ . Note that

(2.3) The element θ acts on $M_{l,p}$ by multiplication by $\zeta^{(p-1)(l+p-1)}$.

Proof. — It is sufficient to compute the action of θ on $e_1^{(l,p)}$. Note that $E^i e_1^{(l,p)} = 0$ as soon as $i \ge 1$. Therefore, for computing $\theta e_1^{(l,p)}$ using the formula (1.12), only the terms corresponding to i = 0 remain. Consequently,

$$\begin{split} \omega_{l,p}(\theta) &= \frac{1}{d} \sum_{j,k=0}^{d-1} \zeta^{-jk} \zeta^{(2p+l-1)k} \zeta^{(p+l-1)(j-k-1)} \\ &= \frac{\zeta^{1-l-p}}{d} \sum_{k=0}^{d-1} \zeta^{pk} \Big(\sum_{j=0}^{d-1} \zeta^{(p+l-1-k)j} \Big) \end{split}$$

The term inside the big parenthesis is equal to *d* if $p+l-1-k \equiv 0 \mod d$, and is equal to 0 otherwise. The result follows.

2.C. Projective modules. — If $\lambda \in \Lambda(d)$, we denote by P_{λ} an indecomposable projective cover of M_{λ} . The next result is proved in [EGST1, Corollary 2.25].

Theorem **2.4** (Erdmann-Green-Snashall-Taillefer). — Let $\lambda \in \Lambda(d)$. Then:

(a) If $\lambda \in \Lambda^{\#}(d)$, then $\dim_{\mathbb{C}}(P_{\lambda}) = 2d$, $\operatorname{Rad}^{3}(P_{\lambda}) = 0$ and the Loewy structure of P_{λ} is given by:

$$\begin{array}{rcl} P_{\lambda} / \operatorname{Rad}(P_{\lambda}) &\simeq & M_{\lambda} \\ \operatorname{Rad}(P_{\lambda}) / \operatorname{Rad}^{2}(P_{\lambda}) &\simeq & M_{\iota(\lambda)} \oplus M_{\iota(\lambda)} \\ \operatorname{Rad}^{2}(P_{\lambda}) &\simeq & M_{\lambda} \end{array}$$

(b) $P_{d,p} = M_{d,p}$ has dimension d.

3. Tensor structure

We mainly refer here to the work of Erdmann, Green, Snashall and Taillefer [EGST1], [EGST2]. Since D(B) is a finite dimensional Hopf algebra, the category D(B)-mod inherits a structure of a tensor category. We will compute here some tensor products between simple modules. For simplifying, we will denote by M_l the simple module $M_{l,0}$.

3.A. Invertible modules. — We denote by $V_{\xi} = \mathbb{C}v_{\xi}$ the one-dimensional D(B)-module associated with the morphism $\varepsilon_{\xi} : D(B) \rightarrow \mathbb{C}$ defined in §1.D:

$$b v_{\xi} = \varepsilon_{\xi}(b) v_{\xi}$$

for all $b \in D(B)$. We have

 $(3.1) V_{\zeta^p} \simeq M_{1,p}.$

An immediate computation using the comultiplication Δ shows that

$$(3.2) M_{l,p} \otimes V_{\zeta q} \simeq V_{\zeta q} \otimes M_{l,p} \simeq M_{l,p+q}$$

as D(B)-modules. The V_{ξ} 's are (up to isomorphism) the only invertible objets in the tensor category D(B)-mod.

3.B. Tensor product with M_2 . — For simplifying, we set $e_i = e_i^{(2,0)}$ for $i \in \{1,2\}$, so that (e_1, e_2) is the standard basis of M_2 . The next result is a particular case of [EGST1, Theorem 4.1].

Theorem 3.3 (Edrmann-Green-Snashall-Taillefer). — Let $\lambda \in \Lambda(d)$ and let (l, p) be a representative of λ in $\{1, 2, ..., d\} \times \mathbb{Z}/d\mathbb{Z}$. Then:

- (a) If $l \leq d-1$ (i.e. if $\lambda \in \Lambda^{\#}(d)$), then $M_2 \otimes M_{l,p} \simeq M_{l+1,p} \oplus M_{l-1,p+1}$.
- (b) $M_2 \otimes M_{d,p} \simeq P_{d-1,p}$.

4. Grothendieck rings

We denote by Gr(D(B)) the Grothendieck ring of the category of (left) D(B)-modules.

4.A. Structure. — Since D(B) is a braided Hopf algebra (with universal *R*-matrix *R*),

(4.1) The ring
$$Gr(D(B))$$
 is commutative.

If *M* is a D(B)-module, we denote by [M] the class of *M* in Gr(D(B)). We set

$$\mathbf{m}_{\lambda} = [M_{\lambda}], \quad \mathbf{m}_{l} = [M_{l,0}] \quad \text{and} \quad \mathbf{v}_{\xi} = [V_{\xi}] \in \operatorname{Gr}(D(B)),$$

Recall that $\mathbf{v}_{\zeta p} = \mathbf{m}_{1,p}$. Then it follows from (3.2) and Theorem 3.3 that

(4.2)
$$\mathbf{v}_{\zeta q} \mathbf{m}_{l,p} = \mathbf{m}_{l,p+q}$$
 and $\mathbf{m}_2 \mathbf{m}_{l,p} = \begin{cases} \mathbf{m}_{l+1,p} + \mathbf{m}_{l-1,p+1} & \text{if } l \leq d-1, \\ 2(\mathbf{m}_{d-1,p} + \mathbf{m}_{1,p-1}) & \text{if } l = d. \end{cases}$

Proposition 4.3. — The Grothendieck ring Gr(D(B)) is generated by \mathbf{v}_{ζ} and \mathbf{m}_2 .

Proof. — We will prove by induction on *l* that $\mathbf{m}_{l,p} \in \mathbb{Z}[\mathbf{v}_{\zeta}, \mathbf{m}_2]$. Since $\mathbf{m}_{1,p} = (\mathbf{v}_{\zeta})^p$, this is true for l = 1. Since $\mathbf{m}_{2,p} = (\mathbf{v}_{\zeta})^p \mathbf{m}_2$, this is also true for l = 2. Now the induction proceeds easily by using (4.2).

4.B. Some characters. — If $b \in D(B)$ is group-like, then the map

$$\begin{array}{ccc} \operatorname{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ [M] & \longmapsto & \operatorname{Tr}(b|_M) \end{array}$$

is a morphism of rings. Here, **Tr** denotes the usual trace (not the quantum trace) of an endomorphism of a finite dimensional vector space. Recall from Lemma 1.7 that the only group-like elements of D(B) are the $K^i z^j$, where $(i, j) \in \Lambda(d)$. We set

$$\begin{array}{cccc} \chi_{i,j} \colon & \operatorname{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & & [M] & \longmapsto & \operatorname{Tr}(K^i z^j|_M). \end{array}$$

An easy computation yields

(4.4)
$$\chi_{i,j}(\mathbf{m}_{l,p}) = \zeta^{pi+(2p+l-1)j} \cdot (l)_{\zeta^i}$$

Note that the $\chi'_{i,i}s$ are not necessarily distinct:

Lemma 4.5. — Let λ and λ' be two elements of $\Lambda(d)$. Then $\chi_{\lambda} = \chi_{\lambda'}$ if and only if λ and λ' are in the same ι -orbit.

Proof. — Let us write $\lambda = (i, j)$ and $\lambda' = (i', j')$. The "if" part follows directly from (4.4). Conversely, assume that $\chi_{i,j} = \chi_{i',j'}$. By applying these two characters to \mathbf{v}_{ζ} and \mathbf{m}_{2} , we get:

$$\begin{cases} \zeta^{i+2j} = \zeta^{i'+2j'}, \\ \zeta^{j}(1+\zeta^{i}) = \zeta^{j'}(1+\zeta^{i'}). \end{cases}$$

So the result follows by applying exactly the same argument as in the proof of Theorem 2.2. $\hfill \Box$

4.C. Stable category. — As *B* is a Hopf algebra, it is Frobenius and so its stable category *B* st (namely the quotient of *B*-mod by the full subcategory *B*-proj consisting of projective objects) is triangulated. Similarly, the category D(B)-stab is triangulated. Note also that a *B*-module (resp. a D(B)-module) is projective if and only if it is injective. Since the tensor product of a projective D(B)-module by any D(B)-module is still projective [**EGNO**, Proposition 4.2.12], it inherits a structure of monoidal category (such that the canonical functor D(B)-mod $\rightarrow D(B)$ -stab is monoidal). In particular, its Grothendieck group (as a triangulated category), which will be denoted by $\operatorname{Gr}^{\mathrm{st}}(D(B))$, is a ring and the natural map

$$\begin{array}{ccc} \operatorname{Gr}(D(B)) & \longrightarrow & \operatorname{Gr}^{\operatorname{st}}(D(B)) \\ \mathbf{m} & \longmapsto & \mathbf{m}^{\operatorname{st}} \end{array}$$

is a morphism of rings. If *M* is a D(B)-module, we denote by $[M]_{st}$ its class in $\operatorname{Gr}^{st}(D(B))$.

Also, it follows Theorem 2.4 that

(4.6) $\mathbf{m}_{d,p}^{\text{st}} = 0$ and $2(\mathbf{m}_{l,p}^{\text{st}} + \mathbf{m}_{d-l,p+l}^{\text{st}}) = 0$

if $l \leq d-1$.

4.D. A further quotient. — We denote by D(B)-PROJ the full subcategory of D(B)-mod whose objects are the D(B)-modules M such that $\operatorname{Res}_{B}^{D(B)}M$ is a projective B-module. Since D(B) is a free B-module (of rank d^{2}), D(B)-proj is a full subcategory of D(B)-PROJ. We denote by D(B)-STAB the quotient of the category D(B)-mod by the full subcategory D(B)-PROJ: it is also the quotient of D(B)-stab by the image of D(B)-PROJ in D(B)-stab.

Lemma 4.7. — *The image of* D(B)-PROJ *in* D(B)-stab *is a thick triangulated subcategory. In particular,* D(B)-STAB *is triangulated.*

Proof. — If *M* is a *D*(*B*)-module, we denote by $\pi_M : P(M) \rightarrow M$ (resp. $i_M : M \rightarrow I(M)$) a projective cover (resp. injective hull) of *M*. We just need to prove the following facts:

- (a) If *M* belongs to D(B)-PROJ, then Ker(π_M) and $I(M)/\text{Im}(i_M)$ also belong to D(B)-PROJ.
- (b) If $M \oplus N$ belongs to D(B)-PROJ, then M and N also belong to D(B)-PROJ.
- (c) If *M* and *N* belong to D(B)-PROJ and $f: M \to N$ is a morphism of D(B)-modules, then the cone of *f* also belong to D(B)-PROJ.

(a) Assume that *M* belongs to D(B)-PROJ. Since *M* is a projective *B*-module, there exists a morphism of *B*-modules $f : M \to P(M)$ such that $\pi_M \circ f = \text{Id}_M$. In particular, $P(M) \simeq \text{Ker}(\pi_M) \oplus M$, as a *B*-module. So $\text{Ker}(\pi_M)$ is a projective *B*-module.

On the other hand, I(M) is a projective D(B)-module since D(B) so it is a projective *B*-module and so it is an injective *B*-module. So, again, $I(M) \simeq M \oplus I(M)/\operatorname{Im}(i_M)$, so $I(M)/\operatorname{Im}(i_M)$ is a projective *B*-module. This proves (a).

(b) is obvious.

(c) Let *M* and *N* belong to D(B)-PROJ and $f : M \to N$ be a morphism of D(B)-modules. Let $\Delta_f : M \to I(M) \oplus N$, $m \mapsto (i_M(m), f(m))$. Then the cone of *f* is isomorphic in D(B)-stab to $(I(M) \oplus N)/\operatorname{Im}(\Delta_f)$. But Δ_f is injective, *M* is an injective *B*-module and so $I(M) \simeq M \oplus (I(M) \oplus N)/\operatorname{Im}(\Delta_f)$ as a *B*-module, which shows that $(I(M) \oplus N)/\operatorname{Im}(\Delta_f)$ is a projective *B*-module.

Since D(B)-STAB is triangulated, we can define its Grothendieck group, which will be denoted by $\operatorname{Gr}^{\operatorname{ST}}(D(B))$. Also, if M belongs to D(B)-PROJ and Nis any D(B)-module, then $M \otimes N$ and $N \otimes M$ are projective as B-modules [**EGNO**, Proposition 4.2.12], so D(B)-STAB inherits a structure of monoidal category, compatible with the triangulated structure. This endows $\operatorname{Gr}^{\operatorname{ST}}(D(B))$ with a ring structure. The natural map $\operatorname{Gr}(D(B)) \to \operatorname{Gr}^{\operatorname{ST}}(D(B))$ will be denoted by $\mathbf{m} \mapsto \mathbf{m}^{\operatorname{ST}}$: it is a morphism of rings. Of course, this morphism factors through $\operatorname{Gr}^{\operatorname{st}}(D(B))$.

If $\lambda \in \Lambda^{\#}(d)$, then it follows from [EGST2, Property 1.4] that there exists a D(B)-module P_{λ}^{B} which is projective as a *B*-module and such that there is an exact sequence

$$0 \longrightarrow M_{\iota(\lambda)} \longrightarrow P_{\lambda}^{B} \longrightarrow M_{\lambda} \longrightarrow 0.$$

It then follows that

(4.8) $\mathbf{m}_{\lambda}^{\mathrm{ST}} + \mathbf{m}_{\iota(\lambda)}^{\mathrm{ST}} = 0.$

Also, we still have

 $\mathbf{m}_{d,p}^{\mathrm{ST}} = \mathbf{0}.$

The next Theorem follows from (4.8), (4.9) and Proposition 4.3:

Theorem 4.10. — The ring $\operatorname{Gr}^{\operatorname{ST}}(D(B))$ is generated by $\mathbf{v}_{\zeta}^{\operatorname{ST}}$ and $\mathbf{m}_{2}^{\operatorname{ST}}$. Moreover,

$$\operatorname{Gr}^{\operatorname{ST}}(D(B)) = \bigoplus_{\lambda \in [\Lambda^{\#}(d)/\iota]} \mathbb{Z}\mathbf{m}_{\lambda}^{\operatorname{ST}}$$

and $\operatorname{Gr}^{\operatorname{ST}}(D(B))$ is a free \mathbb{Z} -module of rank d(d-1)/2.

Recall that Lemma 4.5 shows that, through the $\chi_{i,j}$'s, only d(d + 1)/2 different characters of the ring Gr(D(B)) have been defined. It is not clear if $\mathbb{C}Gr(D(B))$ is semisimple in general but, for d = 2, it can be checked that it is semisimple (of dimension 4), so that there is a fourth character $Gr(D(B)) \rightarrow \mathbb{C}$ which is not obtained through the $\chi_{i,j}$'s.

Now, a character χ : Gr(D(B)) $\rightarrow \mathbb{C}$ factors through GrST(D(B)) if and only if its kernel contains the \mathbf{m}_{λ_0} 's (where λ_0 runs over $\Lambda^0(d)$) and the $\mathbf{m}_{\lambda} + \mathbf{m}_{\iota(\lambda)}$'s (where λ runs over $\Lambda^{\#}(d)$). This implies the following result:

Theorem 4.11. — The character $\chi_{\lambda} : \operatorname{Gr}(D(B)) \to \mathbb{C}$ factors through $\operatorname{Gr}^{\operatorname{ST}}(D(B))$ if and only if $\lambda \in \Lambda^{\#}(d)$. So the $(\chi_{\lambda})_{\lambda \in [\Lambda^{\#}(d)/\iota]}$ are all the characters of $\operatorname{Gr}^{\operatorname{ST}}(D(B))$ and the \mathbb{C} -algebra $\mathbb{C}\operatorname{Gr}^{\operatorname{ST}}(D(B))$ is semisimple. **4.E.** Complements. — If \mathscr{C} is a monoidal category, we denote by $\mathbf{Z}(\mathscr{C})$ its *Drinfeld center* (see [Ka, §XIII.4]) and we denote by $\mathbf{For}_{\mathscr{C}} : \mathbf{Z}(\mathscr{C}) \to \mathscr{C}$ the forgetful functor.

It turns out that the category $\mathbb{Z}(B \text{-mod})$ is naturally equivalent to D(B)-mod and that, through this equivalence, the forgetful functor just becomes the restriction functor $\operatorname{Res}_B^{D(B)}$. We will denote by $\operatorname{can}_{\operatorname{st}}^{D(B)} : D(B) \text{-mod} \to D(B)$ -stab, $\operatorname{can}_{\operatorname{st}}^B : B \text{-mod} \to B$ -stab and $\operatorname{can}_{\operatorname{ST}} : D(B) \text{-mod} \to D(B)$ -STAB the canonical functors. Then the functor

$$\operatorname{can}_{\operatorname{st}}^B \circ \operatorname{Res}_B^{D(B)} : D(B) \operatorname{-----} B \operatorname{-----} stab$$

factors through **Z**(*B*-stab) so that we get a commutative diagram of functors



But any D(B)-module which is projective as a *B*-module is sent to the zero object of **Z**(*B*-stab) through \mathscr{F} , so \mathscr{F} factors through D(B)-PROJ and we get a commutative diagram of functors



Question. Is $\overline{\mathscr{F}}$: D(B)-PROJ $\rightarrow \mathbb{Z}(B$ -stab) an equivalence of categories?

5. Fusion datum

5.A. Quantum traces. — The element $R \in D(B) \otimes D(B)$ defined in §1.F is a universal *R*-matrix which endows D(B) with a structure of braided Hopf

algebra, the category D(B)-mod is braided as follows: if M and N are two D(B)-modules, then the braiding $c_{M,N}: M \otimes N \xrightarrow{\sim} N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = \tau(R)(n \otimes m).$$

Recall that $\tau : D(B) \otimes D(B) \xrightarrow{\sim} D(B) \otimes D(B)$ is given by $\tau(a \otimes b) = b \otimes a$. In particular,

(5.1)
$$c_{N,M}c_{M,N}M \otimes N \longrightarrow M \otimes N$$
 is given by the action by $\tau(R)R$.

For each $i \in \mathbb{Z}$, we have $S^2(b) = (z^{-i}K)b(z^{-i}K)^{-1}$ for all $b \in D(B)$ and $z^{-i}K$ is group-like, so the algebra D(B) is *pivotal* with pivot $z^{-i}K$. This endows the tensor category D(B)-mod with a structure of pivotal category (see Section A) whose associated traces $\operatorname{Tr}^{(i)}_+$ and $\operatorname{Tr}^{(i)}_-$ are given as follows: if M is a D(B)-module and $f \in \operatorname{End}_{D(B)}(M)$, then

$$\operatorname{Tr}_{+}^{(i)}(f) = \operatorname{Tr}(z^{-i}Kf)$$
 and $\operatorname{Tr}_{-}^{(i)}(f) = \operatorname{Tr}(fK^{-1}z^{i}).$

Recall that **Tr** denotes the "classical" trace for endomorphisms of a finite dimensional vector space. So the pivotal structure depends on the choice of *i* (modulo *d*). The corresponding twist is $\theta_i = z^i \theta$, which endows D(B)-mod with a structure of balanced braided category (depending on *i*).

Hypothesis and notation. From now on, and until the end of this paper, we assume that the Hopf algebra D(B) is endowed with the pivotal structure whose pivot is $z^{-1}K$. The structure of balanced braided category is given by $\theta_1 = z\theta$ and the associated quantum traces $\operatorname{Tr}^{(1)}_{\pm}$ are denoted by Tr_{\pm} . If M is a D(B)-module, we set $\dim_{\pm}(M) = \operatorname{Tr}_{\pm}(\operatorname{Id}_M)$.

We define

$$\dim(D(B)) = \sum_{M \in \operatorname{Irr} D(B)} \dim_{-}(M) \dim_{+}(M).$$

Then

(5.2)
$$\dim(D(B)) = \frac{2d^2}{(1-\zeta)(1-\zeta^{-1})}.$$

This follows easily from the fact that

(5.3)
$$\dim_{+} M_{l,p} = \zeta^{1-l-p}(l)_{\zeta} \text{ and } \dim_{-} M_{l,p} = \zeta^{p+l-1}(l)_{\zeta^{-1}} = \zeta^{p}(l)_{\zeta}.$$

5.B. Characters of Gr(D(B)) via the pivotal structure. — As in Section A, these structures (braiding, pivot) allow to define characters of Gr(D(B)) associated with (endo)simple modules. So, if $\lambda \in \Lambda(d)$, then we set

These are morphism of rings (see Proposition A.4). The main result of this section is the following:

Theorem 5.4. — If $\lambda \in \Lambda(d)$, then

$$s_{M_{\lambda}}^{+} = \chi_{-\lambda}$$
 and $s_{M_{\lambda}}^{-} = \chi_{(0,1)-\lambda}$

Proof. — Write $\lambda = (l, p)$. For simplification, we set

$$\boldsymbol{\gamma}_{i,j,k} = \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_{\zeta}}$$

Then

$$\tau(R)R = \frac{1}{d^2} \sum_{i,i',j,j',k,k'=0}^{d-1} \boldsymbol{\gamma}_{i,j,k} \boldsymbol{\gamma}_{i',j',k'} \zeta^{i(k'-j')} (z^{-k'} F^{i'} E^{i} K^{k'+j}) \otimes (z^{-k} E^{i'} F^{i} K^{j'+k})$$

We need to compute the endomorphism of $M_{l,p}$ equal to $(\mathrm{Id}_{M_{l,p}} \otimes \mathrm{Tr}^{M}_{+})(\tau(R)R|_{M_{l,p} \otimes M})$. Since $M_{l,p}$ is simple, this endomorphism is the multiplication by a scalar ϖ , and so it is sufficient to compute the action on $e_1^{(l,p)} \in M_{l,p}$. Therefore, all the terms (in the big sum giving $\tau(R)R$) corresponding to $i \neq 0$ disappear (because $E e_1^{(l,p)} = 0$). Also, since we are only interested in the coefficient on $e_1^{(l,p)}$ of the result (because the coefficients on other vectors will be zero), all the terms corresponding to $i' \neq 0$ also disappear. Therefore,

$$\varpi = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} \operatorname{Tr}(z^{-1}K z^{-k} K^{j'+k}|_M).$$

So it remains to compute the element

$$b = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} z^{-k-1} K^{j'+k+1}$$

of D(B). But,

$$b = \frac{1}{d^2} \sum_{j',k \in \mathbb{Z}/d\mathbb{Z}} \Big(\sum_{j,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{l(p+l-1-k)} \zeta^{k'(-p-j')} \Big) z^{-k-1} K^{j'+k+1}.$$

So, only the terms corresponding to k = l + p - 1 and j' = -p remain and so

$$b = z^{-l-p} K^l,$$

as expected.

The other formula is obtained through a similar computation.

We denote by $\mathbb{S}^{\pm} = (\mathbb{S}_{\lambda,\lambda'}^{\pm})_{\lambda,\lambda' \in \Lambda(d)}$ the square matrix defined by

$$\mathbb{S}_{\lambda,\lambda'}^{\pm} = \operatorname{Tr}_{\pm}(c_{M_{\lambda'},M_{\lambda}} \circ c_{M_{\lambda},M_{\lambda'}}).$$

Similary, we define \mathbb{T}^{\pm} to be the diagonal matrix (whose rows and columns are indexed by $\lambda \in \Lambda(d)$) and whose λ -entry is

$$\mathbb{T}_{\lambda}^{\pm} = \omega_{\lambda}(\theta_{1}^{\pm 1}).$$

Let us first give a formula for $\mathbb{S}_{\lambda,\lambda'}^{\pm}$ and $\mathbb{T}_{\lambda}^{\pm}$.

Corollary 5.5. — Let $(l, p), (l', p') \in \Lambda(d)$. Then

$$\mathbb{S}^{+}_{(l,p),(l',p')} = \frac{\zeta}{1-\zeta} \zeta^{-l\,l'-l\,p'-p\,l'-2p\,p'}(1-\zeta^{l\,l'}), \qquad \mathbb{T}^{+}_{(l,p)} = \zeta^{p(p+l)},$$

$$\mathbb{S}^{-}_{(l,p),(l',p')} = \frac{\zeta^{2p+l+2p'+l'-1}}{1-\zeta} \zeta^{-ll'-lp'-pl'-2pp'}(1-\zeta^{ll'}) \quad and \quad \mathbb{T}^{-}_{(l,p)} = \zeta^{-p(p+l)} \zeta^{-p(p+l)}(1-\zeta^{ll'})$$

Proof. — This follows immediately from (2.3), (5.3), (4.4) and Theorem 5.4. \Box

5.C. Fusion datum associated with D(B)-STAB. — Let \mathscr{E} denote a set of representatives of ι -orbits in $\{1, 2, ..., d-1\} \times \mathbb{Z}/d\mathbb{Z}$. We then define arbitrarily

$$\dim^{\mathrm{ST}}(D(B)) = \sum_{(l,p)\in\mathscr{E}} \dim_{-}(M_{l,p}) \dim_{+}(M_{l,p}).$$

We have no theoretical justification for this definition... Then

$$\dim^{\mathrm{ST}}(D(B)) = \frac{1}{2}\dim(D(B)) = \frac{d^2}{(1-\zeta)(1-\zeta^{-1})}$$

So dimST D(B) is a positive real number and we denote by $\sqrt{\dim^{ST} D(B)}$ its positive square root. Since $1 - \zeta^{-1} = -\zeta^{-1}(1 - \zeta)$, there exists a unique square root $\sqrt{-\zeta}$ of $-\zeta$ such that

$$\sqrt{\dim^{\mathrm{ST}}(D(B))} = \frac{d\sqrt{-\zeta}}{1-\zeta}.$$

We denote by $\mathbb{S}^{ST} = (\mathbb{S}_{\lambda,\lambda'})_{\lambda,\lambda'\in\mathscr{E}}$ the square matrix defined by

$$\mathbb{S}_{\lambda,\lambda'}^{\mathrm{ST}} = \frac{\mathbb{S}_{\lambda,\lambda'}^+}{\sqrt{\dim^{\mathrm{ST}}(D(B))}}$$

Then it follows from Corollary 5.5 that

(5.6)
$$\mathbb{S}_{(l,p),(l',p')}^{\text{ST}} = \frac{\sqrt{-\zeta}}{d} \zeta^{-ll'-lp'-pl'-2pp'} (\zeta^{ll'}-1).$$

6. Comparison with Malle Z-fusion datum

We refer to [Ma] and [Cu] for most of the material of this section. We denote by $\mathscr{E}(d)$ the set of pairs (i, j) of natural numbers such that $0 \le i < j \le d-1$.

6.A. Set-up. — Let $Y = \{0, 1, ..., d\}$ and let $\pi : Y \rightarrow \{0, 1\}$ be the map defined by

$$\pi(i) = \begin{cases} 1 & \text{if } i \in \{0, 1\}, \\ 0 & \text{if } i \ge 2. \end{cases}$$

We denote by $\Psi(Y,\pi)$ the set of maps $f: Y \to \{0,1,\ldots,d-1\}$ such that f is strictly increasing on $\pi^{-1}(0) = \{2,3,\ldots,d\}$ and strictly increasing on $\pi^{-1}(1) = \{0,1\}$. Since f is injective on $\{2,3,\ldots,d\}$, there exists a unique element $\mathbf{k}(f) \in \{0,1,\ldots,d-1\}$ which does not belong to $f(\{2,3,\ldots,d\})$. Note that, since f

is strictly increasing on $\{2,3,\ldots,d\}$, **k**(*f*) determines the restriction of *f* to $\{2,3,\ldots,d\}$. So the map

(6.1)
$$\begin{array}{ccc} \Psi(Y,\pi) & \longrightarrow & \mathscr{E}(d) \times \{0,1,\ldots,d-1\} \\ f & \longmapsto & (f(0),f(1),\mathbf{k}(f)) \end{array}$$

is bijective. For $f \in \Psi(Y, \pi)$, we set

$$\varepsilon(f) = (-1)^{|\{(y,y') \in Y \times Y \mid y < y' \text{ and } f(y) < f(y')\}|}.$$

We denote by $V = \bigoplus_{i=0}^{d-1} \mathbb{C} v_i$ a vector space of dimension d endowed with a basis $(v_i)_{0 \leq i \leq d-1}$ and we denote by \mathscr{S} the square matrix $(\zeta^{ij})_{0 \leq i,j \leq d-1}$, which will be viewed as an automorphism of V. Note that \mathscr{S} is the character stable of the cyclic group μ_d . We set $\delta(d) = (-1)^{d(d-1)/2} \det(\mathscr{S}) = \prod_{0 \leq i < j \leq d-1} (\zeta^i - \zeta^j)$. Recall that

$$\delta(d)^2 = (-1)^{(d-1)(d-2)/2} d^d.$$

If $f \in \Psi(Y, \pi)$, let

$$\mathbf{v}_f = (v_{f(0)} \wedge v_{f(1)}) \otimes (v_{f(2)} \wedge v_{f(3)} \wedge \dots \wedge v_{f(d)}) \in (\bigwedge^2 V) \otimes (\bigwedge^{d-1} V).$$

Then $(\mathbf{v}_f)_{f \in \Psi(Y,\pi)}$ is a \mathbb{C} -basis of $(\bigwedge^2 V) \otimes (\bigwedge^{d-1} V)$. If $f' \in \Psi(Y,\pi)$, we write

$$\left(\left(\bigwedge^{2}\mathscr{S}\right)\otimes\left(\bigwedge^{d-1}\mathscr{S}\right)\right)(\mathbf{v}_{f'})=\sum_{f\in\Psi(Y,\pi)}\mathbf{S}_{f,f'}\mathbf{v}_{f}$$

In other words, $(\mathbf{S}_{f,f'})_{f,f'\in\Psi(Y,\pi)}$ is the matrix of the automorphism $(\bigwedge^2 \mathscr{S}) \otimes (\bigwedge^{d-1} \mathscr{S})$ of $(\bigwedge^2 V) \otimes (\bigwedge^{d-1} V)$ in the basis $(\mathbf{v}_f)_{f\in\Psi(Y,\pi)}$.

Lemma 6.2. — Let $f, f' \in \Psi(Y, \pi)$. We define

$$i = f(0), \quad j = f(1), \quad k = \mathbf{k}(f),$$

 $i' = f'(0), \quad j' = f'(1), \quad k' = \mathbf{k}(f').$

Then

$$\mathbf{S}_{f,f'} = (-1)^{k+k'} \frac{\delta(d)}{d} \zeta^{-kk'} (\zeta^{ii'+jj'} - \zeta^{ij'+ji'}).$$

Proof. — The computation of the action of $\bigwedge^2 \mathscr{S}$ is easy, and gives the term $\zeta^{ii'+jj'} - \zeta^{ij'+ji'}$. So it remains to show that the determinant of the matrix $\mathscr{S}(k, k')$ obtained from \mathscr{S} by removing the *k*-th row and the *k'*-th column is equal to $(-1)^{k+k'}\zeta^{-kk'}\delta(d)/d$. For this, let $\mathscr{S}'(k)$ denote the matrix whose *k*-th row is equal to $(1, t, t^2, ..., t^{d-1})$ (where *t* is an indeterminate) and whose other rows coincide with those of \mathscr{S} . Then $(-1)^{k+k'} \det(\mathscr{S}(k, k'))$ is equal to

the coefficient of t^k in the polynomial det($\mathscr{S}'(k)$). This is a Vandermonde determinant and

$$det(\mathscr{S}'(k)) = \prod_{\substack{0 \leq i < j \leq d-1 \\ i \neq k, j \neq k}} (\zeta^j - \zeta^i) \cdot \prod_{i=0}^{k-1} (t - \zeta^i) \cdot \prod_{i=k+1}^{d-1} (\zeta^i - t)$$
$$= \delta(d) \prod_{\substack{i=0 \\ i \neq k}}^{d-1} \frac{(t - \zeta^i)}{(\zeta^k - \zeta^i)}.$$

But

$$\prod_{\substack{i=0\\i\neq k}}^{d-1} (t-\zeta^i) = \frac{t^d-1}{t-\zeta^k} = \sum_{i=0}^{d-1} t^i \zeta^{(d-1-i)k}.$$

Therefore,

$$\det(\mathscr{S}(k,k')) = (-1)^{k+k'} \delta(d) \frac{\zeta^{(d-1-k')k}}{d\zeta^{(d-1)k}} = (-1)^{k+k'} \frac{\delta(d)}{d} \zeta^{-kk'},$$

as desired.

6.B. Malle Z-fusion datum. — Let

$$\Psi^{\#}(Y,\pi) = \{ f \in \Psi(Y,\pi) \mid \sum_{y \in Y} f(y) \equiv \frac{d(d-1)}{2} \mod d \}$$

and, if $f \in \Psi^{\#}(Y, \pi)$, we define

$$\operatorname{Fr}(f) = \zeta_*^{d(1-d^2)} \prod_{y \in Y} \zeta_*^{-6(f(y)^2 + df(y))},$$

where ζ_* is a primitive 12*d*-th root of unity such that $\zeta_*^{12} = \zeta$.

We denote by \mathbb{T} diagonal matrix (whose rows and column are indexed by $\Psi^{\#}(Y,\pi)$) equal to diag(Fr(f))_{$f \in \Psi^{\#}(Y,\pi)$}. We denote by $\mathbb{S} = (\mathbb{S}_{f,g})_{f,g \in \Psi^{\#}(Y,\pi)}$ the square matrix defined by

$$\mathbb{S}_{f,g} = \frac{(-1)^{d-1}}{\delta(d)} \varepsilon(f) \varepsilon(g) \, \mathbf{S}_{f,g}.$$

Note that $\mathbb{S}_{f,f_{0,1}} \neq 0$ for all $f \in \Psi^{\#}(Y,\pi)$ (see Lemma 6.2).

Proposition 6.3 (Malle [Ma]). — With the previous notation, we have:

- (a) $\mathbb{S}^4 = (\mathbb{ST})^3 = [\mathbb{S}^2, \mathbb{T}] = 1.$
- (b) ${}^{t}\mathbb{S} = \mathbb{S} and {}^{t}\overline{\mathbb{S}} \mathbb{S} = 1.$

(c) For all $f, g, h \in \Psi^{\#}(Y, \pi)$, the number

$$N_{f,g}^h = \sum_{i \in \Psi^{\#}(Y,\pi)} \frac{\mathbb{S}_{i,f} \mathbb{S}_{i,g} \mathbb{S}_{i,h}}{\mathbb{S}_{i,f_{0,1}}}$$

belongs to \mathbb{Z} .

The pair (S, T) is called the *Malle* \mathbb{Z} -fusion datum.

6.C. Comparison. — We wish to compare the \mathbb{Z} -fusion datum (\mathbb{S} , \mathbb{T}) with the ones obtained from the tensor categories D(B)-mod and D(B)-stab. For this, we will use the bijection (6.1) to characterize elements of $\Psi^{\#}(Y,\pi)$ (here, if $k \in \mathbb{Z}$, then k^{res} denotes the unique element in $\{0, 1, ..., d-1\}$ such that $k \equiv k^{\text{res}} \mod d$):

Lemma 6.4. — Let $f \in \Psi(Y, \pi)$. Then $f \in \Psi^{\#}(Y, \pi)$ if and only if $\mathbf{k}(f) = (f(0) + f(1))^{\text{res}}$. Consequently, the map

$$\begin{array}{rcl} \Psi^{\#}(Y,\pi) & \longrightarrow & \mathscr{E}(d) \\ f & \longmapsto & (f(0),f(1)) \end{array}$$

is bijective

Proof. — We have

$$\sum_{y \in Y} f(y) = f(0) + f(1) + \frac{d(d-1)}{2} - \mathbf{k}(f)$$

and the result follows.

If $(i, j) \in \Lambda(d)$, we denote by $f_{i,j}$ the unique element of $\Psi^{\#}(Y, \pi)$ such that $f_{i,j}(0) = i$ and $f_{i,j}(1) = j$. We have

(6.5)
$$\operatorname{Fr}(f_{i,i}) = \zeta^{ij}$$

and, if $(i, j), (i', j') \in \Lambda(d)$, then

(6.6)
$$\mathbb{S}_{f_{i,j},f_{i',j'}} = \frac{(-1)^{(i+j)^{\text{res}}+(i'+j')^{\text{res}}}}{d} \varepsilon(f_{i,j}) \varepsilon(f_{i',j'}) (\zeta^{-ij'-ji'} - \zeta^{-ii'-jj'}).$$

Proof. — The second equality follows immediately from Lemmas 6.2 and 6.4. Let us prove the first one. By definition, $Fr(f_{i,j}) = \zeta_*^{\alpha}$, where

$$\alpha = d(1-d^2) - 6\sum_{y \in Y} (f_{i,j}(y)^2 + df_{i,j}(y)).$$

The construction of $f_{i,j}$ shows that

$$\alpha = d(1-d^2) - 6(i^2 + di) - 6(j^2 + dj) - 6\sum_{k=0}^{d-1} (k^2 + dk) + 6(((i+j)^{\text{res}})^2 + d(i+j)^{\text{res}}).$$

Write $i + j = (i + j)^{\text{res}} + \eta d$, with $\eta \in \{0, 1\}$. Then $\eta^2 = \eta$ and so

$$(i+j)^2 + d(i+j) = ((i+j)^{\text{res}})^2 + d(i+j)^{\text{res}} + 2\eta d(i+j) + 2\eta d^2$$

$$\equiv ((i+j)^{\text{res}})^2 + d(i+j)^{\text{res}} \mod 2d.$$

Therefore,

$$\alpha \equiv 12ij + d(1-d^2) - 6\sum_{k=0}^{d-1} (k^2 + dk) \mod 12d$$

$$\equiv 12ij \mod 12d.$$

So $Fr(f_{i,j}) = \zeta_*^{12ij} = \zeta_*^{ij}$.

Let

$$\begin{array}{rcl} \varphi : & \mathscr{E}(d) & \longrightarrow & \Lambda^{\#}(d) \\ & & (i,j) & \longmapsto & (j-i,i). \end{array}$$

The $\pi(\mathscr{E}(d))$ is a set of representatives of ι -orbits in $\Lambda^{\#}(d)$ and the pairs of matrices (\mathbb{S}^{ST} , \mathbb{T}^{ST}) and (\mathbb{S} , \mathbb{T}) are related by the following equality (which follows immediately from Corollary 5.5, (6.5) and (6.6)):

(6.7)
$$\mathbb{S}_{f_{i,j},f_{i',j'}} = \sqrt{-\zeta} \ \mathbb{S}^{\mathrm{ST}}_{\varphi(i,j),\varphi(i',j')} \quad \text{and} \quad \mathbb{T}_{f_{i,j}} = \mathbb{T}^{\mathrm{ST}}_{\varphi(i,j)}.$$

Therefore:

Theorem 6.8. — Malle \mathbb{Z} -fusion datum (\mathbb{S} , \mathbb{T}) can be categorified by the monoidal category D(B)-STAB, endowed with the pivotal structure induced by the pivot $z^{-1}K$ and the balanced structure induced by $z\theta$.

Remark 6.9. — M. Broué, G. Malle & J. Michel have associated to a class of exceptional reflection groups (the *spetsial* ones) a set of "*unipotent characters*" and a partition of these unipotent characters into families. To each family, they have also associated a \mathbb{Z} -fusion datum (*S*, *T*). It turns out that some of these \mathbb{Z} -fusion data can by categorified by Hopf quotients of the algebra D(B) (this is investigated by A. Lacabanne in his Ph.D. Thesis).

A. Appendix. Recollection of *S*-matrices

We follow closely [EGNO, Chapters 4 and 8].

Let \mathscr{C} be a *tensor category* over \mathbb{C} , as defined in [**EGNO**, Definition 4.1.1]: \mathscr{C} is a locally finite \mathbb{C} -linear rigid monoidal category (whose unit object is denoted by **1**) such that the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is \mathbb{C} -bilinear on morphisms and $\operatorname{End}_{\mathscr{C}}(1) = \mathbb{C}$. If *X* is an object in \mathscr{C} , its left (respectively right) dual is denoted by *X*^{*} (respectively **X*) and we denote by

 $\operatorname{coev}_X : \mathbf{1} \longrightarrow X \otimes X^*$ and $\operatorname{ev}_X : X^* \otimes X \longrightarrow \mathbf{1}$

the coevealuation and evaluation morphisms respectively.

We assume that \mathscr{C} is *braided*, namely that it is endowed with a bifunctorial family of isomorphisms $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that

(A.1)
$$c_{X,Y\otimes Y'} = (\mathrm{Id}_Y \otimes c_{X,Y'}) \circ (c_{X,Y} \otimes \mathrm{Id}_{Y'})$$

and

(A.2)
$$c_{X \otimes X', Y} = (c_{X, Y} \otimes \operatorname{Id}_{X'}) \circ (\operatorname{Id}_X \otimes c_{X', Y}).$$

for all objects X, X', Y and Y' in \mathscr{C} (we have omitted the associativity constraints).

Finally, we also assume that \mathscr{C} is *pivotal* [EGNO, Definition 4.7.8], i.e. that it is equipped with a family of functorial isomorphisms $a_X : X \to X^{**}$ (for X running over the objects of \mathscr{C}) such that $a_{X \otimes Y} = a_X \otimes a_Y$. If $f \in \text{End}_{\mathscr{C}}(X)$, the pivotal structure allows to define two *traces*:

 $\operatorname{Tr}_{+}(f) = \operatorname{ev}_{X^*} \circ (a_X f \otimes \operatorname{Id}_{X^*}) \circ \operatorname{coev}_X \in \operatorname{End}_{\mathscr{C}}(1) = \mathbb{C}$

and

$$\operatorname{Tr}_{-}(f) = \operatorname{ev}_X \circ (\operatorname{Id}_{X^*} \otimes f a_X^{-1}) \circ \operatorname{coev}_{X^*} \in \operatorname{End}_{\mathscr{C}}(1) = \mathbb{C}.$$

We will sometimes write $\operatorname{Tr}_{+}^{X}(f)$ or $\operatorname{Tr}_{-}^{X}(f)$ for $\operatorname{Tr}_{+}(f)$ and $\operatorname{Tr}_{-}(f)$. We define two *dimensions*

$$\dim_+(X) = \operatorname{Tr}_+(\operatorname{Id}_X)$$
 and $\dim_-(X) = \operatorname{Tr}_-(\operatorname{Id}_X)$.

To summarize, we will work under the following hypothesis:

Hypothesis and notation. We fix in this section a braided pivotal tensor category \mathscr{C} as above. We denote by $Gr(\mathscr{C})$ its Grothendieck ring. If X is an object in \mathscr{C} , we denote by [X] its class in $Gr(\mathscr{C})$. The set of isomorphism classes of simple objects in \mathscr{C} will be denoted by $Irr(\mathscr{C})$. If $X \in Irr(\mathscr{C})$ and Y is any object in \mathscr{C} , we denote by [Y : X] the multiplicity of X in a Jordan-Hölder series of Y. **A.A.** *S***-matrices.** — If *X*, *Y* are objects in \mathcal{C} , we set

 $s_{X,Y}^+ = (\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y)(c_{Y,X}c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X).$

and $s_{X,Y}^- = (\mathrm{Id}_X \otimes \mathrm{Tr}_-^Y)(c_{Y,X}c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X).$

It is clear that they induce two morphisms of abelian groups

$$s_X^+$$
: $\operatorname{Gr}(\mathscr{C}) \longrightarrow \operatorname{End}_{\mathscr{C}}(X)$
 $[Y] \longmapsto s_{X,Y}^+$ and s_X^- : $\operatorname{Gr}(\mathscr{C}) \longrightarrow \operatorname{End}_{\mathscr{C}}(X)$
 $[Y] \longmapsto s_{X,Y}^-$.

Definition A.3. — An object X in \mathscr{C} is said endominimal if $\operatorname{End}_{\mathscr{C}}(X) = \mathbb{C}$.

For instance, a simple object is endominimal (and 1 is also endominimal, but 1 is simple in a tensor category [EGNO, Theorem 4.3.1]). Note also that an endominimal module is indecomposable. So if \mathscr{C} is moreover semisimple, then an object is endominimal if and only if it is simple.

If *X* is endominimal, then we will view $s_{X,Y}^+$ and $s_{X,Y}^-$ as elements of $\mathbb{C} = \text{End}_{\mathscr{C}}(X)$.

Proposition A.4. — If X is endominimal, then s_X^+ : $Gr(\mathscr{C}) \to \mathbb{C}$ and s_X^- : $Gr(\mathscr{C}) \to \mathbb{C}$ are morphisms of rings.

Proof. — Assume that *X* is endominimal. We will only prove the result for s_X^+ , which amounts to show that

(*) $s_{X,Y\otimes Y'}^+ = s_{X,Y}^+ s_{X,Y'}^+.$

First, note that the following equality

$$c_{Y \otimes Y', X} c_{X, Y \otimes Y'} = (c_{Y, X} \otimes \mathrm{Id}_{Y'}) \circ (\mathrm{Id}_Y \otimes c_{Y', X} c_{X, Y'}) \circ (c_{X, Y} \otimes \mathrm{Id}_{Y'})$$

holds by (A.1) and (A.2). Taking $Id_X \otimes Id_Y \otimes Tr_+^{Y'}$ on the right-hand side, one gets $s_{X,Y'}^+ c_{Y,X} c_{X,Y} \in End_{\mathscr{C}}(X \otimes Y)$ (because *X* is endominimal). Applying now $Id_X \otimes Tr_+^Y$, one get $s_{X,Y'}^+ s_{X,Y}^+ Id_X$. Since

$$(\mathrm{Id}_X \otimes \mathrm{Tr}^Y_+) \circ (\mathrm{Id}_X \otimes \mathrm{Id}_Y \otimes \mathrm{Tr}^{Y'}_+) = \mathrm{Id}_X \otimes \mathrm{Tr}^{Y \otimes Y'}_+,$$

this proves (*).

Proposition A.5. — Let X be endominimal and let X' be an endominimal subquotient of X. Then

 $s_X^+ = s_{X'}^+$ and $s_X^- = s_{X'}^-$.

Proof. — This is just because the endomorphism $(Id_X \otimes Tr^Y_+)(c_{Y,X}c_{X,Y})$ of *X* is a multiplication by a scalar, and this scalar can be computed on any non-trivial subquotient of *X*.

Corollary A.6. — Let X and X' be two endominimal objects in \mathcal{C} belonging to the same block. Then

$$s_X^+ = s_{X'}^+$$
 and $s_X^- = s_{X'}^-$.

Proof. — By Proposition A.5, we may asume that *X* and *X'* are simple. We may also assume that *X* is not isomorphic to *X'* and that $\text{Ext}^1_{\mathscr{C}}(X, X') = 0$. So let **X** $\in \mathscr{C}$ be such that there exists a non-split exact sequence

$$0 \longrightarrow X' \longrightarrow \mathbf{X} \longrightarrow X \longrightarrow 0.$$

Since $X \not\simeq X'$, we have $\operatorname{End}_{\mathscr{C}}(\mathbf{X}) = \mathbb{C}$ and so **X** is endominimal. Then it follows from Proposition A.5 that

$$s_{\mathbf{X}}^+ = s_X^+ = s_{X'}^+$$
 and $s_{\mathbf{X}}^- = s_X^- = s_{X'}^-$,

as desired.

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