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Folded hypercubes with cycles embedding in hybrid conditionally faulty

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ABSTRACT A network is defined as g -conditionally faulty if there are g fault-free neighbors is found in every vertex at least, where $g \geq 1$. An folded hypercube FQ_n with n -dimension, a famous variation of an n -dimensional hypercube Q_n , can be established from Q_n through putting in an edge to every pair of vertices which has complementary addresses. Any odd n for FQ_n is bipartite. Let FF_v represents the faulty vertex set and FF_e represents the faulty edge set in FQ_n , as well as let $F_{FQ_n}(e)$ represents the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n - 7$. We prove the properties of embedding fault-tolerant cycles in $FQ_n - FF_v - FF_e$ as follows: 1) For $n \geq 4$ and $|F_{FQ_n}(e)| < n - 2$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every even length from 4 to $2^n - 2|FF_v|$; 2) For $n = 4$ and $n \geq 8$ where n is even, and $|F_{FQ_n}(e)| \leq n - 3$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every odd length from $n + 1$ to $2^n - 2|FF_v| - 1$. This study has been submitted to HAL an open archive for the sustainability (<https://hal.archives-ouvertes.fr/hal-01579266>).

INDEX TERMS Conditionally faulty, fault-free, folded hypercubes, hypercubes, interconnection networks

I. INTRODUCTION

To choose an appropriate interconnection network (referred to as network) is one of significant works for the design in parallel computing and distributed systems. At present, many network topologies are presented in the literature [1-3]. The hypercube proposed by Bhuyan and Agrawal [4] is a famous network model with several outstanding characteristics including regularity, symmetry, low degree, short mean internode distance, small diameter, smaller edge complexity, and recursive structure. These characteristics are highly useful for the development and design of large-scale parallel or distributed systems, see Leighton [5]. Therefore, many variants of hypercube are presented including El-Amawy and Latifi [6], Esfahanian *et al.* [7], and Preparata and Vuillemin [8]. The folded hypercube is one of the variants that has become a focus of research. Folded hypercube can be established from a hypercube through putting in an edge to every pair of vertices which has the longest distance, i.e., a pair of vertices has complementary addresses. It has been proved helpful for improving the performance of the system on conventional hypercube in numerous measurements, for

examples, connectivity, diameter, faulty diameter, and many more. (Please refer to El-Amawy and Latifi [6] and Wang [9])

The ability of efficiently simulate algorithms for the design of other architectures is a major characteristic of an interconnection network. We can formulate the such simulation as network embedding. Let G represents *guest network* and H represents *host network*. To embed a G into a H is defined as a one-to-one mapping f from the vertex set G to the vertex set H . Under f , an edge in G is corresponded to a path in H [5]. According to the embedding strategy, we can simulate the influence for a guest network on a host network. Therefore, we can develop the algorithms for a guest network and applied them to the host network.

Cycles (rings) are considered as the most basic networks available for parallel and distributed computation. When we want to design simple algorithms with low communication costs, cycles are suitable one. There are many valid algorithms designed on cycles to solve all kinds of algebra and graph problems, see Szyld [10] and Leighton [5]. In arbitrary networks, cycles are able to be employed for distributed computing in control/data flow structures. These usages encourage the embedding of cycles for networks.

Because the vertices and/or edges in the network may be occasionally broken, the network's fault tolerance must be considered. The literature has shown a lot of studies for the issue of fault-tolerant cycle embedding in an n -dimensional folded hypercube FQ_n in [3, 9, 11-20]. Let FF_v represents the faulty vertex set and FF_e represents the faulty edge set in FQ_n . In 2001, Wang proposed that $FQ_n - FF_e$ consists of a Hamiltonian cycle of length 2^n if $|FF_e| \leq n - 1$ [9]. In 2006, Xu and Ma presented that every edge of FQ_n exists on the cycle for every even length from 4 to 2^n ; if n is even, every edge of FQ_n also exists on the cycle for every odd length from $n + 4$ to $2^n - 1$ [12]. In addition, Xu *et al.* in 2006 stretched his result as aforementioned to show that every fault-free edge of $FQ_n - FF_e$ exists on the cycle for every even length from 4 to 2^n ; if n is even, every fault-free edge of $FQ_n - FF_e$ also exists on the cycle for every odd length from $n + 1$ to $2^n - 1$, where $|FF_e| < n - 1$ [12]. Let $f \in FF_v$ be any faulty vertex in FQ_n . Hsieh *et al.* in 2009 presented that $FQ_n - \{f\}$ consists of the fault-free cycle for every even length from 4 to $2^n - 2$ if $n \geq 3$, and if $n \geq 2$ where n is even, $FQ_n - \{f\}$ consists of the fault-free cycle of every odd length from $n + 1$ to $2^n - 1$ [17]. Furthermore, Cheng *et al.* in 2013 presented that every fault-free edge of $FQ_n - \{f\}$ exists on the cycle for every odd length from $n + 1$ to $2^n - 3$ for $n \geq 2$ where n is even. Kuo in 2015 spread Cheng *et al.*'s result [19] to get that every fault-free edge of $FQ_n - \{f\}$ exists on a cycle for every even length from 4 to $2^n - 2$ if $n \geq 3$, and if $n \geq 2$ where n is even, every fault-free edge of $FQ_n - \{f\}$ also exists on the cycle for every odd length from $n + 1$ to $2^n - 1$ [3]. However, the independent reliability is owned by each component in a network. If a component of a network is independently broken, the probability is low for all breakdowns. Due to this reason, Harary in 1983 first presented the opinion of conditional connectivity [21]. Subsequently, Latifi *et al.* in 1994 determined the conditional vertex-faults which requires that each vertex of a network contains at least g fault-free neighbors, $g \geq 1$ [22]. For this thesis, we focus on $g = 1$ and define that a network is 4-conditionally faulty if its every vertex contains at least four fault-free neighbors. Let $F_{FQ_n}(e)$ represents the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n - 7$. We prove the properties of embedding fault-tolerant cycles in $FQ_n - FF_v - FF_e$, as follows:

- 1) For $n \geq 4$ and $|F_{FQ_n}(e)| < n - 2$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every even length from 4 to $2^n - 2|FF_v|$, where:
- 2) For $n = 4$ and $n \geq 8$ where n is even, and $|F_{FQ_n}(e)| \leq n - 3$, $FQ_n - FF_v - FF_e$ consists of the

fault-free cycle for every odd length from $n + 1$ to $2^n - 2|FF_v| - 1$.

Please note, the terms of network, node, and edge is interchangeable for graph, vertex, and link, respectively used throughout this paper. The following gives the organization of remainder for this paper. Some necessary definitions and notations are presented in Section 2. The major result is shown in Section 3. In the last, concluding remarks are concluded in Section 4.

II. PRELIMINARIES

Let a graph is defined as $G = (V, E)$. $G = (V, E)$ is an ordered pair which V is the *vertex set* and is a finite set, and E is the *edge set* and is a subset of $\{(u, v) | (u, v) \text{ is an unordered pair of } V\}$. The *vertex set* and the *edge set* can be also represents $V(G)$ and $E(G)$, respectively. When $(u, v) \in E$, the vertices u and v are *adjacent*. For the edge $e = (u, v)$, u and v are called the *end-vertices* of e . We call u adjacent to v , and vice versa. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $(V_0 \cap V_1) = \emptyset$; and $E \subseteq \{(x, y) | x \in V_0 \text{ and } y \in V_1\}$. A path $P[v_0, v_k] = \langle v_0, v_1, \dots, v_k \rangle$ is a sequence of different vertices with any two follow-up vertices are adjacent. v_0 and v_k are called as the *end-vertices* of the path. Furthermore, a *subpath* may be involved by a path, represented as $\langle v_0, v_1, \dots, v_i, P[v_i, v_j], v_j, v_{j+1}, \dots, v_k \rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \dots, v_{j-1}, v_j \rangle$. The number of edges on the path represents the length of the path. When $v_0 = v_k$ and v_0, v_1, \dots, v_{k-1} are different, a path $\langle v_0, v_1, \dots, v_k \rangle$ forms a cycle. A vertex is thought *fault-free* if it is not broken. An edge is thought *fault-free* if the two end-vertices and their edge are not broken. Vertex u is a *fault-free* neighbor of v if u and (u, v) are not faulty. A path (cycle) is *fault-free* if it has no faulty edges and faulty vertices. The faulty vertex and/or faulty edge set incident to the end-vertices of any edge $e \in E(G)$ can be denoted as $F_G(e)$. Other graph-theoretic terminologies and notations are not described here can refer to West *et al.* in 2001 [23].

An n -dimensional hypercube Q_n (n -cube for short) is denoted as an undirected graph. $V(Q_n)$ contains 2^n vertices labelled as binary strings of length n from $00 \dots 0_n$ to $11 \dots 1_n$. Each edge $e = (u, v) \in E(Q_n)$ connects two vertices u and v if and only if u and v differ in exactly one bit of their labels, i.e., $u = b_n b_{n-1} \dots b_k \dots b_1$ and $v = b_n b_{n-1} \dots \bar{b}_k \dots b_1$, where \bar{b}_k is the one's complement of b_k , i.e., $\bar{b}_k = 1 - b_k$ if and only if $b_k = i$ for $i \in \{0, 1\}$. e is called as an edge of *dimension* k . Obviously, each vertex connects to exactly n other vertices. Furthermore, it exists 2^{n-1} edges in each dimension and $|E(Q_n)| = n \cdot 2^{n-1}$. Figure 1 shows a 2-dimensional hypercube Q_2 and a 3-dimensional hypercube Q_3 .

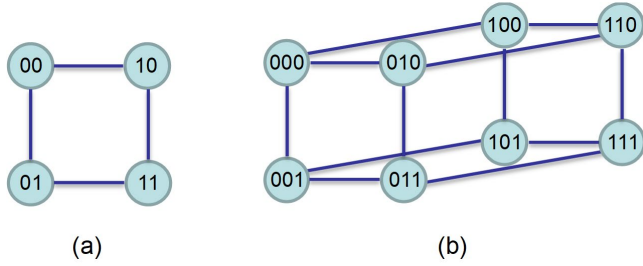


FIGURE 1. (a) 2-dimensional hypercube Q_2 and (b) 3-dimensional hypercube Q_3 .

Let $x = x_n x_{n-1} \dots x_1$ and $y = y_n y_{n-1} \dots y_1$ be two n -bit binary strings; and let $y = x^{(k)}$, where $1 \leq k \leq n$, if $y_k = 1 - x_k$ and $y_i = x_i$ for all $i \neq k, 1 \leq i \leq n$. In addition, let $y = \bar{x}$ if $y_i = 1 - x_i$ for all $1 \leq i \leq n$. The *Hamming distance* $d_H(x, y)$ between vertex x and vertex y is the number of different bits in the corresponding strings of the vertices. The *Hamming weight* $hw(x)$ of x is the number of i 's such that $x_i = 1$. Note that Q_n is a bipartite graph with two partite sets $\{x | hw(x) \text{ is odd}\}$ and $\{x | hw(x) \text{ is even}\}$. Let $d_{Q_n}(x, y)$ be the distance between vertex x and vertex y in graph Q_n . Clearly, $d_{Q_n}(x, y) = d_H(x, y)$.

An n -dimensional folded hypercube FQ_n can be established from an n -cube by putting in an *complementary edge* to every pair of vertices which has the longest distance, i.e., for a vertex whose address is $b = b_n b_{n-1} \dots b_1$, it now has one more edge to vertex $\bar{b} = \bar{b}_n \bar{b}_{n-1} \dots \bar{b}_1$, except its original n edges. Thus, FQ_n has 2^{n-1} more edges than Q_n . These augmented edges skips are called to distinguish them from edges in Q_n , and employ E_s to represent the skips set. So the complete edge set of a folded hyper cube $E(FQ_n)$ can be represented as $E(Q_n) \cup E_s$. Therefore, the edges of FQ_n can be formally defined as that $E(FQ_n) = E(Q_n) \cup E_s = \{e = (u, v) | d_H(u, v) = 1 \in E(Q_n) \text{ or } d_H(u, v) = n \in E_s\}$. Examples of 2-dimensional folded hypercube FQ_2 and 3-dimensional folded hypercube FQ_3 are illustrated in Figure 2. It has been indicated that FQ_n is $(n+1)$ -regular, $(n+1)$ -connected, vertex-transitive, and edge-transitive in Xu *et al.* [12]. Furthermore, FQ_n has been indicated that for any odd $n \geq 3$ is bipartite in Lewinter and Widulski [24].

For convenience, FQ_n can be denoted as $** \dots **_n = *^n$, where $*$ $\in \{0, 1\}$ means the “don't care” symbol. A regular hypercube Q_n can be partitioned into two subcubes Q_{n-1}^i along dimension i , where $1 \leq i \leq n$. The subcubes are defined as $Q_{n-1}^0 = *^{n-i} 0^{i-1}$ and $Q_{n-1}^1 = *^{n-i} 1^{i-1}$, in which the values of the i th bits of the vertices are 0 and 1, respectively. Formally, Q_{n-1}^0 (respectively, Q_{n-1}^1) is a subgraph of FQ_n induced by $\{x_n \dots x_i \dots x_1 \in V(FQ_n) | x_i = 0\}$ (respectively, $\{x_n \dots x_i \dots x_1 \in V(FQ_n) | x_i = 1\}$).

Definition 1 Hsieh and Kuo in 2007 [25] An i -partition on $FQ_n = *^n$, where $1 \leq i \leq n$, partitions FQ_n along dimension i into two $(n-1)$ -cubes $*^{n-i} 0^{i-1} (Q_{n-1}^0)$ and

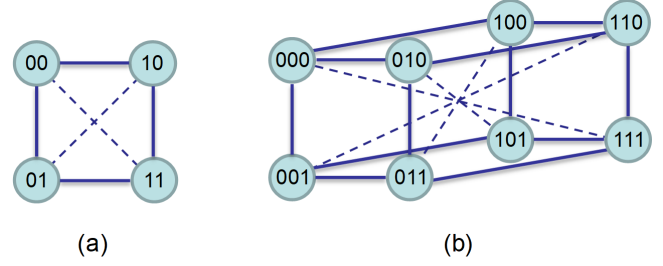


FIGURE 2. (a) 2-dimensional folded hypercube FQ_2 and (b) 3-dimensional folded hypercube FQ_3 , in which edges in E_s are denoted as dashed lines.

$*^{n-i} 1^{i-1} (Q_{n-1}^1)$. Furthermore, all edges in E_s are between Q_{n-1}^0 and Q_{n-1}^1 .

Let F_v (respectively, FF_v) and F_e (respectively, FF_e) represent the faulty vertex set and the faulty edge set in Q_n (respectively, FQ_n). By Definition 1, if we perform an i -partition on FQ_n to form two $(n-1)$ -cubes Q_{n-1}^0 and Q_{n-1}^1 , we derived that $F_v^0 = FF_v \cap V(Q_{n-1}^0)$, $F_v^1 = FF_v \cap V(Q_{n-1}^1)$, $F_e^0 = FF_e \cap E(Q_{n-1}^0)$ and $F_e^1 = FF_e \cap E(Q_{n-1}^1)$. Finally, some previously outcomes of path (cycle) embedding in hypercubes and folded hypercubes are considered in the remainder of this section. These results are beneficial for our method.

Lemma 1 Saad and Schultz in 1988 [26] Let u and v be any two vertices in Q_n and $d_{Q_n}(u, v) = d$. Then, there exist n internally disjoint paths joining u and v in Q_n , where d paths of them are of length d and lie in a d -dimensional subcube.

Lemma 2 Ma *et al.* in 2007 [27] Let u and v be any two fault-free vertices in Q_n . Then, $Q_n - F_v - F_e$ contains a fault-free path of every length l with $d_{Q_n}(u, v) + 2 \leq l \leq 2^n - 2|F_v| - 1$ and $2|(l - d_{Q_n}(u, v))|$, where $|F_v| + |F_e| \leq n - 2$ and $n \geq 3$.

Lemma 3 Xu and Ma in 2006 [28] For $n \geq 3$, every edge of FQ_n lies on a cycle of every even length from 4 to 2^n ; and for $n \geq 2$ where n is even, every edge of FQ_n lies on a cycle of every odd length from $n+1$ to $2^n - 1$.

Lemma 4 Hsieh *et al.* in 2009 [17] For $n \geq 3$, $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2$; and for $n \geq 2$ where n is even, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 1$, where $|FF_v| = 1$.

Lemma 5 Xu *et al.* in 2006 [12] For $n \geq 3$, every edge of $FQ_n - FF_e$ lies on a fault-free cycle of every even length from 4 to 2^n ; and for $n \geq 2$ where n is even, every edge of $FQ_n - FF_e$ lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 1$, where $|FF_e| = n - 1$.

Lemma 6 Cheng and Guo in 2013 [29] Let $F_{Q_n}(e)$ denote the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(Q_n)$. Suppose that Q_n is 3-conditionally faulty and $|F_{Q_n}(e)| \leq n - 2$. Then, every fault-free edge of $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ if $|F_v| + |F_e| \leq 2n - 7$, where $n \geq 5$.

The proof of Lemma 7 is given in the Appendix section.

Lemma 7 Suppose that Q_n is 3-conditionally faulty and $|F_v| \leq 2n - 7$, where $n \geq 7$. Then, Q_n can be partitioned along some dimension $j \in \{1, 2, \dots, n\}$ to form two $(n - 1)$ -cubes Q_{n-1}^0 and Q_{n-1}^1 such that both Q_{n-1}^0 and Q_{n-1}^1 are 2-conditionally faulty with $|F_v^0| \leq 2n - 9$ and $|F_v^1| \leq 2n - 9$.

Lemma 8 Suppose that FQ_n is 4-conditionally faulty and $|FF_v| \leq 2n - 7$, where $n \geq 8$. Then, FQ_n can be partitioned along some dimension $j \in \{1, 2, \dots, n\}$ to form two $(n - 1)$ -cubes Q_{n-1}^0 and Q_{n-1}^1 such that both Q_{n-1}^0 and Q_{n-1}^1 are 3-conditionally faulty, $|F_v^0| \leq 2n - 9$ and $|F_v^1| \leq 2n - 9$.

Proof: According to the definition of FQ_n , $E(FQ_n) = E(Q_n) \cup E_s$ and $V(FQ_n) = V(Q_n)$. If we eliminate all edges in E_s , then $FQ_n - E_s \cong Q_n$. Since FQ_n is 4-conditionally faulty, $FQ_n - E_s$ would be certainly 3-conditionally faulty. By Lemma 7, $FQ_n - E_s \cong Q_n$ can be partitioned along some dimension $j \in \{1, 2, \dots, n\}$ to form two $(n - 1)$ -cubes Q_{n-1}^0 and Q_{n-1}^1 such that both Q_{n-1}^0 and Q_{n-1}^1 are 3-conditionally faulty, $|F_v^0| \leq 2n - 9$ and $|F_v^1| \leq 2n - 9$. Then, the lemma holds. \square

III. CYCLES EMBEDDING IN A FAULTY FOLDED HYPERCUBE

Let $F_{FQ_n}(e)$ represent the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|F_{FQ_n}(e)| \leq n - 3$. We show that

- 1) For $n \geq 4$, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \leq l \leq 2^n - 2|FF_v|$, where $|FF_v| + |FF_e| \leq 2n - 7$;
- 2) For $n \geq 4$ where n is even, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length l with $n + 1 \leq l \leq 2^n - 2|FF_v| - 1$, where $|FF_v| + |FF_e| \leq 2n - 7$.

Lemma 9 Suppose that FQ_n is 4-conditionally faulty, $|FF_v| + |FF_e| \leq 2n - 7$ and $|F_{FQ_n}(e)| \leq n - 2$ for $n \geq 4$. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \leq l \leq 2^n - 2|FF_v|$.

Proof: The cases for $n = 4$ and $n \geq 5$ are considered.

CASE 1. $n = 4$.

In this case, $|FF_v| + |FF_e| \leq 1$. If $|FF_v| = |FF_e| = 0$, by Lemma 3, FQ_4 contains a cycle of every even length l with $4 \leq l \leq 16$. If $|FF_v| = 1$ and $|FF_e| = 0$, by Lemma 4,

$FQ_4 - FF_v$ contains a fault-free cycle of every even length l with $4 \leq l \leq 14$. If $|FF_v| = 0$ and $|FF_e| = 1$, by Lemma 5, $FQ_4 - FF_e$ contains a fault-free cycle of every even length l with $4 \leq l \leq 16$.

CASE 2. $n \geq 5$.

According to the definition of FQ_n , $E(FQ_n) = E(Q_n) \cup E_s$ and $V(FQ_n) = V(Q_n)$. If we eliminate all edges in E_s , then $FQ_n - E_s \cong Q_n$. Note that FQ_n is 4-conditionally faulty, $FQ_n - E_s \cong Q_n$ would be certainly 3-conditionally faulty. Since $|F_v| + |F_e| \leq |FF_v| + |FF_e| \leq 2n - 7$ and $|F_{Q_n}(e)| \leq |F_{FQ_n}(e)| \leq n - 2$, by Lemma 6, $Q_n - F_v - F_e$ contains a fault-free cycle of every even length l with $4 \leq l \leq 2^n - 2|F_v|$, which implies that $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \leq l \leq 2^n - 2|FF_v|$. By integrating the above two cases, the proof is completed. \square

Lemma 10 Suppose that FQ_n is 4-conditionally faulty, $|FF_v| + |FF_e| \leq 2n - 7$ and $|F_{FQ_n}(e)| \leq n - 3$ for $n = 4$ and $n \geq 8$ where n is even. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length l with $n + 1 \leq l \leq 2^n - 2|FF_v| - 1$.

Proof: The cases for $n = 4$ and $n \geq 8$ where n is even are considered.

CASE 1. $n = 4$.

In this case, $|FF_v| + |FF_e| \leq 1$. If $|FF_v| = |FF_e| = 0$, by Lemma 3, FQ_4 contains a cycle of every odd length l with $5 \leq l \leq 15$. If $|FF_v| = 1$ and $|FF_e| = 0$, by Lemma 4, $FQ_4 - FF_v$ contains a fault-free cycle of every odd length l with $5 \leq l \leq 13$. If $|FF_v| = 0$ and $|FF_e| = 1$, by Lemma 5, $FQ_4 - FF_e$ contains a fault-free cycle of every odd length l with $5 \leq l \leq 15$.

CASE 2. $n \geq 8$ where n is even.

If we assume that every faulty edge e in FF_e is regarded as one of the end-vertices of e is faulty, then $|FF_v| + |FF_e| \leq 2n - 7$ in the worst case. Since $|FF_v| + |FF_e| \leq 2n - 7$ and FQ_n is 4-conditionally faulty, by Lemma 8, FQ_n can be partitioned along some dimension $j \in \{1, 2, \dots, n\}$ to form two $(n - 1)$ -cubes Q_{n-1}^0 and Q_{n-1}^1 such that both Q_{n-1}^0 and Q_{n-1}^1 are 3-conditionally faulty, $|F_v^0| \leq 2n - 9$ and $|F_v^1| \leq 2n - 9$ which implies that $|F_v^0| + |F_e^0| \leq 2n - 9$ and $|F_v^1| + |F_e^1| \leq 2n - 9$. Without loss of generality, we may assume that $j = n$ and $|F_v^0| + |F_e^0| \geq |F_v^1| + |F_e^1|$. Therefore, $|F_v^0| + |F_e^0| \leq 2n - 9$ and $|F_v^1| + |F_e^1| \leq n - 4$.

Since $|F_{Q_n}(e)| \leq |F_{FQ_n}(e)| \leq n - 3$, we know that $|F_{Q_{n-1}^0}(e)| \leq n - 3$ and $|F_{Q_{n-1}^1}(e)| \leq n - 3$. Then, let (x, y) be any fault-free edge in Q_{n-1}^0 such that either $(x^{(n)}, y)$ or $(\bar{x}, y^{(n)})$ is fault-free in Q_{n-1}^1 . (If no such an edge exists, then $|FF_v| + |FF_e| \geq \left\lceil \frac{2^{n-1}}{2} \right\rceil = 2^{n-2} > 2n - 7$ for $n \geq 8$ where n is even, which contradicts to the assumption that $|FF_v| + |FF_e| \leq 2n - 7$.) Without loss of

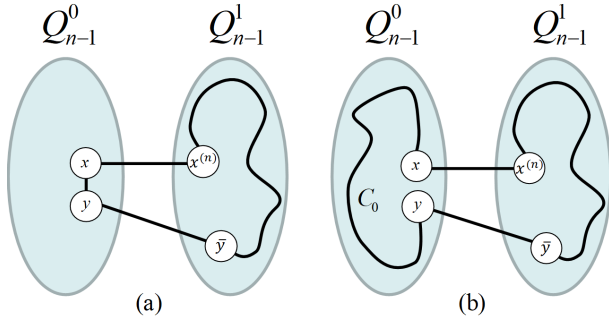


FIGURE 3. An illustration of Case 2 in the proof of Lemma 10. (a) Case 2.2; (b) Case 2.3.

generality, we may assume that $(x^{(n)}, \bar{y})$ in Q_{n-1}^1 is fault-free. Then, we consider the cycle of every odd length l with $n+1 \leq l \leq 2^n - 2|FF_v| - 1$ in the following subcases.

Case 2.1. $l = n + 1$.

In Q_{n-1}^1 , since $|F_v^1| + |F_e^1| \leq n - 4$ and $d_H(x^{(n)}, \bar{y}) = n - 2$, by Lemma 1, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of length $n - 2$. Then, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, x \rangle$ forms a cycle of odd length $l = n + 1$ in $FQ_n - FF_v - FF_e$.

Case 2.2. $l = n + 3$.

In Q_{n-1}^1 , since $|F_v^1| + |F_e^1| \leq n - 4$ and $d_H(x^{(n)}, \bar{y}) = n - 2$, by Lemma 2, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of length $n - 2 + 2 = n$. Then, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, x \rangle$ forms a cycle of odd length $l = n + 3$ in $FQ_n - FF_v - FF_e$. (see Figure 3(a))

Case 2.3. $n + 5 \leq l \leq 2^n - 2|FF_v| - 1$.

Since $|F_v^0| + |F_e^0| \leq 2n - 9$, $|F_{Q_{n-1}^0}(e)| \leq n - 3$, and Q_{n-1}^0 is 3-conditionally faulty, by Lemma 6, (x, y) can lie on a fault-free cycle C_0 of every even length from 4 to $2^{n-1} - 2|F_v^0|$ in Q_{n-1}^0 . Then, C_0 can be denoted as $\langle x, y, P[x^{(n)}, \bar{y}], x \rangle$ in Q_{n-1}^0 . Furthermore, since $|F_v^1| + |F_e^1| \leq n - 4$ and $d_H(x^{(n)}, \bar{y}) = n - 2$, by Lemma 2, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of every even length from $n - 2 + 2 = n$ to $2^{n-1} - 2|F_v^1| - 2$ in Q_{n-1}^1 . Therefore, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, P[y, x], x \rangle$ forms a cycle of every odd length l with $4 - 1 + 2 + n \leq l \leq (2^{n-1} - 2|F_v^0|) - 1 + 2 + (2^{n-1} - 2|F_v^1| - 2)$ which implies $n + 5 \leq l \leq (2^n - 2|FF_v|) - 1$ in $FQ_n - FF_v - FF_e$. (see Figure 3(b))

By integrating the above cases, the proof is completed. \square

By Lemmas 9 and 10, the following theorem is obtained.

Theorem 1 Let $F_{FQ_n}(e)$ denote the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n - 7$. Then, for $n \geq 4$ and $|F_{FQ_n}(e)| \leq n - 2$, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$; and

furthermore, for $n = 4$ and $n \geq 8$ where n is even, and $|F_{FQ_n}(e)| \leq n - 3$, $FQ_n - FF_v - FF_e$ also contains a fault-free cycle of every odd length from $n + 1$ to $2^n - 2|FF_v| - 1$.

IV. CONCLUDING REMARKS

Fault tolerance is one of the important research topics in the field of multi-process computer systems. Many studies focus on vertex-fault-tolerant or edge-fault-tolerant properties for certain specific networks. In this thesis, the 4-conditionally faulty folded hypercube with $|FF_v| + |FF_e| \leq 2n - 7$ is considered. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$ for $n \geq 4$ and $|F_{FQ_n}(e)| \leq n - 2$; and $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length from $n + 4$ to $2^n - 2|FF_v| - 1$ for $n = 4$ and $n \geq 8$ where n is even, and $|F_{FQ_n}(e)| \leq n - 3$ are shown. Finally, this study has been submitted to HAL an open archive for the sustainability (<https://hal.archives-ouvertes.fr/hal-01579266>).

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