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# Large system analysis of a GLRT for detection with large sensor arrays in temporally white noise

Sonja Hiltunen, Philippe Loubaton, *Fellow, IEEE*, and Pascal Chevalier

**Abstract**—This paper addresses the behaviour of a classical multi-antenna GLRT test that allows to detect the presence of a known signal corrupted by a multi-path propagation channel and by an additive temporally white Gaussian noise with unknown spatial covariance matrix. The paper is focused on the case where the number of sensors  $M$  is large, and of the same order of magnitude as the sample size  $N$ , a context which is modeled by the large system asymptotic regime  $M \rightarrow +\infty$ ,  $N \rightarrow +\infty$  in such a way that  $M/N \rightarrow c$  for  $c \in (0, +\infty)$ . The purpose of this paper is to study the behaviour of a GLRT statistics in this regime, and to show that the corresponding theoretical analysis allows to accurately predict the performance of the test when  $M$  and  $N$  are of the same order of magnitude.

**Index Terms**—Multichannel detection, asymptotic analysis, GLRT, random matrix theory

## I. INTRODUCTION

Due to the spectacular development of sensor networks and acquisition devices, it has become common to be faced with multivariate signals of high dimension. Very often, the sample size that can be used in practice in order to perform statistical inference cannot be much larger than the signal dimension. In this context, it is well established that a number of fundamental existing statistical signal processing methods fail. It is therefore of crucial importance to revisit certain classical problems in the high-dimensional signals setting. Previous works in this direction include e.g. [16] and [22] in source localization using a subspace method, or [3],[15],[17],[18] in the context of unsupervised detection.

In the present paper, we address the problem of detecting the presence of a known signal using a large array of sensors. We assume that the observations are corrupted by a temporally white, but spatially correlated (with unknown spatial covariance matrix) additive complex Gaussian noise, and study the generalized likelihood ratio test (GLRT). Although our results can be used in more general situations, we focus on the detection of a known synchronization sequence transmitted by a single transmitter in an unknown

multipath propagation channel. The behaviour of the GLRT in this context has been extensively addressed in previous works, but for the low dimensional signal case (see e.g. [1],[4],[7],[13],[14],[23],[25]). The asymptotic behaviour of the relevant statistics has thus been studied in the past, but it has been assumed that the number of samples of the training sequence  $N$  converges towards  $+\infty$  while the number of sensors  $M$  remains fixed. This is a regime which in practice makes sense when  $M \ll N$ . When the number of sensors  $M$  is large, this regime is however often unrealistic, since in order to avoid wasting resources, the size  $N$  of the training sequence is usually chosen of the same order of magnitude as  $M$ . Therefore, we consider in this paper the asymptotic regime in which both  $M$  and  $N$  converge towards  $\infty$  at the same rate.

We consider both the case where the number of paths  $L$  remains fixed, and the case where  $L$  converges towards  $\infty$  at the same rate as  $M$  and  $N$ . When  $L$  is fixed, we prove that the GLRT statistics  $\eta_N$  converges under hypothesis  $H_0$  towards a Gaussian distribution with mean  $L \log \frac{1}{1-M/N}$  and variance  $\frac{L}{N} \frac{M/N}{1-M/N}$ . This is in contrast with the standard asymptotic regime  $N \rightarrow +\infty$  and  $M$  fixed in which the distribution of  $\eta_N$  converges towards a  $\chi^2$  distribution. Under hypothesis  $H_1$ , we prove that  $\eta_N$  has a similar behaviour than in the standard asymptotic regime  $N \rightarrow +\infty$  and  $M$  fixed, except that the terms  $L \log \frac{1}{1-M/N}$  and  $\frac{L}{N} \frac{M/N}{1-M/N}$  are added to the asymptotic mean and the asymptotic variance, respectively. When  $L$  converges towards  $\infty$  at the same rate as  $M$  and  $N$ , we use existing results (see [2] and [24]) characterizing the behaviour of linear statistics of the eigenvalues of large multivariate  $F$ -matrices, and infer that the distribution of  $\eta_N$  under  $H_0$  is also asymptotically Gaussian. The asymptotic mean converges towards  $\infty$  at the same rate as  $L, M, N$  while the asymptotic variance is a  $O(1)$  term. The asymptotic behaviour of  $\eta_N$  under hypothesis  $H_1$  when  $L$  scales with  $M, N$  is not covered by the existing literature. The derivation of the corresponding new mathematical results would need an extensive work that is not in the scope of the present paper. We rather propose a pragmatic approximate distribution for  $\eta_N$ , motivated by the additive structure of its asymptotic mean and variance in the regime where  $L$  is fixed.

We evaluate the accuracy of the various Gaussian approximations by numerical simulations, by comparing the asymptotic means and variances with their empirical counterparts evaluated by Monte-Carlo simulations. Further, we compare the ROC curves corresponding to the various approximations with the empirical ones. The numerical

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results show that the standard approximations obtained when  $N \rightarrow +\infty$  and  $M$  is fixed completely fail if  $\frac{M}{N}$  is greater than  $\frac{1}{8}$ . The large system approximations corresponding to a fixed  $L$  and  $L \rightarrow +\infty$  appear reliable for small values of  $\frac{M}{N}$ , and, of course, for larger values of  $\frac{M}{N}$ . For the values of  $L, M, N$  that are considered, the approximations obtained in the regime  $L \rightarrow +\infty$  at the same rate as  $M$  and  $N$  appear to be the most accurate, and the corresponding ROC-curves are shown to be good approximations of the empirical ones. Therefore, the proposed Gaussian approximations allow to reliably predict the performance of the GLRT when the number of array elements is large.

This paper is organized as follows. In section II, we provide the signal model under hypotheses  $H_0$  and  $H_1$ , recall the expression of the statistics  $\eta_N$  corresponding to the GLRT, and explain that, in order to study  $\eta_N$ , assuming that the additive noise is spatially white and that the training sequence matrix is orthogonal is not a restriction. In section III, we recall the asymptotic behaviour of  $\eta_N$  in the traditional asymptotic regime  $N \rightarrow +\infty$  and  $M$  fixed. The main results of this paper, concerning the asymptotic behaviour of  $\eta_N$  in the regime  $M, N$  converge towards  $\infty$  at the same rate, are presented in section IV. In this section, we only give outlines of the proofs, while providing the remaining technical details in Appendices. Section V is devoted to the numerical results, and section VI concludes the paper.

**General notations.** For a complex matrix  $\mathbf{A}$ , we denote by  $\mathbf{A}^T$  and  $\mathbf{A}^*$  its transpose and its conjugate transpose, and by  $\text{Tr}(\mathbf{A})$  and  $\|\mathbf{A}\|$  its trace and spectral norm.  $\mathbf{I}$  will represent the identity matrix and  $\mathbf{e}_n$  will refer to a vector having all its components equal to 0 except the  $n$ -th which is equal to 1.

The real normal distribution with mean  $m$  and variance  $\sigma^2$  is denoted  $\mathcal{N}_{\mathbb{R}}(m, \sigma^2)$ . A complex random variable  $Z = X + iY$  follows the distribution  $\mathcal{N}_{\mathbb{C}}(\alpha + i\beta, \sigma^2)$  if  $X$  and  $Y$  are independent with respective distributions  $\mathcal{N}_{\mathbb{R}}(\alpha, \frac{\sigma^2}{2})$  and  $\mathcal{N}_{\mathbb{R}}(\beta, \frac{\sigma^2}{2})$ .

For a sequence of random variables  $(X_n)_{n \in \mathbb{N}}$  and a random variable  $X$ , we write

$$X_n \rightarrow X \text{ a.s. and } X_n \rightarrow_{\mathcal{D}} X$$

when  $X_n$  converges almost surely and in distribution, respectively, to  $X$  when  $n \rightarrow +\infty$ . Finally, if  $(a_n)_{n \in \mathbb{N}}$  is a sequence of positive real numbers,  $X_n = o_P(a_n)$  will stand for the convergence of  $(X_n/a_n)_{n \in \mathbb{N}}$  to 0 in probability, and  $X_n = \mathcal{O}_P(a_n)$  denotes boundedness in probability (i.e. tightness) of the sequence  $(X_n/a_n)_{n \in \mathbb{N}}$ .

## II. PRESENTATION OF THE PROBLEM.

In the following, we assume that a single transmitter sends a known synchronization sequence  $(s_n)_{n=1, \dots, N}$  through a fixed channel with  $L$  paths, and that the corresponding signal is received on a receiver with  $M$  sensors. The received  $M$ -dimensional signal is denoted by  $(\mathbf{y}_n)_{n=1, \dots, N}$ . When the transmitter and the receiver are perfectly synchronized,  $\mathbf{y}_n$  is

assumed to be given for each  $n = 1, \dots, N$  by

$$\mathbf{y}_n = \sum_{l=0}^{L-1} \mathbf{h}_l s_{n-l} + \mathbf{v}_n \quad (1)$$

where  $(\mathbf{v}_n)_{n \in \mathbb{Z}}$  is an additive independent identically distributed complex Gaussian noise verifying

$$\begin{aligned} \mathbb{E}(\mathbf{v}_n) &= 0 \\ \mathbb{E}(\mathbf{v}_n \mathbf{v}_n^T) &= 0 \\ \mathbb{E}(\mathbf{v}_n \mathbf{v}_n^*) &= \mathbf{R} = \sigma^2 \tilde{\mathbf{R}} \end{aligned} \quad (2)$$

where  $\mathbf{R} > 0$  and  $\frac{1}{M} \text{Tr}(\tilde{\mathbf{R}}) = 1$ . Denoting by  $\mathbf{H}$  the  $M \times L$  matrix  $\mathbf{H} = (\mathbf{h}_0, \dots, \mathbf{h}_{L-1})$ , the received signal matrix  $\mathbf{Y} = (\mathbf{y}_1, \dots, \mathbf{y}_N)$  in the presence of a useful signal can be written as

$$\mathbf{Y} = \mathbf{H}\mathbf{S} + \mathbf{V} \quad (3)$$

where  $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_N)$  and where  $\mathbf{S}$  represents the known signal matrix. We assume from now on that the size  $N$  of the training sequence satisfies  $N > M + L$ . We remark that the forthcoming results are valid as soon as the matrix collecting the observations can be written as in Eq. (3). In particular, by appropriately modifying the matrices  $\mathbf{H}$  and  $\mathbf{S}$ , this system model can equivalently be used for a link with multiple transmit antennas.

Furthermore, in the absence of a useful signal, the received signal matrix is given by

$$\mathbf{Y} = \mathbf{V}. \quad (4)$$

In this paper, we study the classical problem of testing the hypothesis  $H_1$  characterized by Equation (3) against the hypothesis  $H_0$  defined by equation (4), in the aim of testing whether there is a useful signal present in the received signal. The hypotheses are

$$\begin{aligned} H_0 : \mathbf{Y} &= \mathbf{V} \\ H_1 : \mathbf{Y} &= \mathbf{H}\mathbf{S} + \mathbf{V}, \end{aligned} \quad (5)$$

where we assume from now on that  $\mathbf{H}$  and  $\mathbf{R}$  are unknown at the receiver side. In the following, we will review the expression of the corresponding generalized maximum likelihood test (GLRT) derived in [4]. The generalized likelihood ratio  $r_N$  is defined by [14]

$$r_N = \frac{\max_{\mathbf{R}, \mathbf{H}} p_{H_1}(\mathbf{Y} | \mathbf{S}, \mathbf{H}, \mathbf{R})}{\max_{\mathbf{R}} p_{H_0}(\mathbf{Y} | \mathbf{R})}. \quad (6)$$

The probability density functions are given by

$$\begin{aligned} p_{H_0}(\mathbf{Y} | \mathbf{R}) &= \frac{1}{\pi^{NM} (\det(\mathbf{R}))^N} e^{-\text{Tr}[\mathbf{Y}^* \mathbf{R}^{-1} \mathbf{Y}]} \\ p_{H_1}(\mathbf{Y} | \mathbf{S}, \mathbf{H}, \mathbf{R}) &= \frac{1}{\pi^{NM} (\det(\mathbf{R}))^N} \cdot e^{-\text{Tr}[(\mathbf{Y} - \mathbf{H}\mathbf{S})^* \mathbf{R}^{-1} (\mathbf{Y} - \mathbf{H}\mathbf{S})]}. \end{aligned} \quad (7)$$

The first step to calculate  $r_N$  is to determine  $\hat{\mathbf{R}}_1$  and  $\hat{\mathbf{H}}$ , the  $\mathbf{R}$  and  $\mathbf{H}$  that maximize the numerator, and  $\hat{\mathbf{R}}_0$ , the  $\mathbf{R}$  that maximizes the denominator, of equation (6). Straightforward

calculations show that  $\hat{\mathbf{H}} = \frac{\mathbf{Y}\mathbf{S}^*}{N}(\frac{\mathbf{S}\mathbf{S}^*}{N})^{-1}$  and  $\hat{\mathbf{R}}_1 = \frac{\mathbf{Y}\mathbf{Y}^*}{N} - (\frac{\mathbf{Y}\mathbf{S}^*}{N})(\frac{\mathbf{S}\mathbf{S}^*}{N})^{-1}(\frac{\mathbf{S}\mathbf{Y}^*}{N})$ . Similarly,  $\hat{\mathbf{R}}_0$  is given by  $\hat{\mathbf{R}}_0 = \frac{\mathbf{Y}\mathbf{Y}^*}{N}$ .

Inserting these estimates into equation (6) leads to  $r_N = (\det(\hat{\mathbf{R}}_1\hat{\mathbf{R}}_0^{-1}))^{-N}$ . Therefore, the log-likelihood ratio  $\eta_N$ , defined by  $\eta_N = \frac{\log r_N}{N}$ , is given by

$$\eta_N = -\log \det \left[ \mathbf{I}_M - \hat{\mathbf{R}}_0^{-1/2} \frac{\mathbf{Y}\mathbf{S}^*}{N} \left( \frac{\mathbf{S}\mathbf{S}^*}{N} \right)^{-1} \frac{\mathbf{S}\mathbf{Y}^*}{N} \hat{\mathbf{R}}_0^{-1/2} \right] \quad (8)$$

or, using the identity  $\det(\mathbf{I} - \mathbf{A}\mathbf{B}) = \det(\mathbf{I} - \mathbf{B}\mathbf{A})$ , by

$$\eta_N = -\log \det [\mathbf{I}_L - \mathbf{T}_N] \quad (9)$$

where  $\mathbf{T}_N$  is the  $L \times L$  matrix defined by

$$\mathbf{T}_N = \left( \frac{\mathbf{S}\mathbf{S}^*}{N} \right)^{-1/2} \frac{\mathbf{S}\mathbf{Y}^*}{N} \left( \frac{\mathbf{Y}\mathbf{Y}^*}{N} \right)^{-1} \frac{\mathbf{Y}\mathbf{S}^*}{N} \left( \frac{\mathbf{S}\mathbf{S}^*}{N} \right)^{-1/2} \quad (10)$$

The generalized maximum likelihood test consists then in comparing  $\eta_N$  to a threshold.

In order to study the behaviour of the test in Eq. (9), we study the limit distribution of  $\eta_N$  under each hypothesis. For this, we remark that it is possible to assume without restriction that  $\frac{\mathbf{S}\mathbf{S}^*}{N} = \mathbf{I}_L$  is verified and that  $\mathbb{E}(\mathbf{v}_n \mathbf{v}_n^*) = \sigma^2 \mathbf{I}$ , i.e.  $\tilde{\mathbf{R}}$  is reduced to the identity matrix. If this is not the case, we denote by  $\tilde{\mathbf{S}}$  the matrix

$$\tilde{\mathbf{S}} = \left( \frac{\mathbf{S}\mathbf{S}^*}{N} \right)^{-1/2} \mathbf{S} \quad (11)$$

and by  $\tilde{\mathbf{Y}}$  and  $\tilde{\mathbf{V}}$  the whitened observation and noise matrices

$$\begin{aligned} \tilde{\mathbf{Y}} &= \tilde{\mathbf{R}}^{-1/2} \mathbf{Y}, \\ \tilde{\mathbf{V}} &= \tilde{\mathbf{R}}^{-1/2} \mathbf{V} \end{aligned} \quad (12)$$

It is clear that  $\frac{\tilde{\mathbf{S}}\tilde{\mathbf{S}}^*}{N} = \mathbf{I}_L$  and that  $\mathbb{E}(\tilde{\mathbf{v}}_n \tilde{\mathbf{v}}_n^*) = \sigma^2 \mathbf{I}$ . Moreover, under  $H_0$ , it holds that  $\tilde{\mathbf{Y}} = \tilde{\mathbf{V}}$ , while under  $H_1$ ,  $\tilde{\mathbf{Y}} = \tilde{\mathbf{H}}\tilde{\mathbf{S}} + \tilde{\mathbf{V}}$  where the channel matrix  $\tilde{\mathbf{H}}$  is defined by

$$\tilde{\mathbf{H}} = \tilde{\mathbf{R}}^{-1/2} \mathbf{H} (\mathbf{S}\mathbf{S}^*/N)^{1/2} \quad (13)$$

Finally, it holds that the statistics  $\eta_N$  can also be written as

$$\eta_N = -\log \det \left[ \mathbf{I}_L - \frac{\tilde{\mathbf{S}}\tilde{\mathbf{Y}}^*}{N} \left( \frac{\tilde{\mathbf{Y}}\tilde{\mathbf{Y}}^*}{N} \right)^{-1} \frac{\tilde{\mathbf{Y}}\tilde{\mathbf{S}}^*}{N} \right] \quad (14)$$

This shows that it is possible to replace  $\mathbf{S}$ ,  $\tilde{\mathbf{R}}$  and  $\mathbf{H}$  by  $\tilde{\mathbf{S}}$ ,  $\mathbf{I}$ , and  $\tilde{\mathbf{H}}$  without modifying the value of statistics  $\eta_N$ . Therefore, without restriction, we assume from now on that

$$\frac{\mathbf{S}\mathbf{S}^*}{N} = \mathbf{I}_L, \quad \tilde{\mathbf{R}} = \mathbf{I}_M \quad (15)$$

In the following, we denote by  $\mathbf{W}$  a  $(N-L) \times N$  matrix for which the matrix  $\Theta = (\mathbf{W}^T, \frac{\mathbf{S}^T}{\sqrt{N}})^T$  is unitary and define the  $M \times (N-L)$  and  $M \times L$  matrices  $\mathbf{V}_1$  and  $\mathbf{V}_2$  by

$$(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}\Theta^* = (\mathbf{V}\mathbf{W}^*, \mathbf{V}\frac{\mathbf{S}^*}{\sqrt{N}}) \quad (16)$$

It is clear that  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are complex Gaussian random matrices with independent identically distributed  $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$

entries, and that the entries of  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are mutually independent. We notice that since  $N > M + L$ , the matrix  $\frac{\mathbf{V}_1\mathbf{V}_1^*}{N}$  is invertible almost surely. We now express the statistics  $\eta_N$  in terms of  $\mathbf{V}_1$  and  $\mathbf{V}_2$ . We observe that

$$\frac{\mathbf{V}\mathbf{V}^*}{N} = \frac{\mathbf{V}_1\mathbf{V}_1^*}{N} + \frac{\mathbf{V}_2\mathbf{V}_2^*}{N} \quad (17)$$

and that

$$\frac{\mathbf{V}\mathbf{S}^*}{N} = \frac{1}{\sqrt{N}} (\mathbf{V}_1, \mathbf{V}_2) \begin{pmatrix} \mathbf{W} \\ \mathbf{S} \end{pmatrix} \frac{1}{\sqrt{N}} \quad (18)$$

coincides with  $\frac{\mathbf{V}_2\mathbf{S}^*}{\sqrt{N}}$  because  $\mathbf{W}\frac{\mathbf{S}^*}{\sqrt{N}} = 0$ . Therefore, under hypothesis  $H_0$ ,  $\eta_N$  can be written as

$$\eta_N = -\log \det \left( \mathbf{I} - \frac{\mathbf{V}_2^*}{\sqrt{N}} \left( \frac{\mathbf{V}_1\mathbf{V}_1^*}{N} + \frac{\mathbf{V}_2\mathbf{V}_2^*}{N} \right)^{-1} \frac{\mathbf{V}_2}{\sqrt{N}} \right) \quad (19)$$

Using the identity

$$\begin{aligned} \mathbf{A}^* (\mathbf{B}\mathbf{B}^* + \mathbf{A}\mathbf{A}^*)^{-1} &= \\ \mathbf{A}^* (\mathbf{B}\mathbf{B}^*)^{-1} \mathbf{A} (\mathbf{I} + \mathbf{A}^* (\mathbf{B}\mathbf{B}^*)^{-1} \mathbf{A})^{-1} & \end{aligned} \quad (20)$$

we obtain that, under hypothesis  $H_0$ ,  $\eta_N$  can be written as

$$\eta_N = \log \det \left( \mathbf{I}_L + \mathbf{V}_2^*/\sqrt{N} (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} \mathbf{V}_2/\sqrt{N} \right) \quad (21)$$

Similarly, it is easy to check that, under  $H_1$ ,  $\eta_N$  is given by

$$\eta_N = \log \det (\mathbf{I}_L + \mathbf{G}_N) \quad (22)$$

where the matrix  $\mathbf{G}_N$  is defined by

$$\mathbf{G}_N = \left( \mathbf{H} + \mathbf{V}_2/\sqrt{N} \right)^* (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} \left( \mathbf{H} + \mathbf{V}_2/\sqrt{N} \right) \quad (23)$$

### III. STANDARD ASYMPTOTIC ANALYSIS OF $\eta_N$ .

In order to give a better understanding of the similarities and differences with the more complicated case where  $M$  and  $N$  converge towards  $+\infty$  at the same rate, we first recall some standard results concerning the asymptotic distribution of  $\eta_N$  under  $H_0$  and  $H_1$  when  $N \rightarrow +\infty$  but  $M$  remains fixed.

#### A. Hypothesis $H_0$ .

A general result concerning the GLRT, known as Wilk's theorem (see e.g. [14], [21] Chapter 8-5), implies that  $N\eta_N$  converges in distribution towards a  $\chi^2$  distribution with  $2ML$  degrees of freedom. For the reader's convenience, we provide an informal justification of this claim. We use (21) and remark that when  $N \rightarrow +\infty$  and  $M$  and  $L$  remain fixed, the matrices  $\mathbf{V}_1\mathbf{V}_1^*/N$  and  $\frac{1}{N}\mathbf{V}_2^* (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} \mathbf{V}_2$  converge a.s. towards  $\sigma^2 \mathbf{I}$  and the zero matrix respectively. Moreover,

$$\frac{1}{N}\mathbf{V}_2^* (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} \mathbf{V}_2 = \frac{1}{\sigma^2}\mathbf{V}_2^*\mathbf{V}_2/N + o_P\left(\frac{1}{N}\right) \quad (24)$$

and a standard second order expansion of  $\eta_N$  leads to

$$\eta_N = \frac{1}{\sigma^2} \text{Tr} (\mathbf{V}_2^*\mathbf{V}_2/N) + o_P\left(\frac{1}{N}\right) \quad (25)$$

This implies immediately that the limit distribution of  $N\eta_N$  is a chi-squared distribution with  $2ML$  degrees of freedom. Informally, this implies that  $\mathbb{E}(\eta_N) \simeq L\frac{M}{N}$  and  $\text{Var}(\eta_N) \simeq \frac{L}{N}\frac{M}{N}$ .

### B. Hypothesis $H_1$ .

Under hypothesis  $H_1$ ,  $\eta_N$  is given by (22). When  $N \rightarrow +\infty$  and  $M$  and  $L$  remain fixed, the matrix  $\mathbf{V}_1\mathbf{V}_1^*/N$  converges a.s. towards  $\sigma^2\mathbf{I}$  and it is easily seen that

$$\eta_N = \log \det \left( \mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2} \right) + \text{Tr} \left[ \left( \mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2} \right)^{-1} \mathbf{\Delta}_N \right] + \mathcal{O}_P(1/N) \quad (26)$$

where the matrix  $\mathbf{\Delta}_N$  is given by

$$\mathbf{\Delta}_N = \mathbf{H}^* \mathbf{\Upsilon}_N \mathbf{H} + \frac{1}{\sigma^2} \left( \frac{\mathbf{V}_2^*}{\sqrt{N}} \mathbf{H} + \mathbf{H}^* \frac{\mathbf{V}_2}{\sqrt{N}} \right) \quad (27)$$

with  $\mathbf{\Upsilon}_N = (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} - \mathbf{I}/\sigma^2$ . Standard calculations show that

$$\sqrt{N} \left( \eta_N - \log \det \left( \mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2} \right) \right) \rightarrow \mathcal{N}(0, \kappa_1) \quad (28)$$

where  $\kappa_1$  is given by

$$\kappa_1 = \text{Tr} \left[ \mathbf{I} - \left( \mathbf{I} + \frac{\mathbf{H}^* \mathbf{H}}{\sigma^2} \right)^{-2} \right] \quad (29)$$

Note that in [14] and [25], the asymptotic distribution of  $\eta_N$  is studied under the assumption that the entries of the matrix  $\mathbf{H}$  are  $\mathcal{O}(\frac{1}{\sqrt{N}})$  terms. In that context,  $\eta_N$  behaves as a non-central  $\chi^2$  distribution.

## IV. MAIN RESULTS.

In this section, we present the main results of this paper related to the asymptotic behaviour of  $\eta_N$  when  $M$  and  $N$  converge towards  $\infty$  at the same rate. The analysis of  $\eta_N$  in the asymptotic regime  $M$  and  $N$  converge towards  $\infty$  at the same rate differs deeply from the standard regime studied in section III. In particular, it is no longer true that the empirical covariance matrix  $\mathbf{V}_1\mathbf{V}_1^*/N$  converges in the spectral norm sense towards  $\sigma^2\mathbf{I}$ . This, of course, is due to the fact that the number of entries of this  $M \times M$  matrix is of the same order of magnitude than the number of available scalar observations (i.e.  $M(N-L) = \mathcal{O}(MN)$ ). We also note that for any deterministic  $M \times M$  matrix  $\mathbf{A}$ , the diagonal entries of the  $L \times L$  matrix  $\frac{1}{N} \mathbf{V}_2^* \mathbf{A} \mathbf{V}_2$  converge towards 0 when  $N \rightarrow +\infty$  and  $M$  remains fixed, while this does not hold when  $M$  and  $N$  are of the same order of magnitude (see Proposition 4 in Appendix I). It turns out that the asymptotic regime where  $M$  and  $N$  converge towards  $\infty$  at the same rate is more complicated than the conventional regime of section III. As the proofs of the following theorems are rather technical, we just provide in this section the outlines of the approaches that are used to establish them. The detailed proofs are given in the Appendix II.

### A. Asymptotic behaviour of $\eta_N$ when the number of paths $L$ remains fixed when $M$ and $N$ increase.

All along this section, we assume that:

**Assumption 1.** •  $M$  and  $N$  converge towards  $+\infty$  in such a way that  $c_N = \frac{M}{N} < 1 - \frac{L}{N}$  converges towards  $c$ , where  $0 < c < 1$

- the number of paths  $L$  remains fixed when  $M$  and  $N$  increase.

In the asymptotic regime defined by Assumption 1,  $M$  can be interpreted as a function  $M(N)$  of  $N$ . Therefore,  $M$ -dimensional vectors or matrices where one of the dimensions is  $M$  will be indexed by  $N$  in the following. Moreover, in order to simplify the exposition,  $N \rightarrow +\infty$  should be interpreted in this section as the asymptotic regime defined by Assumption 1.

As  $M$  is growing, we have to be precise with how the power of the useful signal component  $\mathbf{H}\mathbf{S}$  is normalized. In the following, we assume that the norms of vectors  $(\mathbf{h}_l)_{l=0, \dots, L-1}$  remain bounded when the number of sensors  $M$  increases. This implies that the signal to noise ratio at the output of the matched filter  $\mathbf{S}^* \mathbf{H}^* \mathbf{Y} / \sqrt{N}$ , i.e.  $\text{Tr}((\mathbf{H}^* \mathbf{H})^2) / (\sigma^2 \text{Tr}(\mathbf{H}^* \mathbf{H}))$ , is a  $\mathcal{O}(1)$  term in our asymptotic regime. We mention however that the received signal to noise ratio  $\text{Tr}(\mathbf{H}^* \mathbf{H}) / (M\sigma^2)$  converges towards 0 at rate  $\frac{1}{N}$  when  $N$  increases.

### 1) Asymptotic behaviour of $\eta_N$ under hypothesis $H_0$ :

Under hypothesis  $H_0$ , the following theorem holds.

**Theorem 1.** *It holds that*

$$\eta_N - L \log \left( \frac{1}{1 - c_N} \right) \rightarrow 0 \text{ a.s.} \quad (30)$$

and that

$$\frac{\sqrt{N}}{\sqrt{\frac{Lc_N}{1-c_N}}} \left( \eta_N - L \log \left( \frac{1}{1 - c_N} \right) \right) \rightarrow_{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0, 1) \quad (31)$$

Informally, Theorem 1 leads to  $\mathbb{E}(\eta_N) \simeq -L \log(1 - c_N)$  and  $\text{Var}(\eta_N) \simeq \frac{L}{N} \frac{c_N}{1 - c_N}$ . We recall that if  $M$  is fixed,  $N\eta_N$  behaves like a  $\chi^2$  distribution with  $2ML$  degrees of freedom. In that context,  $\mathbb{E}(\eta_N) \simeq Lc_N$  and  $\text{var}(\eta_N) \simeq \frac{L}{N} c_N$ . Therefore, the behaviour of  $\eta_N$  in the two asymptotic regimes deeply differ. However, if  $c_N \rightarrow 0$ ,  $-\log(1 - c_N) \simeq c_N$ , and the asymptotic means and variances of  $\eta_N$  tend to coincide.

**Outline of the proof.** We denote by  $\mathbf{F}_N$  the  $L \times L$  matrix

$$\mathbf{F}_N = \mathbf{V}_2^* / \sqrt{N} (\mathbf{V}_1 \mathbf{V}_1^* / N)^{-1} \mathbf{V}_2 / \sqrt{N} \quad (32)$$

and remark that under  $H_0$ , (21) leads to

$$\eta_N = \log \det (\mathbf{I}_L + \mathbf{F}_N) \quad (33)$$

First step: proof of (30). As  $L$  does not increase with  $M$  and  $N$ , it is sufficient to establish that

$$\mathbf{F}_N - \frac{c_N}{1 - c_N} \mathbf{I}_L \rightarrow 0 \text{ a.s.} \quad (34)$$

Our approach is based on the observation that if  $\mathbf{A}_N$  is a  $M \times M$  deterministic Hermitian matrix verifying  $\sup_N \|\mathbf{A}_N\| < a < +\infty$ , then,

$$\mathbb{E}_{\mathbf{V}_2} \left| \left( \mathbf{V}_2^* / \sqrt{N} \mathbf{A}_N \mathbf{V}_2 / \sqrt{N} \right)_{k,l} - \frac{\sigma^2}{N} \text{Tr}(\mathbf{A}_N) \delta(k - l) \right|^4 \leq \frac{C(a)}{N^2} \quad (35)$$

where  $C(a)$  is a constant term depending on  $a$ , and where  $\mathbb{E}_{\mathbf{V}_2}$  represents the mathematical expectation operator w.r.t.  $\mathbf{V}_2$ . This is a consequence of Proposition 4 in the Appendix I. Assume for the moment that there exists a deterministic constant  $a$  such that

$$\|(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1}\| \leq a \quad (36)$$

for each  $N$  greater than a non random integer  $N_0$ . Then, as  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are independent, it is possible to use (35) for  $\mathbf{A}_N = (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1}$  and to take the mathematical expectation w.r.t.  $\mathbf{V}_1$  of (35) to obtain that

$$\mathbb{E} \left| (\mathbf{F}_N)_{k,l} - \frac{\sigma^2}{N} \text{Tr}(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \delta(k-l) \right|^4 \leq \frac{C(a)}{N^2} \quad (37)$$

for each  $N > N_0$ , and, using the Borel-Cantelli lemma, that

$$\mathbf{F}_N - \frac{\sigma^2}{N} \text{Tr}(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{I}_L \rightarrow 0 \text{ a.s.} \quad (38)$$

In order to conclude, we use known results related to the almost sure convergence of the eigenvalue distribution of matrix  $\mathbf{V}_1 \mathbf{V}_1^*/N$  towards the so-called Marcenko-Pastur distribution (see Eq. (77) in the Appendix I) which imply that

$$\frac{1}{N} \text{Tr}(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} - \frac{c_N}{\sigma^2(1-c_N)} \rightarrow 0 \quad (39)$$

almost surely. This, in conjunction with (38), leads to (34) and eventually to (30).

However, there does not exist a deterministic constant  $a$  satisfying (36) for each  $N$  greater than a non random integer. In order to solve this issue, it is sufficient to replace matrix  $(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1}$  by a convenient regularized version. It is well known (see Proposition 1 in the Appendix I) that the smallest and the largest eigenvalue of  $\mathbf{V}_1 \mathbf{V}_1^*/N$  converge almost surely towards  $\sigma^2(1-\sqrt{c})^2 > 0$  and  $\sigma^2(1+\sqrt{c})^2$  respectively. This implies that if  $\mathcal{E}_N$  is the event defined by

$$\mathcal{E}_N = \{\text{one of the eigenvalues of } \mathbf{V}_1 \mathbf{V}_1^*/N \text{ escapes from } [\sigma^2(1-\sqrt{c})^2 - \epsilon, \sigma^2(1+\sqrt{c})^2 + \epsilon]\} \quad (40)$$

(where  $\epsilon$  is chosen such that  $\sigma^2(1-\sqrt{c})^2 - \epsilon > 0$ ) then, almost surely, for  $N$  larger than a random integer, it holds that  $\mathbb{1}_{\mathcal{E}_N} = 1$ . Therefore, almost surely, for  $N$  large enough, it holds that  $\eta_N = \eta_N \mathbb{1}_{\mathcal{E}_N}$ . These two random variables thus share the same almost sure asymptotic behaviour. Moreover, it is clear that  $\eta_N \mathbb{1}_{\mathcal{E}_N}$  coincides with  $\log \det(\mathbf{I} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N})$ . In order to study the almost sure behaviour of  $\eta_N \mathbb{1}_{\mathcal{E}_N}$ , it is thus sufficient to evaluate the behaviour of matrix  $\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N}$ , which has the same expression than  $\mathbf{F}_N$ , except that matrix  $(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1}$  is replaced by  $(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbb{1}_{\mathcal{E}_N}$ . The latter matrix verifies

$$\|(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbb{1}_{\mathcal{E}_N}\| \leq \frac{1}{\sigma^2(1-\sqrt{c})^2 - \epsilon} \quad (41)$$

for each integer  $N$  almost surely. Therefore, the regularized matrix  $(\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbb{1}_{\mathcal{E}_N}$  satisfies (36) almost surely for each integer  $N$  for  $a = \frac{1}{\sigma^2(1-\sqrt{c})^2 - \epsilon}$ . This immediately leads to the conclusion that  $\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N}$  has the same almost sure behaviour than  $\frac{c_N}{1-c_N} \mathbf{I}_L \mathbb{1}_{\mathcal{E}_N}$ , or equivalently than  $\frac{c_N}{1-c_N} \mathbf{I}_L$ . This, in turn, implies (30).

Second step: proof of (31). As  $\eta_N = \eta_N \mathbb{1}_{\mathcal{E}_N}$  almost surely for  $N$  large enough, the asymptotic distributions of  $\sqrt{N}[\eta_N - L \log(\frac{1}{1-c_N})]$  and  $\sqrt{N}[\eta_N \mathbb{1}_{\mathcal{E}_N} - L \log(\frac{1}{1-c_N})]$  coincide. We thus study the latter sequence of random variables because the presence of the regularization factor  $\mathbb{1}_{\mathcal{E}_N}$  allows to simplify a lot the derivations.

A standard second order expansion of  $\log \det(\mathbf{I} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N})$  leads to

$$\sqrt{N}[\eta_N \mathbb{1}_{\mathcal{E}_N} - L \log(\frac{1}{1-c_N})] = (1-c_N)\sqrt{N} \left( \text{Tr}(\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N} - \frac{c_N}{1-c_N} \mathbf{I}) \right) + o_P(1) \quad (42)$$

It is thus sufficient to evaluate the asymptotic behaviour of the characteristic function  $\psi_{N,0}$  of random variable  $\beta_{0,N} = (1-c_N)\sqrt{N} \left( \text{Tr}(\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N} - \frac{c_N}{1-c_N} \mathbf{I}) \right)$  defined by  $\psi_{N,0}(u) = \mathbb{E}(e^{iu\beta_{N,0}})$ . For this, we first evaluate  $\mathbb{E}_{\mathbf{V}_2}(e^{iu\beta_{N,0}})$ , and using Proposition 2 and Proposition 4 in Appendix I, we establish that  $\mathbb{E}_{\mathbf{V}_2}(e^{iu\beta_{N,0}})$  has the same asymptotic behaviour as

$$\exp \left[ -\frac{u^2}{2} \sigma^4 L (1-c_N)^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-2} \mathbb{1}_{\mathcal{E}_N} \right] \quad (43)$$

It is known that  $\frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-2}$  behaves almost surely as  $\frac{1}{\sigma^4(1-c_N)^3}$  (see Eq. (78) in the Appendix I). From this, we obtain immediately that

$$\psi_{N,0}(u) - \exp \left( -\frac{u^2}{2} \frac{Lc_N}{1-c_N} \right) \rightarrow 0 \quad (44)$$

for each  $u$ , which, in turn, establishes (31).

2) *Asymptotic behaviour of  $\eta_N$  under hypothesis  $H_1$ :* The behaviour of  $\eta_N$  under hypothesis  $H_1$  is given by the following result.

**Theorem 2.** *It holds that*

$$\eta_N - \bar{\eta}_{N,1} \rightarrow 0 \text{ a.s.} \quad (45)$$

where  $\bar{\eta}_{N,1}$  is defined by

$$\bar{\eta}_{N,1} = L \log \frac{1}{1-c_N} + \log \det(\mathbf{I} + \mathbf{H}^* \mathbf{H} / \sigma^2) \quad (46)$$

Moreover,

$$\frac{\sqrt{N}}{\left( \frac{Lc_N}{1-c_N} + \kappa_1 \right)^{1/2}} (\eta_N - \bar{\eta}_{N,1}) \rightarrow_{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0, 1) \quad (47)$$

where  $\kappa_1$  is defined by (29).

**Remark 1.** *Interestingly, it is seen that the asymptotic mean and variance of  $\eta_N$  are equal to the sum of the asymptotic mean and variance of  $\eta_N$  in the standard regime  $N \rightarrow +\infty$  and  $M$  fixed, with the extra terms  $L \log \left( \frac{1}{1-c_N} \right)$  and  $\frac{Lc_N}{N(1-c_N)}$ , which coincide with the asymptotic mean and variance of  $\eta_N$  under  $H_0$ .*

**Outline of the proof.** We recall that, under  $H_1$ ,  $\eta_N$  is given by (22). As in the proof of Theorem 1, it is sufficient to study the regularized statistics  $\eta_N \mathbb{1}_{\mathcal{E}_N}$  which is also equal to

$$\eta_N \mathbb{1}_{\mathcal{E}_N} = \log \det(\mathbf{I}_L + \mathbb{1}_{\mathcal{E}_N} \mathbf{G}_N) \quad (48)$$

First step: proof of (45). In order to evaluate the almost sure behaviour of  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$ , we expand  $\mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c}$  as

$$\begin{aligned} \mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c} &= \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} + \\ &\quad (\mathbf{V}_2/\sqrt{N})^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} + \\ &\quad \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} (\mathbf{V}_2/\sqrt{N}) \mathbb{1}_{\mathcal{E}_N^c} \end{aligned} \quad (49)$$

The first term of the righthandside of (49) is known to behave as  $\frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)}$  (see (81) in the Appendix I) while the independance between  $\mathbf{V}_1$  and  $\mathbf{V}_2$  implies that the third and the fourth terms converge almost surely towards the zero matrix. This is because the fourth-order moments w.r.t.  $\mathbf{V}_2$  of their entries are  $\mathcal{O}(\frac{1}{N^2})$  terms.

Second step: proof of (47). Using a standard second order expansion, we obtain immediately that

$$\sqrt{N} (\eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1}) = \sqrt{N} \text{Tr}(\mathbf{D}_N \mathbf{\Delta}_N) + o_P(1) \quad (50)$$

where  $\mathbf{\Delta}_N$  and  $\mathbf{D}_N$  are defined by

$$\mathbf{\Delta}_N = \mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c} - \left( \frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)} + \frac{c_N}{1-c_N} \mathbf{I} \right) \quad (51)$$

and

$$\mathbf{D}_N = (1-c_N)(\mathbf{I}_L + \mathbf{H}^* \mathbf{H}/\sigma^2)^{-1} \quad (52)$$

In order to establish (47), it is therefore sufficient to evaluate the asymptotic behaviour of the characteristic function  $\psi_{N,1}$  of random variable  $\beta_{N,1} = \sqrt{N} \text{Tr}(\mathbf{D}_N \mathbf{\Delta}_N)$ . We define  $\kappa_N$  and  $\omega_N$  by

$$\kappa_N = \text{Tr} \left( \mathbf{C}_N (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \right) \quad (53)$$

and

$$\begin{aligned} \omega_N &= \text{Tr} [\mathbf{D}_N \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c}] + \\ &\quad \text{Tr} \left[ \mathbf{D}_N (\mathbf{V}_2/\sqrt{N})^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} \right] + \\ &\quad \text{Tr} \left[ \mathbf{D}_N \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} (\mathbf{V}_2/\sqrt{N}) \mathbb{1}_{\mathcal{E}_N^c} \right] \end{aligned} \quad (54)$$

where  $\mathbf{C}_N$  the  $M \times M$  matrix given by

$$\mathbf{C}_N = (1-c_N) \mathbf{H} (\mathbf{I}_L + \mathbf{H}^* \mathbf{H}/\sigma^2)^{-1} \mathbf{H}^* \quad (55)$$

Then,  $\beta_{N,1}$  can be written as

$$\begin{aligned} \beta_{N,1} &= \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) + \\ &\quad \sqrt{N} \left( \omega_N - \frac{c_N}{1-c_N} \text{Tr}(\mathbf{D}_N) \right) \end{aligned} \quad (56)$$

Using the equation above as well as Proposition 2 and Proposition 4 from Appendix I, we establish that  $\mathbb{E}_{\mathbf{V}_2}(e^{iu\beta_{N,1}})$  behaves as

$$\exp \left( iu \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) \right) \exp \left( -\frac{u^2}{2} \zeta \right) \quad (57)$$

where  $\zeta = \frac{c_N}{(1-c_N)^3} \text{Tr}(\mathbf{D}_N^2) + 2 \frac{c_N}{(1-c_N)} \text{Tr}(\mathbf{D}_N^2 \mathbf{H}^* \mathbf{H})$ . In order to obtain the limiting behaviour of  $\psi_{N,1}(u)$ , it is thus sufficient to evaluate the limit of

$$\mathbb{E}_{\mathbf{V}_1} \left[ \exp \left( iu \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) \right) \right] \quad (58)$$

This technical point is addressed in Proposition 3 in Appendix I.

**Remark 2.** It is useful to recall that the expression of the asymptotic mean and variance of  $\eta_N$  provided in Theorem 2 assumes that  $\tilde{\mathbf{R}} = \mathbf{I}$  and that  $\frac{\mathbf{S}\mathbf{S}^*}{N} = \mathbf{I}$ . If this is not the case, we have to replace  $\mathbf{H}$  by  $\tilde{\mathbf{R}}^{-1/2} \mathbf{H} (\mathbf{S}\mathbf{S}^*/N)^{1/2}$  in Theorem 2.

**Remark 3.** We note that Theorem 2 allows to quantify the influence of an overdetermination of  $L$  on the asymptotic distribution of  $\eta_N$  under  $\mathbf{H}_1$ . This analysis is interesting from a practical point of view, since it is not always possible to know the exact number of paths and their delays. If  $L$  is overestimated, i.e. if the true number of paths is  $L_1 < L$ , then, matrix  $\mathbf{H}$  can be written as  $\mathbf{H} = (\mathbf{H}_1, 0)$ . We also denote by  $\mathbf{S}_1$  and  $\mathbf{S}_2$  the  $L_1 \times N$  and  $(L-L_1) \times N$  matrices such that  $\mathbf{S} = (\mathbf{S}_1^T, \mathbf{S}_2^T)^T$ . It is easy to check that the second term of  $\bar{\eta}_{N,1}$ , i.e.

$$\log \det \left( \mathbf{I}_L + (\mathbf{S}\mathbf{S}^*/N)^{1/2} \mathbf{H}^* \tilde{\mathbf{R}}^{-1} \mathbf{H} (\mathbf{S}\mathbf{S}^*/N)^{1/2} \right) \quad (59)$$

coincides with

$$\log \det \left( \mathbf{I}_{L_1} + (\mathbf{S}_1 \mathbf{S}_1^*/N)^{1/2} \mathbf{H}_1^* \tilde{\mathbf{R}}^{-1} \mathbf{H}_1 (\mathbf{S}_1 \mathbf{S}_1^*/N)^{1/2} \right) \quad (60)$$

and is thus not affected by the overdetermination of  $L$ . Therefore, choosing  $L > L_1$  increases  $\bar{\eta}_{N,1}$  by the factor  $(L-L_1) \log \left( \frac{1}{1-c_N} \right)$ . As for the asymptotic variance, it is also easy to verify that  $\kappa_1$  is not affected by the overdetermination of the number of paths, and that the asymptotic variance is increased by the factor  $(L-L_1) \frac{c_N}{1-c_N}$ . It is interesting to notice that the standard asymptotic analysis of subsection III-B does not allow to predict any influence of the overdetermination of  $L$  on the asymptotic distribution of  $\eta_N$ .

**B. Asymptotic behaviour of  $\eta_N$  when the number of paths  $L$  converges towards  $\infty$  at the same rate as  $M$  and  $N$ .**

The asymptotic regime considered in section IV-A is relevant when the number of paths  $L$  is much smaller than  $M$  and  $N$ . This hypothesis may however be restrictive, so that it is of potential interest to study the following regime:

**Assumption 2.**  $L, M$  and  $N$  converge towards  $+\infty$  in such a way that  $c_N = \frac{M}{N}$  and  $d_N = \frac{L}{N}$  converge towards  $c$  and  $d$ , where  $0 < c + d < 1$

As explained below in Paragraph IV-B.1, the behaviour of  $\eta_N$  under  $\mathbf{H}_0$  in this regime is a consequence of existing results. The behaviour of  $\eta_N$  under  $\mathbf{H}_1$  is however not covered by the existing literature. The derivation of the corresponding new mathematical results needs extensive work that is not in the scope of the present paper. Motivated by the additive structure of the asymptotic mean and variance of  $\eta_N$  under  $\mathbf{H}_1$  under assumption 1, we propose in Paragraph IV-B.2 a pragmatic Gaussian approximation of the distribution of  $\eta_N$  under  $\mathbf{H}_1$

1) Asymptotic behaviour of  $\eta_N$  under hypothesis  $H_0$ :

**Theorem 3.** We define  $\tilde{\eta}_N$  by

$$\begin{aligned} \tilde{\eta}_N = & -N((1-c_N)\log(1-c_N) \\ & + (1-d_N)\log(1-d_N)) \\ & + N(1-c_N-d_N)\log(1-c_N-d_N) \end{aligned} \quad (61)$$

and  $\tilde{\delta}_N$  by

$$\tilde{\delta}_N = -\log\left(\frac{2\sqrt{a_N^2 - b_N^2}}{a_N + \sqrt{a_N^2 - b_N^2}}\right) \quad (62)$$

where

$$a_N = \left(1 - \frac{c_N}{1-d_N}\right)^2 + \frac{d_N}{1-d_N} \left(1 + \frac{c_N(1-c_N)}{d_N(1-d_N)}\right) \quad (63)$$

$$b_N = 2\frac{d_N}{1-d_N} \sqrt{\frac{c_N(1-c_N)}{d_N(1-d_N)}} \quad (64)$$

Then, it holds that  $\mathbb{E}(\eta_N) = \tilde{\eta}_N + \mathcal{O}(\frac{1}{N})$  and that

$$\frac{1}{\sqrt{\tilde{\delta}_N}}(\eta_N - \tilde{\eta}_N) \rightarrow_{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0, 1) \quad (65)$$

**Justification.** The eigenvalues of  $\mathbf{F}_N$  coincide with the non-zero eigenvalues of  $(\mathbf{V}_2\mathbf{V}_2^*)/N (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}$ . Therefore,  $\eta_N$  appears a linear statistics of the eigenvalues of this matrix.  $(\mathbf{V}_2\mathbf{V}_2^*)/N (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}$  is a multivariate  $F$ -matrix. The asymptotic behaviour of the empirical eigenvalue distribution of this kind of random matrix as well as the corresponding central limit theorems are well established (see e.g. Theorem 4-10 and Theorem 9-14 in [2] as well as [24]) when the dimensions of  $V_1$  and  $V_2$  converge towards  $+\infty$  at the same rate. Theorem 3 follows from these results.

**Remark 4.** We notice that the results of Theorem 3 differ deeply from the results of Theorem 1. We first remark that  $\tilde{\eta}_N$ , and thus  $\mathbb{E}(\eta_N)$ , converge towards  $\infty$  at the same rate that  $L, M, N$ . Moreover,  $\eta_N - \mathbb{E}(\eta_N)$  is an  $\mathcal{O}_P(1)$  term under assumption 2, while it is an  $\mathcal{O}_P(\frac{1}{\sqrt{N}})$  term when  $L$  does not scale with  $M, N$ . However, it is possible to informally obtain the expressions of the asymptotic mean and variance of  $\eta_N$  in Theorem 1 from (61) and (62). For this, we remark that a first order expansion w.r.t.  $d_N = \frac{L}{N}$  of  $\tilde{\eta}_N$  and  $\tilde{\delta}_N$  leads to

$$\tilde{\eta}_N = L \left( \log\left(\frac{1}{1-c_N}\right) + \mathcal{O}(L/N) \right) \quad (66)$$

and to

$$\tilde{\delta}_N = \frac{L}{N} \frac{c_N}{1-c_N} + \mathcal{O}((L/N)^2) \quad (67)$$

which, of course, is in accordance with Theorem 1.

2) Asymptotic behaviour of  $\eta_N$  under hypothesis  $H_1$ :

Under  $H_1$ ,  $\eta_N$  is a linear statistics of the eigenvalues of matrix

$$\left(\mathbf{H} + \mathbf{V}_2/\sqrt{N}\right) \left(\mathbf{H} + \mathbf{V}_2/\sqrt{N}\right)^* (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1} \quad (68)$$

To the best of our knowledge, the asymptotic behaviour of the linear statistics of the eigenvalues of this matrix has not yet been studied in the asymptotic regime where  $L, M, N$

converge towards  $\infty$  at the same rate. It is rather easy to evaluate an approximation of the empirical mean of  $\eta_N$  under  $H_1$  using the results of [8]. However, to establish the asymptotic gaussianity of  $\eta_N$  and the expression of the corresponding variance, we need to establish a central limit theorem for linear statistics of the eigenvalues of non-zero mean large  $F$ -matrices. This needs an important work that is not in the scope of the present paper, which is why we propose the following pragmatic approximation of the distribution of  $\eta_N$ .

**Claim 1.** It is relevant to approximate the distribution of  $\eta_N$  under  $H_1$  by a real Gaussian distribution with mean  $\tilde{\eta}_N + \log \det(\mathbf{I} + \mathbf{H}^*\mathbf{H}/\sigma^2)$  and variance  $\tilde{\delta}_N + \kappa_1/N$ .

**Justification of Claim 1.** As mentioned in Remark 1, when  $M, N \rightarrow \infty$  and  $L$  is fixed, under  $H_1$ , the asymptotic mean  $\bar{\eta}_{N,1}$  is the sum of the asymptotic mean under  $H_0$  given by (30) and the second term  $\log \det(\mathbf{I} + \mathbf{H}^*\mathbf{H}/\sigma^2)$ . Thus, in the regime where  $N, M, L \rightarrow \infty$ , it seems reasonable to approximate the asymptotic mean of  $\eta_N$  by the sum of  $\tilde{\eta}_N$  defined by (61) with the second term  $\log \det(\mathbf{I} + \mathbf{H}^*\mathbf{H}/\sigma^2)$ . We can reason similarly with the variance. The asymptotic variance under  $H_1$ , (47), is the sum of the asymptotic variance under  $H_0$ , outlined in Theorem 1, and the extra term  $\frac{\kappa_1}{N}$ . Therefore, the asymptotic variance under  $H_1$  in the regime where  $N, M, L \rightarrow \infty$  can be approximated by the asymptotic variance under  $H_0$  for the same regime, plus the extra term  $\frac{\kappa_1}{N}$ . The results provided by this approximation are evaluated numerically in section V.

For the reader's convenience, the main results of this paper are summarized in Table I, where  $\tilde{\delta}_N$  is given by equation (62),  $\kappa_1$  by equation (29) and  $\tilde{\eta}_N$  by equation (61).

## V. NUMERICAL RESULTS.

In this section, we validate the relevance of the Gaussian approximations of section IV. In our numerical experiments, we have calculated the asymptotic expected values and variances as well as their empirical counterparts, evaluated by Monte Carlo simulations with 100.000 trials. In this section, to refer to the different approximations, we use the (a), (b) and (c) defined in table I.

The fixed channel  $\mathbf{H}$  is equal to  $\mathbf{H} = \frac{1}{(\text{Tr}(\mathbf{H}\mathbf{H}^*))^{1/2}} \bar{\mathbf{H}}$  where  $\bar{\mathbf{H}}$  is a realization of a  $M \times L$  Gaussian random matrix with i.i.d.  $\mathcal{N}_c(0, \frac{1}{M})$  entries. We remark that  $\text{Tr}(\mathbf{H}\mathbf{H}^*) = 1$ .

The rows of the training sequence matrix  $\mathbf{S}$  are chosen as cyclic shifts of a Zadoff-Chu sequence of length  $N$  [5]. Due to the autocorrelation properties of Zadoff-Chu sequences, designed so that the correlation between any shift of the sequence with itself is zero, we have  $\mathbf{S}\mathbf{S}^*/N = \mathbf{I}_L$  if  $L \leq N$ .

*A. Influence of  $c_N = \frac{M}{N}$  on the asymptotic means and variances.*

We first evaluate the behaviour of the means and variances of the three Gaussian approximations in terms of  $c_N = \frac{M}{N}$ . We only show the results for the asymptotic variance under  $H_1$ , but note that the results are similar for the expected



Assumption on parameters	Distribution under $H_0$	Distribution under $H_1$
(a) Classical, $N \rightarrow \infty$	$\eta_N \sim \frac{1}{N} \chi_{2ML}^2$ $(\mathbb{E}[\eta_N] = Lc_N, \text{Var}[\eta_N] = Lc_N \cdot \frac{1}{N})$	$\eta_N \sim \mathcal{N}_{\mathbb{R}}\left(\log \det \left(\mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2}\right), \frac{\kappa_1}{N}\right)$
(b) Proposed, $M, N \rightarrow \infty$	$\eta_N \sim \mathcal{N}_{\mathbb{R}}\left(L \log \frac{1}{1-c_N}, \frac{Lc_N}{1-c_N} \cdot \frac{1}{N}\right)$	$\eta_N \sim \mathcal{N}_{\mathbb{R}}\left(L \log \frac{1}{1-c_N} + \log \det \left(\mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2}\right), \frac{\kappa_1}{N} + \frac{Lc_N}{1-c_N} \cdot \frac{1}{N}\right)$
(c) Proposed, $L, M, N \rightarrow \infty$	$\eta_N \sim \mathcal{N}_{\mathbb{R}}(\tilde{\eta}_N, \tilde{\delta}_N)$	$\eta_N \sim \mathcal{N}_{\mathbb{R}}(\tilde{\eta}_N + \log \det \left(\mathbf{I} + \frac{\mathbf{H}\mathbf{H}^*}{\sigma^2}\right), \frac{\kappa_1}{N} + \tilde{\delta}_N)$

TABLE I  
ASYMPTOTIC DISTRIBUTION OF  $\eta_N$  FOR DIFFERENT ASSUMPTIONS, UNDER  $H_0$  AND  $H_1$

values and under hypothesis  $H_0$ . Figure 1 compares the theoretical variances with the empirical variances obtained by simulation, under hypothesis  $H_1$ , as a function of  $c_N$ , the ratio between  $M$  and  $N$ . In this simulation,  $M = 10$ ,  $L = 5$  and  $N = 20, 40, 60, 80, 160, 320$ . When  $c_N$  is small, the three approximations (a), (b) and (c) give the same variance, as expected, and are very close to the empirical variance. When  $c_N \geq \frac{1}{8}$ , the assumption that  $M$  is small compared to  $N$  is no longer valid, and the classical asymptotic analysis (a) fails. The two large system approximations (b) and (c) provide similar results when  $c_N \leq \frac{1}{4}$ , i.e. when  $N = 40$ , or equivalently when  $\frac{L}{N} \leq \frac{1}{8}$ . However, when  $N = 20$ , i.e.  $\frac{L}{N} = \frac{1}{4}$ , (c), the approximation corresponding to the regime where  $L, M, N$  converge towards  $\infty$  leads to a much more accurate prediction of the empirical variance. We remark that the approximation (c) is also reliable for rather small values of  $L, M, N$ , i.e.  $L = 5, M = 10, N = 20$ . We also remark that the regimes (b) and (c) where  $M, N$  are of the same order of magnitude capture the actual performance even when  $c_N$  is small, which, by extension, implies that the standard asymptotic analysis (a) always performs worse compared to the two large system approximations. If  $N, M$  increase while  $c_N$  stays the same, the results will be even closer to the theoretical values, since the number of samples is larger.

In the simulations that follow, we will use  $c_N = 1/2$  with  $N = 300$ ,  $M = 150$  and  $L = 10$ , if not otherwise stated.

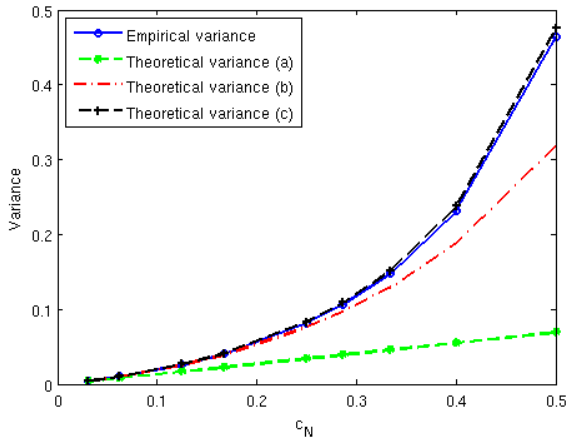


Fig. 1. Proposed asymptotic analysis with standard asymptotic analysis

### B. Comparison of the asymptotic means and variances of the approximations of $\eta_N$ under $H_0$

We first compare in figures 2 and 3 the asymptotic expected values and variances with the empirical ones when  $L$  increases from  $L = 1$  to  $L = 30$  while  $M = 150$  and  $N = 300$ , i.e.  $c_N = 1/2$ . The figures show that the standard asymptotic analysis of section III completely fails for all values of  $L$ . This is expected, given the value of  $\frac{M}{N}$ . As  $L$  increases, the assumption that  $L$  is small becomes increasingly invalid, and the only model that functions well in this regime is the model (c). This is valid both for the expected value and variance, and the theoretical values are very close to their empirical counterparts. We remark that the approximation (c), valid when  $L \rightarrow +\infty$ , also allows to capture the actual empirical performance when  $L$  is small.

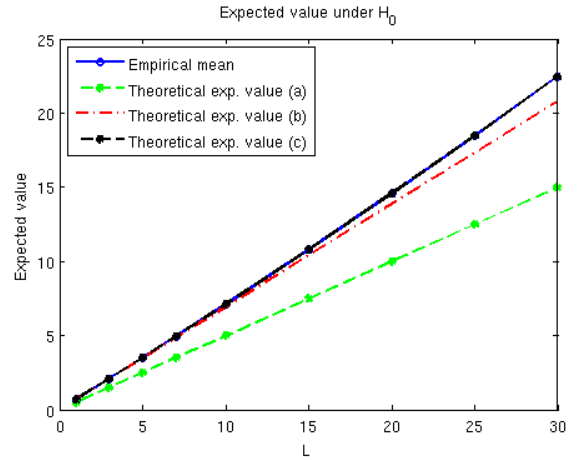


Fig. 2.  $H_0$ : Asymptotic expected values as a function of  $L$

### C. Validation of asymptotic distribution under $H_0$

Although the expected values and variances can be very accurate, this does not necessarily mean that the empirical distribution is Gaussian. Therefore, we need to validate also the distribution under  $H_0$ . The asymptotic distribution under  $H_0$  can be validated by analyzing its accuracy when calculating a threshold used to obtain ROC-curves. Note that this analysis also shows the applicability of the results for a practical case of timing synchronization.

We calculate the ROC curves in two different ways. The first is the ROC curve calculated empirically. We determine a threshold  $s$  from the empirical distribution under  $H_0$  which gives a given probability of false alarm as  $P_{fa} = \mathbb{P}(\eta_N > s)$ .

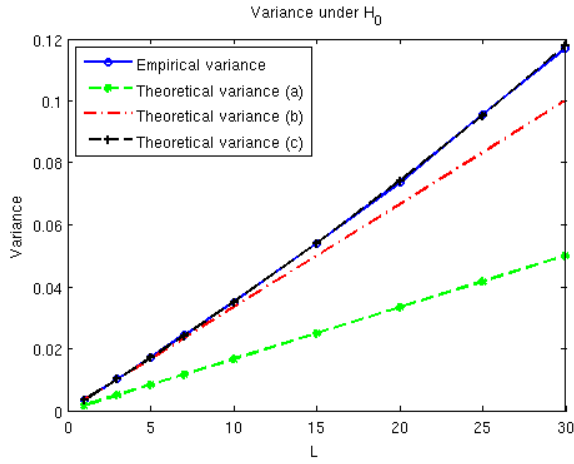


Fig. 3.  $H_0$ : Asymptotic variances as a function of  $L$

Its corresponding probability of non-detection,  $P_{nd}$ , is then obtained as the probability that the empirical values of the synchronization statistics under  $H_1$  pass this threshold. The other ROC-curves are obtained by calculating the threshold  $s$  from the asymptotic Gaussian distributions under  $H_0$ , and using this theoretical threshold to calculate the  $P_{nd}$  from the empirical distribution under  $H_1$ .

Figure 4 shows the ROC-curves obtained with the approaches mentioned above when  $L = 10$ ,  $M = 150$ ,  $N = 300$ . Since the standard asymptotic analysis (a) gives very bad results, its results are omitted. It is clear that ROC-curve obtained by using the asymptotic distribution (b), obtained with the assumption that  $L$  is small, differs greatly from the results from the approximation (c), even for this relatively small value of  $L$ . This is because the theoretical threshold depends greatly on the expected value, and if it is not precisely evaluated, it gives erroneous results. In (c), the model where  $N, M, L \rightarrow \infty$ , the expected value and variance are very close to their empirical counterparts, and the resulting threshold can be used to precisely predict the synchronization performance for the set of parameters used when  $P_{fa} \geq 10^{-3}$  and  $P_{nd} > 10^{-3}$ . Figure 5 shows, for the regime (c), the ROC

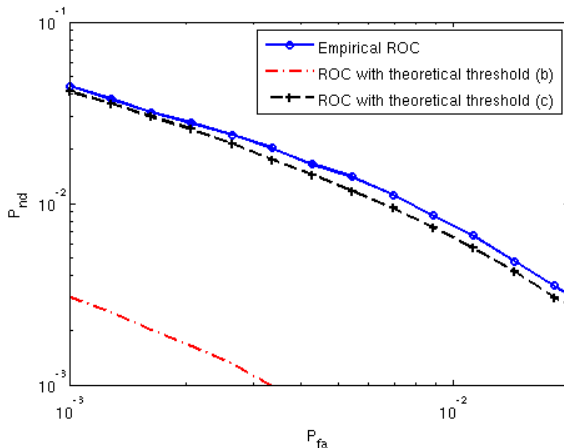


Fig. 4. ROC curve obtained with theoretical threshold plotted with the empirical ROC curve

curves obtained with the theoretical threshold, together with the empirical results. In the figure,  $L$  goes from 1 to 20, while  $M = 15L$  goes from 15 to 300 and  $N = 30L$  goes from 30 to 600. It is seen that when the three parameters grow, the distance between the theoretical and empirical ROC curves decreases.

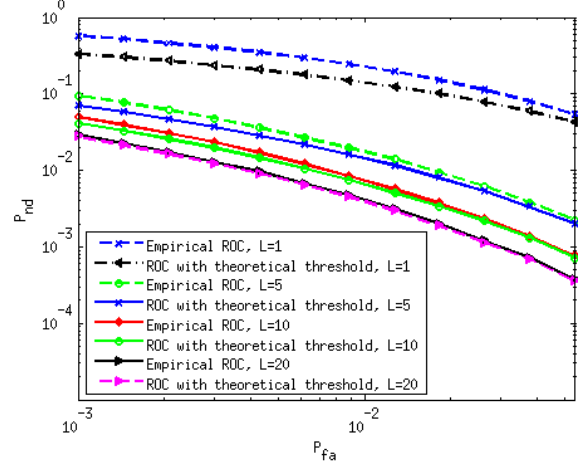


Fig. 5. ROC curve obtained with theoretical threshold, for proportionally growing  $N, M, L$ , model (c)

#### D. Comparison of the asymptotic means and variances of the approximations of $\eta_N$ under $H_1$ .

In this section, we will proceed to validate the expected value and variance under  $H_1$ .

Figures 6 and 7 validate the asymptotic expected values and variances under  $H_1$ . Similarly to hypothesis  $H_0$ , the theoretical expected values and variances are poorly evaluated using the standard asymptotic analysis (a). We note that the asymptotic expected values deduced for the regime (c) are very close to the empirical expected values and variances. For an  $L$  sufficiently small, however, also the regime (b) gives asymptotic expected values and variances that are close to their empirical counterparts.

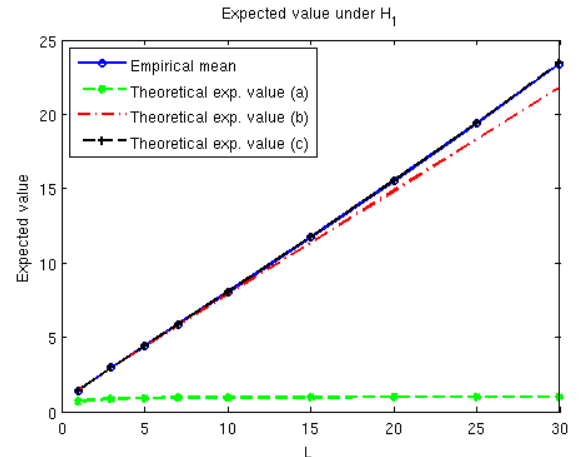


Fig. 6.  $H_1$ : Asymptotic expected values as a function of  $L$

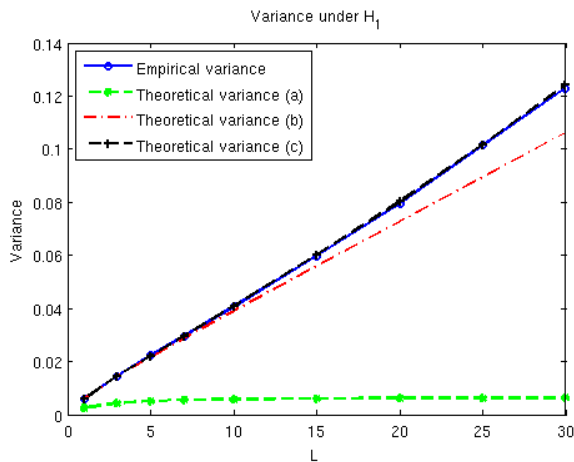


Fig. 7.  $H_1$ : Asymptotic variances as a function of  $L$

### E. Validation of asymptotic distribution under $H_1$

To validate the asymptotic distributions under  $H_1$ , we calculate theoretical ROC-curves using both asymptotic distributions. For each  $P_{fa}$ , a threshold  $s$  is calculated from the theoretical Gaussian distribution under  $H_0$ . This threshold is then used to calculate the  $P_{nd}$  from the theoretical Gaussian distribution under  $H_1$ , using  $P_{nd} = 1 - \mathbb{P}_{H_1}(\eta_N > s)$ . Figure 8 shows these theoretical ROC curves plotted together with the empirical ROC curve. Here,  $L = 10$ ,  $M = 150$  and  $N = 300$ . It is seen that the approximation corresponding to the regime  $N, M, L \rightarrow \infty$  provides, as in the context of hypothesis  $H_0$ , a more accurate theoretical ROC curve. It is seen that the ROC curve associated with the regime small  $L$  (b) is closer from the empirical ROC curve than in the context of hypothesis  $H_0$ . This is because the corresponding asymptotic means are, for both  $H_0$  and  $H_1$ , less than the actual empirical means. These two errors tend to compensate in the theoretical ROC curves (b), which explains why the theoretical ROC curve (b) of figure 8 is more accurate than the corresponding ROC curve of figure 4, for small  $L$ .

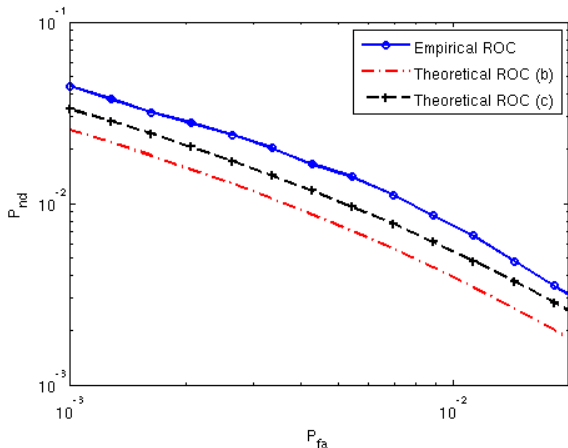


Fig. 8. Theoretical ROC curves plotted with the empirical ROC curve

We now evaluate the behaviour of the ROC curves when  $N, M, L$  grow at the same rate. In figure 9,  $L$  goes from 1

to 20, while  $M = 15L$  goes from 15 to 300 and  $N = 30L$  goes from 30 to 600. The results show that as  $N, M, L$  grow proportionally, the theoretical results tend to approach the empirical values, but that, in contrast with the context of figure 5, a residual error remains. It would be interesting to evaluate more accurately the asymptotic behaviour of  $\eta_N$  under  $H_1$  in the regime  $L \rightarrow +\infty$ , and to check if the residual error tends to diminish. However, as mentioned in Paragraph IV-B.2, this needs to establish a central limit theorem for linear statistics of the eigenvalues of non zero mean large F-matrices, which is a non trivial task.

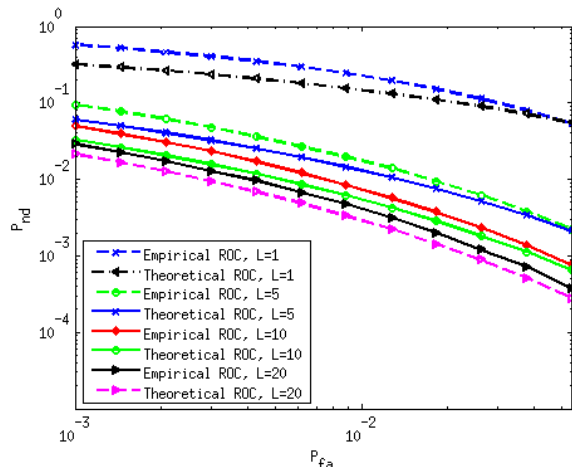


Fig. 9. Theoretical ROC curves for proportionally growing  $N, M, L$ , model (c)

## VI. CONCLUSION.

In this paper, we have studied the behaviour of the multi-antenna GLR detection test statistics  $\eta_N$  of a known signal corrupted by a multi-path deterministic channel and an additive white Gaussian noise with unknown spatial covariance. We have addressed the case where the number of sensors  $M$  and the number of samples  $N$  of the training sequence converge towards  $\infty$  at the same rate. When the number of paths  $L$  does not scale with  $M$  and  $N$ , we have established that  $\eta_N$  has a Gaussian behaviour with asymptotic mean  $L \log \frac{1}{1-M/N}$  and variance  $\frac{L}{N} \frac{M/N}{1-M/N}$ . This is in contrast with the standard regime  $N \rightarrow +\infty$  and  $M$  fixed where  $\eta_N$  has a  $\chi^2$  behaviour. Under hypothesis  $H_1$ ,  $\eta_N$  has still a Gaussian behaviour. The corresponding asymptotic mean and variance are obtained as the sum of the asymptotic mean and variance in the standard regime  $N \rightarrow +\infty$  and  $M$  fixed, and  $L \log \frac{1}{1-M/N}$  and  $\frac{L}{N} \frac{M/N}{1-M/N}$  respectively, i.e. the asymptotic mean and variance under  $H_0$ . We have also considered the case where the number of paths  $L$  converges towards  $\infty$  at the same rate as  $M$  and  $N$ . Using known results of [2] and [24], concerning the behaviour of linear statistics of the eigenvalues of large F-matrices, we have deduced that in the regime where  $L, M, N$  converge to  $\infty$  at the same rate,  $\eta_N$  still has a Gaussian behaviour under  $H_0$ , but with a different mean and variance. The analysis of  $\eta_N$  under  $H_1$  when  $L, M, N$  converge to  $\infty$  needs to establish a central limit theorem for linear statistics of the eigenvalues of

large non zero-mean F-matrices, a difficult task that we will address in a future work. Motivated by the results obtained in the case where  $L$  remains finite, we have proposed to approximate the asymptotic distribution of  $\eta_N$  by a Gaussian distribution whose mean and variance are the sum of the asymptotic mean and variance under  $H_0$  when  $L \rightarrow +\infty$  with the asymptotic mean and variance under  $H_1$  in the standard regime  $N \rightarrow +\infty$  and  $M$  fixed. Numerical experiments have shown that the Gaussian approximation corresponding to the standard regime  $N \rightarrow +\infty$  and  $M$  fixed completely fails as soon as  $\frac{M}{N}$  is not small enough. The large system approximations provide better results when  $\frac{M}{N}$  increases, while also allowing to capture the actual performance for small values of  $\frac{M}{N}$ . We have also observed that, for finite values of  $L, M, N$ , the Gaussian approximation obtained in the regime  $L, M, N$  converge towards  $\infty$  is more accurate than the approximation in which  $L$  is fixed. In particular, the ROC curves that are obtained using the former large system approximation are accurate approximations of the empirical ones in a reasonable range of  $P_{fa}, P_{nd}$ . We therefore believe that our results can be used to reliably predict the performance of the GLRT, and that the tools that are developed in this paper are useful in the context of large antenna arrays.

#### APPENDIX I USEFUL TECHNICAL RESULTS.

In this appendix, we provide some useful technical results concerning the behaviour of certain large random matrices. In the remainder of this appendix,  $\Sigma_N$  represents a  $M \times N$  matrix with  $\mathcal{N}_{\mathbb{C}}(0, \frac{\sigma^2}{N})$  i.i.d. elements. We of course assume in this section that  $M$  and  $N$  both converge towards  $+\infty$  in such a way that  $c_N = \frac{M}{N} < 1$  converges towards  $c < 1$ . In the following, we give some results concerning the behaviour of the eigenvalues  $\hat{\lambda}_{1,N} \leq \hat{\lambda}_{2,N} \dots \leq \hat{\lambda}_{M,N}$  of the matrix  $\Sigma_N \Sigma_N^*$  as well as on its resolvent  $\mathbf{Q}_N(z)$  defined for  $z \in \mathbb{C} - \mathbb{R}^+$  by

$$\mathbf{Q}_N(z) = (\Sigma_N \Sigma_N^* - z \mathbf{I}_M)^{-1} \quad (69)$$

We first state the following classical result (see e.g. [2], Theorem 5.11).

**Proposition 1.** *When  $N \rightarrow +\infty$ ,  $\hat{\lambda}_{1,N}$  converges almost surely towards  $\sigma^2(1 - \sqrt{c})^2$  while  $\hat{\lambda}_{M,N}$  converges a.s. to  $\sigma^2(1 + \sqrt{c})^2$ .*

In the following, we denote by  $\mathcal{I}_\epsilon$  the interval defined by

$$\mathcal{I}_\epsilon = [\sigma^2(1 - \sqrt{c})^2 - \epsilon, \sigma^2(1 + \sqrt{c})^2 + \epsilon] \quad (70)$$

(with  $\epsilon$  chosen in such a way that  $\sigma^2(1 - \sqrt{c})^2 - \epsilon > 0$ ) and by  $\mathcal{E}_N$  the event defined by

$$\mathcal{E}_N = \{\text{one of the } (\hat{\lambda}_{k,N})_{k=1, \dots, M} \text{ escapes from } \mathcal{I}_\epsilon\} \quad (71)$$

and remark that the almost sure convergence of  $\hat{\lambda}_{1,N}$  and  $\hat{\lambda}_{M,N}$  implies that

$$\mathbb{1}_{\mathcal{E}_N^c} = 1 \text{ almost surely for each } N \text{ larger than a random integer} \quad (72)$$

Proposition 1 implies that the resolvent  $\mathbf{Q}_N(z)$  is almost surely defined on  $\mathbb{C} - \mathcal{I}_\epsilon$  for  $N$  large enough, and in particular for  $z = 0$ .

Another important property is the almost sure convergence of the empirical eigenvalue distribution  $\hat{\mu}_N = \frac{1}{M} \sum_{k=1}^M \delta_{\hat{\lambda}_{k,N}}$  of  $\Sigma_N \Sigma_N^*$  towards the Marcenko-Pastur distribution (see e.g. [2] and [20] and the references therein). Formally, this means that the Stieltjes transform  $\hat{m}_N(z)$  of  $\hat{\mu}_N$  defined by

$$\hat{m}_N(z) = \int_{\mathbb{R}} \frac{d\hat{\mu}_N(\lambda)}{\lambda - z} = \frac{1}{M} \text{Tr}(\mathbf{Q}_N(z)) \quad (73)$$

satisfies

$$\lim_{N \rightarrow +\infty} (\hat{m}_N(z) - m_{c_N}(z)) = 0 \quad (74)$$

almost surely for each  $z \in \mathbb{C} - \mathbb{R}^+$  (and uniformly on each compact subset of  $\mathbb{C} - \mathbb{R}^+$ ), where  $m_{c_N}(z)$  represents the Stieltjes transform of the Marcenko-Pastur distribution of parameter  $c_N$ , denoted by  $\mu_{c_N}$  in the following.  $m_{c_N}(z)$  satisfies the following fundamental equation

$$m_{c_N}(z) = \frac{1}{-z(1 + \sigma^2 c_N m_{c_N}(z)) + \sigma^2(1 - c_N)} \quad (75)$$

for each  $z \in \mathbb{C}$ .  $\mu_{c_N}$  is known to be absolutely continuous, its support is the interval  $[\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$ , and its density is given by

$$\frac{\sqrt{(x - x_{c_N}^-)(x_{c_N}^+ - x)}}{2\sigma^2 c_N \pi x} \mathbb{1}_{[x_{c_N}^-, x_{c_N}^+]}(x). \quad (76)$$

with  $x_{c_N}^- = \sigma^2(1 - \sqrt{c_N})^2$  and  $x_{c_N}^+ = \sigma^2(1 + \sqrt{c_N})^2$ . As  $\mu_{c_N}$  is supported by  $[\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$ , the almost sure convergence (74) holds not only on  $\mathbb{C} - \mathbb{R}^+$ , but also for each  $z \in \mathbb{C} - \mathcal{I}_\epsilon$ . In particular, (74) is valid for  $z = 0$ . Solving the equation (75) for  $z = 0$  leads immediately to  $m_{c_N}(0) = \frac{1}{\sigma^2(1 - c_N)}$ , and to

$$\lim_{N \rightarrow +\infty} \frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-1} - \frac{1}{\sigma^2(1 - c_N)} = 0 \quad (77)$$

almost surely. Taking the derivative of (74) w.r.t.  $z$  at  $z = 0$ , and using that  $m'_{c_N}(0) = \frac{1}{\sigma^4(1 - c_N)^3}$ , we also obtain that

$$\lim_{N \rightarrow +\infty} \frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-2} - \frac{1}{\sigma^4(1 - c_N)^3} = 0 \quad (78)$$

almost surely. Moreover, it is possible to specify the convergence speed in (77) and (78). The following proposition is a direct consequence of Theorem 9.10 in [2].

**Proposition 2.** *It holds that*

$$\frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-1} - \frac{1}{\sigma^2(1 - c_N)} = \mathcal{O}_P\left(\frac{1}{N}\right) \quad (79)$$

$$\frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-2} - \frac{1}{\sigma^4(1 - c_N)^3} = \mathcal{O}_P\left(\frac{1}{N}\right) \quad (80)$$

Theorem 9.10 in [2] implies that the left hand side of (79), renormalized by  $N$ , converges in distribution towards a Gaussian distribution, which, in turn, leads to (79). (80) holds for the same reason.

**Remark 5.** As  $c_N \rightarrow c$ , the previous results of course imply that  $\frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-1}$  (resp.  $\frac{1}{M} \text{Tr}(\Sigma_N \Sigma_N^*)^{-2}$ ) converge towards  $\frac{1}{\sigma^2(1-c)}$  (resp.  $\frac{1}{\sigma^4(1-c)^3}$ ). However, the rate of convergence is not a  $\mathcal{O}_P(\frac{1}{N})$  term if the convergence speed of  $c_N$  towards  $c$  is less than  $\mathcal{O}(\frac{1}{N})$ . Therefore, it is more relevant to approximate the left hand sides of (79) and (80) by  $\frac{1}{\sigma^2(1-c_N)}$  and  $\frac{1}{\sigma^4(1-c_N)^3}$ .

The above results allow to characterize the asymptotic behaviour of the normalized trace of  $(\Sigma_N \Sigma_N^*)^{-1}$  and  $(\Sigma_N \Sigma_N^*)^{-2}$ . However, it is also useful to obtain similar results on the bilinear forms of these matrices.

**Proposition 3.** We consider two deterministic  $M$ -dimensional unit norm vectors  $\mathbf{u}_N$  and  $\mathbf{v}_N$ . Then, it holds that

$$\lim_{N \rightarrow +\infty} \mathbf{u}_N^* (\Sigma_N \Sigma_N^*)^{-1} \mathbf{v}_N - \frac{\mathbf{u}_N^* \mathbf{v}_N}{\sigma^2(1-c_N)} = 0 \quad (81)$$

and that

$$\lim_{N \rightarrow +\infty} \mathbf{u}_N^* (\Sigma_N \Sigma_N^*)^{-2} \mathbf{v}_N - \frac{\mathbf{u}_N^* \mathbf{v}_N}{\sigma^4(1-c_N)^3} = 0 \quad (82)$$

almost surely. Moreover,

$$\mathbf{u}_N^* (\Sigma_N \Sigma_N^*)^{-1} \mathbf{v}_N - \frac{\mathbf{u}_N^* \mathbf{v}_N}{\sigma^2(1-c_N)} = \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right) \quad (83)$$

Finally, if  $\mathbf{C}_N$  is a positive  $M \times M$  matrix such that  $\text{Rank}(\mathbf{C}_N) = K$  is independent of  $N$ , and satisfying for each  $N$   $0 < d_1 \leq \text{Tr}(\mathbf{C}_N^2) < d_2 < \infty$  for some constants  $d_1$  and  $d_2$ , then, we consider the sequence of random variables  $(\kappa_N)_{N \geq 1}$  defined by

$$\kappa_N = \text{Tr}(\mathbf{C}_N (\Sigma_N \Sigma_N^*)^{-1}) \quad (84)$$

Define by  $\theta_N$  the term

$$\theta_N = \frac{\text{Tr}(\mathbf{C}_N^2)}{\sigma^4(1-c_N)^3} \quad (85)$$

Then, it holds that

$$\mathbb{E} \left[ \exp \left( iu\sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) \right) \right] - \exp \left( -\frac{\theta_N u^2}{2} \right) \rightarrow 0 \quad (86)$$

for each  $u \in \mathbb{R}$ , and that

$$\frac{\sqrt{N}}{\sqrt{\theta_N}} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) \rightarrow_{\mathcal{D}} \mathcal{N}(0, 1) \quad (87)$$

The almost sure convergence result (81) is well known (see e.g. [12] in the context of a more general matrix model), while (82) can be established by differentiating the behaviour of the bilinear forms of  $\mathbf{Q}_N(z)$  w.r.t.  $z$ . Moreover, (83) is a consequence of (87) used for the rank 1 matrix  $\mathbf{C}_N = \mathbf{v}_N \mathbf{u}_N^*$ . (86) and (87) are new and need to be established.

A technical difficulty appears in the present context because we consider the resolvent of the matrix  $\Sigma_N \Sigma_N^*$  at  $z = 0$  while in previous works,  $z$  is supposed to belong to  $\mathbb{C} - \mathbb{R}^+$ . To solve this issue, we use the regularization technic introduced in a more general context in [11]. For the proof, we refer the reader to Appendix III.

We finish this appendix by a standard result whose proof is omitted.

**Proposition 4.** We consider a  $M \times L$  random matrix  $\Gamma_N$  with  $\mathcal{N}_{\mathbb{C}}(0, \frac{\sigma^2}{N})$  i.i.d. entries, as well as the following deterministic matrices:  $\mathbf{A}_N$  is  $M \times M$  and hermitian,  $\mathbf{B}_N$  is  $M \times L$  and satisfies  $\sup_N \|\mathbf{B}_N\| < +\infty$  while  $\mathbf{D}_N$  is a positive  $L \times L$  matrix and also verifies  $\sup_N \|\mathbf{D}_N\| < +\infty$ . Then, if  $(\omega_N)_{N \geq 1}$  represents the sequence of random variables defined by

$$\omega_N = \text{Tr}[\mathbf{D}_N (\Gamma_N^* \mathbf{A}_N \Gamma_N + \Gamma_N^* \mathbf{B}_N + \mathbf{B}_N^* \Gamma_N)] \quad (88)$$

it holds that

$$\mathbb{E}(\omega_N) = \sigma^2 \frac{1}{N} \text{Tr}(\mathbf{A}_N) \text{Tr}(\mathbf{D}_N), \quad (89)$$

$$\text{Var}(\omega_N) = \frac{1}{N} \zeta_N$$

where  $\zeta_N$  is defined by

$$\zeta_N = \sigma^4 \frac{1}{N} \text{Tr}(\mathbf{A}_N^2) \text{Tr}(\mathbf{D}_N^2) + 2\sigma^2 \frac{1}{N} \text{Tr}(\mathbf{D}_N^2 \mathbf{B}_N^* \mathbf{B}_N) \quad (90)$$

Moreover,

$$\mathbb{E}|\omega_N - \mathbb{E}(\omega_N)|^4 \leq \frac{a_1}{N^2} + \frac{a_2}{N^2} \left( \frac{1}{N} \text{Tr}(\mathbf{A}_N^2) \right)^2 + \frac{a_3}{N^3} \frac{1}{N} \text{Tr}(\mathbf{A}_N^8) \quad (91)$$

where  $a_1, a_2, a_3$  are constant terms depending on  $L, \sup_N \|\mathbf{B}_N\|$  and  $\sup_N \|\mathbf{D}_N\|$ . Finally, if  $\limsup_N \zeta_N < +\infty$ , it holds that

$$\mathbb{E} \left( \exp iu\sqrt{N} (\omega_N - \mathbb{E}(\omega_N)) \right) - e^{-\frac{u^2 \zeta_N}{2}} \rightarrow 0 \quad (92)$$

for each  $u \in \mathbb{R}$ .

## APPENDIX II PROOFS OF THEOREMS 1 AND 2

**Proof of Theorem 1.** In order to establish Theorem 1, we use the results of Appendix I for the matrix  $\Sigma_N = \frac{1}{\sqrt{N}} \mathbf{V}_1$ . We note that  $\frac{1}{\sqrt{N}} \mathbf{V}_1$  is a  $M \times (N-L)$  matrix while the results of Appendix I have been presented in the context of a  $M \times N$  matrix. In principle, it should be necessary to exchange  $N$  by  $N-L$  in Propositions 1 to 3. However,  $c_N - \frac{M}{N-L} = \mathcal{O}(\frac{1}{N})$ , so that it is possible to use the results of the above propositions without exchanging  $N$  by  $N-L$ .

We first verify (30). For this, we introduce the event  $\mathcal{E}_N$  defined by (71). We first remark that  $\eta_N - \eta_N \mathbb{1}_{\mathcal{E}_N^c} \rightarrow 0, a.s.$  It is thus sufficient to study the behaviour of  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$  which is also equal to

$$\eta_N \mathbb{1}_{\mathcal{E}_N^c} = \log \det(\mathbf{I} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c}) \quad (93)$$

We now study the behaviour of each entry  $(k, l)$  of matrix  $\mathbb{1}_{\mathcal{E}_N^c} \mathbf{F}_N$ . For this, we use Proposition 4 for  $\mathbf{D}_N = \mathbf{e}_k \mathbf{e}_l^T$ ,  $\Gamma_N = \frac{\mathbf{V}_2}{\sqrt{N}}$  and  $\mathbf{A}_N = \mathbb{1}_{\mathcal{E}_N^c} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1}$ .  $\mathbf{A}_N$  is of course not deterministic, but as  $\mathbf{V}_2$  and  $\mathbf{V}_1$  are independent, it is possible to use the results of Proposition 4 by replacing the mathematical expectation operator by the mathematical

expectation operator  $\mathbb{E}_{\mathbf{V}_2}$  w.r.t.  $\mathbf{V}_2$ . We note that the present matrix  $\mathbf{A}_N$  verifies

$$\mathbf{A}_N \leq \frac{\mathbf{I}}{\sigma^2(1-\sqrt{c})^2 - \epsilon} \quad (94)$$

because  $\mathbb{1}_{\mathcal{E}_N^c} \neq 0$  implies that all the eigenvalues of  $\frac{\mathbf{V}_1 \mathbf{V}_1^*}{N}$  belong to  $\mathcal{L}_\epsilon = [\sigma^2(1-\sqrt{c})^2 - \epsilon, \sigma^2(1+\sqrt{c})^2 + \epsilon]$ . Therefore, (91) immediately implies that

$$\mathbb{E}_{\mathbf{V}_2} |\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c} - \mathbb{E}_{\mathbf{V}_2} (\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c})|^4 \leq \frac{a}{N^2} \quad (95)$$

where  $a$  is a deterministic constant. Taking the mathematical expectation of the above inequality w.r.t.  $\mathbf{V}_1$ , and using the Borel-Cantelli Lemma lead to

$$\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c} - \mathbb{E}_{\mathbf{V}_2} (\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c}) \rightarrow 0 \text{ a.s.} \quad (96)$$

or equivalently, to

$$\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c} - \delta(k-l) \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \rightarrow 0 \text{ a.s.} \quad (97)$$

(77) implies that  $\mathbf{F}_{N,k,l} \mathbb{1}_{\mathcal{E}_N^c} - \delta(k-l) \frac{c_N}{1-c_N} \rightarrow 0$  almost surely, or equivalently that

$$\mathbf{F}_N - \frac{c_N}{1-c_N} \mathbf{I} \rightarrow 0 \text{ a.s.} \quad (98)$$

This eventually leads to (30).

We now establish (31). For this, we first remark that (72) implies that  $\eta_N = \eta_N \mathbb{1}_{\mathcal{E}_N^c} + \mathcal{O}_P(\frac{1}{N^p})$  for each integer  $p$ . Therefore, the asymptotic behaviour of the distribution of the left hand side of (31) is not modified if  $\eta_N$  is replaced by  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$  given by (93). We denote by  $\mathbf{\Delta}_N$  the matrix defined by

$$\mathbf{\Delta}_N = \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} - \frac{c_N}{1-c_N} \mathbf{I} \quad (99)$$

We first prove that  $\mathbf{\Delta}_N = \mathcal{O}_P(\frac{1}{\sqrt{N}})$ . For this, we express  $\mathbf{\Delta}_N$  as

$$\begin{aligned} \mathbf{\Delta}_N &= \left( \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} - \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \mathbb{1}_{\mathcal{E}_N^c} \mathbf{I} \right) + \\ &\quad \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \mathbb{1}_{\mathcal{E}_N^c} \mathbf{I} - \frac{c_N}{1-c_N} \mathbf{I} \end{aligned} \quad (100)$$

The first term of the right hand side of (100) is  $\mathcal{O}_P(\frac{1}{\sqrt{N}})$  because the fourth-order moments of its entries are  $\mathcal{O}(\frac{1}{N^2})$  terms. As for the second term, (79) implies that it is a  $\mathcal{O}_P(\frac{1}{N})$ . A standard second order expansion of  $\log \det(\mathbf{I} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c})$  leads to

$$\eta_N \mathbb{1}_{\mathcal{E}_N^c} = L \log \frac{1}{1-c_N} + (1-c_N) \text{Tr}(\mathbf{\Delta}_N) + \mathcal{O}_P\left(\frac{1}{N}\right) \quad (101)$$

Therefore, it holds that

$$\begin{aligned} \sqrt{N} \left( \eta_N \mathbb{1}_{\mathcal{E}_N^c} - L \log \frac{1}{1-c_N} \right) &= \sqrt{N} (1-c_N) \text{Tr}(\mathbf{\Delta}_N) \\ &\quad + \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right), \end{aligned} \quad (102)$$

or, using (100), that

$$\begin{aligned} \sqrt{N} \left( \eta_N \mathbb{1}_{\mathcal{E}_N^c} - L \log \frac{1}{1-c_N} \right) &= \\ \sqrt{N} (1-c_N) \text{Tr} \left( \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} - \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \mathbb{1}_{\mathcal{E}_N^c} \right) & \\ + \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right) & \quad (103) \end{aligned}$$

As

$$\mathbb{E}_{\mathbf{V}_2} \left( \text{Tr} (\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c}) \right) = \sigma^2 c_N \frac{1}{M} \text{Tr} \left( \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \mathbb{1}_{\mathcal{E}_N^c} \right), \quad (104)$$

Proposition 4 used for  $\mathbf{A}_N = \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-1} \mathbb{1}_{\mathcal{E}_N^c}$ ,  $\mathbf{B}_N = 0$  and  $\mathbf{D}_N = (1-c_N) \mathbf{I}$  leads to

$$\begin{aligned} \mathbb{E}_{\mathbf{V}_2} \left( \exp iu \sqrt{N} \left( \eta_N - L \log \frac{1}{1-c_N} \right) \right) - \\ \exp \left[ -\frac{u^2}{2} \sigma^4 L (1-c_N)^2 c_N \frac{1}{M} \text{Tr} \left( \frac{\mathbf{V}_1 \mathbf{V}_1^*}{N} \right)^{-2} \mathbb{1}_{\mathcal{E}_N^c} \right] \rightarrow 0 \end{aligned} \quad (105)$$

a.s. for each  $u \in \mathbb{R}$ . (78) and the dominated convergence theorem finally implies that

$$\begin{aligned} \mathbb{E} \left( \exp iu \sqrt{N} \left( \eta_N - L \log \frac{1}{1-c_N} \right) \right) - \\ \exp \left[ -\frac{u^2}{2} \frac{L c_N}{1-c_N} \right] \rightarrow 0 \end{aligned} \quad (106)$$

This establishes (31).

**Proof of Theorem 2** We recall that, under  $H_1$ ,  $\eta_N$  is given by (22). As in the proof of Theorem 1, it is sufficient to study the regularized statistics  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$  which is also equal to

$$\eta_N \mathbb{1}_{\mathcal{E}_N^c} = \log \det (\mathbf{I}_L + \mathbb{1}_{\mathcal{E}_N^c} \mathbf{G}_N) \quad (107)$$

In order to evaluate the almost sure behaviour of  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$ , we expand  $\mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c}$  as

$$\begin{aligned} \mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c} &= \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^* / N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} + \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} + \\ &\quad (\mathbf{V}_2 / \sqrt{N})^* (\mathbf{V}_1 \mathbf{V}_1^* / N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} + \\ &\quad \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^* / N)^{-1} (\mathbf{V}_2 / \sqrt{N}) \mathbb{1}_{\mathcal{E}_N^c} \end{aligned} \quad (108)$$

By (81), the first term of the right hand side of (108) behaves almost surely as  $\frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)}$ , while it has been shown before that the second term converges a.s. towards  $\frac{c_N}{1-c_N} \mathbf{I}$ . To address the behaviour of entry  $(k, l)$  of the sum of the third and the fourth terms, we use Proposition 4 for  $\mathbf{\Gamma}_N = \frac{\mathbf{V}_2}{\sqrt{N}}$ ,  $\mathbf{A}_N = 0$ ,  $\mathbf{B}_N = (\mathbf{V}_1 \mathbf{V}_1^* / N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c}$  and  $\mathbf{D}_N = \mathbf{e}_k \mathbf{e}_l^T$ . (91) implies that entry  $(k, l)$  converges almost surely towards 0. Therefore, we have proved that

$$\mathbf{G}_N - \left( \frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)} + \frac{c_N}{1-c_N} \mathbf{I} \right) \rightarrow \mathbf{0} \text{ a.s.} \quad (109)$$

from which (45) follows immediately.

The proof of (47) is similar to the proof of (31), thus we do not provide all the details. We replace  $\eta_N$  by  $\eta_N \mathbb{1}_{\mathcal{E}_N^c}$ , and remark that the matrix  $\mathbf{\Delta}_N$ , given by

$$\mathbf{\Delta}_N = \mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c} - \left( \frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)} + \frac{c_N}{1-c_N} \mathbf{I} \right) \quad (110)$$

verifies  $\mathbf{\Delta}_N = \mathcal{O}_P(\frac{1}{\sqrt{N}})$ . To check this, it is sufficient to use the expansion (49), and to recognize that:

- by (83),

$$\mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} - \frac{\mathbf{H}^* \mathbf{H}}{\sigma^2(1-c_N)} = \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right), \quad (111)$$

- by Proposition 4 and (91),

$$\begin{aligned} & (\mathbf{V}_2/\sqrt{N})^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c} + \\ & \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} (\mathbf{V}_2/\sqrt{N}) \mathbb{1}_{\mathcal{E}_N^c} = \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right) \end{aligned} \quad (112)$$

- it has been shown before that

$$\mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c} - \frac{c_N}{1-c_N} \mathbf{I} = \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right). \quad (113)$$

Using a standard linearization of  $\log \det(\mathbf{I} + \mathbf{G}_N \mathbb{1}_{\mathcal{E}_N^c})$ , this implies that

$$\eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1} = \text{Tr}(\mathbf{D}_N \mathbf{\Delta}_N) + \mathcal{O}_P(1/N) \quad (114)$$

where  $\mathbf{D}_N$  is the  $L \times L$  matrix given by

$$\mathbf{D}_N = (1-c_N)(\mathbf{I}_L + \mathbf{H}^* \mathbf{H}/\sigma^2)^{-1} \quad (115)$$

We define  $\kappa_N$  and  $\omega_N$  by

$$\kappa_N = \text{Tr}\left(\mathbf{C}_N (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1}\right) \quad (116)$$

and

$$\begin{aligned} \omega_N &= \text{Tr}\left[\mathbf{D}_N \mathbf{F}_N \mathbb{1}_{\mathcal{E}_N^c}\right] + \\ & \text{Tr}\left[\mathbf{D}_N (\mathbf{V}_2/\sqrt{N})^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c}\right] + \\ & \text{Tr}\left[\mathbf{D}_N \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} (\mathbf{V}_2/\sqrt{N}) \mathbb{1}_{\mathcal{E}_N^c}\right] \end{aligned} \quad (117)$$

where  $\mathbf{C}_N$  the  $M \times M$  matrix given by

$$\mathbf{C}_N = (1-c_N) \mathbf{H}(\mathbf{I}_L + \mathbf{H}^* \mathbf{H}/\sigma^2)^{-1} \mathbf{H}^* \quad (118)$$

Using (114), we obtain that

$$\begin{aligned} \eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1} &= \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} + \\ \omega_N - \frac{c_N}{1-c_N} \text{Tr}(\mathbf{D}_N) &+ \mathcal{O}_P\left(\frac{1}{N}\right) \end{aligned} \quad (119)$$

We also remark that (79) used for  $\mathbf{\Sigma}_N = \frac{1}{\sqrt{N}} \mathbf{V}_1$  implies that

$$\omega_N - \mathbb{E}_{\mathbf{V}_2}(\omega_N) = \omega_N - \frac{c_N}{1-c_N} \text{Tr}(\mathbf{D}_N) + \mathcal{O}_P\left(\frac{1}{N}\right) \quad (120)$$

Therefore, it holds that

$$\sqrt{N}(\eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1}) = \sqrt{N}(\text{Tr}(\mathbf{D}_N \mathbf{\Delta}_N)) \quad (121)$$

can be written as

$$\begin{aligned} \sqrt{N}(\eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1}) &= \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right) + \\ & \sqrt{N}(\omega_N - \mathbb{E}_{\mathbf{V}_2}(\omega_N)) + \mathcal{O}_P\left(\frac{1}{\sqrt{N}}\right) \end{aligned} \quad (122)$$

We denote by  $\zeta_N$  the term

$$\begin{aligned} \zeta_N &= \sigma^4 \frac{1}{N} \text{Tr}\left((\mathbf{V}_1 \mathbf{V}_1^*/N)^{-2} \mathbb{1}_{\mathcal{E}_N^c}\right) \text{Tr}(\mathbf{D}_N^2) + \\ & 2\sigma^2 \frac{1}{N} \text{Tr}\left(\mathbf{D}_N^2 \mathbf{H}^* (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c}\right) \end{aligned} \quad (123)$$

We use Proposition 4 and (92) for  $\mathbf{\Gamma}_N = \mathbf{V}_2/\sqrt{N}$ ,  $\mathbf{A}_N = (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbb{1}_{\mathcal{E}_N^c}$  and  $\mathbf{B}_N = (\mathbf{V}_1 \mathbf{V}_1^*/N)^{-1} \mathbf{H} \mathbb{1}_{\mathcal{E}_N^c}$ , and obtain that

$$\begin{aligned} & \mathbb{E}_{\mathbf{V}_2} \left[ \exp\left(iu \sqrt{N}(\eta_N \mathbb{1}_{\mathcal{E}_N^c} - \bar{\eta}_{N,1})\right) \right] - \\ & \exp\left(iu \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1-c_N)} \right)\right) \exp\left(-\frac{u^2}{2} \zeta_N\right) \rightarrow 0 \end{aligned} \quad (124)$$

a.s.  $\zeta_N$  has almost surely the same behaviour as  $\zeta$  given by

$$\zeta = \frac{c_N}{(1-c_N)^3} \text{Tr}(\mathbf{D}_N^2) + 2 \frac{c_N}{(1-c_N)} \text{Tr}(\mathbf{D}_N^2 \mathbf{H}^* \mathbf{H}) \quad (125)$$

which implies that

$$\exp\left(-\frac{u^2}{2} \zeta_N\right) - \exp\left(-\frac{u^2}{2} \zeta\right) \rightarrow 0 \text{ a.s.} \quad (126)$$

Therefore, taking the mathematical expectation of (124) w.r.t  $\mathbf{V}_1$  and using the dominated convergence theorem as well as (86), lead, after some calculations, to

$$\begin{aligned} & \mathbb{E} \left[ \exp\left(iu \sqrt{N}(\eta_N - \bar{\eta}_{N,1})\right) \right] \\ & - \exp\left[-\frac{u^2}{2} \left( \frac{Lc_N}{1-c_N} + \kappa_1 \right)\right] \rightarrow 0 \end{aligned} \quad (127)$$

for each  $u$ . As  $\inf_N \left(\frac{Lc_N}{1-c_N} + \kappa_1\right) > 0$ , (47) follows from (127) (see Proposition 6 in [10]).

### APPENDIX III PROOF OF (87)

To establish (87), we follow the approach of [10] which is based on the joint use of the integration by parts formula and of the Poincaré-Nash inequality (see section III-B of [10]). However, the approach of [10] allows to manage functionals of the resolvent  $\mathbf{Q}_N(z)$  for  $z \in \mathbb{C} - \mathbb{R}^+$ . For this, the inequality  $\|\mathbf{Q}_N(z)\| \leq \frac{1}{\text{dist}(z, \mathbb{R}^+)}$  plays a fundamental role. For  $z = 0$ ,  $\|\mathbf{Q}_N(0)\|$  coincides with  $\frac{1}{\lambda_{1,N}}$  which is not upper-bounded by a deterministic positive constant for  $N$  greater than a non random integer. This issue was solved before using the regularization term  $\mathbb{1}_{\mathcal{E}_N^c}$ . However, the use of the integration by parts formula and the Poincaré-Nash inequality needs to consider smooth enough functions of  $\mathbf{\Sigma}_N$ . Motivated by [11], we consider the regularization term  $\chi_N$  defined by

$$\chi_N = \det[\phi(\mathbf{\Sigma}_N \mathbf{\Sigma}_N^*)] \quad (128)$$

where  $\phi$  is a smooth function such that

$$\begin{aligned} \phi(\lambda) &= 1 \text{ if } \lambda \in \mathcal{I}_\epsilon = [\sigma^2(1 - \sqrt{c})^2 - \epsilon, \sigma^2(1 + \sqrt{c})^2 + \epsilon] \\ \phi(\lambda) &= 0 \text{ if } \lambda \in [\sigma^2(1 - \sqrt{c})^2 - 2\epsilon, \sigma^2(1 + \sqrt{c})^2 + 2\epsilon]^c \\ \phi &\in [0, 1] \text{ elsewhere} \end{aligned}$$

In the following, we need to use the following property: for each  $\epsilon > 0$ , it holds that

$$P(\mathcal{E}_N) = \mathcal{O}\left(\frac{1}{N^p}\right) \quad (129)$$

where  $\mathcal{E}_N$  is defined by (71). Property (129) is not mentioned in Theorem 5.11 of [2] which addresses the non Gaussian case. However, (129) follows directly from Gaussian concentration arguments.

It is clear that

$$(\boldsymbol{\Sigma}_N \boldsymbol{\Sigma}_N^*)^{-1} \chi_N \leq \frac{\mathbf{I}}{\sigma^2((1 - \sqrt{c})^2 - 2\epsilon)} \quad (130)$$

Lemma 3-9 of [11] also implies that, considered as a function of the entries of  $\boldsymbol{\Sigma}_N$ ,  $\chi_N$  is continuously differentiable. Moreover, it follows from Proposition 1 that almost surely, for  $N$  large enough,  $\chi_N = 1$  and  $\kappa_N = \kappa_N \chi_N$ . Therefore, it holds that  $\kappa_N \chi_N = \kappa_N + \mathcal{O}_P\left(\frac{1}{N^p}\right)$ , and that

$$\begin{aligned} \sqrt{N} \left( \kappa_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1 - c_N)} \right) \\ = \sqrt{N} \left( \kappa_N \chi_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1 - c_N)} \right) + \mathcal{O}_P\left(\frac{1}{N^p}\right) \end{aligned} \quad (131)$$

for each  $p \in \mathbb{N}$ . In order to establish (86), it is thus sufficient to prove that

$$\begin{aligned} \mathbb{E} \left[ \exp \left( iu \sqrt{N} \left( \kappa_N \chi_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1 - c_N)} \right) \right) \right] \\ - \exp \left( -\frac{\theta_N u^2}{2} \right) \rightarrow 0 \end{aligned} \quad (132)$$

for each  $u$ . To obtain (87), we remark that, as  $\inf_N \theta_N > 0$ , it follows from (132) that

$$\frac{\sqrt{N}}{\sqrt{\theta_N}} \left( \kappa_N \chi_N - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1 - c_N)} \right) \rightarrow_{\mathcal{D}} \mathcal{N}_{\mathbb{R}}(0, 1)$$

(see Proposition 6 in [10]). (87) eventually appears as a consequence of (131).

The above regularization trick thus allows to replace the matrix  $(\boldsymbol{\Sigma}_N \boldsymbol{\Sigma}_N^*)^{-1}$  by  $(\boldsymbol{\Sigma}_N \boldsymbol{\Sigma}_N^*)^{-1} \chi_N$ , which verifies (130). In order to establish (132), it is sufficient to prove that

$$\mathbb{E}(\kappa_N \chi_N) - \frac{\text{Tr}(\mathbf{C}_N)}{\sigma^2(1 - c_N)} = o\left(\frac{1}{\sqrt{N}}\right) \quad (133)$$

and that

$$\begin{aligned} \mathbb{E} \left[ \exp \left( iu \sqrt{N} (\kappa_N \chi_N - \mathbb{E}(\kappa_N \chi_N)) \right) \right] \\ - \exp \left( -\frac{\theta_N u^2}{2} \right) \rightarrow 0 \end{aligned} \quad (134)$$

for each  $u$ .

In the rest of this section, to simplify the notations, we omit to write the dependance on  $N$  of the various terms  $\boldsymbol{\Sigma}_N$ ,

$\mathbf{Q}_N(0)$ ,  $\chi_N, \dots$ , and denote them by  $\boldsymbol{\Sigma}, \mathbf{Q}(0), \chi, \dots$ . However, we keep the notation  $c_N$ , in order to avoid confusion between  $c_N$  and  $c$ . Furthermore, the matrix  $\mathbf{Q}(0)$  is denoted by  $\mathbf{Q}$ . If  $x$  is a random variable,  $x^\circ$  represents the zero mean variable  $x^\circ = x - \mathbb{E}(x)$ . In the following, we denote by  $\delta$  the random variable defined by

$$\delta = \sqrt{N} \kappa \chi$$

and by  $\psi^\circ(u)$  the characteristic function of  $\delta^\circ$  defined by

$$\psi^\circ(u) = \mathbb{E}(\exp iu \delta^\circ)$$

We first establish the following Proposition.

**Proposition 5.** *It holds that*

$$(\psi^\circ(u))' = -u \mathbb{E}(\text{Tr}(\mathbf{C}^2 \mathbf{Q}^2 \chi)) \psi^\circ(u) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad (135)$$

where  $'$  represents the derivative w.r.t. the variable  $u$ .

**Proof.** We consider the characteristic function  $\psi(u)$  of  $\delta$ , and evaluate

$$\psi'(u) = i\sqrt{N} \mathbb{E}(\text{Tr}(\mathbf{Q} \mathbf{C} \chi) e^{iu\delta})$$

We remark that  $\mathbf{Q} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^* = \mathbf{I}$  so that

$$\mathbb{E}(\mathbf{Q} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^* \chi e^{iu\delta}) = \mathbb{E}(\chi e^{iu\delta}) \mathbf{I}$$

We claim that

$$\mathbb{E}(\chi e^{iu\delta}) = \psi(u) + \mathcal{O}\left(\frac{1}{N^p}\right) \quad (136)$$

for each  $p$ . We remark that

$$|\mathbb{E}(e^{iu\delta}(1 - \chi))| \leq 1 - \mathbb{E}(\chi)$$

We recall that the event  $\mathcal{E}$  is defined by (71) and that  $P(\mathcal{E}) = \mathcal{O}\left(\frac{1}{N^p}\right)$  for each  $p$  (see (129)).  $\mathbb{1}_{\mathcal{E}^c} \leq \chi$  leads to  $1 - \mathbb{E}(\chi) \leq P(\mathcal{E})$ . This justifies (136). Therefore, it holds that

$$\mathbb{E}(\mathbf{Q} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^* \chi e^{iu\delta}) = \left( \psi(u) + \mathcal{O}\left(\frac{1}{N^p}\right) \right) \mathbf{I} \quad (137)$$

for each  $p$ . We now evaluate each entry of the lefthandside of (137) using the integration by parts formula. For this, we denote by  $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_N)$  the columns of  $\boldsymbol{\Sigma}$ . It holds that

$$(\mathbf{Q} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^*)_{r,s} = \sum_{j=1}^N (\mathbf{Q} \boldsymbol{\xi}_j)_r \bar{\boldsymbol{\Sigma}}_{s,j}$$

and that

$$\mathbb{E}[(\mathbf{Q} \boldsymbol{\xi}_j)_r \bar{\boldsymbol{\Sigma}}_{s,j} \chi e^{iu\delta}] = \sum_{t=1}^M \mathbb{E}(\mathbf{Q}_{r,t} \boldsymbol{\Sigma}_{t,j} \bar{\boldsymbol{\Sigma}}_{s,j} \chi e^{iu\delta})$$

The integration by parts formula leads to

$$\mathbb{E}(\mathbf{Q}_{r,t} \bar{\boldsymbol{\Sigma}}_{s,j} \chi e^{iu\delta} \boldsymbol{\Sigma}_{t,j}) = \frac{\sigma^2}{N} \mathbb{E} \left[ \frac{\partial (\mathbf{Q}_{r,t} \bar{\boldsymbol{\Sigma}}_{s,j} \chi e^{iu\delta})}{\partial \boldsymbol{\Sigma}_{t,j}} \right]$$



After some algebra, we obtain that

$$\begin{aligned} \mathbb{E}(\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j} \chi e^{iu\delta} \Sigma_{t,j}) &= \frac{\sigma^2}{N} \mathbb{E}(\mathbf{Q}_{r,t} \chi e^{iu\delta}) \delta(t=s) \\ &\quad - \frac{\sigma^2}{N} \mathbb{E}((\mathbf{Q}\xi_j)_r \mathbf{Q}_{t,t} \bar{\Sigma}_{s,j} \chi e^{iu\delta}) \\ &\quad - \frac{i\sigma^2 u}{\sqrt{N}} \mathbb{E}(\mathbf{Q}_{r,t} (\mathbf{Q}\mathbf{C}\mathbf{Q}\xi_j)_t \bar{\Sigma}_{s,j} \chi e^{iu\delta}) \\ &\quad + \frac{\sigma^2}{N} \mathbb{E}\left(\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j} e^{iu\delta} \frac{\partial \chi}{\partial \bar{\Sigma}_{t,j}}\right) \end{aligned} \quad (138)$$

We now need to study more precisely the properties of the derivative of  $\chi$  w.r.t.  $\bar{\Sigma}_{t,j}$ . For this, we give the following Lemma

**Lemma 1.** *We denote by  $\mathcal{A}$  the event:*

$$\begin{aligned} \mathcal{A} &= \{\text{one of the } \hat{\lambda}_{k,N} \text{ escapes from } \mathcal{L}_\epsilon\} \\ &\quad \cap \{(\hat{\lambda}_{l,N})_{l=1,\dots,M} \in \text{supp}(\phi)\} \end{aligned} \quad (139)$$

Then, it holds that

$$\frac{\partial \chi}{\partial \bar{\Sigma}_{t,j}} = 0 \text{ on } \mathcal{A}^c \quad (140)$$

and that

$$\mathbb{E}\left|\frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}}\right|^2 = \mathcal{O}\left(\frac{1}{N^p}\right) \quad (141)$$

for each  $p$ .

**Proof.** Lemma 1 follows directly from Lemma 3.9 of [11] and from the calculations in the proof of Proposition 3.3 of [11].

Lemma 1 implies that the last term of (138) is  $\mathcal{O}(\frac{1}{N^p})$  for each  $p$ . To check this, we remark that

$$\mathbb{E}\left(\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j} e^{iu\delta} \frac{\partial \chi}{\partial \bar{\Sigma}_{t,j}}\right) = \mathbb{E}\left(\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j} e^{iu\delta} \mathbb{1}_{\mathcal{A}} \frac{\partial \chi}{\partial \bar{\Sigma}_{t,j}}\right)$$

The Schwartz inequality leads to

$$\begin{aligned} &\left|\mathbb{E}\left(\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j} e^{iu\delta} \mathbb{1}_{\mathcal{A}} \frac{\partial \chi}{\partial \bar{\Sigma}_{t,j}}\right)\right|^2 \\ &\leq \mathbb{E}\left(|\mathbf{Q}_{r,t} \bar{\Sigma}_{s,j}|^2 \mathbb{1}_{\mathcal{A}}\right) \mathbb{E}\left|\frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}}\right|^2 \end{aligned}$$

On event  $\mathcal{A}$ , all the eigenvalues of  $\Sigma\Sigma^*$  belong to  $[\sigma^2(1 - \sqrt{c})^2 - 2\epsilon, \sigma^2(1 + \sqrt{c})^2 + 2\epsilon]$ . Therefore,  $|\mathbf{Q}_{r,t} \mathbb{1}_{\mathcal{A}}$  is bounded and (141) implies that the last term of (138) is  $\mathcal{O}(\frac{1}{N^p})$  for each  $p$ . Summing (138) over  $t$ , we obtain that

$$\begin{aligned} \mathbb{E}((\mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) &= \frac{\sigma^2}{N} \mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta}) \\ &\quad - \sigma^2 c_N \mathbb{E}(\hat{m}(0) (\mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) \\ &\quad - \frac{i\sigma^2 u}{\sqrt{N}} \mathbb{E}((\mathbf{Q}^2 \mathbf{C} \mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) + \mathcal{O}\left(\frac{1}{N^p}\right) \end{aligned} \quad (142)$$

where we recall that  $\hat{m}(0) = \frac{1}{M} \text{Tr}(\mathbf{Q})$  represents the Stieltjes transform of the empirical eigenvalue distribution  $\hat{\mu}$  of  $\Sigma\Sigma^*$

at  $z = 0$ . Using that  $(1 - \chi) \leq \mathbb{1}_{\mathcal{E}}$ , it is easy to check that for each  $p$ , it holds that

$$\begin{aligned} \mathbb{E}\left((\mathbf{Q}\xi_j)_r \hat{m}(0) \bar{\Sigma}_{s,j} \chi e^{iu\delta}\right) &= \\ &\quad \mathbb{E}((\mathbf{Q}\xi_j)_r \hat{m}(0) \bar{\Sigma}_{s,j} \chi^2 e^{iu\delta}) + \mathcal{O}\left(\frac{1}{N^p}\right) \end{aligned}$$

We denote by  $\beta$  the term  $\beta = \hat{m}(0)\chi$ , and express  $\beta$  as  $\beta = \alpha + \beta^\circ$ . Replacing  $\chi$  by  $\chi^2$  in the second term of the righthandside of (142) and plugging  $\beta = \alpha + \beta^\circ$  into (142), we obtain that immediately that

$$\begin{aligned} \mathbb{E}((\mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) &= \frac{\sigma^2}{N(1 + \sigma^2 c_N \alpha)} \mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta}) \\ &\quad - \frac{i\sigma^2 u}{\sqrt{N}(1 + \sigma^2 c_N \alpha)} \mathbb{E}((\mathbf{Q}^2 \mathbf{C} \mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) \\ &\quad - \frac{\sigma^2 c_N}{1 + \sigma^2 c_N \alpha} \mathbb{E}(\beta^\circ (\mathbf{Q}\xi_j)_r \bar{\Sigma}_{s,j} \chi e^{iu\delta}) + \mathcal{O}\left(\frac{1}{N^p}\right) \end{aligned} \quad (143)$$

Summing over  $j$ , we get that

$$\begin{aligned} \mathbb{E}((\mathbf{Q}\Sigma\Sigma^*)_r \chi e^{iu\delta}) &= \frac{\sigma^2}{1 + \sigma^2 c_N \alpha} \mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta}) \\ &\quad - \frac{i\sigma^2 u}{\sqrt{N}(1 + \sigma^2 c_N \alpha)} \mathbb{E}((\mathbf{Q}^2 \mathbf{C} \mathbf{Q}\Sigma\Sigma^*)_r \chi e^{iu\delta}) \\ &\quad - \frac{\sigma^2 c_N}{1 + \sigma^2 c_N \alpha} \mathbb{E}(\beta^\circ (\mathbf{Q}\Sigma\Sigma^*)_r \chi e^{iu\delta}) + \mathcal{O}\left(\frac{1}{N^p}\right) \end{aligned} \quad (144)$$

or, using that  $\mathbf{Q}\Sigma\Sigma^* = \mathbf{I}$ ,

$$\begin{aligned} \mathbb{E}(\chi e^{iu\delta}) \delta(r=s) &= \frac{\sigma^2}{1 + \sigma^2 c_N \alpha} \mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta}) \\ &\quad - \frac{i\sigma^2 u}{\sqrt{N}(1 + \sigma^2 c_N \alpha)} \mathbb{E}((\mathbf{Q}^2 \mathbf{C})_{r,s} \chi e^{iu\delta}) \\ &\quad - \frac{\sigma^2 c_N}{1 + \sigma^2 c_N \alpha} \mathbb{E}(\beta^\circ \chi e^{iu\delta}) \delta(r=s) + \mathcal{O}\left(\frac{1}{N^p}\right) \end{aligned} \quad (145)$$

In order to evaluate  $\alpha$ , we take  $u = 0$  and sum over  $r = s$  in (145), and obtain that

$$\alpha = \frac{1}{\sigma^2(1 - c_N)} + \frac{1}{1 - c_N} \mathbb{E}(\beta^\circ \chi) + \mathcal{O}\left(\frac{1}{N^p}\right)$$

$\mathbb{E}(\beta^\circ \chi)$  coincides with  $\mathbb{E}(\beta^\circ \chi^\circ)$ . Using (141), the Poincaré-Nash inequality leads immediately to  $\mathbb{E}((\chi^\circ)^2) = \mathcal{O}(\frac{1}{N^p})$ , and to

$$\alpha = \frac{1}{\sigma^2(1 - c_N)} + \mathcal{O}\left(\frac{1}{N^p}\right) \quad (146)$$

for each  $p$ . As a consequence, we also get that

$$\mathbb{E}(\mathbf{Q}_{r,s} \chi) = \frac{1}{\sigma^2(1 - c_N)} \delta(r=s) + \mathcal{O}\left(\frac{1}{N^p}\right) \quad (147)$$

We now use (145) in order to evaluate  $\mathbb{E}((\mathbf{Q}_{r,s} \chi)^\circ \chi e^{iu\delta})$ . For this, we first establish that the use of (130) and of the Poincaré-Nash inequality implies that

$$\text{Var}(\beta) = \mathbb{E}((\beta^\circ)^2) = \mathcal{O}\left(\frac{1}{N^2}\right) \quad (148)$$

To check this, we use the Poincaré-Nash inequality:

$$\text{Var}(\beta) \leq \frac{\sigma^2}{N} \mathbb{E}\left(\sum_{i,j} \left|\frac{\partial \beta}{\partial \bar{\Sigma}_{i,j}}\right|^2 + \left|\frac{\partial \beta}{\partial \bar{\Sigma}_{i,j}}\right|^2\right)$$

We just evaluate the terms corresponding to the derivatives with respect to the terms  $(\bar{\Sigma}_{i,j})_{i=1,\dots,M,j=1,\dots,N}$ . It is easily seen that

$$\frac{\partial \beta}{\partial \bar{\Sigma}_{i,j}} = -\frac{1}{M} (\mathbf{e}_i^T \mathbf{Q}^2 \boldsymbol{\xi}_j) \chi + \frac{1}{M} \text{Tr}(\mathbf{Q}) \frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}}$$

Therefore, it holds that

$$\left| \frac{\partial \beta}{\partial \bar{\Sigma}_{i,j}} \right|^2 \leq 2 \frac{1}{M^2} \boldsymbol{\xi}_j^* \mathbf{Q}^2 \mathbf{e}_i \mathbf{e}_i^T \mathbf{Q}^2 \boldsymbol{\xi}_j \chi^2 + 2 \frac{1}{M} \text{Tr}(\mathbf{Q}) \left| \frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}} \right|^2$$

Using the identity  $\mathbf{Q}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^* = \mathbf{I}$  as well that  $\frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}} = \mathbb{1}_{\mathcal{A}} \frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}}$  (see (140)), we obtain that

$$\frac{\sigma^2}{N} \sum_{i,j} \left( \mathbb{E} \left| \frac{\partial \beta}{\partial \bar{\Sigma}_{i,j}} \right|^2 \right) \leq 2\sigma^2 \frac{1}{MN} \mathbb{E} \left( \frac{1}{M} \text{Tr}(\mathbf{Q}^3) \chi \right) + 2 \frac{\sigma^2}{N} \mathbb{E} \left( \frac{1}{M} \text{Tr}(\mathbf{Q}) \mathbb{1}_{\mathcal{A}} \sum_{i,j} \left| \frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}} \right|^2 \right)$$

On the set  $\mathcal{A}$ , the eigenvalues of  $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^*$  are located into  $[\sigma^2(1 - \sqrt{c})^2 - 2\epsilon, \sigma^2(1 + \sqrt{c})^2 + 2\epsilon]$ . Therefore, we get that

$$\frac{1}{M} \text{Tr}(\mathbf{Q}) \mathbb{1}_{\mathcal{A}} \leq \frac{1}{\sigma^2(1 - \sqrt{c})^2 - 2\epsilon}$$

Using (141), we obtain that

$$2 \frac{\sigma^2}{N} \mathbb{E} \left( \frac{1}{M} \text{Tr}(\mathbf{Q}) \mathbb{1}_{\mathcal{A}} \sum_{i,j} \left| \frac{\partial \chi}{\partial \bar{\Sigma}_{i,j}} \right|^2 \right) = \mathcal{O} \left( \frac{1}{N^p} \right)$$

for each  $p$ . Moreover, (130) implies that

$$\frac{1}{M} \text{Tr}(\mathbf{Q}^3) \chi \leq \frac{1}{(\sigma^2(1 - \sqrt{c})^2 - 2\epsilon)^3}$$

and that

$$2\sigma^2 \frac{1}{MN} \mathbb{E} \left( \frac{1}{M} \text{Tr}(\mathbf{Q}^3) \chi \right) = \mathcal{O} \left( \frac{1}{N^2} \right)$$

This establishes (148).

Therefore, the Schwartz inequality leads to  $\mathbb{E}(\beta^\circ \chi e^{iu\delta}) = \mathcal{O}(\frac{1}{N})$ . Writing  $\mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta})$  as

$$\begin{aligned} \mathbb{E}(\mathbf{Q}_{r,s} \chi e^{iu\delta}) &= \mathbb{E}(\mathbf{Q}_{r,s} \chi^2 e^{iu\delta}) + \mathcal{O} \left( \frac{1}{N^p} \right) = \\ &= \mathbb{E}(\mathbf{Q}_{r,s} \chi) \mathbb{E}(\chi e^{iu\delta}) + \mathbb{E}((\mathbf{Q}_{r,s} \chi)^\circ \chi e^{iu\delta}) + \mathcal{O} \left( \frac{1}{N^p} \right) = \\ &= \mathbb{E}(\mathbf{Q}_{r,s} \chi) \mathbb{E}(\chi e^{iu\delta}) + \mathbb{E}((\mathbf{Q}_{r,s} \chi)^\circ e^{iu\delta}) + \mathcal{O} \left( \frac{1}{N^p} \right) \end{aligned}$$

(146), (147) and (145) lead to

$$\mathbb{E}((\mathbf{Q}_{r,s} \chi)^\circ e^{iu\delta}) = \frac{iu}{\sqrt{N}} \mathbb{E}((\mathbf{Q}^2 \mathbf{C})_{r,s} \chi e^{iu\delta}) + \mathcal{O} \left( \frac{1}{N} \right) \quad (149)$$

or equivalently to

$$\mathbb{E}(\delta^\circ e^{iu\delta}) = iu \mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi e^{iu\delta}) + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right)$$

Using the Nash-Poincaré inequality, it can be checked that

$$\text{Var}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) = \mathcal{O} \left( \frac{1}{N} \right)$$

Therefore, the Schwartz inequality leads to

$$\mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi e^{iu\delta}) = \mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) \mathbb{E}(e^{iu\delta}) + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right)$$

and we get that

$$\mathbb{E}(\delta^\circ e^{iu\delta}) = iu \mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) \mathbb{E}(e^{iu\delta}) + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) \quad (150)$$

Plugging  $\delta = \delta^\circ + \mathbb{E}(\delta)$  into (150) eventually leads to

$$\mathbb{E}(\delta^\circ e^{iu\delta^\circ}) = iu \mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) \mathbb{E}(e^{iu\delta^\circ}) + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) \quad (151)$$

which is equivalent to (135). This, in turn, establishes Proposition 5.

We now complete the proof of (134). We integrate (135), and obtain that

$$\psi^\circ(u) = \exp \left[ -\frac{u^2}{2} \mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) \right] + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right)$$

(see section V-C of [10] for more details). (82) implies that

$$\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) - \frac{\text{Tr}(\mathbf{C}^2)}{\sigma^4(1 - c_N)^3} \rightarrow 0 \text{ a.s.}$$

As  $\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi - \text{Tr}(\mathbf{Q}^2 \mathbf{C}^2)$  also converges to 0 almost surely, we obtain that

$$\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi - \frac{\text{Tr}(\mathbf{C}^2)}{\sigma^4(1 - c_N)^3} \rightarrow 0 \text{ a.s.}$$

As matrix  $\mathbf{Q}^2 \chi$  is bounded and  $\sup_N \text{Tr}(\mathbf{C}^2) < +\infty$ , it is possible to use the Lebesgue dominated convergence theorem and to conclude that

$$\mathbb{E}(\text{Tr}(\mathbf{Q}^2 \mathbf{C}^2) \chi) - \frac{\text{Tr}(\mathbf{C}^2)}{\sigma^4(1 - c_N)^3} \rightarrow 0$$

This proves (134).

It remains to establish (133). For this, we use (147), and obtain that

$$\mathbb{E}(\text{Tr}(\mathbf{Q} \mathbf{C}) \chi) - \frac{\text{Tr}(\mathbf{C})}{\sigma^2(1 - c_N)} = \mathcal{O} \left( \frac{1}{N^p} \right)$$

for each  $p$ . This, of course, implies (133).

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