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Deformation and flow of amorphous solids: a review of mesoscale elastoplastic models

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The deformation and flow of disordered solids, such as metallic glasses and concentrated emulsions, involves swift localized rearrangements of particles that induce a long-range deformation field. To describe these heterogeneous processes, elastoplastic models handle the material as a collection of ‘mesoscopic’ blocks alternating between an elastic behavior and plastic relaxation, when they are too loaded. Plastic relaxation events redistribute stresses in the system in a very anisotropic way. We review not only the physical insight provided by these models into practical issues such as strain localization, creep and steady-state rheology, but also the fundamental questions that they address with respect to criticality at the yielding point and the statistics of avalanches of plastic events. Furthermore, we discuss connections with concurrent mean-field approaches and with related problems such as the plasticity of crystals and the depinning of an elastic line.

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FREQUENTLY USED NOTATIONS

Σ	Macroscopic shear stress
Σ_y	Macroscopic yield stress
σ	Local shear stress
σ_y	Local yield stress
γ	Shear strain
$\dot{\gamma}$	Shear rate
EPM	Elastoplastic model
MD	Molecular dynamics
rhs (lhs)	right-hand side (left-hand side)

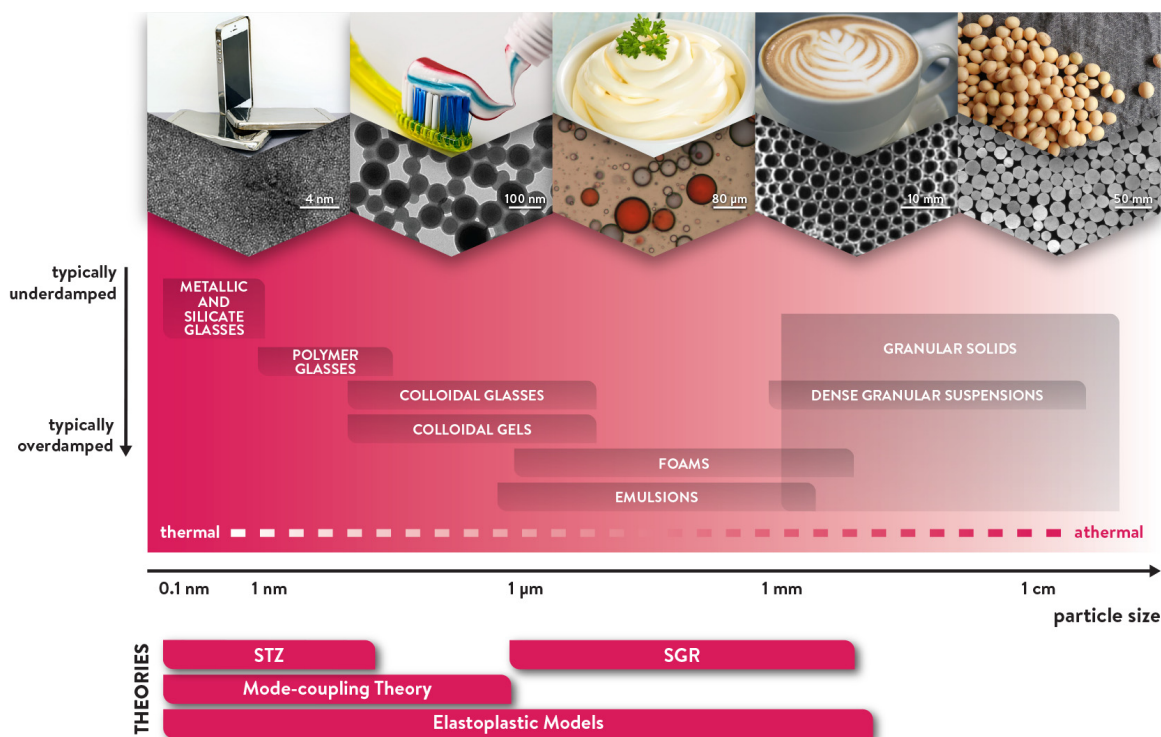


FIG. 1 Overview of amorphous solids. *From left to right, top row*: cellular phone case made of metallic glass (1); toothpaste (2); mayonnaise (3); coffee foam (4); soya beans (5). *Second row*: a transmission electron microscopy (TEM) image of a fractured bulk metallic glass ($\text{Cu}_{50}\text{Zr}_{45}\text{Ti}_5$) by X. Tong et. al (Shanghai University, China); TEM image of blend (PLLA/PS) nanoparticles obtained by miniemulsion polymerization, from L. Becker Peres et al. (UFSC, Brazil); emulsion of water droplets in silicon oil observed with an optical microscope by N. Bremond (ESPCI Paris); a soap foam filmed in the lab by M. van Hecke (Leiden University, Netherlands); thin nylon cylinders of different diameters pictured with a camera, from T. Miller et al. (University of Sydney, Australia). The white scale bars are approximate. *Just below*, a chart of different amorphous materials, classified by the size and the damping regime of their elementary particles. *At the bottom*: some popular modeling approaches, arranged according to the length scales of the materials for which they were originally developed. STZ stands for the shear transformation zone theory of Langer (2008), and SGR for the soft glassy rheology theory of Sollich *et al.* (1997).

INTRODUCTION

19th-century French Chef Marie-Antoine Carême (1842) claims that ‘mayonnaise’ comes from the French verb ‘*manier*’ (‘to handle’), because of the continuous whipping that is required to make the mixture of egg yolk, oil, and vinegar thicken. This etymology may be erroneous, but what is certain is that the vigorous whipping of these liquid ingredients can produce a viscous substance, an emulsion consisting of oil droplets dispersed in a water-based phase. At high volume fraction of oil, mayonnaise even acquires some resistance to changes of shape, like a solid; it no longer yields to small forces, such as its own weight. Similar materials, sharing solid and liquid properties, pervade our kitchens and fridges: Chantilly cream, heaps of soya grains or rice are but a couple of examples. They also abound on our bathroom shelves (shaving foam, tooth paste, hair gel), and in the outside world (sand heaps, clay, wet concrete), see Fig. 1 for further examples. All these materials will deform, and may flow, if they are pushed hard enough, but will preserve their shape otherwise. Generically known as amorphous (or disordered) solids, they have no more in common than what the etymology implies: their structure is disordered, that is to say, deprived of regular pattern at “any” scale, as liquids, but they are nonetheless solid. So heterogeneous a categorization may make one frown, but has proven useful in framing a unified theoretical description (Barrat and de Pablo, 2007). In fact, the absence of long range order or of a perceptible microstructure makes the steady-state flow of amorphous solids simpler, and much less dependent on the preparation and previous deformation history, than that of their crystalline counterparts. A flowing amorphous material is therefore a relatively simple realization of a state of matter driven far from equilibrium by an external action, a topic of current interest in statistical physics.

A matter of clear industrial interest, the prediction of the mechanical response of such materials under exter-

nal/applied stresses or deformations is a challenge for Mechanical Engineering, too. This problem naturally brings in its wake many questions of fundamental physics. Obviously, it is not exactly solvable in closed form, since it involves the coupled mechanics equations of the $N \gg 1$ elementary constituents of the macroscopic material; this is a many-body problem with intrinsic disorder and very few symmetries. Two paths can be considered as alternatives: (i) searching for empirical laws in the laboratory, and/or (ii) proposing approximate, coarse-grained mathematical models for the materials. The present review is a pedagogical journey along the second path.

Along this route, substantial assumptions are made to simplify the problem. The prediction capability of models hinges on the accuracy of these assumptions. Following their distinct interests and objectives, different scientific communities have adopted different modeling approaches. Material scientists tend to include a large number of parameters, equations and rules, in order to reproduce simultaneously different aspects of the material behavior. Statistical physicists aspire for generality and favor minimal models, or even toy models, in which the parameter space is narrowed down to a few variables. The distinctive family of “elasto-plastic” models (EPM) is located more or less at the interface (or in a region of overlap) between the foregoing approaches: as simple models, they aim to describe a general phenomenology for all amorphous materials, but they may also include enough meaningful physical parameters to address material particularities, in view of potential applications. To do so, they operate at a mesoscopic scale and rely on simple assumptions to connect the microscopic phenomenology to the macroscopic behavior, therefore occupying a central position in the endeavor to bridge scales in the field (Rodney *et al.*, 2011). An appropriate comparison could be made with classical lattice models of magnetic systems, which permit the exploration of a number of fundamental and practical issues, by retaining a few key features such as local exchange and long range dipolar interactions, spin dynamics, local symmetries, etc., without incorporating directly the more microscopic ingredients about the electronic structure.

In this review, we shall describe the current status of these EPM, starting in Sec. I with the microscopic observations that guided the coarse-graining efforts. We will discuss several possible practical implementations of coarse-grained systems of interacting elastoplastic elements, considering the possible attributes of the building blocks (Sec. II) and the more technical description of their mutual interactions (Sec. III). Section IV is then concerned with the widespread approximations of the effect of the stress fluctuations resulting from these interactions. In Sec. V and VI, we describe the understanding of the macroscopic response of amorphous solids to a shear deformation that can be gained from the study of EPM. Section VII focuses on more microscopic and statistical features, notably the temporal and spatial organization of stress fluctuations in ‘avalanches’, while Sec. VIII gives a short perspective on the much less studied phenomena of creep and aging. The review ends on a discussion of the relation between EPM and several other descriptions of mechanical response in disordered systems, in Sec. IX, and some final outlooks.

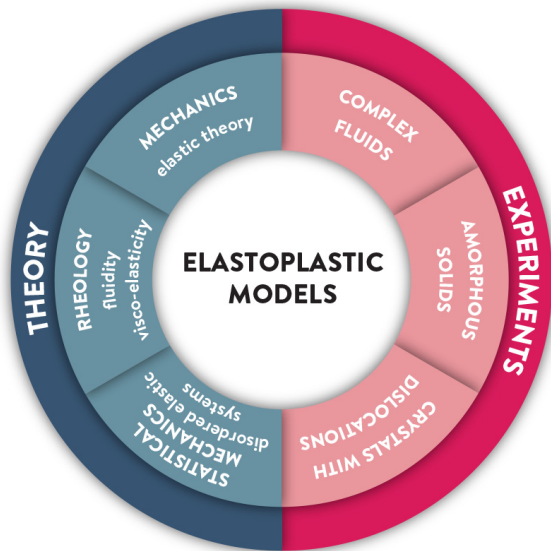


FIG. 2 Scientific position of elastoplastic modeling.

I. GENERAL PHENOMENOLOGY

A. What are amorphous solids?

From a mechanical perspective, amorphous solids are neither perfect solids nor simple liquids.

Albeit solid, some of these materials are made of liquid to a large extent and appear soft. Nevertheless, at rest they preserve a solid structure, for example, challenging gravity, or offering elastic resistance to deformations, and will flow meekly only if a sufficient load is applied to them. Accordingly, in the rheology of complex fluids (Bonn *et al.*, 2017), they are often referred to as “yield stress materials”. Foams and emulsions, that is, densely packed bubbles or droplets dispersed in a continuous liquid phase, owe their solidity to the action of surface tension, which strives to restore the equilibrium shape of their constituent bubbles or droplets upon deformation. Their elastic moduli are then approximately given by the surface tension divided by the bubble or droplet size, which may range from tens

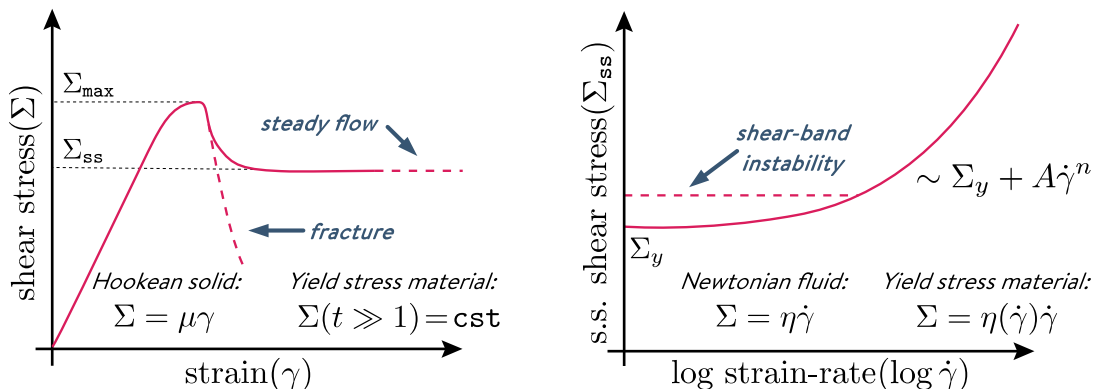


FIG. 3 Schematic macroscopic response of amorphous solids to deformation. (Left) evolution of the shear stress Σ with the imposed shear strain γ , with a stress overshoot Σ_{\max} . In the event of fracture, the stress dramatically drops down. (Right) Steady-state flow curve, i.e., dependence of the steady-state shear stress Σ_{ss} on the shear rate $\dot{\gamma}$, represented with semi-logarithmic axes. If the material shear bands, a stress plateau is generally observed.

of microns to several millimeters; a few hundred Pascal would be a good order of magnitude. Colloidal glasses, on the other hand, are dense suspension of solid particles of less than a micron in size, which makes them light enough for Brownian agitation to impede sedimentation. They rely on entropic forces to maintain their reference structure; values between 10 and 100 Pa are often encountered for their shear moduli.

Poles apart from these soft solids, “hard” amorphous solids comprise oxide or metallic glasses, as well as glassy polymers. They are typically made of much smaller particles than their soft counterparts. Indeed, very roughly speaking, the elastic moduli are inversely proportional to the size of the constituents. (Granular media, in which the elastic moduli depend on the material composing the grains and the applied pressure, are obviously an exception to this rule of thumb.) For instance, the atoms that compose the metallic or silica glasses live in the Angström scale, and these materials have very large Young moduli, of order 100 GPa (somewhat below for silicate glasses, sometimes above for metallic glasses). These atomic glasses are obtained from liquids when temperature is lowered below the glass transition temperature while crystallization is avoided. To do so, high cooling rates of typically $10^5 - 10^6 \text{K} \cdot \text{s}^{-1}$ are required for metallic glasses (Greer, 1995; Greer and Ma, 2007), whereas values below $1 \text{K} \cdot \text{s}^{-1}$ may be used for oxide glasses. After a certain amount of deformation, brittle materials will break without incurring significant plastic (irretrievable) deformation (the typical example would be silica glasses, although this has been questioned (Lacroix *et al.*, 2012)), whereas ductile materials will deform plastically before breaking.

B. What controls the dynamics of amorphous solids?

Another distinction regards the nature of the excitations that can alter the structural configuration of the system.

1. Athermal systems

When the elementary constituent sizes are large enough ($\gtrsim 1\mu\text{m}$) to neglect Brownian effects (thermal fluctuations), the materials are said to be athermal. Dry granular packings, dense granular suspensions, foams, and emulsions (see Fig. 1) belong in this category. An external force is required to activate their dynamics and generate configurational changes. Typical protocols for externally driving the system include: shearing it by rotating the wall of a rheometer (Barnes *et al.*, 1989), deforming it by applying pressure in a given direction, or simply making use of gravity if the material lies on a tilted plane (Coussot and Boyer, 1995). Rheometers control either the applied torque T or the angular velocity Ω of the rotating part. In the former case, the applied macroscopic shear stress is kept fixed, at a value $\Sigma = \frac{T}{2\pi h R^2}$ on a rotating cylinder of radius R and height h (Fardin *et al.*, 2014), while one monitors the resulting shear strain γ or shear rate $\dot{\gamma}$ if the material flows steadily. Conversely, strain-controlled experiments consist in imposing $\gamma(t)$ or $\dot{\gamma}$ and monitoring the stress response $\Sigma(t)$.

Now, how do amorphous solids respond to such external forces? For small applied stresses Σ , the deformation is elastic, *i.e.*, mostly reversible (see Fig.3[left]). Submitted to larger stresses, the material shows signs of plastic (irreversible) deformation; but the latter ceases rapidly, unless Σ overcomes a critical threshold Σ_y known as yield

stress. For $\Sigma > \Sigma_y$, the material yields. This process can culminate in macroscopic fracture; for brittle materials like silica glass, it always does so. Contrariwise, most soft amorphous solids will finally undergo a stationary plastic flow. The ensuing flow curve $\Sigma = f(\dot{\gamma})$ in the steady state is often fitted by a Herschel-Bulkley law

$$\Sigma = \Sigma_y + A\dot{\gamma}^n, \quad (1)$$

with $n > 0$ (see Fig. 3[right]).

The transition between the solid-like elastic response and the irreversible plastic deformation is known as *yielding* transition. Statistical physicists often regard it as an example of a dynamical phase transition, an out-of-equilibrium phenomenon with characteristics similar to equilibrium phase transitions (Jaiswal *et al.*, 2016; Lin *et al.*, 2015, 2014b).

2. Thermal systems

On the other hand, thermal fluctuations may play a role in materials with small enough ($\lesssim 1\mu m$) elementary constituents, such as colloidal and polymeric glasses, colloidal gels, silicate and metallic glasses. Still, these materials are out of thermodynamic equilibrium and they do not sample the whole configuration space under the influence of thermal fluctuations. It follows that different preparation routes (and in particular different cooling rates) tend to produce systems with different mechanical properties. Even the waiting time between the preparation and the experiment matters, because the system's configuration evolves meanwhile, through activated events: this is the aging phenomenon. In particular, the high cooling rates used for quenching generate a highly heterogeneous internal stress field in the material (Ballauff *et al.*, 2013). In some regions, particles manage to rearrange geometrically, minimizing in part the interaction forces among them, but many other regions are frozen in a highly strained configuration. Slow rearrangements will take place at finite temperature and tend to relax locally strained configurations (“particles break out of the cages made by their neighbors”), along with the stress accumulated in them.

That being said, the elastic moduli are usually only weakly affected by the preparation route, i.e., the cooling rate (Ashwin *et al.*, 2013) and the waiting time (Divoux *et al.*, 2011b), while other key features of the transient response to the applied shear are often found to depend on it. This sensitivity to preparation particularly affects the overshoot in the stress *vs.* strain curve, depicted in Fig. 3 and used to define the static yield stress Σ_{\max} , and is observed in experiments (Divoux *et al.*, 2011b) as well as numerical simulations (Patinet *et al.*, 2016; Rottler and Robbins, 2005). In soft materials that can undergo stationary flow, this issue may be deemed secondary; the flow creates a nonequilibrium stationary state, and the memory of the initial preparation state is erased after a finite deformation. On the other hand, in systems that break at finite deformation, the amount of deformation before failure is of paramount importance, and so is its possible sensitivity to the preparation scheme, due to different abilities of the system to localize deformations (see Sec. V).

3. Potential Energy Landscape

The Potential Energy Landscape (PEL) picture offers an illuminating perspective to understand the changes associated with aging in thermal systems (Doliwa and Heuer, 2003a,b; Goldstein, 1969). The whole configuration of the system (particle coordinates and, possibly, velocities) is considered as a “state point” Γ that evolves on top of a hypersurface $V(\Gamma)$ representing the total potential energy. Despite the high dimension of such a surface (proportional to the number N of particles), it can be viewed as a rugged landscape, with hills and nested valleys; the number of local minima generally grows exponentially with N (Wales and Bogdan, 2006). Contrary to crystals, glassy (disordered) states do not minimize the free energy of the system; aging thus consists in an evolution towards lower-energy states (on average) through random, thermally activated jumps over energy barriers, or more precisely saddle points of the PEL. As the state point reaches deeper valleys, the jumps become rarer and rarer; the structure stabilizes, even though some plasticity is still observed locally (Ruta *et al.*, 2012).

Under external loading, the system responds on much shorter time scales than for aging. Accordingly, some thermal systems may be treated as athermal, for all practical purposes. Nonetheless, interesting physical behavior emerges when these two time scales compete (either because the temperature is high enough, or because the system is driven very slowly) (Chattoraj *et al.*, 2010; Johnson and Samwer, 2005; Rottler and Robbins, 2005; Shi and Falk, 2005; Vandembroucq and Roux, 2011).

C. Jagged stress-strain curves and localized rearrangements

The contrasting inelastic material responses to shear, ranging from failure to flow, may give the impression that there is a chasm between “hard” and “soft” materials. They are indeed often seen as different fields, plasticity for hard solids versus rheology for soft materials. Nevertheless, the gap is not so wide as it looks. Indeed, some hard solids may flow plastically to some extent without breaking, while soft solids retain prominent solid-like features under flow at low enough shear rates, unlike simple liquids.

To start with, consider the macroscopic response to a constant stress Σ (or shear rate $\dot{\gamma}$) of a foam (Lauridsen *et al.*, 2002) or a metallic glass (Wang *et al.*, 2009): Instead of a smooth deformation, the strain evolution with time $\gamma(t)$ (or stress evolution $\Sigma(t)$, respectively) is often found to be jagged. The deeper the material lies in its solid phase, the more “serrated” the curves (Dalla Torre *et al.*, 2010; Sun *et al.*, 2012). Serrated curves are not specific features of the deformation of amorphous solids; they are observed in all “stick-slip” phenomena. Indeed, the system is repetitively loaded until a breaking point, where an abrupt discharge (energy release) occurs. Interestingly, this forms the basis of the elastic rebound theory proposed by Reid (1910) after the 1906 Californian earthquake. Other elementary examples include pulling a particle with a spring of finite stiffness in a periodic potential, a picture often used in crystalline solids to describe the motion of dislocations - the elementary mechanism of plasticity. In the plastic flow of amorphous solids, potential energy is accumulated in the material in the form of elastic strain, until some rupture threshold is passed. At this point, a plastic event occurs, with a release of the stored energy and a corresponding stress drop. The precise nature of the plastic event that gives rise to the stress drop, however, will strongly depend on the scale of observation.

Once again, the PEL perspective is enlightening: in this perspective, the external driving imposes a (usually time-dependent) constraint on the regions that the state point can visit. Mathematically, this is enforced by means of a Lagrange multiplier, which effectively tilts the potential $V(\Gamma)$ into

$$\tilde{V}(\Gamma, \gamma) \equiv V(\Gamma) - \Omega_0 \Sigma \gamma, \quad (2)$$

where Ω_0 is the volume of the system and Σ the macroscopic stress. As the imposed macroscopic strain γ increases, some of the barriers surrounding the state point subside, until one of them flattens so much that the system can slide into another valley without energy cost. This marks the onset of a plastic event; for smooth potentials, close to the topological change at $\gamma = \gamma_c$, Gagnon *et al.* (2001); Maloney and Lacks (2006) demonstrated that the barrier height scales as

$$\Delta V \sim (\gamma_c - \gamma)^{3/2}. \quad (3)$$

Note that this instability can be anticipated if thermal fluctuations are present. In summary, in the PEL, deformation is a succession of barrier-climbing phases (elastic loading) and descents. (For a discussion on the properties of the PEL of a model glass, see, *e.g.*, (Doliwa and Heuer, 2003a)). The first step in building a microscopic understanding of the flow process is to identify the nature of these plastic events.

But what can be said about the microscopic deformation of atomic or molecular glasses when the motion of atoms or molecules remains virtually invisible to direct experimental techniques? In the 1970s, inspiration was brought by the better known realm of crystals. As early as 1934, with the works of Orowan, Polanyi and G. I. Taylor, it was known that the motion of crystalline defects (dislocations) is the main lever of their (jerky) deformation. Could similar static structural defects be identified in the absence of a regular structure? The question has been vivid to the present day, so that it is fair to say that, at least, they are much more elusive than in crystals. But the main inspiration drawn from research on crystals was in fact of more pragmatic nature: Bragg and Nye (1947) showed that “bubble rafts”, *i.e.*, monolayers of bubbles, could serve as upscaled models of crystalline metals and provide insight into the structure of the latter. The lesson was simple: If particles in crystals are too small to be seen, let us make them larger. Some thirty years later, the idea was translated to disordered systems by Argon and Kuo (1979), who used bidisperse bubble rafts as models for metallic glasses. Most importantly, they observed the prominence of singularities in the deformation, more precisely, rapid rearrangements involving a few bubbles. Princen and Kiss (1986) suggested that the elementary rearrangement in such systems was a local topological change of the foam structure known as T1 event and involving four bubbles (see Fig. 4a).

Evidence

Since then, evidence for such swift localized rearrangements has been amassed in very diverse systems, both experimentally and numerically. Here, we shall simply list some of the works that followed Argon and Kuo (1979)’s and

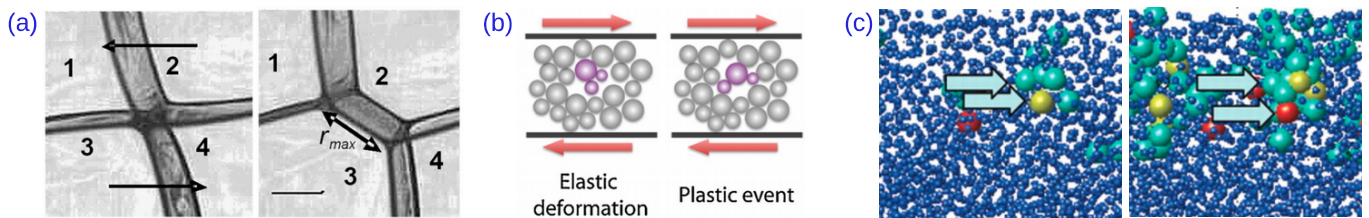


FIG. 4 Localized rearrangements. (a) T1 event in a strained bubble cluster. From (Biance *et al.*, 2009). (b) Sketch of a rearrangement. From (Bocquet *et al.*, 2009). (c) Instantaneous changes of neighbors in a slowly sheared colloidal glass. Adapted from (Schall *et al.*, 2007). Particles are magnified and colored according to the number of nearest neighbors that they lose.

early investigations on foams and emulsions (Princen, 1983, 1985; Princen and Kiss, 1986, 1989):

- simple numerical glass models like Lennard-Jones glasses (Falk and Langer, 1998; Maloney and Lemaître, 2004; Maloney and Lemaître, 2006; Tanguy *et al.*, 2006) and other systems (Gartner and Lerner, 2016),
- numerical models of metallic glasses (Rodney and Schuh, 2009; Srolovitz *et al.*, 1983),
- numerical models of silicon glasses (amorphous silicon) (Albaret *et al.*, 2016; Fusco *et al.*, 2014),
- numerical models of polymer glasses (Papakonstantopoulos *et al.*, 2008; Smessaert and Rottler, 2013)
- dense colloidal suspensions (Chikkadi and Schall, 2012; Jensen *et al.*, 2014; Schall *et al.*, 2007),
- dense emulsions (Desmond and Weeks, 2015),
- dry and wet foams (Biance *et al.*, 2009, 2011; Debrégeas *et al.*, 2001; Kabla and Debrégeas, 2003),
- granular matter (Amon *et al.*, 2012, 2013; Denisov *et al.*, 2016; Le Bouil *et al.*, 2014).

Admittedly, the details of these rearrangements do vary between the systems (*see below*). But, in all cases, they are the essential events whereby the macroscopic deformation is transcribed into the material structure, beyond the elastic response; their essential characteristic - as compared to the crystalline case - is their strong spatial localization. In the following, we shall refer to these events, which are the building bricks of EPM, as “plastic events”¹. Since these rearrangements must contribute to the deformation, they will retain part of the symmetry of the externally imposed shear and can thus be modeled as localized shear deformations (or “shear transformations”), if variations are overlooked, whether correctly or not.

Quantitative description

Although these rearrangements can sometimes be spotted visually, a more objective and quantitative criterion for their detection is desirable. Making use of the inelastic nature of these transformations, Falk and Langer (1998) pioneered the use of D_{\min}^2 , a quantity that measures how non-affine the local displacements around a particle are. More precisely, the relative displacement of neighboring particles between successive configurations is computed, and compared to the one that would result from a locally affine deformation; D_{\min}^2 is the deviation obtained by optimizing the parameters of the local affine deformation to minimize the deviation. This quantity has been used heavily since then (Chikkadi and Schall, 2012; Chikkadi *et al.*, 2011; Jensen *et al.*, 2014; Schall *et al.*, 2007). Generally speaking, a very strong localization of events is observed, with spatial maps of D_{\min}^2 that consist of a few active regions of limited spatial extension, separated by regions of (locally) affine and elastic deformation.

Other indicators of nonaffine transformations have also been used. For instance, different observables, including the strain component along a neutral direction (say, ϵ_{yz} if the applied strain is along ϵ_{xy} in a 3D system) (Schall *et al.*, 2007), the field of deviations from the uniform transformation of particle positions, the count of nearest-neighbor losses (Chikkadi and Schall, 2012) or the identification of regions with large (marginal) particle velocities, are also good options to detect rearrangements. Up to differences in their intensities, these methods were shown to provide similar information about shear transformations in slow flows of colloidal suspensions (Chikkadi and Schall, 2012). Alternative methods take advantage of the irreversibility of plastic rearrangements, by reverting every strain increment $\delta\gamma$ imposed on the system ($\gamma \rightarrow \gamma + \delta\gamma \rightarrow \gamma$) in a quasistatic shear protocol and comparing the reverted configuration with the original one (Albaret *et al.*, 2016). Differences will be seen in the rearrangement cores (which underwent plasticity) and their surroundings (which were elastically deformed by the former). To specifically target

¹ The reader should however be warned that the expression was also used in the literature to refer to cascades of such localized rearrangements (Fusco *et al.*, 2014; Maloney and Lemaître, 2006; Tanguy *et al.*, 2006).

the anharmonic forces active in the core, shear can be reverted partially, to harmonic order, by following the Hessian upstream instead of performing a full nonlinear shear reversal (Lemaître, 2015).

Some reservations should now be made with respect to the picture of clearly separated localized transformations. First, the validity of the binary picture distinguishing elastic and plastic regions has been challenged for hard particles, such as grains (Bouzid *et al.*, 2015a). More generally, near the jamming transition, the complexity of particle motion and the spatial extent of low-energy vibration modes may jeopardize the accuracy of this vision (Andreotti *et al.*, 2012). It is also clear that as the temperature or the shear rate are increased and the material departs from solidity, thermal or mechanical noise may wash out the picture of well isolated, localized events. Nevertheless, it has recently been argued that localized rearrangements can still be identified at relatively high temperatures. For instance, these rearrangements leave an elastic imprint in supercooled liquids via the elastic field that each of them induce; this imprint is revealed when one studies suitable stress or strain correlation functions (Chattoraj and Lemaître, 2013; Illing *et al.*, 2016; Lemaître, 2014).

Variations

These quantitative indicators of microscale plasticity have brought to light substantial variations and differences between actual rearrangements and idealized shear transformations. Even though EPM will generally turn a blind eye to this variability, let us shortly mention some of its salient features:

First, the size of plastically rearranging regions varies from a handful of particles in foams, emulsions and colloidal suspensions (for instance, about 4 particles in a sheared colloidal glass, according to Schall *et al.* (2007)) to a couple of dozens or hundreds in metallic glasses (10 to 30 in the numerical simulations of Fan *et al.* (2015), 25 for the as-cast glass and 34 for its annealed counterpart in the indentation experiment of Choi *et al.* (2012), 200 to 700 in the shearing experiments of Pan *et al.* (2008)). Note that, for metallic glasses, the indicated sizes are not backed out by direct experimental evidence, but are based on activation energy calculations.

Albaret *et al.* (2016) proposed a detailed numerical characterization of plastic rearrangements in their amorphous silicon model by fitting the particle displacements during plastic events with the expected (Eshelby) elastic fields around ellipsoidal transformations zones with spontaneous deformation ϵ^* (where the tensor ϵ^* was fitted). Although rearrangements seem to have a typical linear size, around 3\AA , they found that the most robust quantity is actually the product of ϵ^* with the inclusion volume V_{in} . Furthermore, the diagonal components of ϵ^*V_{in} (dilation or compression) only represent about 5% of the deviatoric components (shear), which confirms the prevalent shear nature of the transformation. It should also be mentioned that the diagonal components were either of positive or negative, i.e., either of dilational or of compressional nature depending on the specificities of the implemented potential: Plastic rearrangements are not always dilational. Finally, the authors of the study were able to reproduce the stress vs. strain curve on the basis of the (strain-dependent) shear modulus and the fitted local elastic strain releases ϵ^* . This confirms that localized plastic rearrangements emitting an Eshelby field are the unique elementary blocks of the plastic response.

Secondly, the shape of the rearrangements is also subject to variations. In quiescent systems rearrangements through string-like motion of particles seem to be more accessible (Keys *et al.*, 2011), even though shear transformations have also been claimed to be at the core of structural relaxation in deeply supercooled liquids (Lemaître, 2014). The application of a macroscopic shear clearly favors the latter type of rearrangements. Albeit facilitated by the driving, these transformations may nonetheless be predominantly activated by thermal fluctuations in thermal systems (Schall *et al.*, 2007). There is some (limited) indication that the characteristics of the rearranging regions change as one transits from mechanically triggered events to thermally activated ones, for instance with a visible increase in the size of the region in metallic glass models (Cao *et al.*, 2013).

Thirdly, owing to the granularity of the rearranging region (which is not a continuum!), the displacements of the individual particles in the region do not strictly coincide with a shear transformation, i.e., $\mathbf{r} \rightarrow \mathbf{r} + \epsilon \cdot (\mathbf{r} - \mathbf{r}_c)$ (where \mathbf{r} generically refers to a particle position); incidentally, this is the major reason why the observable D_{\min}^2 detects plastic rearrangements.

Structural origins of rearrangements

What determines a region's propensity to undergo a rearrangement? Microstructural properties underpinning the weakness of a region (i.e., how prone to rearranging it is) have long been searched. In the first half of the 20th century efforts were made to connect viscosity with the available free volume per particle, notably by using (contested)

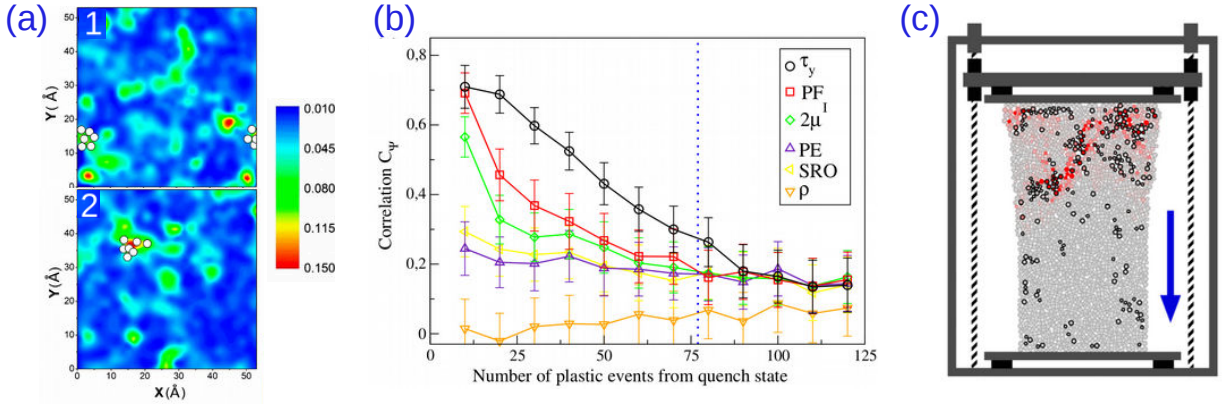


FIG. 5 Local structural determinants for the onset of a rearrangement. (a) Contour maps of the particle-based participation ratio in the 1% softest vibrational modes for two numerical samples of Cu₆₄Zr₃₆ metallic glass (Ding *et al.*, 2014). (b) Correlation between the locations of future plastic rearrangements and diverse local properties in an instantaneously quenched binary glass model. The following properties have been considered: local yield stress (τ_y), participation in the soft vibrational modes (PF), lowest shear modulus ($2\mu^I$), local potential energy (PE), short-range order (SRO), local density (ρ). From (Patinet *et al.*, 2016). (c) Snapshot of the configuration of a compressed granular pillar, with particles colored from gray to red according to their D_{\min}^2 value. Particles identified as soft by the SVM have thick black contours. From (Cubuk *et al.*, 2015).

experimental evidence from polymeric materials (Batschinski, 1913; Doolittle, 1951; Fox Jr and Flory, 1950; Williams *et al.*, 1955). The idea that local variations of free volume V_f control the local weakness have then been applied widely to systems of hard particles (metallic glasses, colloidal suspensions, granular materials) (Spaepen, 1977). Falk and Langer (1998)’s Shear Transformation Zone theory originally proposed to distinguish weak zones prone to shear transformations on the basis of this criterion. Hassani *et al.* (2016) have invalidated criteria based on the strictly local free volume but showed that a nonlocal definition distinctly correlates with the deformation field, as do potential-energy based criteria (Shi *et al.*, 2007). Paying closer attention to the microstructure, Ding *et al.* (2014) proved the existence of correlations between rearrangements and geometrically unfavored local configurations (whose Voronoi cell strongly differs from an icosahedra) in model binary alloys.

Linear properties have also been considered, with the hope that regions that are soft in terms of their linear response would also be weak in their nonlinear response. Regions with low elastic shear moduli were indeed shown to concentrate most of the plastic activity (Tsamados *et al.*, 2009), even though no yielding criterion based on the local stress or strain is valid uniformly throughout the material (Tsamados *et al.*, 2008). Focusing on vibrational properties, Widmer-Cooper *et al.* (2008) provided evidence that in supercooled liquids the particles that vibrate most in the M lowest energy modes (i.e., those with a high participation fraction in the M softest modes, where M is arbitrarily fine-tuned), are more likely to rearrange. This holds true at zero temperature (Manning and Liu, 2011) and also for metallic and polymer glasses (Smessaert and Rottler, 2015) (see Fig. 5). Note that this enhanced likelihood should be understood as a statistical correlation, beating random guesses by a factor of 2 or 3 or up to 7 in some cases, rather than as a systematic criterion. However, in the cases where the rearrangement spot is correctly predicted, the soft-mode-based prediction for the direction of motion during the rearrangement is fairly reliable (Rottler *et al.*, 2014).

If one is allowed to probe nonlinear local properties, then Patinet *et al.* (2016) showed that predictions based on the local yield stress, numerically measured by deforming the outer medium affinely, outperform criteria relying on the microstructure and the linear properties, as indicated in Fig. 5c.

Leaving behind traditional approaches, a couple of recent papers showed that it is possible to train an algorithm to recognize the atomic-scale patterns characteristic of a glassy state and spot its ”soft” regions (Cubuk *et al.*, 2015, 2016; Schoenholz *et al.*, 2016). In this Machine Learning approach, rather than focusing on typical structure indicators, a large number of ’features’ quantified for each particle is used, concretely $M=166$ ’structure functions’, indicating e.g. the radial and angular correlations between an atom and its neighbors (Behler and Parrinello, 2007). Adopting both an experimental frictional granular packing and a bidisperse glass model, the authors focused on the identification of local softness and its relation with the dynamics of the glass transition. First, with computationally costly shear simulations and measurements of nonaffine displacements via D_{\min}^2 , the particles that ’move’ (i.e., break out of the cages formed by their neighbors) are identified as participating in a plastic rearrangement and used to train a Support Vector Machine (SVM) algorithm. Each particle’s environment is handled as a point in the high-

dimensional vector space parameterized by the structure functions and the algorithm identifies the hyperplane that best separates environments associated with ‘moving’ particles and those associated with ‘stuck ones’ in the training set. Once trained, the algorithm is able to predict with high accuracy if a particle will ‘move’ or not when the material is strained, depending on its environment in the *quiescent* configuration, prior to shear.

D. Nonlocal effects

Once a rearrangement is triggered, it will deform the medium over long distances, in the same way as an earthquake is felt a large distance away from its epicenter. This may trigger other rearrangements at a distance, which rationalizes the presence of nonlocal effects in the flow of disordered solids. Importantly, this mechanism relies on the solidity of the medium, which is key to the transmission of elastic waves.

These long-range interactions and the avalanches that they may generate justify the somewhat hasty connection sketched above between the serrated macroscopic stress curves and the abrupt localized events at the microscale. The problem is that in the thermodynamic limit any one of these micro-events should go unnoticed macroscopically. For sure, the thermodynamic limit is not reached in some materials, notably those with large constituents, such as foams and grains, but also in nanoscale experiments on metallic glasses and numerical simulations. On the other hand, in the bulk, without collective effects and avalanches involving a large number of plastic events, the impact of microscopic events on the macroscopic response could not be explained. Since mesoscale plasticity models intend to capture these collective effects, a description of the interactions at play is required.

Idealized elastic propagator

Let us start by focusing on the consequence of a single shear transformation. Its rotational part can be overlooked because its effect is negligible in the far field, as compared to deformation, represented by the linear strain tensor $\boldsymbol{\epsilon} = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^\top}{2}$, where \mathbf{u} stands for the displacement. Recall that a shear deformation, say $\boldsymbol{\epsilon}(\mathbf{r} \approx \mathbf{0}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in two dimensions, consists of a stretch along the direction $\theta = \frac{\pi}{4} [\pi]$, in polar coordinates, and a contraction along the perpendicular direction. The induced displacement field \mathbf{u} simply mirrors this symmetry, with displacements that point outwards along $\theta = \frac{\pi}{4} [\pi]$ and inwards along $\theta = \frac{3\pi}{4} [\pi]$. This leads to a dipolar azimuthal dependence for \mathbf{u} and a four-fold (‘quadrupolar’) one for its symmetrized gradient $\boldsymbol{\epsilon}$. More precisely, by imposing mechanical equilibrium on the stress $\boldsymbol{\Sigma}$, viz.,

$$\nabla \cdot \boldsymbol{\Sigma} = 0$$

in an incompressible medium ($\nabla \cdot \mathbf{u} = 0$) with a linear elastic law, $\boldsymbol{\Sigma}^{\text{dev}} \propto \boldsymbol{\epsilon}^{\text{dev}}$ (where the superscript denotes the deviatoric part), Picard *et al.* (2004) derived the induced strain field in two dimensions,

$$\epsilon_{xy}(r, \theta) \propto \frac{\cos(4\theta)}{r^2}. \quad (4)$$

Here, only one of the strain components is expressed, but the derivation is straightforwardly extended to a tensorial form (Budrikis *et al.*, 2017; Nicolas and Barrat, 2013a). Experiments on colloidal suspensions (Jensen *et al.*, 2014; Schall *et al.*, 2007) and emulsions (Desmond and Weeks, 2015) as well as numerical works (Kabla and Debrégeas, 2003; Maloney and Lemaître, 2006; Tanguy *et al.*, 2006) have confirmed the relevance of Eq. 4, as illustrated in Fig. 6.

Exact induced field and variations

The strain field of Eq. 4 is valid in the far field, or for a strictly pointwise shear transformation. Yet, the response can be calculated in the near field following Eshelby (1957)’s works, by modeling the shear transformation as an elastic inclusion bearing an *eigenstrain* $\boldsymbol{\epsilon}^*$, *i.e.*, spontaneously evolving towards the deformed configuration $\boldsymbol{\epsilon}^*$. This handling adds near-field corrections to Eq. 4.

Describing a *plastic* rearrangement with an *elastic* eigenstrain is imperfect in principle, but the difference mostly affects the dynamics of stress relaxation (Nicolas and Barrat, 2013a). In fact, Eshelby’s expression perfectly reproduces the *average* displacement field induced by an ideal circular shear transformation in a 2D binary Lennard-Jones glass (Puosi *et al.*, 2014), although significant fluctuations around this mean response arise because of elastic heterogeneities.

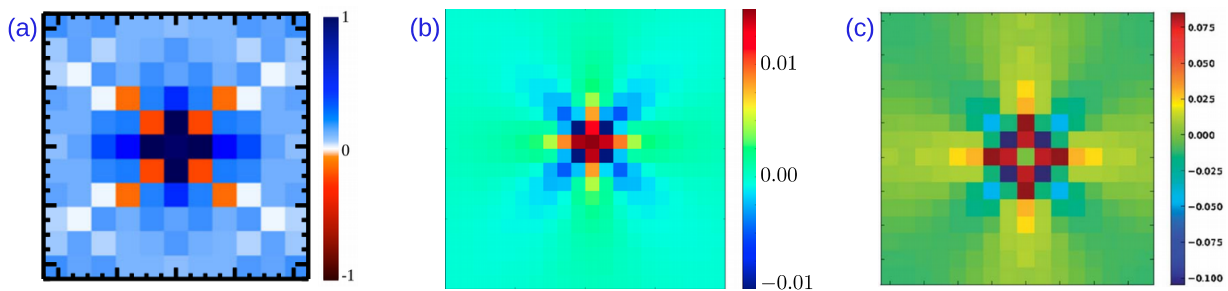


FIG. 6 *Average stress redistribution around a shear transformation.* (a) Experimental measurement in very dense emulsions. Adapted from (Desmond and Weeks, 2015). (b) Average response to an imposed shear transformation obtained in atomistic simulations with the binary Lennard-Jones glass used by Puosi *et al.* (2014). (c) Simplified theoretical form, given by Eq. 4. From (Martens *et al.*, 2012). Note that the absolute values are not directly comparable between the graphs and that, in subfigures b and c, the central block is artificially colored.

The numerical study was extended to the deformation of a spherical inclusion in 3D, and to the nonlinear regime, by Priezjev (2015).

Besides elastic heterogeneity, further deviations from the Eshelby response result from the difference between an actual plastic rearrangement and the idealized shear transformation considered here. For instance, Cao *et al.* (2013) report differences between the medium or far-field response to rearrangements in the shear-driven regime as opposed to the thermal regime and that only the former quantitatively obey Eshelby’s formula. It might be that the dilatational component of the rearrangement, discarded in the ideal shear transformation, is important in the thermal regime.

The salient points discussed above in the rheology of amorphous solids seem to build a coherent scenario, consisting of periods of elastic loading interspersed with swift localized rearrangements of particles. These plastic events may interact via the long-range anisotropic elastic deformations that they induce. These elements are the phenomenological cornerstones of the EPM described in the following section.

II. THE BUILDING BLOCKS OF ELASTOPLASTIC MODELS (EPM)

A. General philosophy of the models

The simplicity and genericity of the basic flow scenario described above has led to the emergence of multiple, largely phenomenological, coarse-grained models. These models are generally described as “elasto-plastic” or sometimes “discrete automaton” or “mesoscopic” models for amorphous plasticity. They incorporate the basic flow scenario by decomposing the system into “mesoscopic” blocks (presumably of the typical size of a rearrangement) in which the elastic behavior is interrupted by plastic events. With a few exceptions, they are implemented on a regular lattice, so they are effectively a subclass of discrete automata evolving according to predefined rules. Schematically, a model can be specified by the following set of rules (Rodney *et al.*, 2011):

R1.: a (default) linear elastic response of each mesoscopic block,

R2.: a local yield criterion that determines the onset of a plastic event ($n : 0 \rightarrow 1$),

R3.: a redistribution of the stress during plasticity that gives rise to long range interactions among blocks,

R4.: a recovery criterion that fixes the end of a plastic event ($n : 1 \rightarrow 0$),

where the activity n is defined as $n = 0$ if the block is purely elastic, and $n = 1$ otherwise.

Rules R2 and R4 determine the dynamical rules controlling the manner in which a region switches from elastic to plastic and conversely, for instance by specifying the rates associated with the transitions

$$n : 0 \leftrightarrow 1,$$

whereas R1 and R3 define the mechanical response of the material for a given set of plastically active regions (ongoing plastic events). This is specified by an equation of evolution of the stress σ_i carried by block i (where i is a d -dimensional vector denoting the lattice coordinates of the block); to fix ideas, if the sample is subjected to simple shear at a rate $\dot{\gamma}$, this equation might read, in scalar version,

$$\dot{\sigma}_i = \underbrace{\mu\dot{\gamma}}_{\text{driving}} - \underbrace{|\mathcal{G}_{ii}|n_i \frac{\sigma_i}{\tau}}_{\text{local relaxation}} + \underbrace{\sum_{j \neq i} \mathcal{G}_{ij} n_j \frac{\sigma_j}{\tau}}_{\text{nonlocal contributions}}. \quad (5)$$

Here, the stress increment $\dot{\sigma}_i$ per unit time is the sum of an elastic contribution from the external driving, obeying Hooke's law (viz., $\dot{\sigma} = \mu\dot{\gamma}$ with μ the shear modulus), a local plastic relaxation if the site is currently plastic (*i.e.*, if $n_i = 1$), and a redistribution of the stress released by nonlocal plastic events (at positions j), mediated by the propagator \mathcal{G}_{ij} . Locally, the plastic deformation of an active block ($n_i = 1$) is constrained by the elastic deformation of the embedding medium, hence a relaxation efficiency $0 < |\mathcal{G}_{ii}| < 1$. Note that, in translationally invariant geometry, the propagator becomes independent of space, viz., $\mathcal{G}_{ji} \rightarrow \mathcal{G}_{i-j}$ and $\mathcal{G}_{ii} \rightarrow \mathcal{G}_0$.

In essence, EPM aspire to follow in the footsteps of the successes of simplified lattice models in describing complex collective phenomena in condensed matter and statistical physics. The assumption is that most microscopic details are irrelevant with respect to the main rheological properties and that the physics can be condensed into a few relevant parameters or processes. Several reasons could be put forward to favor their use over more realistic modeling approaches, e.g.,

1. to assess the validity of a proposed theoretical scenario and, in particular, to identify the key physical processes in the rheology,
2. to provide an efficient simulation tool giving access to (otherwise inaccessible) large statistics or long-time runs,
3. to provide a simple route towards the derivation of macroscopic equations and to bridge the gap between rheological models (constitutive laws) and statistical physics models (sandpile models, depinning models, Ising-like models).

The substantial variations in the physical ingredients incorporated in distinct EPM, notably with respect to R2 (yield criterion) and R4 (plastic event duration), seem to be a strong blow to the first objective. But one should bear in mind that these materials are so diverse that a given process (*e.g.*, thermal activation) may be negligible in some of them, and paramount in others. Perhaps less intuitive is the role played by the experimental conditions and the observables under consideration in determining the physical ingredients that need to be retained in an EPM. Let us bolster this statement with a couple of examples. Keeping track of previous configurations (*e.g.*, past yield stresses) might be vital in order to describe oscillatory shear experiments in which the system performs a cycle between a few configurations (Fiocco *et al.*, 2014), whereas it is irrelevant for steady shear. Also, potentially universal aspects of the yielding transition are expected to be relatively insensitive to the precise rules, while the details of the flow pattern will obviously be more affected. Thus, as noted in (Bonn *et al.*, 2017), one should not only select the relevant ingredients in a model only in light of the *intrinsic* importance of these effects (as quantified for instance by dimensionless numbers), but also depending on their bearing on the investigated properties.

In the following, we list the physical processes that are put in the limelight in the diverse EPM and indicate for what type of materials and in what conditions they are of primary importance.

B. Thermal fluctuations

How relevant are thermal fluctuations and their effect on the motion of particles in the description? This question boils down to the distinction between thermal materials and athermal ones exposed in Section I.B.

It is widely believed that thermal fluctuations largely contribute to the activation of plastic events in metallic and molecular glasses, as well as in colloidal systems made of small enough colloids. In the latter systems, Schall *et al.* (2007)'s estimation of the activation energy indicates that transformations are mostly thermally activated, but with a stress-induced bias towards one direction. This will impact the choice of the yield criterion (R2 above). EPM focusing on thermal materials (Bulatov and Argon, 1994a; Ferrero *et al.*, 2014) will set a yield rate based on a stress-biased Arrhenius law for thermal activation, *viz.*,

$$\nu(\sigma) = \nu_0 e^{\frac{-\tilde{V}(\sigma)}{k_B T}}, \quad (6)$$

where ν_0 is an attempt frequency, $\tilde{V}(\sigma)$ represents the height of the (smallest) potential barrier impeding the rearrangement, tilted by the application of an external stress σ . Recalling Eq. 2, $\tilde{V}(\sigma) = V - \Omega_0 \sigma \gamma$, one immediately

recovers the expression of $\nu(\sigma)$ used by Eyring (1935) to calculate the viscosity of liquids if σ and γ are treated as independent parameters, whereas locally imposing a Hookean relation between stress and elastic strain, viz., $\sigma \propto \gamma$, leads to the γ^2 -scaling of the tilt used, e.g., in Sollich *et al.* (1997)'s Soft Glassy Rheology model.

On the other hand, thermal activation plays virtually no role in foams (Ikeda *et al.*, 2013) and granular materials. Consequently, EPM designed for athermal materials (Chen *et al.*, 1991; Hébraud and Lequeux, 1998) will favor a pure threshold-based yield criterion, viz.,

$$\nu(\sigma) = \nu_0 \Theta(-\tilde{V}(\sigma)) \text{ or equivalently } \nu(\sigma) = \nu_0 \Theta(\sigma - \sigma_y),$$

where σ_y is the local stress threshold for yielding; a deterministic yield criterion is recovered in the limit $\nu_0 \rightarrow \infty$. Incidentally, note that, in this one-dimensional tilt vision, directional considerations in the PEL are handled somewhat light-heartedly; in theory, there is no reason why the loading should push the system towards the saddle point.

As far as rheology is concerned, the athermal approximation is conditioned by the possibility to neglect the activation rate with respect to the driving rate. If we consider a sample sheared at rate $\dot{\gamma}$, schematically this condition reads

$$\nu(\sigma) \ll \frac{\dot{\gamma}}{\gamma_y}, \quad (7)$$

which in essence is similar to the limit of large Péclet number $Pe \equiv \dot{\gamma} a^2 / D$, where a is the particle size and D is the single-particle diffusivity in the dilute limit (Ikeda *et al.*, 2013). The latter condition is however more (and, possibly, excessively) stringent because in the dense system the diffusivity is much smaller than in the dilute limit.

Now, some subtleties ought to be mentioned. An athermal system may very well be sensitive to temperature variations, through changes in their material properties (*e.g.*, dilation): For example, the fact that Divoux *et al.* (2008) reported creep motion for a granular heap submitted to cyclic temperature variations does not mean that thermal activation is important, but rather points to dilational effects. Secondly, as already stressed, the relevance of thermal fluctuations may depend on the considered level of detail: It has been argued that they may precipitate the emergence of avalanches by breaking nano-contacts between grains in very slowly sheared systems (Zaitsev *et al.*, 2014), but it is very dubious that this may impact steady-shear granular rheology.

C. Driving

Suppose that the material deforms under the action of some external driving; how important are the specificities of the driving conditions?

1. Stress or strain driven

As reported in Section I.B, the driving may be stress (fixed σ) or strain-controlled (fixed $\dot{\gamma}$). Numerical simulations have mostly considered strain-driven situations. In EPM, this affects R1(elastic response) and R3(stress redistribution), *i.e.*, the mechanical response for a fixed set of plastic blocks. In a strain-driven protocol, the elastic response (R1) is generally obtained by converting the macroscopic driving into local stress increments $\mu \dot{\gamma}(t) dt$, where dt is the time step and $\dot{\gamma}(t)$ is the current macroscopic strain rate, and the stress redistribution operated by the elastic propagator $\mathcal{G}_{r \rightarrow r'}$ keeps the macroscopic strain fixed. Stress-controlled setups have been somewhat less studied in the framework of EPM but examples can be found in (Homer and Schuh, 2009; Lin *et al.*, 2015, 2014b; Picard *et al.*, 2004). In this case, the propagator should keep the macroscopic stress constant. In practice, this often comes down to changing its 0-Fourier mode, which controls the mean value.

2. Time-dependence

EPM often focus on steady shear situations, in which case $\dot{\gamma}(t) = \text{cst}$. But time-dependent driving protocols $\dot{\gamma} = f(t)$ (or $\sigma = f(t)$) are also encountered, in particular step shear $\dot{\gamma}(t) = \gamma_0 \delta(t)$, oscillatory shear $\dot{\gamma}(t) = \gamma_0 \cos(\omega t)$, which gives access to linear rheology for small γ_0 . In creeping flows subjected to $\sigma(t) = \text{cst} < \sigma_y$, the shear rate $\dot{\gamma}(t)$ eventually decays to zero, often as a power law (Leocmach *et al.*, 2014). For further discussions on creep, see Sec. VIII.

3. Symmetry of the driving

Plastic events are biased towards the direction of the external loading. If the latter acts uniformly on the material, it is convenient to focus on only one stress component, thus reducing the stress and strain tensors to scalars. In particular, for simple shear conditions, with a displacement gradient $\nabla \mathbf{u} = \begin{pmatrix} 0 & \gamma(t) \\ 0 & 0 \end{pmatrix}$ (in the linear approximation in two dimensions), one may settle with the ϵ_{xy} component. It differs from pure shear, $\nabla \mathbf{u} = \begin{pmatrix} 0 & \gamma(t)/2 \\ \gamma(t)/2 & 0 \end{pmatrix}$ in that the latter is rotationless, whereas the former involves a rotational part $\boldsymbol{\omega} = \begin{pmatrix} 0 & \gamma(t)/2 \\ -\gamma(t)/2 & 0 \end{pmatrix}$, but has the same principal strains (eigenvalues) $\pm\gamma(t)/2$. These deformations are encountered *locally* whenever volume changes can be neglected; the cone-and-plate, plate-plate, and Taylor-Couette rheometers (Larson, 1999) used to probe the flow of yield-stress fluids fall in this category. For metallic glasses and other hard materials, uniaxial compression tests (*i.e.*, $\boldsymbol{\sigma}(t) = \sigma(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the bulk, with $\sigma(t) < 0$) and tension ($\sigma(t) > 0$) are often performed.

Even though in several of these situations the macroscopic is more or less uniform and acts mostly on one component of the (suitably defined) stress tensor, the other components reach finite values because of stress redistribution. Full tensorial approaches may then be justified (Bulatov and Argon, 1994a; Homer and Schuh, 2009; Sandfeld and Zaiser, 2014). Recently, the influence of a tensorial, rather than scalar, description on the flow and avalanche properties in these cases was evaluated; it was found to be insignificant overall (Budrikis *et al.*, 2017; Nicolas *et al.*, 2014b), and the effect of dimensionality to be weak (Liu *et al.*, 2016). The reader is referred to Sec. VII for more details. However, there exist a wide range of experimental setups in which the loading is intrinsically heterogeneous, in particular the bending, torsion, and indentation tests on hard glasses (see (Budrikis *et al.*, 2017) for an implementation of these tests in a finite-element-based EPM) or the microchannel flows of dense emulsions (Nicolas and Barrat, 2013a). The exploitation of EPM in heterogeneous driving conditions appears to be a promising new avenue.

4. Driving rate

To resolve the flow temporally, the simplest approach is a Eulerian method, which computes the strain increments on all blocks at each time step from Eq. 5. Kinetic Monte-Carlo methods have also been employed and are particularly efficient in stress-controlled slow flows, insofar as the long elastic loading phases without plastic events are bypassed: Activation rates ν_i are calculated for all the blocks i using a refined version of Eq. 6 and the time lapse before the next plastic event is deduced from the cumulative rate $\nu = \sum_i \nu_i$ (Homer and Schuh, 2009). If the flow is even slower, *rate* effects may deliberately be overlooked. Indeed, a number of models consider the limit of vanishing shear rate $\dot{\gamma} \rightarrow 0$, where the material is essentially a quiescent solid undergoing intermittent plastic events due of the loading (Baret *et al.*, 2002; Lin *et al.*, 2014b; Talamali *et al.*, 2012). In these *extremal models*, the algorithm identifies the least stable site at each step and increases the applied stress enough to destabilize it. From this single destabilization an avalanche of plastic events may ensue. Connection to real time is lost. Extremal models are the lattice-based counterpart of quasistatic atomistic simulations, in which a small strain increment is applied at each step and the system then relaxes athermally to the local energy minimum (Maloney and Lemaître, 2004).

D. Rearrangement duration and material time scales

In experiments as well as atomistic simulations, rearrangements are seen to occur very rapidly, so much so that they are considered instantaneous in several rheological models, *i.e.*, $\tau \rightarrow 0$ in Eq. 5. On the contrary, in various models the finite duration of plastic events plays a major role in the $\dot{\gamma}$ -dependence of the rheology (Liu *et al.*, 2016; Martens *et al.*, 2012; Nicolas *et al.*, 2014a; Picard *et al.*, 2005) or in the intrinsic relaxation of the system (Ferrero *et al.*, 2014).

Suppose that, under slow driving, a rearrangement takes a typical time τ_{pl} . For overdamped dynamics, one expects this time scale to be the ratio between an effective microscopic viscosity η_{eff} and the elastic shear modulus μ (Nicolas and Barrat, 2013a), *viz.*,

$$\tau_{\text{pl}} \sim \eta_{\text{eff}}/\mu,$$

while for underdamped systems τ_{pl} is associated with the persistence time of localized vibrations. If the time scale τ_{pl} competes with the driving time scale $\tau_{\dot{\gamma}} \equiv \frac{\gamma_y}{\dot{\gamma}}$, where γ_y is the local yield strain, then plastic events will be disrupted by

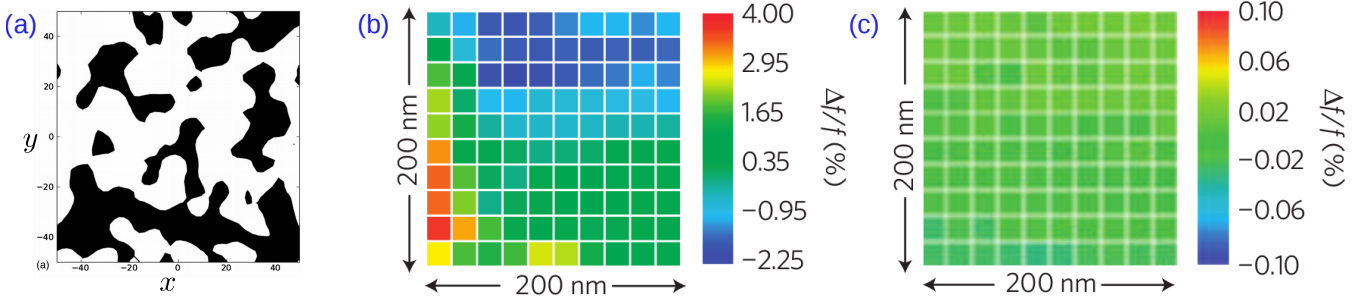


FIG. 7 *Spatial variations of the mechanical and configurational properties of glasses.* (a) Maps of the weaker local shear modulus in a 2D Lennard-Jones glass. Black (white) represents values larger (smaller) than the mean value. Distances are in particle size (Lennard-Jones) units. From (Tsamados *et al.*, 2009). (b) and (c) Maps of the local contact-resonance frequency, which is related to the indentation modulus, measured by atomic force acoustic microscopy in (b) a bulk metallic glass PdCuSi and in (c) a crystalline sample of SrTiO₃. The latter clearly appears more homogeneous in terms of mechanical properties. The radius of contact is of order 10 nm. From (Wagner *et al.*, 2011).

the driving. The rate dependence of the macroscopic stress may then stem from this disruption (Nicolas *et al.*, 2014a). On the contrary, if the driving is too slow to allow such competition, *viz.*, $\tau_{pl} \ll \tau_{\dot{\gamma}}$, then individual rearrangements can be considered instantaneous as far as the rheology is concerned, but the *avalanches* of rearrangements (*i.e.*, the series of plastic events that would still be triggered by an initial event, were the driving turned off) might still take a finite time, controlled by the signal propagation time τ_{pr} within the avalanche. Since the size of avalanches is expected to diverge as $\dot{\gamma} \rightarrow 0$ in the athermal limit, they may be affected by the driving when $\dot{\gamma} \rightarrow 0$. While the delays due to shear wave propagation are generally left aside in EPM, some works have bestowed them a central role in the finite shear-rate rheology (Lemaître and Caroli, 2009; Lin *et al.*, 2014b) and endeavored to represent this propagation in a more realistic way (Karimi and Barrat, 2016; Nicolas *et al.*, 2015). Sec. VI and VII will provide more details on these aspects. Note that the true quasistatic limit is reached when

$$\frac{\tau_{pl}}{\tau_{\dot{\gamma}}} \rightarrow 0 \text{ and } \frac{\tau_{pr}}{\tau_{\dot{\gamma}}} \rightarrow 0$$

and the athermal criterion of Eq. 7 is satisfied, *i.e.*, the Pclet number is very large. In that case, the material remains in mechanical equilibrium at all times and its trajectory in the PEL is rate-independent.

































E. Spatial disorder in the mechanical properties

Glasses, and more generally amorphous solids, are known to display heterogeneous mechanical properties. Indeed, there have been both experimental and numerical reports on the heterogeneity of the local elastic moduli (see Fig. 7) and the energy barriers (Tsamados *et al.*, 2008; Zargar *et al.*, 2013) on the mesoscale. Yet, the extent to which this disorder impacts the rheology remains unclear. This uncertainty is reflected in EPM. Some models feature no such heterogeneity (Hébraud and Lequeux, 1998; Picard *et al.*, 2005), while it is central in others (Langer, 2008). In the latter case, heterogeneity is generally implemented in the form of a disorder on the yield stresses or energy barriers. Let us mention a couple of examples. In Sollich *et al.* (1997)'s Soft Glassy Rheology model, energy barriers are exponentially distributed and the exponential dependence of activation rates on the energy barrier (Eq. 6) leads to a transition from Newtonian to non-Newtonian rheology for broader energy distributions (see Sec. IV.D.1 for more details on the model). In their EPM centered on metallic glasses, Li *et al.* (2013) modify the free energy required for the activation of an event depending on the free volume created during previous rearrangements. Amorphous composite materials, *i.e.*, materials featuring meso/macro-inclusions of another material, can be described as a patchwork of regions of high and low yield stresses (Tyukodi *et al.*, 2016a) or high and low elastic moduli (Chen and Schuh, 2015). In the latter case, macroscopic effective shear and bulk moduli can be derived.

More generally, for single phase materials, the survey of the above results gives the impression that disorder has bearing on the rheology when thermal activation plays an important role. On the other hand, the impact of a yield stress disorder may be less important in athermal systems. In fact, using the Hébraud-Lequeux model as a paradigm, Agoritsas *et al.* (2015) showed that disorder is irrelevant in the mean-field description of athermal plasticity originally proposed by Hébraud and Lequeux (1998), in the low shear rate limit; more precisely, it only affects the coefficients

in the rheological law.

TABLE I Classification of some of the main EPM in the literature.

Yielding	Reference	Features	Remarks	Proposed for
Activated	Bulatov and Argon (1994a) <i>et seq.</i>	  	Propagator computed on hexagonal lattice	AMORPHOUS SOLIDS, IN PARTICULAR GLASSES AND GLASS-FORMING LIQUIDS
	Homer and Schuh (2009) <i>et seq.</i>	  	Stress redistribution computed with Finite Elements	METALLIC GLASSES
	Ferrero <i>et al.</i> (2014)	 	Pl. events of finite duration	AMORPHOUS SOLIDS
	Sollich <i>et al.</i> (1997)	 	Effective activation temperature accounts for mechanical noise	SOFT MATERIALS (FOAMS, EMULSIONS, ETC.)
Threshold	Chen <i>et al.</i> (1991)	  	Propagator computed on square spring network	EARTHQUAKES
	Baret <i>et al.</i> (2002); Budrikis <i>et al.</i> (2017); Vandembroucq and Roux (2011)	  	Uniform distribution of barriers; extremal dynamics	AMORPHOUS SOLIDS, NOTABLY METALLIC GLASSES
	Dahmen <i>et al.</i> (2011)	 	‘Narrow’ distribution of thresholds; static and dynamic thresholds differ; mean-field approach	GRANULAR MATTER AND AKIN
	Hébraud and Lequeux (1998)	 	Finite yield rate above threshold; stress redistributed as white noise	SOFT MATERIALS (DENSE SUSPENSIONS)
	Martens <i>et al.</i> (2012); Picard <i>et al.</i> (2005)	 	Finite yield rate above threshold; pl. events of finite duration	AMORPHOUS SOLIDS
	Nicolas <i>et al.</i> (2014a) <i>et seq</i>	 	Pl. events end after finite <i>strain</i>	SOFT ATHERMAL AMORPHOUS SOLIDS
	Lin <i>et al.</i> (2014b)	  	Stress-controlled	SOFT AMORPHOUS SOLIDS
‘Continuous’ approaches	Onuki (2003b)	 	Dynamical evolution on a periodic potential; dipolar propagator due to opposite dislocations	2D CRYSTALLINE AND GLASSY SOLIDS
	Jagla (2007)	 	Dynamical evolution on random potential; propagator computed via compatibility condition	AMORPHOUS SOLIDS
	Marmottant and Graner (2013)		overdamped evolution in a periodic potential; Pl. events of finite duration; no stress redistribution	FOAMS

Legend – Barrier distribution:  Single value.  Distributed (exponentially, unless otherwise specified).

 Instantaneous plastic events.  Quadrupolar elastic propagator.

F. Spatial resolution of the model

On a related note, one may wonder how important it is to resolve spatially an EPM, or equivalently, in what cases one may settle with a mean-field approach blind to spatial information. Clearly, there are situations in which mean field makes a bad candidate, in particular when the driving or flow is macroscopically heterogeneous, when the focus is on spatial correlations (Nicolas *et al.*, 2014c) or on critical properties (Lin *et al.*, 2014b; Liu *et al.*, 2016). But a mean field analysis could suffice in many other situations. Indeed, Martens *et al.* (2012) showed that the flow curve obtained with their spatially resolved EPM can be predicted on the basis of mean-field reasoning. Thus, the details of the spatial correlations only had limited effect on the macroscopic stress. Similarly, Ferrero *et al.* (2014)’s EPM-based simulations confirmed mean-field predictions by Bouchaud and Pitard (2001) regarding thermal relaxation of amorphous solids in some regimes; but not without finding discrepancies in others. In the latter regimes, spatial correlations thus seemed to play a significant role.

The discussion about whether spatial resolution is required to describe global quantities is not settled yet. It has been argued that, owing to the long range of the elastic propagator (which decays radially r^{-d} in d dimensions), mean-field arguments should generically hold in amorphous solids (Dahmen *et al.*, 1998, 2009). However, it has been realized that the non-convex nature of the propagator (alternatively positively and negatively along the azimuthal direction) undermines this argument (Budrikis and Zapperi, 2013) and results in much larger fluctuations than the ones produced by a uniform stress redistribution (Lin *et al.*, 2014a; Nicolas *et al.*, 2014b; Talamali *et al.*, 2011). Mean-

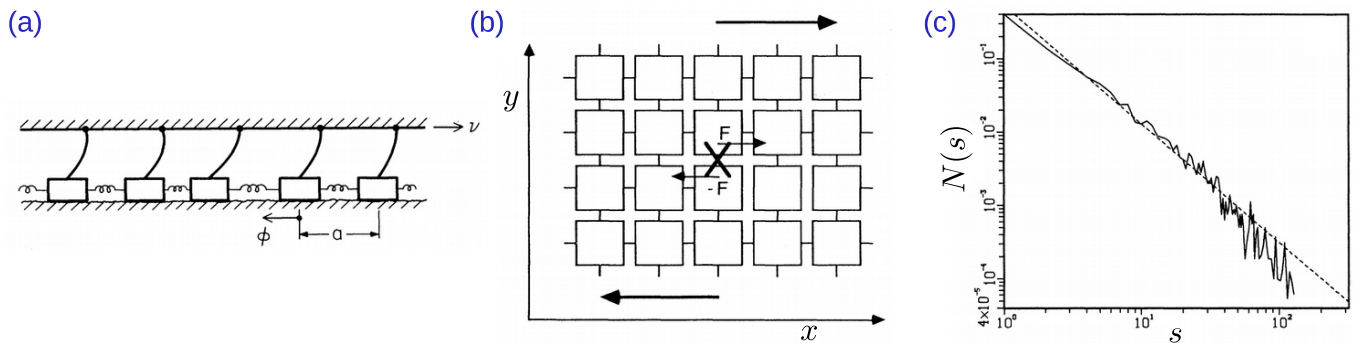


FIG. 8 (a) Sketch of the discrete 1D Burridge and Knopoff model. From (Carlson *et al.*, 1994). (b) and (c) Chen *et al.* (1991)'s spring network model. (b) Sketch of the effect of a bond rupture in the model. (c) Distribution of avalanche sizes, in terms of number of broken bonds. Adapted from (Chen *et al.*, 1991).

field predictions have been tested against the results of lattice-based models simulations of a sheared amorphous solid close to (or in) the limit of vanishing driving, with a focus on the statistics of stress-drops or avalanches, and *non-mean-field* exponents were found for the power-law distribution of avalanche sizes (Budrikis and Zapperi, 2013; Lin *et al.*, 2014b; Liu *et al.*, 2016; Talamali *et al.*, 2011). This question is addressed in greater depth in Sec. VII.

In this review, we will put the spotlight on spatially resolved models, which are not exactly solvable in general and require a numerical treatment. When relevant, we will discuss how a mean-field treatment can be performed to obtain analytical results.

G. Bird's eye view of the various models

To conclude this section, some of the main EPM are classified in Table I.

III. ELASTIC COUPLINGS AND THE INTERACTION KERNEL

A key feature of EPM is to allow plastic events to interact via an elastic deformation field, which can generate avalanches. In this respect, the choice of the elastic interaction kernel may significantly impact the results of the simulations (Budrikis and Zapperi, 2013; Martens *et al.*, 2012). This fairly technical section presents the various idealizations of the interaction kernel that have been used in the literature on amorphous solids, by increasing order of sophistication. We endeavor to relate this level of sophistication with the nature of the developments that were sought.

A. Sandpile models and first-neighbor stress redistribution

The watershed between the models for earthquakes and general avalanches and the more recent EPM is often fuzzy. In fact, the latter literally burgeoned on avalanche and earthquake-ridden scientific grounds.

As a paradigmatic earthquake model, consider the celebrated model by Burridge and Knopoff (1967), whose main features are concisely reviewed in (Carlson *et al.*, 1994). It focuses on the fault separating two slowly moving tectonic plates. This region is structurally weak because of the gouge (crushed rock powder) it is made of; thus, failure tends to localize along this fault. In the model, the contact points across the fault are represented by massive blocks and the compressive and shear forces acting along it are modeled as springs, as sketched in Fig. 8a. Due to these forces, the initially pinned (stuck) blocks may slide during avalanches. More precisely, in the continuous, nondimensional 1D form, the displacement $U(x, t)$ at time t of the material at position x reads

$$\ddot{U} = \xi^2 \frac{\partial^2 U}{\partial x^2} + vt - U - \phi(\dot{U}). \quad (8)$$

Here, the left-hand side (lhs) is related to inertia, the second derivative on the right-hand side (rhs) is of compressive origin, and the loading term vt due to the motion of the plate as well as the displacement $-U$ contribute to a shear

term. Finally, $\phi(\dot{U})$ is a velocity-dependent frictional term. Had Coulomb's law of friction been used, it would have been constant for $|\dot{U}| \neq 0$, but the original model assumed velocity weakening, i.e., a decrease of $|\phi(\dot{U})|$ with $|\dot{U}|$. At $\dot{U} = 0$, the function ϕ is degenerate, which allows static friction to exactly cancel the sum of forces on the rhs of Eq. (8), so the blocks remain pinned at a fixed position U until the destabilizing forces $\xi^2 \frac{\partial^2 U}{\partial x^2} + vt$ exceed a certain threshold. Phenomenologically, simulations of the model show frequent small events (with a power-law distribution of cumulative slip) and rare events of large magnitude, in which the destabilization of a number of sites close to instability results in a perturbation of large amplitude (Carlson *et al.*, 1994; Otsuka, 1972).

Important in the above model is the effect of the pinning force ϕ at $\dot{U} = 0$. It entails that the destabilizing action caused by the depinning of a site (via the diffusive term in Eq. 8) is fully screened by its neighbors, unless they yield too. Such first-neighbor redistribution of strain is readily simulated using cellular automata, which can be interpreted as sandpile models: Whenever a column of sand, labelled (i, j) , gets too high with respect to its neighbors (say, for convenience, whenever $\sigma_{i,j} \geq 4$), some grains at its top are transferred to the neighboring columns, with the following discharge rules in two dimensions:

$$\begin{aligned} \sigma_{i,j} \geq 4 : \sigma_{i,j} &\rightarrow \sigma_{i,j} - 4 \\ \sigma_{i\pm 1,j} &\rightarrow \sigma_{i\pm 1,j} + 1 \\ \sigma_{i,j\pm 1} &\rightarrow \sigma_{i,j\pm 1} + 1 \end{aligned} \tag{9}$$

where σ is the height difference between columns. The sandpile is loaded by randomly strewing grains over it in a quasistatic manner. The study of these systems soared in the late 1980s and early 1990s, whence the overbearing concept of self-organized criticality emerged (Bak *et al.*, 1987). According to the latter, the avalanches naturally drive the sandpiles toward marginally stable states, with no characteristic lengthscale for the regions on the verge of instability, hence the observation of scale-free frequency distributions of avalanche sizes. As an aside, let us mention that this approach has not been used only for earthquakes (Bak and Tang, 1989; Carlson and Langer, 1989; Ito and Matsuzaki, 1990; Sornette and Sornette, 1989) and avalanches in sandpiles, it has also been transposed to the study of integrate-and-fire cells (Corral *et al.*, 1995) and forest fires (Chen *et al.*, 1990), *inter alia*.

In seismology, these models have been fairly successful in reproducing the Gutenberg and Richter (1944) statistics of earthquake. This empirical law states that the frequency of earthquakes of (energy) magnitude

$$M_e = \frac{2}{3} \log(E) - 2.9, \tag{10}$$

where E is the energy release, in a given region obeys the power law relation, $\log P(m \geq m_0) \simeq -bm_0 + \text{cst}$, where $b \simeq 0.88$, or equivalently

$$p(E) \sim E^{-\tau}, \text{ with } \tau = 1 + \frac{2}{3}b \approx 1.5.$$

For the sake of accuracy, we ought to say that there exist several earthquake magnitude scales besides that of Eq. 10. They roughly coincide at not too large values; in fact, M_e is not the initial Richter scale. More importantly, the value of the exponent $b \in [0.8, 1.5]$ depends on the considered earthquake catalog, notably on the considered region. For sandpile-like models, various exponents have been reported: $\tau \approx 1$ in 2D and $\tau \approx 1.35$ in 3D, with no effect of a disorder on the yield stresses (Bak and Tang, 1989), whereas the exponent for the mean-field democratic fiber bundle close to global failure is $\tau = 3/2$ (see Section IX.C). More extensive numerical simulations have led to the values $\tau \simeq 1.30$ (Lübeck and Usadel, 1997), or $\tau \simeq 1.27$ (Chessa *et al.*, 1999), for the 2D Bak *et al.* (1987) sandpile model.

Olami *et al.* (1992) modified the model to make the redistribution rule of Eq. 9 non-conservative. In this sandpile picture, this would correspond to a net loss of grains, which seems unphysical; but in Burridge and Knopoff (1967)'s block-and-spring model the non-conservative parameter simply refers to the fraction of strain which is absorbed by the driving plate during an event, instead of being transferred to the neighbors. Interestingly, as non-conservativeness increases, criticality is maintained, insofar as the avalanche distribution $p(E)$ remains scale-free, even though the critical exponent τ gradually gets larger. Only when less than 20% of the strain is transferred to the neighbors does a transition to an exponential distribution occur. The dynamics then become more and more local with increasing dissipation, until the blocks completely stop interacting, when the redistribution is purely dissipative.

However, unlike the redistribution of grains in the sandpile model, elastic interactions are actually long-ranged, as we wrote in Section I.D. In particular, in the deformation of amorphous solids, no pinning of the region surrounding an event can be invoked to justify the restriction of the interaction to the first neighbors.

B. Networks of springs

Accordingly, a more realistic account of the long-ranged elastic propagation is desirable. Unfortunately, the complexity of the *bona fide* Eshelby propagator obtained from Continuum Mechanics hampers its numerical implementation and use, so most studies have relied on simplified propagators, which share some similarities with Eshelby's.

First, in the spirit of the classical description of a solid as an assembly of particles confined to their positions by the interactions with their neighbors, the material was modeled as a system of blocks connected by “springs” of stiffness κ and potential energy

$$\frac{1}{2}\kappa(\mathbf{u}_i - \mathbf{u}_j)^2,$$

where \mathbf{u}_i is the displacement of block i . Note that this expression for the potential energy entails noncentral forces, so that the “springs” can bear shear forces; some details about the difference with respect to networks of conventional springs are presented in Section IX.C. The pioneering steps towards EPM followed from the application of such spring network models to the study of rupture. For this purpose, each bond is endowed with a random threshold, above which it yields and redistributes the force that it used to bear. In their study of a 2D triangular lattice with central forces, Hansen *et al.* (1989) measured the evolution of the applied force F with the displacement u ; this evolution starts with a phase of linear increase, followed by a peak and a smooth decline until global failure. The $F(u)$ curves for different linear lattice sizes L roughly collapsed onto a master curve if F and u were rescaled by $L^{-3/4}$. In addition, just before failure, the distribution of forces in the system was “multifractal”, with no characteristic value.

Chen *et al.* (1991) considered a square lattice of blocks and “springs”, sketched in Fig. 8b. The rupture of a spring triggers the release of a dipole of opposite point forces (generating vorticity) on neighboring blocks. This differs somewhat from the force quadrupole corresponding to an (*irrotational*) local shear (see Section III.C), but also leads to an anisotropic shape. Contrary to Hansen *et al.* (1989), they allowed broken springs to instantly regenerate to an unloaded state, after the redistribution of their load. Physically, this discrepancy parallels a change of focus, from brittle materials to earthquakes, for which the external loading due to tectonic movements is assumed to be by far slower than the healing of bonds. For a quasistatic increase of the load, the model displays intermittent dynamics and scale-free avalanches, and a power-law exponent $\tau = 1.4$ was reported in 2D, in semi-quantitative agreement with the Gutenberg-Richter earthquake statistics.

C. Pointwise idealization of the Eshelby propagator

1. Derivation

An alternative to block-and-spring models, rooted in Continuum Mechanics and popularized by Picard *et al.* (2004), consists in simplifying Eshelby (1957)'s calculations of the elastic propagator by considering the pointwise circular limit of a 2D shear transformation. The latter then contributes to the equation of mechanical equilibrium as a source term $\mathbf{f}'(\mathbf{r})$, viz.,

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{r}) + \mathbf{f}'(\mathbf{r}) = 0. \quad (11)$$

The surrounding medium is supposed to be linear elastic and uniform, so that its elastic stress reads

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\boldsymbol{\epsilon}, \quad (12)$$

where p is the pressure and $\boldsymbol{\epsilon} = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^\top}{2}$ is the linear strain tensor. For simplicity, incompressibility is assumed, $\nabla \cdot \mathbf{u} = 0$. The solution of Eq. 11 is then well known in hydrodynamics and involves the Oseen-Burgers tensor $\mathcal{O}(\mathbf{r}) = \frac{1}{8\pi\mu r} (\mathbb{I} + \frac{\mathbf{r} \otimes \mathbf{r}}{r^2})$, with \mathbb{I} the identity matrix, viz.,

$$\mathbf{u}(\mathbf{r}) = \int \mathcal{O}(\mathbf{r} - \mathbf{r}') \mathbf{f}'(\mathbf{r}'). \quad (13)$$

Finally, to get the source term $\mathbf{f}'(\mathbf{r})$, one assumes that a shear transformation located at the origin locally shifts the unstrained configuration by an amount $\boldsymbol{\epsilon}^{\text{Pl}}$: the configuration that cancels the shear stress in $\mathbf{r} = \mathbf{0}$ is no longer $\boldsymbol{\epsilon}(\mathbf{r}) = \mathbf{0}$, but $\boldsymbol{\epsilon}(\mathbf{r}) = \boldsymbol{\epsilon}^{\text{Pl}} a^d \delta(\mathbf{r})$, in the limit of a rearrangement of vanishing volume $a^d \rightarrow 0$. In this sense, $\boldsymbol{\epsilon}^{\text{Pl}}$ can be regarded as an eigenstrain that generates a source term $\mathbf{f}'(\mathbf{r}) = -2\mu\nabla \cdot [\boldsymbol{\epsilon}^{\text{Pl}} a^d \delta(\mathbf{r})]$ in Eq. 11. In the unbounded 2D

plane, setting coordinates such that $\epsilon^{\text{Pl}} = \begin{pmatrix} 0 & \epsilon_0 \\ \epsilon_0 & 0 \end{pmatrix}$, the response to $\mathbf{f}'(\mathbf{r})$ in terms of xy -component of the stress reads

$$\sigma_{xy}(\mathbf{r}) = 2\mu\epsilon_0 a^2 \mathcal{G}(\mathbf{r}) \quad \text{with the propagator } \mathcal{G}^\infty(\mathbf{r}) \equiv \frac{\cos(4\theta)}{\pi r^2}, \quad (14)$$

where (r, θ) are polar coordinates. This field is shown in Fig. 6c. Reassuringly, in the far field this coincides with the response to a cylindrical Eshelby inclusion.

As a short aside, let us mention a variant to these calculations, which puts in the limelight the connection with deformation processes in a crystal. This variant is reminiscent of Eshelby's a cut-and-glue method, in which an ellipsoid is cut out of the material, deformed, and then reinserted. Following earlier endeavors by Ben-Zion and Rice (1993), Tüzes *et al.* (2017) carved out a square around the rearrangement, instead of an ellipsoid, displaced its edges to mimic shear, and then glued it back. This is tantamount to inserting four edge dislocations in the region and also yields an Eshelby-like quadrupolar field.

Rather than focusing on unbounded media, it is convenient to work in a bounded system with periodic boundary conditions and with a general plastic strain field $\epsilon^{\text{Pl}}(\mathbf{r})$. Switching to Fourier space ($\mathbf{r} \leftrightarrow \mathbf{q} \equiv (q_x, q_y)$), the counterpart of Eq. 14 is then

$$\sigma_{xy}(\mathbf{q}) = 2\mu\mathcal{G}(\mathbf{q}) \epsilon^{\text{Pl}}(\mathbf{q}) \quad \text{where } \mathcal{G}(\mathbf{q}) = -\frac{4q_x^2 q_y^2}{q^4}. \quad (15)$$

Note that the frame is sometimes defined such that $\epsilon^{\text{Pl}} = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & -\epsilon_0 \end{pmatrix}$; in this case, $\mathcal{G}(\mathbf{q}) = \frac{-(q_x^2 - q_y^2)^2}{q^4}$. In practice, the system will generally be discretized into a (square) lattice, which allows one to use a Fast Fourier Transform routine and restrict the considered wavenumbers to $q_x, q_y = \frac{2\pi n}{L}$, $n \in \{[\frac{-L}{2}], \dots, [\frac{L}{2}]\}$.

Besides, because of dissipative forces, quantified by an effective viscosity η_{eff} , the plastic strain rate $\dot{\epsilon}^{\text{Pl}}$ in the shear transformation cannot be infinite and a rearrangement will last for a finite time $\tau_{pl} \sim \eta_{eff}/\mu$ (see Section II.D). Therefore, in each numerical time step dt , the plastic strain ϵ^{Pl} implemented in Eq. 15 will be the strain increment $\delta\epsilon^{\text{Pl}} \equiv \dot{\epsilon}^{\text{Pl}} dt$ during that step. This amounts to saying that, locally, the rearrangement occurs gradually, even though the redistribution of stress to the rest of the medium is instantaneous (because mechanical equilibrium was assumed, so that there is no time dependence in the elastic propagator in Eq. 15).

2. Issues with this approximation and possible remediations

The idealized elastic propagator in Eq. 15 brings on some technical issues. First, its slow ($\propto r^{-d}$) radial decay in space raises convergence problems in periodic space. Indeed, the fields created by the periodic images of each plastic event have to be summed, but the sum converges only conditionally in real space, i.e., depends on the order of summation. This is reflected by the singularity of $\mathcal{G}(\mathbf{q})$ near $\mathbf{q} = \mathbf{0}$. In polar crystals, such a difficulty also arises, when computing the Madelung energy, but may be solved with the Ewald (1921) method. Here, we make use of the conserved quantities to state that $\mathcal{G}(\mathbf{q} = \mathbf{0}) = 0$ in a stress-controlled system and $\mathcal{G}(\mathbf{q} = \mathbf{0}) = -1$ in a strain-controlled system. Another possibility is to sum the images in an arbitrary order that is compatible with convergence. These distinct implementations match in the far field, but differ in the near field, which leads to different organizations for the flow (Budrikis and Zapperi, 2013).

Second, on a periodic lattice, one should in principle compute the periodic sum

$$\mathcal{G}^{sum}(\mathbf{q}) \equiv \sum_{\mathbf{n} \in \mathbb{Z}^d} \mathcal{G}(\mathbf{q} + 2\pi\mathbf{n})$$

if, at the lattice nodes, one wishes the backward discrete Fourier transform of $\mathcal{G}^{sum}(\mathbf{q})$ to coincide with the solution $\mathcal{G}^\infty(\mathbf{r})$ for an unbounded medium. However, the high-frequency components in $\mathcal{G}(\mathbf{q})$, due to the spurious singularity of $\mathcal{G}^\infty(\mathbf{r})$ at $\mathbf{r} = \mathbf{0}$ (Eq. 15), make the periodic sum diverge. In practice, wavenumbers outside the first Brillouin zone $]-\pi, \pi]^d$ are plainly discarded, which comes down to solving Eqs. 11-12 on the periodic lattice, rather than in the continuum. Nevertheless, spurious fluctuations in the response field are sometimes observed; the problem is mitigated by using a finer grid and smoothing the obtained field (Nicolas *et al.*, 2014b).

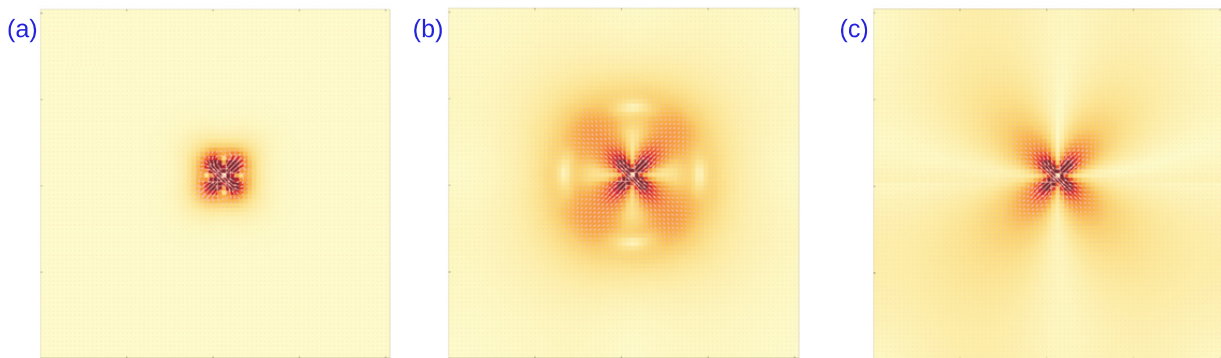


FIG. 9 Average displacement field induced by a shear transformation in an underdamped elastic medium, computed with a basic Finite Element routine. The plotted snapshots correspond to different delays after the transformation was (artificially) triggered at the origin: (a) $\Delta t = 2$, (b) $\Delta t = 10$, (c) $\Delta t = 1000$. Red hues indicate larger displacements. Adapted from (Nicolas *et al.*, 2015)

3. Variations: Soft modes and lattice symmetries; tensoriality; convection

All in all, many technical details of the implementation of the elastic propagator appear to affect the spatial organization of the flow (Talamali *et al.*, 2011), but leave the qualitative picture and (apparently) the scaling laws unaltered. However, an aspect that seems to be crucial is the need to preserve the eigenmodes of the propagator $\mathcal{G}(\mathbf{q})$ associated with eigenvalue (energy) zero. These so called soft modes are the fields ϵ^{pl} such that

$$\forall \mathbf{q}, \mathcal{G}(\mathbf{q}) \epsilon^{pl}(\mathbf{q}) = 0,$$

and, since they cost no elastic energy, their deployment is favored by the dynamics (Tyukodi *et al.*, 2016b). Their importance is further explained in Section IV.C. It turns out that the eigenmodes of $\mathcal{G}(\mathbf{q})$ in Eq. 15 are simply the Fourier modes (plane waves); among these, the soft modes are those with wavevectors that make an angle $\pm\pi/4$ with respect to the principal direction of the plastic strain tensor ϵ^{pl} .

In particular, if the material is loaded under simple shear with velocity direction x and velocity gradient along y , the emergence of a uniform shear band along x should produce no elastic stress in the medium, at least if such a band emerges uniformly. However, misaligned lattice axes, not directed along x or y , would not be compatible with such a shear band (which would then have sawtooth-like edges) and artificially suppress the soft modes (Tyukodi, 2016). More generally, the use of a regular lattice in EPM may be questioned, as it may affect the quantitative correlations in the flow (Budrikis *et al.*, 2017), but the scalings of, e.g., avalanche sizes seem to be mostly insensitive to these details.

On another note, the foregoing calculations focused on the xy -shear stress component, because of the macroscopic stress symmetry, thus promoting a scalar description. It is straightforward to generalize the reasoning to a fully tensorial form; but it turns out that, for the problems under consideration, the tensorial extension has virtually no effect (Budrikis *et al.*, 2017; Nicolas *et al.*, 2014b). Similarly, moving from 2D to 3D does not introduce qualitative changes and scaling relations are preserved (Budrikis *et al.*, 2017; Liu *et al.*, 2016). Also, the periodic boundary conditions can be substituted by no slip boundary conditions at a wall, via the image method (Picard *et al.*, 2004), for instance to model flow in a microchannel (Nicolas and Barrat, 2013a). Translational invariance of the propagator is then broken and plastic events relax stress faster, for a given eigenstrain, if they occur close to the walls. Finally, despite the convenience of using a fixed lattice grid with static elasto-plastic blocks, physically these blocks should be advected by the flow. In a bounded medium, a coarse version of advection can be implemented by incrementally shifting the blocks along the streamlines without altering the global shape of the lattice (Nicolas and Barrat, 2013a). On the other hand, with periodic boundary conditions, the deformation of the frame results in the shift of the periodic images with respect to the simulation cell; advection thus requires to compute the elastic propagator afresh, in the deformed frame (Nicolas *et al.*, 2014b).

D. Finite-Element-based approaches

Albeit computationally more costly, Finite-Element (FE)-based computations of stress redistributions overcome some limitations of the foregoing approaches and offer more flexibility. The FE method solves the continuum me-

chanics equation within each element of a meshgrid by interpolating the local strain and stress from the values of the displacements and point forces at the nodes of the element.

If mechanical equilibrium is maintained at all times, the stress redistributed by a shear transformation can be computed by equilibrating the elastic stress $\mathbf{C}\epsilon^*$ generated by the eigenstrain ϵ^* borne by a given element (where \mathbf{C} is the stiffness tensor). Using a triangular mesh refined around this eigenstrain-bearing element, Sandfeld *et al.* (2015) demonstrated that the computed stress field coincides with the elastic propagator of Eq. 14 in the pointwise rearrangement limit. But these researchers also found that a coarser mesh made of uniform square elements gives results that are almost as good, except in a near-field region of a handful of sites' radius. The flexibility of the method was then exploited to study the quasistatic deformation of the system beyond the periodic boundary conditions, e.g., in a bounded medium and with free surfaces, and with inhomogeneous loading conditions (indentation, bending, etc.). Universal, but non-mean-field, statistics of avalanches of plastic events were reported in these diverse conditions (Budrikis *et al.*, 2017) (also see Section VII).

In an earlier endeavor (Homer *et al.*, 2010; Homer and Schuh, 2009), each shear transformation zone consisted of *several* elements of a triangular mesh which all bore an eigenstrain. As the size of this zone increases, the redistributed stress field accurately converged to the theoretical Eshelby field. Zones made of 13 elements were deemed quite satisfactory in this respect. Homer and Schuh (2010) later extended the approach to 3D. Dynamics were brought into play via the implementation of an event-driven (Kinetic Monte Carlo) scheme determining the thermal activation of shear transformations, in the wake of the pioneering works of Bulatov and Argon (1994a). The cooling of the system, its thermal relaxation and its rheology under applied stress were then studied. Macroscopically homogeneous flows were observed at low stresses and/or high temperatures, whereas the strain localized at low temperature for initially unequilibrated (zero residual stress) systems, which was not necessarily supported by experimental data. More systematic strain localization at low temperature was found by Li *et al.* (2013), who incorporated the processes of free volume creation during plastic rearrangements and subsequent free volume annihilation (see Section V.C.2).

The capabilities of FE methods were further exploited by Nicolas *et al.* (2015) to go beyond the assumption of elastic homogeneity of the material. For this purpose, the mesoscale elastic constants in regions of 5 particles in diameter were measured in an atomistic glass model; the local shear moduli were found to be broadly distributed, with relative fluctuations of around 30% and marked anisotropy (i.e., one direction of shear being much weaker than the other one). Introducing this heterogeneity of shear moduli in the EPM sufficed to capture the sample-to-sample fluctuations of the elastic response to an artificially triggered shear transformation observed in Molecular Dynamics simulations (Puosi *et al.*, 2014); accounting for anisotropy was less critical. Besides, inertial and viscous terms were not omitted in the FE description, so that mechanical equilibration was not instantaneous and shear waves were seen to propagate in the transient, as in the atomistic simulations. The natural inclusion of inertia in FE was also exploited by Karimi *et al.* (2017) to analyze the effect of inertia on the universal avalanche statistics and compare with atomistic simulations directly. (Note that the effect of a delay in signal propagation had already been contemplated in an *effective* way by Lin *et al.* (2014a), while, for the same purpose, Papanikolaou (2016) introduced a pinning delay in his EPM based on the depinning framework.) It was then possible to investigate the influence of the damping strength on the rheology of the elastoplastic system, which was indeed done by Karimi and Barrat (2016). Using a Maxwellian fluid description for blocks in the plastic regime and an unstructured mesh, these researchers found trends qualitatively very similar to what is observed in Molecular Dynamics when the friction coefficient is varied.

E. Continuous approaches based on periodic potentials

Notwithstanding their variable sophistication, all above methods rest on a clearcut distinction between plastic rearrangements and elastic deformations. This binary distinction is relaxed in continuous approaches based on a free energy functional $F[e]$ that depends on the mechanical strain variables, generically denoted by e here. Since the energy increase due to elastic loading is cyclically interrupted by plastic rearrangements (associated with discrete jumps between valleys of the energy landscape), $F[e]$ involves *periodic* functions of the plastic deformation ϵ_p or the total deformation e . For instance, Marmottant and Graner (2013) made use of the following effective potential,

$$U_{\text{eff}}(e, \epsilon_p) = \frac{E}{2} (e - \epsilon_p)^2 + E\epsilon_y \frac{\epsilon_0}{2\pi} \cos\left(\frac{2\pi\epsilon_p}{\epsilon_0}\right),$$

where E is an elastic modulus, ϵ_y is a yield strain and ϵ_0 is the period of the pinning potential. If this prescription is coupled with a dynamical equation of the form

$$\tau \dot{\epsilon}_p = \frac{1}{E} \left(-\frac{\partial U_{\text{eff}}}{\partial \epsilon_p} \right),$$

with τ the characteristic relaxation timescale (leading to the Prandtl–Tomlinson model for stick–slip), a serrated stress *vs.* strain curve is obtained under constant driving. The finite time needed by the plastic deformation ϵ_p to jump between energy valleys implies that, at high driving rates, ϵ_p will not be able to instantaneously jump between, say, $\epsilon_p^{(-)}$ and $\epsilon_p^{(+)}$. Therefore, the elastic strain ϵ will continue increasing in the valley around $\epsilon_p^{(-)}$ for some time, although the criterion for the onset of plasticity has already been met, which is similar to having a finite latency time prior to relaxation once the threshold is exceeded in Picard *et al.* (2005)’s model. Similar equations of motion in a random potential have been proposed for solid friction; the occurrence of stick-slip dynamics owes to the “pinning” of the system in one potential valley, up to some threshold, while there exists another stable position (Tyukodi *et al.*, 2016b).

To go beyond the mean-field level, this type of continuous approach can be resolved spatially. In an inspirational study, Onuki (2003a) introduced an elastic free energy of the form

$$F_{el} = \int d\mathbf{r} K_0 e_1^2(\mathbf{r}) + \mathcal{F}(e_2(\mathbf{r}), e_3(\mathbf{r})), \quad (16)$$

where K_0 is the bulk modulus and the volumetric strain e_1 as well as the shear strains e_2 and e_3 are explicit functions of the displacement field $\mathbf{u}(\mathbf{r})$. Here, \mathcal{F} is an arbitrarily chosen function that is invariant under rotations of the reference frame $\theta \rightarrow \theta + \pi/3$ (because a 2D triangular lattice is assumed) and periodic in its arguments. Introducing F_{el} in the equation of motion

$$\rho \ddot{\mathbf{u}} = -\frac{\delta F_{el}}{\delta \mathbf{u}} + \eta_0 \nabla^2 \dot{\mathbf{u}} + \nabla \cdot \boldsymbol{\sigma}^R, \quad (17)$$

where ρ is the density, η_0 is the viscosity and $\boldsymbol{\sigma}^R$ is a random stress tensor due to thermal fluctuations, suffices to obtain qualitatively realistic stress *vs.* strain curves. The framework was then extended to study the effect of an interplay between the volumetric strain e_1 and the density ρ , and to capture the elastic effects of edge dislocations, if the material is crystalline (Onuki, 2003b).

If the strain components e_1 , e_2 and e_3 are handled as independent primary variables, instead of being functions of \mathbf{u} as in Eq. 17, then their compatibility as components of a strain tensor should be ensured by the Saint-Venant condition

$$S[e_1, e_2, e_3] = 0 \text{ where } S[e_1, e_2, e_3] \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) e_1 - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) e_2 - 2 \frac{\partial^2}{\partial x \partial y} e_3.$$

This constraint is implemented by means of a Lagrange multiplier in the total free energy F , viz., $F \rightarrow F + \lambda S[e_1, e_2, e_3]$. A minimization of F yields equations of motion such as $\dot{e} \propto -\frac{\delta F_{el}}{\delta e}$, for a generic strain component e , that complete the definition of the EPM (Jagla, 2007). If the function $\mathcal{F}(e_2, e_3)$ entering the free energy in Eq. 16 is assumed to depend quadratically on e_2 (as in linear elasticity), while its nonlinearity with respect to e_3 preserves the possibility of plastic relaxation, then it can be shown analytically that, owing to the Saint-Venant condition, a plastic strain along e_3 gradually gives rise to an elastic strain with a quadrupolar structure (Kantha *et al.*, 1995). Once mechanical equilibrium is reached, the final strain field produced by this local eigenstrain coincides in the incompressible limit with the elastic propagator of Eq. 15 (Jagla, 2017). The approach was extended recently by Jagla (2017) to study the influence of different choices for the effective plastic disorder potentials in the flow curve and critical exponents.

IV. MECHANICAL NOISE AND ITS APPROXIMATIONS

The previous section has shed light on the modeling of the elastic propagator, *i.e.*, the effect of a single rearrangement on the surrounding elastic medium. In practice, however, several rearrangements may occur simultaneously, and the rate $\zeta_i(t)$ of stress increment received by a given block (say, site i) at time t is then a sum of contributions from many

sites, i.e., using Eq. 5, $\zeta_i(t) = \sum_{j \neq i} n_j \mathcal{G}_{ij} \frac{\sigma_j}{\tau}$, where n_j denotes the plastic activity of site j . Due to its fluctuating nature, this quantity is often referred to as *mechanical noise*. By rewriting Eq. 5 as

$$\frac{\partial}{\partial t} \sigma_i(t) = \mu \dot{\gamma} - n_i \frac{|\mathcal{G}_0| \sigma_i(t)}{\tau} + \zeta_i(t), \quad (18)$$

one can readily see that, in combination with the external loading and the dynamical rules governing n_i , the mechanical noise signal $\{\zeta_i(t)\}$ fully determines the local stress evolution. All “one-point” properties (such as the flow curve, the density of plastic sites, the distribution of local stresses, etc.) can be obtained by averaging the local properties at i over time. This shows the central role of $\{\zeta_i(t)\}$ in determining these properties. Unfortunately, this signal is complex, as it stems from interacting plastic events throughout the system; nevertheless, mean-field approaches suggest to substitute it with a simpler “mean” field.

A. Uniform redistribution of stress

The mechanical noise can be split into :

- a constant background $\langle \zeta_i \rangle$, which contributes to a drift term $\mu \dot{\gamma}_i^{\text{eff}} \equiv \mu \dot{\gamma} + \langle \zeta_i \rangle$ in Eq. 18, and
- zero-average fluctuations $\delta \zeta_i(t)$.

Owing to the infinite range and slow decay of the elastic propagator ($\propto r^{-d}$ in d-dimensional space, see Sec. I.D), site j is significantly coupled to a large number of other sites. This large connectivity has led some researchers to overlook the fluctuations in favor of the average drift term. Along these lines, in the framework of Picard’s EPM, which features a constant rate τ^{-1} of yield above a uniform threshold and a constant rate τ_{res}^{-1} of elastic recovery, viz.,

$$n : 0 \xrightarrow[\tau_{\text{res}}^{-1}]{\tau^{-1} \Theta(\sigma - \sigma_y)} 1, \quad (19)$$

Martens *et al.* (2012) averaged Eq. 18 over time and found a mean-field analytical expression for the flow curve, which reproduces the simulation results to a large extent. It also correctly predicts the destabilization of the homogeneous flow leading to shear-banding for a range of model parameters, in particular at large τ_{res} .

In fact, the neglect of fluctuations would be rigorously justified if the system were infinite and the non-convexity of the propagator \mathcal{G} were left aside. The latter criterion is for instance fulfilled in a simple quasistatic model in which sites yield past a threshold σ_c and redistribute the released stress ($\delta \sigma_i$) *uniformly* to the other $N - 1 \approx N$ sites (Dahmen *et al.*, 1998), viz.,

$$\begin{aligned} \sigma_i > \sigma_c : \sigma_i &\rightarrow \sigma_i - \delta \sigma_i \\ \sigma_j &\rightarrow \sigma_j + \frac{\delta \sigma_i}{N}, \forall j \neq i. \end{aligned}$$

The simplicity of the model allows analytical progress. A first approach consists in treating the distances $x_i = \sigma_c - \sigma_i$ to the threshold σ_c as independent variables in the system and sorting them in ascending order ($i \rightarrow 1, 2, \dots$). An avalanche will persist as long as the stress increment $\frac{\delta \sigma_1}{N}$ due to the yielding of the most unstable site suffices to make the second most unstable fail, viz., $\frac{\delta \sigma_1}{N} > x_2$. Using an argument along these lines in a model featuring disorder in the yield thresholds ($\sigma_c \rightarrow \sigma_{c,i}$) and post-failure weakening (i.e., when site i yields, the threshold is restored to a lower value $\sigma_{c,i}(t+1) < \sigma_{c,i}(t)$), Dahmen *et al.* (1998) were able to rationalize the existence of a regime of power-law distributed avalanches and a regime of runaway, system-spanning avalanches.

Alternatively, owing to the similarity of the simplified problem with force-driven depinning, one can make use of the machinery developed in the latter field. Transversal scaling arguments and renormalization group expansions (Fisher *et al.*, 1997) then allow one to derive scalings for different properties of the system in the quasistatic limit, such as the size of avalanches. Note that this method was initially applied to the depinning problem and to earthquakes. Only later on was it claimed to be much more general and to have bearing on very diverse systems exhibiting intermittent dynamics or “crackling noise” (Sethna *et al.*, 2001), in particular the yielding transition of amorphous solids (see Sec. VII). Recently, these mean-field scaling predictions about avalanche sizes, shapes, and dynamics have been used to fit experimental data, in metallic glasses subjected to extremely slow uniaxial compression (Antonaglia *et al.*, 2014; Dahmen *et al.*, 2009) as well as in compacted granular matter (Denisov *et al.*, 2016).

B. Random stress redistribution

1. Deviations from uniform mean field

Notwithstanding this success, the underpinning of the foregoing mean-field approach has been called into question. Theoretically, the argument based on the long range of the interactions is undermined by the fact that these interactions are sometimes positive and sometimes negative (Budrikis and Zapperi, 2013). A crude estimate of the ratio of fluctuations over mean value of the stress increments points to a divergence of this ratio at low shear rates $\dot{\gamma}$ and therefore to the failure of the mean-field theory, according to Ginzburg and Landau's criterion (Nicolas *et al.*, 2014b). Numerically, some lattice-based simulations do indeed reveal departures from mean-field predictions for the critical exponents (Budrikis and Zapperi, 2013; Lin *et al.*, 2014a; Liu *et al.*, 2016). For instance, in these simulations, near $\dot{\gamma} \rightarrow 0$, the distribution of avalanche sizes S follows a power law $P(S) \sim S^{-\tau}$ with an exponent τ that deviates from the $\tau = 3/2$ value predicted by mean field (see Sec. VII for details).

2. The Hébraud-Lequeux model

To improve on the hypothesis of a constant mean field $\dot{\gamma}^{\text{eff}}$, fluctuations of the mechanical noise need to be accounted for. In the crudest approximation, they can be substituted by random white noise $\zeta^{(w.n.)}(t)$, with $\langle \zeta^{(w.n.)} \rangle = 0$. This turns Eq. 18 into a biased Brownian walk for the local stresses, in the elastic regime $n_i = 0$. Hébraud and Lequeux (1998)'s model was developed along these lines. The ensuing stochastic equation (Eq. 18 with $\zeta_i(t) \rightarrow \dot{\gamma}^{\text{eff}} + \zeta^{(w.n.)}(t)$ and $\tau \rightarrow 0$) can be recast into a probabilistic Fokker-Planck-like equation operating on the distribution $P(\sigma, t)$ of local stresses σ , viz.,

$$\frac{\partial P(\sigma, t)}{\partial t} = -\mu\dot{\gamma} \frac{\partial P(\sigma, t)}{\partial \sigma} + D(t) \frac{\partial^2 P(\sigma, t)}{\partial \sigma^2} - \frac{\Theta(|\sigma| - \sigma_c)}{\tau_{\text{liq}}} P(\sigma, t) + \Gamma(t) \delta(\sigma), \quad (20)$$

where the diffusive term $D \frac{\partial^2 P}{\partial \sigma^2}$ on the rhs arises from the fluctuations acting on σ_i , with a coefficient $D(t)$ assumed to be proportional to the number of plastic sites $\Gamma(t) \equiv \tau_{\text{liq}}^{-1} \int_{|\sigma'| > \sigma_y} P(\sigma', t) d\sigma'$, viz., $D(t) = \alpha \Gamma(t)$. The first term on the rhs of Eq. 20 is a drift term, which amalgamates $\dot{\gamma}^{\text{eff}}$ with $\dot{\gamma}$; the last two terms correspond to the failure of overloaded sites (above σ_c) on a timescale τ_{liq} and their rebirth at $\sigma = 0$ due to stress relaxation. The resulting mean-field equations can be solved in the limit of vanishing shear rates $\dot{\gamma}$ (Agoritsas *et al.*, 2015; Olivier, 2011). For a coupling constant $\alpha < 1/2$, diffusion vanishes at low shear rates, with $D \propto \dot{\gamma}$, a yield stress $\Sigma_Y > 0$ is obtained and the average stress obeys $\Sigma \simeq \Sigma_Y + k\dot{\gamma}^{1/2}$, with $k > 0$, in the limit of slow shear rates. For $\alpha > 1/2$, the system behaves like a Newtonian liquid.

3. Fraction of sites close to yielding

The diffusive term introduced in Eq. 20 impacts the distribution of sites close to yield, i.e., at distances $x \ll 1$ from the yield threshold σ_c , where $x \equiv |\sigma| - \sigma_c$. On these short distances, or, equivalently, in the limit of short time scales Δt , the back-and-forth diffusive motion over typical distances $\propto \sqrt{\Delta t}$ prevails over the drift in the random walk. Therefore, for $\dot{\gamma} \rightarrow 0$, determining the distribution $P(x)$ is tantamount to finding the concentration of Brownian particles near an absorbing boundary at $x = 0$ (yield): the well-known solution is a linear vanishment of the concentration near $x = 0$, viz., $P(x) \sim x$ for $x \approx 0$ (Lin *et al.*, 2014a; Lin and Wyart, 2016). This result ought to be compared with $P(x) \sim x^0$ for drift-dominated problems, such as depinning. Lin *et al.* (2014a) further claim that this discrepancy is at the origin of the differences in scaling behavior between the depinning transition ($v \propto (F - F_c)^\beta$) with $\beta < 1$) and the flow of disordered solids ($\dot{\gamma} \propto (\sigma - \sigma_c)^\beta$) with $\beta > 1$ generally).

Still, the singularity of the propagator $\mathcal{G} \sim \frac{\cos 4\theta}{r^d}$ casts doubt on the Gaussian nature of the random stress increments $\delta\zeta$ and would rather suggest a broad density function

$$\rho(\delta\zeta) \sim \delta\zeta^{-2}$$

in the limit of sparse plastic events, with an upper cut-off $\delta\zeta_M$ proportional to the volume of a rearranging region. For such a heavy-tailed distribution, the biased random Brownian walk of σ_j is replaced by a Lévy flight of index $\mu = 1$ for σ_j . On the basis of a simple extremal model, Lemaître and Caroli (2007) demonstrated that this change altered the avalanche statistics as well as the distribution of distances to yield $P(x)$. To be explicit, their model was based on

plastic yield above a uniform yield strain γ_c , which resets the local stress to zero and increments the stresses at other sites by random values drawn from $\rho(\delta\zeta)$. Lin and Wyart (2016) explored a related, but more general (Hébraud-Lequeux-like) model analytically and confirmed the impact of the non-Gaussianity of the noise. These authors derived an asymptotic expression for $P(x)$, which scales as x^θ for $x \ll 1$. The exponent $\theta > 0$ is a non-universal exponent that depends on the loading and the amplitude of the noise and supplements the other two exponents characterizing the depinning transition.

C. Validity of the above “mean-field” approximations

The foregoing paragraphs have presented distinct levels of “mean-field” approximations. Now we enquire into their range of validity and record the results in Table II.

1. Uniform mean field

Clearly, the neglect of fluctuations in the constant mean-field approach is sensible only in the drift-dominated regime, *i.e.*, when $|\int_0^{\Delta t} \dot{\gamma}^{\text{eff}}(t') dt'| \gg |\int_0^{\Delta t} \delta\zeta(t') dt'|$ on the considered time scales Δt , with the notations of Sec. IV.A. With interactions that change signs, this excludes vanishing shear rates or too small time windows Δt . But at high shear rates, this approach appears to correctly predict the avalanche scaling exponents in the EPM studied by Liu *et al.* (2016).

2. White-noise fluctuations

Complemented with Gaussian fluctuations, the approximation is valid beyond the drift-dominated regime. In fact, any mechanical noise signal with (i) finite average and finite variance, in particular $\rho(\delta\zeta) = o(\delta\zeta^{-3})$, and (ii) no significant time correlations, can be replaced by Gaussian white noise in Eq. 18 (Lin and Wyart, 2016). Accordingly, the universality class of the Hébraud-Lequeux model encompasses all models based on similar rules for plasticity and where the mechanical noise fulfils the above criteria (i) and (ii). In particular, for coupling parameters α such that the diffusivity $D(t)$ goes to zero at $\dot{\gamma} \rightarrow 0$, their flow curves will follow a Herschel-Bulkley behavior $\Sigma = \Sigma_Y + k\dot{\gamma}^m$ with $m = 1/2$ in the low shear rate limit. This holds true in the presence of disorder on the local yield thresholds σ_c and for plastic events that do not relax the local stress strictly to zero, but to a low random value (Agoritsas *et al.*, 2015). On the other hand, should the shear modulus of elastic blocks or the relaxation time display a shear-rate dependence, the exponent m will deviate from $1/2$ (Agoritsas and Martens, 2017).

Besides, even in an elastic system with sparse plastic events, where the formula for the elastic propagator suggests $\rho(\delta\zeta) = \delta\zeta^{-2}$, the finite-variance constraint (i) on ρ could be fulfilled. Indeed, the elastic propagator is but a far-field approximation and the large $\delta\zeta$ values predicted in the near field have not been observed numerically or experimentally. Consequently, there will always be an upper cut-off in the predicted ζ^{-2} -distribution, which may justify the Gaussian noise approximation on long time scales. In this regard, it is interesting to note that the extremal model studied by Lemaître and Caroli (2007) seemed to capture the avalanche scaling exponent measured in atomistic simulations better with Gaussian fluctuations than with the heavy-tailed fluctuations suggested by the elastic propagator.

3. Heavy-tailed fluctuations

Nevertheless, if it so happens that the fluctuation distribution is heavy-tailed, the breadth of the mechanical noise distribution ρ should not be overlooked, as it modifies the expected distribution $P(x)$ of local distances to yield x . Yet, in dimensions $d = 2$ and $d = 3$, taking into account the heavy tail does not suffice to capture the exponent θ (defined by $P(x) \sim x^\theta$) measured in lattice-based simulations relying on the genuine elastic propagator (Lin and Wyart, 2016). This points to the failure of criterion (ii) above, *i.e.*, the importance of temporal correlations in the mechanical noise signal $\{\zeta_i(t)\}$; these correlations are notably due to avalanches and are lost in the mean-field reasoning. Only for a higher dimension $d = 4$ does the mean-field prediction for θ come closer to the value measured in the lattice-based model, which suggests an upper critical dimension $d = 4$. This claim may be contrasted with Chen *et al.* (1991)’s early speculation of an upper critical dimension of 3 for the applicability of *constant* mean field in their model. Leaving aside mean-field concerns for a minute, we find quite noteworthy that the θ exponents measured in the lattice-based

EPM are quite compatible with their (indirectly) measured value in atomistic simulations in the quasistatic regime, in 2D and 3D (Lin *et al.*, 2014a).

4. Structure of the elastic propagator and soft modes

Coming back to the comparison between EPM and their approximations, we note that the mean-field predictions are recovered (even for $d < 4$) if the elastic propagator is shuffled, that is, if the coupling between sites i and j is given by $\mathcal{G}_{\tau(i)\tau(j)}$, where τ is a random permutation of indices which changes at each time step, instead of being given by \mathcal{G}_{ij} . This shows that the temporal correlations in the mechanical noise signal arise because of the spatial structure of \mathcal{G} . Of particular importance in this structure, claim Talamali *et al.* (2012) and Tyukodi *et al.* (2016b), are the soft deformation modes of the propagator (the uniform shear bands described in Sec. III.C), that create no elastic stress in the material. To clarify this importance, the authors focused on the evolution of the cumulative plastic strain ϵ^{pl} in extremal dynamics and recast the EPM equation of motion (Eq. 5) into a depinning-like equation (also see Sec. IX.B), viz.,

$$\eta \partial_t \epsilon_p = \mathcal{P} \left(\sigma^{ext} + 2\mu \mathcal{G} * \epsilon_p - \sigma_y \right),$$

where $\eta = \mu\tau$ is a viscosity, σ_y is the local stress threshold, and $\mathcal{P}(x) = x$ if $x > 0$ and 0 otherwise. The deformation of a disordered solid in d dimensions is then regarded as the advance of a d -dimensional elastic hypersurface in a $d+1$ -dimensional space, where the additional dimension is ϵ^{pl} . Under the influence of the driving, the elastic hypersurface moves forward along ϵ^{pl} , and, in so doing, gets deformed owing to the disorder in the yield thresholds σ_y seen by different points on the hypersurface.

Despite the similar framework, Tyukodi *et al.* (2016b) showed that EPM will considerably deviate from elastic depinning problems because of the existence of soft modes in the EPM kernel \mathcal{G} , while nontrivial soft modes are prohibited by the definite positiveness of the *depinning* propagator. As time goes on, the width $W^\epsilon \equiv \left\langle \left(\overline{\epsilon^{\text{pl}}} - \overline{\epsilon^{\text{pl}}} \right)^2 \right\rangle$ of the elastic hypersurface (where the overbar denotes a spatial average and the brackets indicate an ensemble average over the disorder) saturates in the depinning problem. This saturation is due to the higher elastic stresses released by regions of higher ϵ^{pl} , which destabilize regions of lower ϵ^{pl} and therefore act as restoring forces to homogenize ϵ^{pl} over the hypersurface. On the contrary, in EPM, W^ϵ (the variance of ϵ^{pl}) grows endlessly by populating the soft modes of plastic deformation, which generate no elastic restoring force, and diverges in a diffusive fashion at long times.

D. A mechanical noise activation temperature?

1. The Soft Glassy Rheology model (SGR)

The Soft Glassy Rheology model of Sollich *et al.* (1997) proposed an alternative way to handle mechanical noise fluctuations $\{\delta\zeta(t)\}$. In the SGR spirit, these random stress “kicks” operate as an effective temperature x that can activate plastic events, in the same way as thermal fluctuations do. Accordingly, the fluctuation-induced diffusive term in Eq. 20 is replaced by an Arrhenius law to describe activated effects. More precisely, in SGR, the material is divided into mesoscopic regions that evolve in a landscape of traps whose depths are randomly drawn from an exponential distribution (Bouchaud, 1992)

$$\rho(E) \propto \exp(-E/E_g).$$

Here, E_g is a material parameter that will be set to unity. To hop from trap to trap, blocks must overcome the energy barrier E . The external driving facilitates these hops by elastically deforming each region at a rate $\dot{l} = \dot{\gamma}$, where l is the local strain, which lowers the local energy barrier: $E \rightarrow E - \frac{1}{2}kl^2$. The stiffness parameter k is such that kl is the local stress. Finally, SGR assumes that the random “kicks” due to mechanical noise activate hops at a rate $\omega(E, l)$ given by an Arrhenius law, viz.,

$$\omega(E, l) = \omega_0 \exp\left(\frac{-E + \frac{1}{2}kl^2}{x}\right), \quad (21)$$

where ω_0 is the attempt frequency and x quantifies the intensity of the mechanical noise. After a hop, l is set back to zero and a new trap depth E is randomly picked from the distribution ρ .

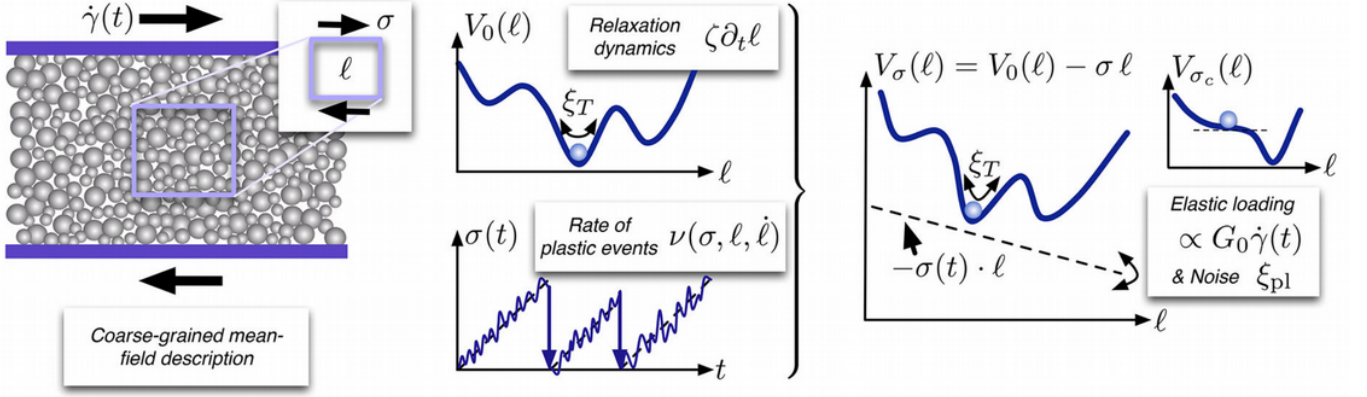


FIG. 10 Sketches illustrating the difference between thermal fluctuations ξ_T and mechanical noise ξ_{pl} . From (Agoritsas *et al.*, 2015)

The low- $\dot{\gamma}$ rheology that emerges from this simple model is quite interesting. As the effective temperature x decreases, the system transits from a Newtonian regime $\Sigma \propto \dot{\gamma}$, for $x > 2$, to a power-law regime $\Sigma \propto \dot{\gamma}^{x-1}$ for $1 < x < 2$, and a yield stress emerges for $x < 1$, where the stress follows the Herschel-Bulkley law $\Sigma - \Sigma_y \propto \dot{\gamma}^{1-x}$. Indeed, for $x < 1$, the ensemble average of the time spent in a trap, viz.,

$$\langle \tau \rangle = \int \rho(E) \omega^{-1}(E, l) dE$$

diverges at $\dot{\gamma} = 0$. The system ages and falls into deeper and deeper traps on average; it follows that there is no typical material time for the relaxation of the cumulated stress. Moreover, the wealth of timescales afforded by the use of an Arrhenius law also leads to interesting linear viscoelastic properties, in accordance with experimental data on colloidal pastes and emulsions.

2. Mechanical noise v. thermal fluctuations

However, the theoretical validity of this hypothesis has been contested in recent years (Agoritsas *et al.*, 2015; Nicolas *et al.*, 2014a; Pons *et al.*, 2015). The bone of contention is that, contrary to thermal fluctuations, mechanical noise fluctuations persistently modify the energy landscape of the region, insofar as the plastic events that trigger them are mostly irreversible. Indeed, with the notations used above (Eq. 18), the evolution of the stress in an elastic region is given by

$$\sigma(t) = \mu \dot{\gamma}^{\text{eff}} t + \underbrace{\int_0^t \delta\zeta(t') dt'}_{\equiv \xi_{pl}(t)} + \text{cst.}$$

This stress is applied at the boundary of the region by the outer medium and effectively tilts its potential energy V into $\tilde{V}(t) \equiv V - \gamma\sigma(t)$, which favors internal deformation (Gagnon *et al.*, 2001). Here, γ is the shear strain associated with the internal configuration and the activation volume is set to unity. Assuming overdamped dynamics with friction coefficient μ , one can write

$$\begin{aligned} 0 &= -\mu \frac{d\gamma}{dt}(t) - \frac{\partial \tilde{V}}{\partial \gamma}(t) + \xi_T(t) \\ &= -\mu \frac{d\gamma}{dt}(t) - \frac{dV}{d\gamma}[\gamma(t)] + \langle \dot{\sigma} \rangle t + \xi_{pl}(t) + \xi_T(t). \end{aligned}$$

Here, ξ_T denotes the thermal fluctuations. Mechanical noise and thermal fluctuations differ in that $\langle \xi_T(t) \xi_T(t') \rangle \propto \delta(t - t')$ (for white noise), whereas, if the autocorrelation function $C(\Delta t) \equiv \langle \delta\zeta(t) \delta\zeta(t + \Delta t) \rangle$ decays quickly to zero,

$$\langle \xi_{pl}(t) \xi_{pl}(t') \rangle = \int_0^t d\tau \int_0^{t'} d\tau' C(\tau - \tau') \sim \min(t, t').$$

Under the sole influence of ξ_{pl} , the energy barrier \tilde{V} flattens out after a time $T \sim (\max dV/d\gamma)^2 \equiv \sigma_y^2$. This purely diffusive case is encountered in Picard's model; the escape occurs much faster than in an activated process.

Numerical simulations of model glasses by Puosi *et al.* (2015) have confirmed that local stress fluctuations grow diffusively with time, at least at very low shear rates, viz.,

$$\langle (\xi_{\text{pl}}(t + \Delta t) - \xi_{\text{pl}}(t))^2 \rangle \propto \Delta t.$$

On the experimental side, in granular matter, the (fluidizing) effect of an actively sheared zone on the mechanical response of a distant region has been vividly demonstrated: It facilitates the penetration of an intruder (Nichol *et al.*, 2010) or the motion of a rodlike probe (Reddy *et al.*, 2011), presumably by agitating the grains in the distant region, as if they were thermally agitated. However, although an Eyring-like activation picture may account for the observations, Bouzid *et al.* (2015b) have argued that this does not suffice to validate the idea of a mechanical noise temperature, insofar as the results can be reproduced by other nonlocal models as well. Studying a related effect, Pons *et al.* (2015) have shown that applying small oscillatory stress modulations to a granular packing loaded below its yield point can dramatically fluidize it. This effect is presumed to stem from a secular enhancement of the fluidity which is due to the stress modulations and varies *algebraically* (and not exponentially) with their intensity in the proposed theory.

E. Connection with the diffusion of tracers

Rather than the local stresses, many experimental works have access to observables related to particle displacements, in particular dynamic light scattering or particle tracking techniques. It is thus interesting to be able to connect the local stress dynamics to the diffusion of tracer particles. Single events as well as plastic avalanches are expected to contribute to the tracers' motion even far away from the plastic zone due to their long-range effects, as sketched in Fig. 11(a) (Lemaître and Caroli, 2009; Nichol *et al.*, 2010).

Using the response to a punctual force in an incompressible elastic medium (Eq. 13), the displacement field induced by a single plastic event can be calculated and is displayed in Fig. 11(b). To mimic diffusion, Martens *et al.* (2011) introduced imaginary tracers that follow the displacement field generated by the ongoing plastic events and were able to rationalize the relation between the nonaffine part of the self-diffusion coefficient and dynamical heterogeneities (characterized by the four-point stress susceptibility), as shown in Fig. 11(c). Quantities comparable to the self-intermediate scattering function in purely relaxing systems are also accessible, as discussed in Sec. VIII.A.

V. STRAIN LOCALIZATION: FROM TRANSIENT HETEROGENEITIES TO PERMANENT SHEAR BANDS

The similarities in the deformation of amorphous solids, exposed in Sec. I, should not mask the widely different macroscopic consequences of applying shear to these materials. The elastoplastic viewpoint helps to understand these differences in a common framework.

TABLE II Synthetic view of the distinct types of fluctuations at play and the methods with which they can be handled.

	Fluctuation-dominated regime			Drift-dominated regime
<i>Mechanical noise fluctuations</i> $\delta\zeta$	Strong correlations and broad distribution	No time correlations but broad distr. $\rho(\delta\zeta) \propto \delta\zeta^{-1-\mu}$	Gaussian white noise $\rho(\delta\zeta) \in o(\delta\zeta^{-3})$	
<i>Dynamics of σ in elastic regime</i>	Correlated evolution	Lévy flight	(biased) Brownian motion	
<i>Applicable 'mean field' treatment</i>	None known so far	Reasoning on $P(\sigma)$ taking into account $\rho(\delta\zeta)$	Eq. on $P(\sigma)$ with diffusive term due to $\delta\zeta$	Uniform mean-field approx. may be valid
<i>$P(x)$ for $x \approx 0$</i>	$\sim x^\theta$ with unrationalized exponent	$\sim x^\theta$ with dimension dependent θ	$\sim x$	$\sim x^0$
<i>References and examples</i>	Lin and Wyart (2016) for $d < 4$, Liu <i>et al.</i> (2016) at low $\dot{\gamma}$	Lin and Wyart (2016)'s shuffled model	Hébraud <i>et al.</i> (1997)	Liu <i>et al.</i> (2016) at high $\dot{\gamma}$

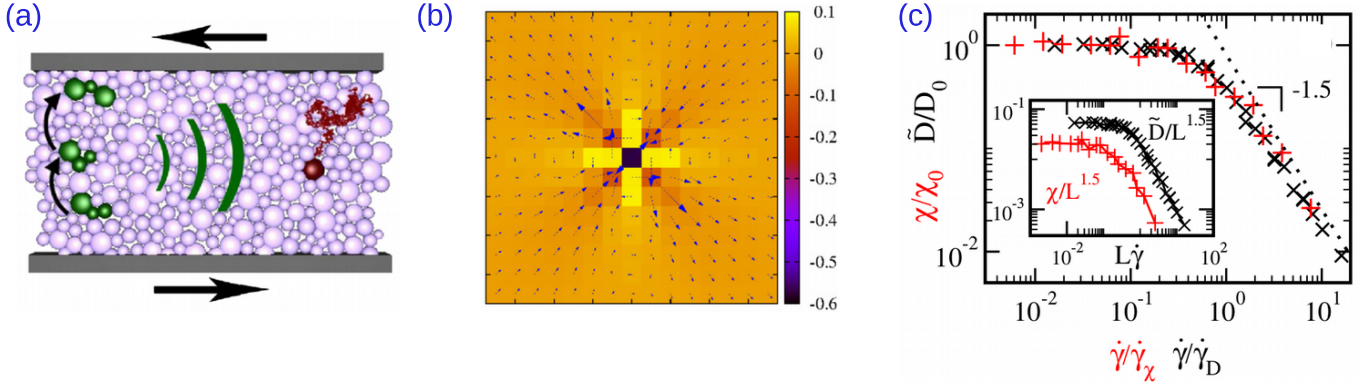


FIG. 11 (a) Schematic illustration of the long-range effects of plastic avalanches (in green) on the diffusion of a tracer (in red). From (Martens *et al.*, 2011) (b) Color map of the stress redistributed by a plastic event located at the origin and associated displacement field (arrows). (c) Comparison of the scaling of the rescaled dynamical susceptibility χ/χ_0 for different system sizes and shear rates with the scaling of the rescaled long-time diffusion coefficient \tilde{D}/D_0 ; the inset shows the individual scalings. From (Martens *et al.*, 2011).

A. Two opposite standpoints

In the common sense, there is a chasm between (i) foams and other soft solids, that flow, and (ii) metallic or silicate glasses that break/fracture after a certain amount of deformation (see Fig. 12b(right)).

To start with the latter category, in daily life, the soda-lime glasses routinely used to make windowpanes, bottles, etc., and more generally silicate glasses exemplify the concept of perfect brittleness, in which the material deforms elastically and then breaks, without going through a stage of plastic deformation. Nevertheless, at small scales plastic deformations accompanying the densification of the material were revealed in indentation experiments with a diamond tip (Yoshida *et al.*, 2007) as well as experiments of uni-axial compression of micropillars of amorphous silica (Lacroix *et al.*, 2012) (which overall behaves comparably to soda-lime glass (Perriot *et al.*, 2011)) and simulations of extended shear (Rountree *et al.*, 2009). However, in many situations, plasticity plays virtually no role, in particular when failure is initiated by a crack: No evidence of plasticity-related cavities was seen by Guin and Wiederhorn (2004) (also see references therein) and, with the help of simulations, Fett *et al.* (2008) claimed that the surface displacements experimentally observed at crack tips are compatible with theoretical predictions discarding plasticity. (It should however be mentioned that a minority of works support the existence of plasticity near the crack tip).

In metallic glasses, global failure is preceded by substantial plastic deformation. The latter is generally localized in thin shear bands, that appear as clear bands in *post-mortem* scanning electron micrographs. These bands are typically 10 to 50nm or even 100 nm-thin (Bokeloh *et al.*, 2011; Schuh *et al.*, 2007), i.e., much thinner than the adiabatic shear bands encountered in crystalline metals and alloys, which are about 10 – 100 μm -thick. Despite these plastic deformations, brittleness remains a major industrial issue for metallic glasses. Added to their cost and the difficulty of obtaining large samples, this drawback may outshine their advantageous mechanical properties, such as their high elastic limit (Wang, 2012). As a consequence, much effort has been devoted to improving their ductility.

By contrast, foams, emulsions and various other soft solids can undergo permanent shear flow without enduring irretrievable damage. This conspicuous discrepancy with hard molecular glasses can however be lessened by noticing that, even among soft solids, the flow sometimes localizes in shear bands (Bécu *et al.*, 2006; Lauridsen *et al.*, 2004). Still, the distinction between hard solids that deform and break and soft solids that deform and flow is overly caricatural. The case of gels, which consist of long entangled (and often cross-linked) chains, demonstrates that soft solids, too, may break upon deformation. But, then, *what distinguishes a material that flows from one that fails? What determines whether the deformation will be macroscopically localized in shear bands or homogeneous (on the macroscopic scale)?*

1. The shear-banding instability from the standpoint of rheology

To start with, let us consider the rheological perspective. Shear-banding in complex fluids is interpreted as the consequence of the presence of an instability in the constitutive curve, i.e., the flow curve $\Sigma_0 = f(\dot{\gamma})$ that would be

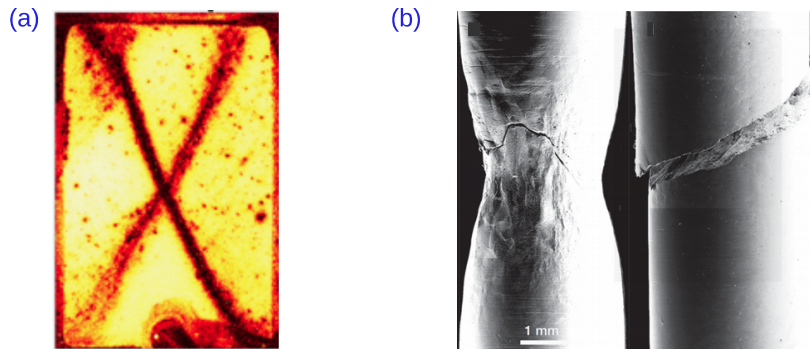


FIG. 12 Experimental observation of shear bands in (a) a granular packing of $\sim 90 \mu\text{m}$ glass beads under biaxial compression and (b, right) a bulk metallic glass sample under uniaxial tension. The composite glass shown in (b, left), reinforced with dendrites, displays a more ductile response to tension. From (Le Bouil *et al.*, 2014) and (Hofmann *et al.*, 2008), respectively.

obtained if the flow were macroscopically *homogeneous*. Indeed, it is easy to show that homogeneous flow in *decreasing* portions of the constitutive curve is unstable to perturbations and is superseded by co-existing bands. The actual flow curve displays a stress plateau $\Sigma(\dot{\gamma}) = \text{cst}$ for $\dot{\gamma}$ between two values $\dot{\gamma}_l$ and $\dot{\gamma}_h$. Shear localization corresponds to the particular case $\dot{\gamma}_l \simeq 0$, i.e., that of a non-flowing band. In other words, it will occur if the constitutive curve already starts decreasing at $\dot{\gamma} = 0$.

Note that, exploiting this negative-slope criterion, purely mean-field calculations can predict shear-banding (e.g., in (Coussot and Ovarlez, 2010)), which is somehow counterintuitive, given the manifest spatial heterogeneity associated with the phenomenon. Nevertheless, such calculations obviously leave aside the spatial organization of the flow (its banded structure), which hinges on the shape of the elastic propagator in simulations: In EPM, with similar dynamical rules, the banded flow structure obtained with the long-ranged elastic propagator of Eq. 15 is not preserved if the propagator is substituted by a stress redistribution to the first-neighbors, even if the latter is anisotropic (Martens *et al.*, 2012).

The simple criterion based on the steady-state constitutive curve needs to be somewhat adjusted for amorphous solids, which often exhibit aging effects. Then, the yield stress of the quiescent material may vary with the waiting time since preparation (Varnik *et al.*, 2003). Consequently, even if the flow curve obtained by ramping down $\dot{\gamma}$ from a high value is strictly monotonic, shear-banding may arise in non presheared samples. This will happen if an initially undeformed band gradually solidifies and thus further resists deformation, while the rest of the material is sheared. The solid band is “trapped” in its solid state because of the aging at play (Martin and Hu, 2012; Moorcroft *et al.*, 2011).

2. The mechanics of bands in a solid

Turning to the viewpoint of solid-state mechanics, as emphasized in Sec. III.C, uniform strain bands inclined by $\pm\pi/4$ with respect to the principal directions of the strain tensor are soft modes of the elastic propagator (Eq. 15), which means that they do not generate elastic stresses in the system. Should there be a weak stripe in the material (in the sense of low elastic moduli or low yield thresholds), it will then be energetically beneficial to accommodate part of the macroscopic strain in it in the form of a slip line. Such an energy-based argument is especially relevant in a quasistatic protocol in which the system always reaches the local energy minimum between strain increments. If the stripe in which the strain localizes displays ideal plasticity, the macroscopic stress-strain curve $\Sigma = f(\epsilon)$ stops increasing due to the banding instability.

But this continuum-based approach ignores the granularity of the material at the scale of plastic rearrangements by postulating the spontaneous and synchronous creation of a strain band all at once. Contrasting with this postulate, some experimental evidence in colloidal glasses (Chikkadi *et al.*, 2011) and granular matter (Amon *et al.*, 2012; Le Bouil *et al.*, 2014) indicates that shear bands actually consist of disconnected, non-simultaneous localized plastic rearrangements, as implemented in EPM. Therefore, only *on average* is a strain band uniform; its granularity (as a patchwork of localized plastic rearrangements) as well as the time fluctuations in its plastic activity have no reason to be overlooked. The sequential emergence of the band may explain its sensitivity to details in the implementation of the elastic propagator (Talamali *et al.*, 2012).

Taking the granularity of the band into account, Dasgupta *et al.* (2013, 2012) proposed to explain the existence

and the direction of shear bands by an argument based on the minimization of the elastic energy of a collection of Eshelby inclusions in a uniform elastic medium over their possible configurations in space. The neglect of the elastic heterogeneity of glasses in the reasoning was justified by the authors by the specific consideration of carefully quenched (hence, more homogeneous) glasses. An additional concern could be raised as regards the use of a global one-step minimization, whereas plastic events occur sequentially and the elastic deformation field in the material evolves during the process.

More generally, the strain bands described in the context of solids probably differ from the long-lived or permanent shear bands observed experimentally in steadily sheared materials. The former might be more accurately referred to as transient “slip lines” and some reports of “shear bands” in atomistic simulations should probably rather be interpreted as slip lines, as already noted by Maloney and Lemaître (2006). However, it has been suggested that the transient banding instability can act as a *precursor* to the formation of a shear band (Fielding, 2014).

In fact, transient banding is a matter of interest *per se*, as it can be long-lived (Divoux *et al.*, 2010). Moorcroft and Fielding (2013) proposed a way to rationalize its occurrence on the basis of a generic banding criterion involving the transient constitutive curves $\Sigma_0 = f(\dot{\gamma}, \gamma)$, where γ is the cumulative strain since shear startup, in a fictitious system constrained to deform homogeneously. The rheological criterion $\frac{d\Sigma_0}{d\dot{\gamma}} < 0$ is recovered at infinite times $\gamma \rightarrow \infty$, while a purely elastic banding instability is predicted if $A \frac{\partial \Sigma_0}{\partial \gamma} + \dot{\gamma} \frac{\partial^2 \Sigma_0}{\partial \gamma^2} < 0$, with a model-dependent prefactor $A > 0$, provided that the material is sheared much faster than it can relax ($\dot{\gamma} \rightarrow \infty$). In the light of this, the authors claim that there is a generic tendency to transient banding in materials that exhibit a stress overshoot in shear startup. This connection has been noticed in EPM in the quasistatic regime (Lin *et al.*, 2015).

However, transient slip lines need not always convert into steady-state banding in a homogeneously sheared disordered solid. What is required for this purpose is a mechanism that explains how the transient “slip lines”, instead of being dispersed, concentrate in the same region of the shear-banded material as time goes on. The distinction between the situation at finite strains and in the steady state should perhaps be emphasized. The first-order yield transition in the statistics of low-energy barriers observed by Karmakar *et al.* (2010) at a *finite strain* γ_c is not necessarily associated with a first-order (banding) transition in the *steady-state* flow curve $\Sigma(\dot{\gamma})$. Similarly, Jaiswal *et al.* (2016) numerically observed that, in a batch of finite-size samples subjected to a strain γ , about half of the samples will have irreversibly yielded when $\gamma = \gamma_c$, while the other half come back to their initial configuration upon unloading; but it is not straightforward to conclude from this interesting observation that, if one stitched a “yielding” sample together with a “recovering” one, a shear band would localize in the “yielding” part at longer times.

B. Spatial correlations in driven amorphous solids

The intermediate scale of EPM helps bridge the time and length scale gap between transient slip lines and permanent shear-banding. A complex flow organization at short to intermediate time scales, featuring strong intermittency and marked spatial correlations between plastic events, is encountered even in driven amorphous solids that do not exhibit macroscopic shear localization, provided that the driving is sufficiently slow.

1. Spatial correlations

The computational efficiency of EPM allows one to quantify the spatial extent of correlations in the flow through the measurement of cooperativity or correlation lengths ξ in bulk flows. The Kinetic Elastoplastic (KEP) Theory of Bocquet *et al.* (2009), an extension of the Hébraud-Lequeux model (see Sec. IV.B.2) that includes heterogeneities, predicts a decrease of ξ with the shear rate as

$$\xi \sim (\Sigma - \Sigma_y)^{-1/2} \sim \dot{\gamma}^{-1/4},$$

in contrast with the theoretical prediction $\xi \sim \dot{\gamma}^{-1/2}$ in 2D of Lemaître and Caroli (2009), beyond which independent avalanches are supposedly triggered.

Simulations of homogeneous shear flow on spatially resolved EPM have shown, in general, results departing from the $\xi \sim \dot{\gamma}^{-1/4}$ scaling. By studying how the strain-rate dependence of the average stress drop $\langle \delta\sigma \rangle$ is affected by changes in the size of the system and collapsing the data onto a master curve, Picard *et al.* (2005) measured a correlation length that scales with $\dot{\gamma}^{-1/2}$ in 2D (see Fig.13a). Nicolas *et al.* (2014b) related this scaling to the average spacing between simultaneous plastic events, which scales as $\dot{\gamma}^{-1/d}$ in d dimensions, and several definitions of correlation lengths were shown to follow this dependence in EPM. The variable sign of the elastic propagator enters the reasoning, insofar

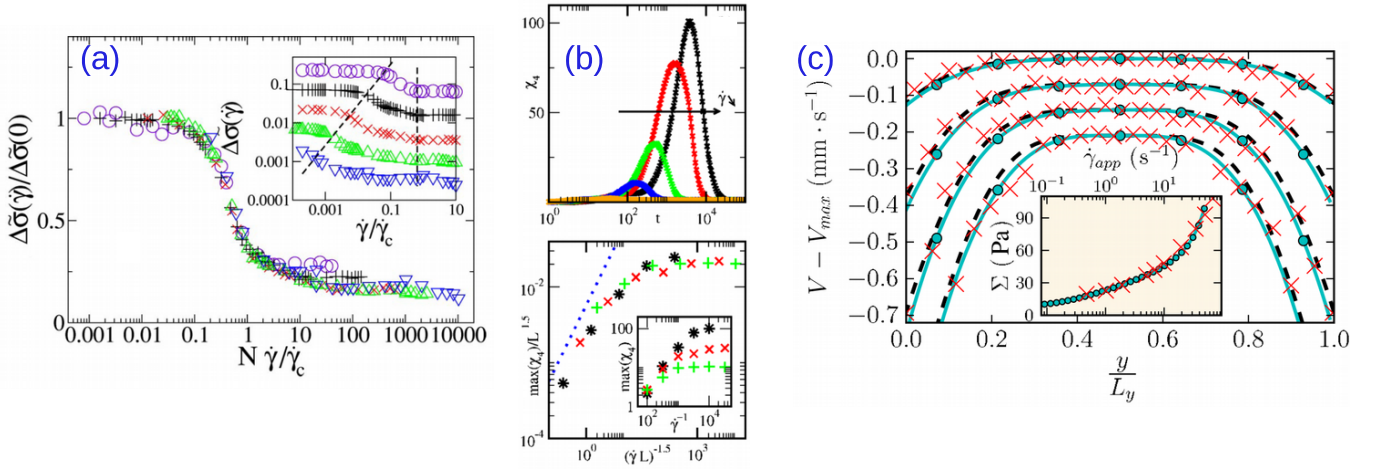


FIG. 13 (a) Mean stress drop (normalized by the average stress) $\Delta\tilde{\sigma}$, as a function of the renormalized dimensionless shear rate, $N\dot{\gamma}/\dot{\gamma}_c$. From (Picard *et al.*, 2005). (b) *Top*: Time evolution of the dynamical stress susceptibility $\chi_4(t, \dot{\gamma})$ at different strain rates. *Bottom*: Finite-size scaling plot for the maxima of the dynamical susceptibilities. Adapted from (Martens *et al.*, 2011). (c) Velocity profiles across the channel for different applied pressures. (Red crosses) Experimental data for an oil-in-water emulsion; (dashed cyan lines) EPM; (solid black lines) bulk rheology predictions. The profiles have been shifted vertically for legibility. Adapted from (Nicolas and Barrat, 2013b).

as plastic events are able to screen each other, because the sign of their contributions may differ. Nevertheless, the $\dot{\gamma}^{-1/d}$ scaling is not generic. In particular, the correlation length derived from the four-point stress correlator $\mathcal{G}_4(\mathbf{r})$, exploited by Martens *et al.* (2011) (see Fig.13b), is more sensitive to the avalanche shape and was shown to depend on the chosen EPM dynamical rules. Below the yield stress, Lin *et al.* (2015) claim that the system is critical, with system-spanning avalanches in the transient, which is supported by a study of the cutoffs in the avalanche size distributions in EPM simulations. This implies a diverging correlation length $\xi = \infty$ in the whole $\Sigma < \Sigma_y$ phase - not unlike what is seen in 2D dislocation systems, at all applied stresses (Ispánovity *et al.*, 2014).

Besides, it should be noted that EPM tend to overestimate the intensity and the extent of the correlations between plastic events, e.g., compared to particle-based simulations (Nicolas *et al.*, 2014c). Furthermore, the quasistatic divergence of ξ observed in athermal EPM, with $\xi \rightarrow \infty$ as $\dot{\gamma} \rightarrow 0$, will be cut off strongly in systems at a finite temperature, where thermal noise stifles the correlations (Hentschel *et al.*, 2010).

2. Cooperative effects under inhomogeneous driving

The correlations measured in macroscopically homogeneous flows generate conspicuous cooperative effects if the loading or the macroscopic flow is inhomogeneous over the correlation length scale.

This is in particular the case in pressure-driven flows through a narrow channel, of width w ($w \approx 10^2 \mu\text{m}$ for microchannels). In this geometry, the streamline-averaged shear stress Σ varies linearly across the channel, which notably implies that it is zero at the center (both in 2D and 3D). Therefore, for yield-stress fluids, a plug flow with an advected, but unsheared central part (where $|\Sigma| < \Sigma_y$) is expected. Also, large stress gradients, of order $\frac{\Sigma_y}{w}$, are generated across the channel. Accordingly, differently-stressed streamlines interact in the flow via the elastic propagator.

Seminal experiments on concentrated oil-in-water emulsions by Goyon *et al.* (2008) have revealed that the actually observed 2D profiles are more rounded than the expected plug flow, and overall the flow is enhanced compared to the predictions from the bulk rheology. Thus, there is no unique relation between the local strain rate and the local stress (Goyon *et al.*, 2010): the rheology is nonlocal. Using a similar system, Jop *et al.* (2012) demonstrated the existence of finite strain rate fluctuations $\delta\dot{\gamma}(\mathbf{r}) \equiv \sqrt{\langle \dot{\gamma}(\mathbf{r})^2 \rangle - \langle \dot{\gamma}(\mathbf{r}) \rangle^2}$ in the plug, which are minimal at the channel center. Numerical simulations of athermal soft disks confirmed the impact of confinement on the rheology: In 2D periodic Poiseuille flow (corresponding to a juxtaposition of Poiseuille flows of alternate directions), the wall stress below which flow stops substantially increases as the channel becomes narrower (Chaudhuri *et al.*, 2012).

This remarkable effect of spatial correlations on an inhomogeneous flow is often rationalized by means of a nonlocal term in the equation controlling the fluidity f , defined here as the inverse viscosity $\dot{\gamma}/\sigma$, a variable which is thought to

be proportional to the rate of plastic events. Owing to the symmetry of the propagator, the leading-order correction to the local fluidity involves the Laplacian $\nabla^2 f$, which yields a steady-state fluidity diffusion equation,

$$\xi^2 \nabla^2 f + [f_b(\Sigma) - f] = 0, \quad (22)$$

where ξ is a cooperative length and f_b is the fluidity in a bulk flow subject to stress Σ . The KEP model of Bocquet *et al.* (2009) provides a formal justification of Eq. 22 to linear order in f , with $\xi \sim (\Sigma - \Sigma_y)^{-1/2}$, by accounting for the mechanical noise generated in the immediate vicinity of plastic events. In fact, using a *constant* value of ξ (for each material) in Eq. 22 already provides very good fits of the experimental curves, not only for concentrated emulsions (Goyon *et al.*, 2008) and lattice-Boltzmann simulations thereof (Benzi *et al.*, 2014), but also for polymer microgels (Carbopol) (Geraud *et al.*, 2013). In the case of emulsions, ξ takes a value of zero below the jamming point, and reaches up to 3 to 5 droplet diameters (20 – 30 μm), in the very dense limit (Goyon *et al.*, 2008). Similarly, for Carbopol samples, ξ is found to measure 2 to 5 structural sizes, as determined from the size of optical heterogeneities (Géraud *et al.*, 2017).

However, the fitting involves the adjustment of a boundary condition, namely, the fluidity f_w at the wall. The high sensitivity of the fits to f_w (Geraud *et al.*, 2013) may limit the accuracy of the experimental measurement of ξ . This difficulty highlights the value of EPM for testing the validity of theoretical predictions. In EPM descriptions of channel flow, the driving term $\mu\dot{\gamma}$ is set to zero in Eq. 5; flow arises on account of the initially imposed transverse stress profile $\Sigma(y)$. The presence of channel walls is accounted for by a no-slip boundary condition, which adds a correction to Eq. 15 for the elastic propagator. This correction can be calculated via a method of images and leads to a faster local relaxation for plastic events near walls (Nicolas and Barrat, 2013a). Combined with appropriate dynamical rules, the model semi-quantitatively reproduces the shear rate fluctuations in the plug observed by Jop *et al.* (2012) as well as the moderate deviations of the velocity profiles from the bulk predictions witnessed with smooth walls, provided that the EPM block size corresponds to around 2 droplet diameters (see Fig. 13c). The fluidity diffusion equation, Eq. 22 either with $\xi = \text{cst}$ or $\xi \sim \dot{\gamma}^{-1/4}$, captures the EPM fluidity profiles reasonably well, albeit imperfectly. Taking a closer look at the decay of the shear rate $\dot{\gamma}(y)$ in a region ($y > 0$) subject to $\Sigma < \Sigma_y$ contiguous to a sheared band ($y < 0$), Gueudré *et al.* (2017) find that EPM results obey a scaling relation involving a length scale $\xi \propto (\Sigma) \sim (\Sigma - \Sigma_y)^{-\nu}$ but that the scaling exponents clearly differ from mean-field predictions and are also inconsistent with KEP-based Eq. 22. In particular, for $\Sigma \approx \Sigma_y$, $\dot{\gamma}(y)$ is argued to decay algebraically with $y > 0$ instead of exponentially. Recalling that the finite size L of a system induces a shift by $\Delta\Sigma \propto L^{-1/\nu}$ of the critical stress for flow initiation in a homogeneous setup (so that $l(\Sigma) = L$ at initiation), the same authors claim that pressure-driven flows display larger finite-size effects than simple shear flows, e.g. with respect to the critical stress $\Sigma_{stop}(L)$ below which flow stops, because in the former setup L should be substituted by the width $(\Sigma_w - \Sigma_{stop})L$ of the sheared band near the wall (where the stress is Σ_w), hence a smaller effective size.

The description of nonlocal effects by Eq. 22 has also been applied to granular matter, which generically display heterogeneous flow and shear bands (Kamrin and Koval, 2012). To do so, the fluidity was redefined as $\dot{\gamma}/\mu$, owing to the fact that the rheology of dry frictional grains is best expressed as a relation $\mathcal{I}(\mu)$ between the inertial number \mathcal{I} (which is a rescaled shear rate) and the friction $\mu \equiv \Sigma/P$ (with P the pressure). The resulting model has proven its ability to capture cooperative effects and account for the global velocity profile observed in discrete element simulations of a simple shear flow with gravity, a gravity-driven flow in a channel (Kamrin and Koval, 2012) as well as the flow of a granular layer on an inclined plate, which is sensitive to the thickness of the layer (Kamrin and Koval, 2012). Nevertheless, the validity of the definition of a “granular fluidity”, which is not an intrinsic state variable (because of the denominator Σ or μ), has been questioned, on the basis that employing another variable would also lead to an exponential decay of the flow away from an actively sheared zone (Bouزيد *et al.*, 2015a). Other definitions for the fluidity variable f that should enter a diffusive equation have thus been put forward: Aranson and Tsimring (2006) considered the ratio between the “static” and the “fluid” part of the stress tensor; Bouزيد *et al.* (2015b) claim that the choice $f = \mathcal{I}$ (inertial number) best matches the results of their discrete-element simulations, in particular the continuity of f at the interface between differently-loaded regions.

3. Cooperative effects due to boundaries

Coming back to emulsions, Goyon *et al.* (2008)’s observations indicate that the flow deviates much more from the bulk predictions, with an enhanced fluidization, when smooth walls are replaced by rough walls. Further experimental studies on regularly patterned surfaces show that the wall fluidization enhancement varies nonmonotonically with the height of the (steplike) asperities, for asperities smaller than the droplet diameter, as does the wall slip velocity (Mansard *et al.*, 2014).

These strong deviations in the presence of rough walls exceed by far what is found in EPM. This points to another physical origin than the coupling to regions subject to higher shear stresses. Since wall slip was experimentally observed, it has been suggested that the “collisions” of droplets against surface asperities, as they slide along the wall, are the missing source of plastic activity; adding sources of mechanical noise along the walls in EPM can indeed capture the experimental features (Nicolas and Barrat, 2013b). Derzsi *et al.* (2017) experimentally confirmed the presence of roughness-induced scrambles at the wall and, with the help of lattice-Boltzmann simulations, the ensuing increase in the rate of plastic rearrangements near rough walls.

C. The ingredients for permanent shear localization or fracture

Several EPM have been able to reproduce permanent strain localization (Bulatov and Argon, 1994b; Coussot and Ovarlez, 2010; Jagla, 2007; Li *et al.*, 2013; Martens *et al.*, 2012; Nicolas *et al.*, 2014b; Vandembroucq and Roux, 2011). In these cases, after a transient, the plastic activity will typically concentrate in a narrow region of space, generally a band, that may slowly diffuse over time. The discriminative observation of localization for certain (but not all) EPM and for a certain range of parameters only, gives hints as to the ingredients responsible for this phenomenon. These ingredients are to be found in the rules for yielding or for elastic recovery. Of course, they are of rather generic nature; relating them to microscopic physical properties is not obvious in general. In fact, very few detailed comparisons between microscopic calculations and the EPM rules are available, so that one remains at the level of a qualitative interpretation.

To start with, one notices that large applied stresses $\Sigma \gg \Sigma_y$ are incompatible with localization. Indeed, the applied stress then exceeds the local yield stresses: Plastic events pervade the system, which globally flows in a viscous manner. In other words, large loadings fluidize the material, consistently with experimental observations (Divoux *et al.*, 2012).

On the other hand, at lower stresses (hence, lower shear rates), plastic events are sparser and may hit the same regions over and over again, provided that the latter are strongly or durably weakened by these events. Meanwhile, in the rest of the material, the driving term in Eq. 5 is compensated by the nonlocal contributions due to a band of plastic events, i.e., a uniform relaxation (Martens *et al.*, 2012). The general cause for localization thus evidenced is the insufficient healing of regions following rearrangements (Nicolas *et al.*, 2014b). In the following, we look into the distinct possible origins of this weakening.

1. Long rearrangements

Coussot and Ovarlez (2010) rationalized shear-banding in jammed systems by considering the formation and breakage of particle clusters. Locally, these events delimit periods of solid and liquid behavior, in which the elastic stress is of order $\mu\dot{\gamma}t$ (with μ the shear modulus and $\dot{\gamma}t$ the local strain) and 0, respectively, while there is a constant viscous stress of order $\eta\dot{\gamma}$. On the basis of a mean-field argument, they showed that if the liquid-like phase lasts longer than η/μ , then the flow curve becomes nonmonotonic, which is the hallmark of shear-banding. The idea was elaborated by Martens *et al.* (2012), who used a spatially resolved EPM of the Picard type with a variable rearrangement (“healing”) time τ_{res} as a parameter, with the notation of Eq. 19. Their findings confirmed the formation of shear bands in space for large τ_{res} , associated with the emergence of nonmonotonicity in the macroscopic flow curve. The banded flow shares many properties with systems at a first-order transition in which different phases coexist; the shear rate is well defined (independent of the driving) inside the band and there is an interface with the nonflowing phase. This spatial organization in the form of a band is intrinsically related to the long range and anisotropy of the propagator, for which bands are soft modes (see Sec. III.C).

Attractive interactions in adhesive colloidal systems (Irani *et al.*, 2014) and directional bonds in molecular systems are tentative candidates for possible microscopic origins of long rearrangements, i.e., long time delays before the destabilized region reaches another stable configuration.

Similarly, the introduction of a pinning delay upon yielding by Papanikolaou (2016), during which potential forces are inactivated (presumably due to inertial effects), enhance strain localization.

2. Structural softening combined with slow recovery (aging)

Both experimental (Rogers *et al.*, 2008) and numerical (Shi and Falk, 2005) data indicate that letting a system “age” in the absence of strain favors strain localization, or even fracture. The EPM proposed by Vandembroucq and Roux (2011) and inspired by the weakening mechanism in Fisher *et al.* (1997)’s model for earthquakes helped interpret

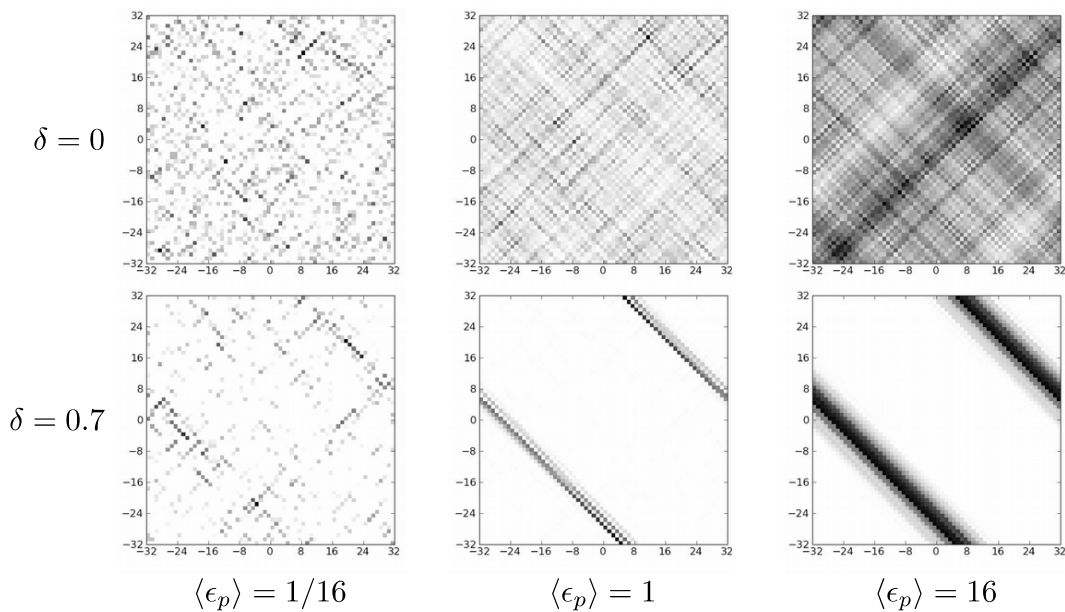


FIG. 14 Maps of cumulated plastic strain ϵ_p obtained in an EPM for different biases δ in the post-yielding distribution of local yield stresses (see main text) and at different rescaled “times” $\langle\epsilon_p\rangle$, as indicated on the figure. Darker colors represent larger values of ϵ_p . The principal strain directions are the horizontal and vertical directions. Adapted from (Vandembroucq and Roux, 2011).

this effect: The distribution $P(\sigma_y)$ used for resetting the local yield stresses σ_y following a plastic event was shifted by an amount δ with respect to the initial $P(\sigma_y)$, to mimic the lower structural temperature of the material prior to shear. For large enough negative δ (strain weakening), the first regions to yield are rejuvenated to a state with lower threshold, so that the system gets trapped in a banded structure. The bands thus created are localized and pinned in space if the elementary slip distance is small; otherwise, larger slip events are created, enhancing nonlocal effects and making bands less stable and more diffusive. Nicolas *et al.* (2014b) introduced a healing process in this picture, by allowing the blocks that have just become elastic again to age and gradually recover higher energy barriers, viz.,

$$\dot{E}_y(t) = k \frac{E_y^\infty - E_y(t)}{E_y^\infty - E_y^{\min}},$$

where k is the rate of recovery at which the energy barrier rises from its post-yielding value E_y^{\min} to the asymptotic value E_y^∞ . For low enough recovery rates k , shear localization was observed. However, the localized behavior tends to fade away when $\dot{\gamma}$ reaches very small values. This may be paralleled with the recovery of a homogeneous flow in the dense colloidal suspension studied by Chikkadi *et al.* (2011) for shear rates below a certain value, which allow the strained system to structurally relax before further deformation.

Along similar lines, Li *et al.* (2013) implemented a process of free volume creation and annihilation in a finite-element-based EPM designed to describe the deformation of the metallic glass Vitreloy 1. In their model, free volume is created by the dilation accompanying a shear transformation and is annihilated gradually in strictly local diffusional events. The activation of shear transformations, in turn, is facilitated by a local excess of free volume. Simulations relying on a kinetic Monte Carlo scheme for the dynamics showed that the deformation localizes in bands at low temperatures and that the variations of free volume are critical for this localization. A parallel can obviously be drawn between the creation of free volume during shear transformations and the lowering of yield stresses in other EPM. There is perhaps an even stronger connection with the plasticity-induced enhancement of the local effective temperature in variants of the Soft Glassy Rheology model (Fielding *et al.*, 2009) and the Shear Transformation Zone theory (Manning *et al.*, 2007), which can also lead to the observation of shear bands.

Another approach to account for the competition between local relaxation and driving-induced plastic events was proposed by Jagla (2007). In his continuous model (see Sec. III.E), the system relaxes via a slow drift of the local energy landscape seen by a given site towards lower energies. Sites whose evolution towards potential minima is not interrupted by plastic deformations benefit from this local ‘structural relaxation’. Their elastic energy decreases

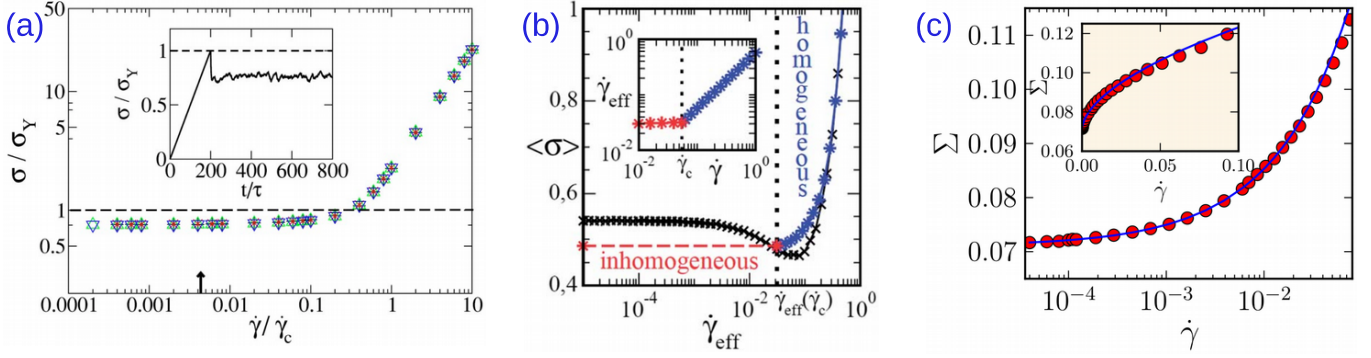


FIG. 15 *Steady-state flow curves obtained in variants of Picard’s EPM.* (a) Rescaled flow curves for different system sizes for the original Picard EPM, in the common logarithmic-linear representation. *Inset:* typical stress-strain curve in response to a small shear rate, starting from a stress-free configuration. From (Picard *et al.*, 2005). (b) Non-monotonic flow curve obtained in Picard’s model with a long local restructuring time. *Inset:* average local shear rate in the flowing regions. For $\dot{\gamma} < \dot{\gamma}_c$ a mechanical instability leads to shear-banding, with a coexistence of a flowing band and a static region. From (Martens *et al.*, 2012). (c) Flow curve obtained in a variant of Picard’s model. The straight line is a fit to a Herschel-Bulkley equation, with an exponent of 0.56. *Inset:* same data, in linear representation. From (Nicolas *et al.*, 2014a).

and the local yield stress increases; while their plastically active counterparts have no time to undergo structural relaxation, and their yield stress remains consequently low. Again, this leads to a nonmonotonic flow curve in a mean-field analysis, and to strain localization at low $\dot{\gamma}$.

To what extent precisely these strain localization mechanisms are connected with the weakening-induced runaway (system-spanning) events observed in Fisher *et al.* (1997)’s model for earthquakes or Papanikolaou *et al.* (2012) and Jagla *et al.* (2014)’s topple-down oscillations due to viscoelastic relaxation between earthquakes remains uncertain.

3. Temperature rise in shear bands

A temperature rise ΔT has been experimentally evidenced during the operation of shear bands in metallic glasses (Lewandowski and Greer, 2006; Zhang *et al.*, 2007). The dominant view is that it is however not the initial cause of the shear banding observed at low strain rates, as ΔT is small in this case. Still, local heating may result in the recrystallisation of the material, with associated changes in its mechanical properties (presumably more brittle behavior). Such effects are obviously not included in EPM, and probably better described at the level of macroscopic equations as a thermomechanical instability. The discussion above is therefore only relevant for the initiation of the instability and for systems in which thermal effects are weak.

A related mechanism leading to a nonmonotonic flow curve, first identified in MD simulations (Nicolas *et al.*, 2016) and then also seen in finite-element-based EPM (Karimi and Barrat, 2016), is at play when inertia is progressively introduced in an initially overdamped system. At a given strain rate, inertial effects create long-lived oscillations that trigger the yielding of neighboring elements more efficiently than if mechanical equilibrium were instantly restored, with an instantaneous stress redistribution. In MD simulations this facilitation was shown to be equivalent to a heating of a more strongly damped system. In a system in which inertia is important, the energy dissipated in the flow remains long enough in the relevant degrees of freedom to effectively modify the local stress evolution. Note that stable shear banding has not been observed in this case, probably due to insufficient system sizes.

VI. STEADY-STATE BULK RHEOLOGY

In this Chapter we redirect the focus to materials that actually flow rather than fail. This is the relevant framework for foams, emulsions, colloidal suspensions, and various other soft glassy systems. There have been several approaches in the EPM literature to capture the steady-state rheology of these materials, which is macroscopically characterized by the flow curve $\Sigma(\dot{\gamma})$. The local yielding and healing dynamics adopted in the EPM description play a crucial role in determining these flow properties, as will be discussed in the following.

A. Activation-based (glassy) rheology v. dissipation-based (jammed) rheology

Differences have been brought to light between the activation-based rheology found in aging systems such as glasses and the dissipation-based rheology of jammed systems such as foams. The particle-based simulations of Ikeda *et al.* (2012, 2013) contributed to disentangling glassy (thermal) and jammed (athermal) rheologies. These researchers identified the time scales of the different processes at play, namely Brownian motion and dissipation, in addition to the driving, and were able to separate the thermal rheology associated with the former from the dissipation-based one. As the shear rate $\dot{\gamma}$ is increased, the driving starts to interrupt the thermally triggered plastic relaxation and dissipation starts to dominate the rheology. The idea of a transition from a thermal to an athermal regime was bolstered by experiments on microgel colloidal suspensions, which are impacted by thermal fluctuations close to the transition to rigidity (glass transition), but obey jamming-like scalings further from that transition (Basu *et al.*, 2014).

The competition between the driving and the realization of plastic events had already been emphasized in the first works about EPM and has since been implemented in different ways in these models.

For flows dominated by thermal relaxation, it makes sense to consider EPM of the type of Bulatov and Argon (1994a)'s and Homer and Schuh (2009)'s, as well as Sollich *et al.* (1997)'s Soft Glassy Rheology (SGR) model (presented in Sec. IV.D.1), in which plastic events are activated at a rate given by an Arrhenius law (Eq. 6). For instance, in SGR, where this rate is controlled by a fixed effective temperature x , as $\dot{\gamma}$ increases, blocks can accumulate more elastic strain before a plastic event is activated. The macroscopic stress thus increases with $\dot{\gamma}$. In fact, it does so in a non-universal way: The flow curve follows a Herschel-Bulkley law (Eq. 1) with an exponent controlled by x .

Most other EPM works on steady-state rheology, however, focus on systems close to the athermal regime, in particular foams and dense emulsions of large droplets (where thermal fluctuations are negligible in the range of $\dot{\gamma}$ of general interest). The first EPM specifically designed for the study of athermal flow at finite driving rates is that of Picard *et al.* (2005), which we already mentioned in Sec. IV.A. Since this model is athermal, elastoplastic blocks can only yield when their stress exceeds a local threshold σ_y ; the yielding process is then stochastic, with a fixed rate τ^{-1} , as is the process of elastic recovery.

Figure 15(a) shows the (monotonic) flow curve obtained for this model with all relevant material time scales set to unity. Superficially, the curve coincides with many experimental flow curves, but more quantitatively it does not follow a Herschel-Bulkley law: The crossover to a Newtonian regime already sets in at macroscopic stresses Σ only slightly above the yield stress Σ_y . This is due to the postulated elastic stress accumulation above the threshold σ_y for a fixed duration τ on average. These seemingly oversimplified yielding and healing rules have been refined since then. To make the picture more realistic, Nicolas *et al.* (2014a) opted for an instantaneous triggering of plastic events at σ_y and introduced a yield stress distribution. In their model, the event lasts for a fixed local strain “duration” γ_c . Therefore, the local dissipation process can be disrupted by the external driving, which contributes to the local strain. The ensuing flow curves are more compatible with experimental ones and are well described by a Herschel-Bulkley law, as shown in Fig. 15(c).

Many experimental soft systems exhibit a qualitative change of their flow behavior when the adhesion properties of their constituents are modified, with e. g. a higher propensity to shear-banding when the surfactant concentration in an emulsion is altered (Bécu *et al.*, 2006). This discovery prompted the idea that there exist different classes of jammed systems depending on microscopic interactions. Coussot and Ovarlez (2010) suggested that adhesion results in longer local restructuring events. Martens *et al.* (2012)'s EPM-based studies confirmed that long plastic events lead to a nonmonotonic constitutive curve and the formation of permanent shear bands in the unstable parts of the flow curve, as discussed in detail in Sec. V.C and shown in Fig. 15(b).

B. Is the flow curve a mean-field, or “local”, property?

In the case of shear-banded flows mean-field predictions of the flow curve that rely on effective local dynamics and overlook the banded structure will fail. But in homogeneous situations is it possible to predict the flow curve by mean-field reasoning? This seems plausible because both particle-based (Roy *et al.*, 2015) and EPM computations of the flow-curve prove much less sensitive to finite-size effects than are correlation lengths (see Fig. 15(a)).

In the quasistatic limit, the deformation of athermal systems is highly intermittent and strongly correlated in space. At finite $\dot{\gamma}$, the external forcing starts perturbing the individual localized rearrangements and the cascades thereof. Defining τ_{pl} as the time scale for an individual rearrangement (of size a), we can estimate the upper bound for the duration of a cascade as the propagation time (if applicable) across the system of linear size L , viz., $\tau_{prop} \approx \tau_{pl} (L/a)^\beta$, where the exponent β depends on the damping regime. The driving time scale $\gamma_y/\dot{\gamma}$ can thus compete with either individual rearrangements (τ_{pl}) or avalanches (τ_{prop}).

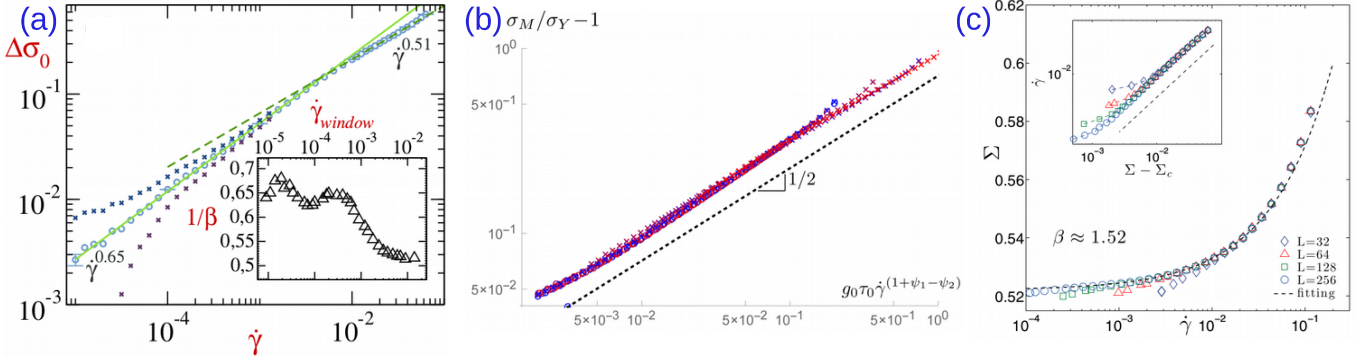


FIG. 16 *Shear-rate dependence of flow curves obtained with EPM.* (a) Dependence on $\dot{\gamma}$ of the difference $\Delta\sigma_0 \equiv \Sigma - \Sigma_y$ between the steady-state stress Σ and Σ_y in Nicolas *et al.* (2014a)'s model. From (Liu *et al.*, 2016). The data clearly show two different scaling regimes: Close to criticality the Herschel-Bulkley exponent is $n = 0.65$, whereas at high $\dot{\gamma}$ the effective exponent tends towards $1/2$. (b) Flow curves for the same EPM at relatively high $\dot{\gamma}$ with a shear-rate-dependent local shear modulus $G_0(\dot{\gamma}) \sim \dot{\gamma}^{\psi_1}$ or a shear-rate-dependent strain “duration” $\gamma_c(\dot{\gamma}) \sim \dot{\gamma}^{-\psi_2}$ for plastic events. From (Agoritsas and Martens, 2017). (c) Flow-curves obtained from stress-imposed EPM simulations in 2D that are well-fitted by an Herschel-Bulkley law with an exponent of $n \approx 0.66$. From (Lin *et al.*, 2014b).

Therefore, at least two scaling regimes could be seen as $\dot{\gamma}$ is varied (Bonn *et al.*, 2017). Two regimes were indeed reported in Liu *et al.* (2016)'s EPM simulations, as shown in Fig. 16(a). At very small shear rates, the rheology should be dominated by the interruption of avalanches (Maloney and Lemaître, 2006). In this regime, a proper scaling description can be helpful in order to relate the various critical exponents close to the yielding transition; Lin *et al.* (2014b)'s proposals along these lines are detailed in Sec. VII. It has been pointed out that the flow exponent can explicitly depend on the plastic disorder potential (Jagla, 2017) and thus on the details in the plastic behavior. On the other side EPM based on the quadrupolar interaction kernel tend to predict similar Herschel-Bulkley exponents in the limit $\dot{\gamma} \rightarrow 0$, independently of their dynamical rules ($n_{2D} \approx 0.65$ and n_{3D} between 0.65 and 0.72). One may hypothesize that this correlation dominated regime is determined mostly by the form of the elastic propagator. In this regime mean-field reasoning will probably fail due to the strongly intermittent and correlated dynamics. Lin and Wyart (2016)'s mean-field approach, which assumes a random distribution of mechanical noise with fat tails, does indeed predict an upper critical dimension $d_c = 4$, not relevant to any experimental situation. Moreover, the Herschel-Bulkley exponent is predicted to be $n = 1$ (Lin and Wyart, 2017) and is thus larger than the typical numerical and experimental ones in 2D or 3D.

Stronger driving decorrelates plastic events. Mechanical noise then results from the superposition of a large number of events and its characteristics thus change as $\dot{\gamma}$ is increased². Accordingly, a crossover in the flow behavior is expected. A suitable mean-field description should capture this transition in the flow curve by implementing the correct noise term, which should be defined self-consistently and have a shear-rate dependence (also see Table II). Existing approaches are either dedicated to the quasistatic regime and rely on a broad random distribution for the mechanical noise (Lin and Wyart, 2016), or actually target the regime of intermediate $\dot{\gamma}$, where mechanical noise is reduced to Gaussian white noise in the local stress dynamics due to the finite density of plastic events (Hébraud and Lequeux, 1998). As discussed in Sec. IV.B.2, at vanishing shear rates the latter type of modeling leads to a Herschel-Bulkley exponent $n = 1/2$, close to many experimental values. This exponent is robust to several variations of the model (Agoritsas *et al.*, 2015), but varies if a shear-rate dependence is introduced in the elastic modulus or the local restructuring time, as can be verified on the simulated EPM flow curves plotted in Fig. 16(b) (Agoritsas and Martens, 2017). Nevertheless, the diffusive effect of the mechanical noise has not been sufficiently justified so far and the approach remains phenomenological.

² E. E. Ferrero *et al.*, in preparation (2017).

C. Strain-driven vs. stress driven protocols

Most EPM works consider strain-controlled protocols (defined in Sec. II.C). Some of the counterexamples are given by Lin *et al.* (2014b) (see Fig. 16(c)) and the recent work by Jagla (2017). Another example of stress-imposed modeling is the numerical work in Liu (2016)'s PhD thesis. In a section dedicated to the transient dynamics prior to fluidization, a stress-controlled EPM is introduced. To this end, the internal stress resulting from plastic events is separated from the externally applied stress field, which can be chosen arbitrarily.

In this type of protocols, depending on the initial condition, two types of stationary solutions are obtained, namely, steady flow and a dynamically frozen state. Under athermal conditions the system may always reach a configuration with large local yield stresses, in which the dynamics gets stuck, even if the applied stress Σ is larger than the dynamical yield stress Σ_y . The smaller Σ and the smaller the system size, the more likely becomes the visiting of such an absorbing state. But if a flowing stationary state is reached for a given time and granted that the mechanical properties do not show history dependence (Narayanan *et al.*, 2017), strain-controlled and stress-controlled protocols yield identical flow curves Liu (2016).

VII. CRITICAL BEHAVIOR AND AVALANCHES AT THE YIELDING TRANSITION

Even though amorphous solids retain complex solid-like properties under continuous flow, the onset of flow is of particular interest from a physical viewpoint owing to the critical behavior that may come along with it. Far from being a weakness, the simplified description provided by EPM (which were originally phenomenological models) represents an asset for the study of this yielding transition. In this section we review the thriving literature about the statistics of avalanches close to the yielding transition.

A. Short introduction to out-of-equilibrium transitions

Statistical physics is largely concerned with phase transitions, whereby some properties of a system abruptly change upon the small variation of a control parameter. The paradigmatic example of an equilibrium phase transition is the Ising model, which consists of spins positioned on a lattice and interacting with their first neighbors. This model describes the ferromagnetic to paramagnetic transition of a magnet as the temperature T rises above a critical temperature T_c ; the transition is marked by the presence of correlated domains of all length scales and the vanishment of the magnetization m (the “order parameter”) as

$$m \sim (T_c - T)^\beta. \quad (23)$$

Quite interestingly, the *critical exponents*, β and its kin, are shared by many other, a priori unrelated systems: The latter are said to belong to the same universality class as the Ising model.

These ideas extend beyond equilibrium, but fewer methods are available to deal with the *dynamical phase transitions* encountered out of equilibrium. In this respect, it is worth noting that the Herschel-Bulkley constitutive law can be recast into an expression analogous to Eq. 23, viz.,

$$\dot{\gamma} \sim (\Sigma - \Sigma_y)^\beta. \quad (24)$$

This yielding transition is receiving more and more attention as an example of transition in a driven system; in particular, there have been some lively discussions as to whether it belongs to the same universality class as the depinning transition for driven elastic lines (Sec. IX.B).

1. Avalanches in sandpile models

As models featuring threshold dynamics and a toppling rule, EPM are also connected to the somewhat simpler sandpile models, introduced in Sec. III.A. Let us clarify some concepts using the latter class of systems.

Simulations of 2D sandpile models display avalanches of grains of duration T (number of iterations to reach stability) and size S (total number of transferred grains). These avalanches are here compact structures, unlike those observed in EPM (where the propagator is long-ranged and inhomogeneous). Without entering into details, at vanishing

deposition rate, the cumulative distributions of S and T exhibit power-law scalings, with a cut-off at large scales due to the finite size of the system, viz.,

$$C(S) = S^{1-\tau} f(S/L^{d_f}) \text{ and } C(T) = T^{1-\tau'} g(T/L^z), \quad (25)$$

where $\tau > 0$ and $\tau' > 0$ are critical exponents, f and g are fast decaying functions, and the positive exponents d_f and z are called the fractal dimension of the avalanches and the dynamical exponent, respectively. This means that small avalanches are more frequent than larger ones, but in such a fashion that no *typical* or characteristic size can be established, which has been called *self-organized criticality*. Let us note that the *extremal dynamics* used to trigger avalanches can be substituted by a very slow (quasistatic) uniform loading of the columns of sand if some randomness is introduced in the stability thresholds. In this sense, self-organized criticality in the sandpile model simply exposes the criticality associated with the dynamical phase transition undergone by the loaded system.

At finite deposition rate, different avalanche regimes appear, depending on the rate or, somewhat equivalently, the frequency at which one probes the power spectrum $S_J(\omega) \equiv \int dt \int d\tau e^{-i\omega\tau} J(t)J(t+\tau)$ of the current J , i.e., the number of grains falling out of the pile per unit time. At large frequencies ω , independent, non-overlapping avalanches are seen, while they start interacting as ω decreases. In this regime, their overlaps cut off the correlation lengths of single avalanches, but due to mass conservation during grain transfer, the scale-free behavior is preserved. On long time scales, i.e., for low ω , the observed features are typical of discharge events, whereby the whole sandpile becomes unstable after having been loaded (Hwa and Kardar, 1992).

2. Stress drops and avalanches in EPM

Similarly to the instabilities in sandpile models, the plastic events occurring in EPM can trigger avalanches of successive ruptures. To facilitate the comparison with experiments or atomistic simulations, these avalanches are usually quantified by looking at the time series of the macroscopic stress $\sigma(t)$ and, more specifically, at the stress drops $\Delta\sigma$ associated with plastic relaxation. Close to criticality, the duration T of these drops and their extensive size $S \equiv \Delta\sigma L^d$, where L is the linear size of the system and d its dimension, most often display distributions formally similar to Eq. 25, viz.

$$P(S) \sim S^{-\tau} f(S/S_{\text{cut}}) \text{ and } P(T) \sim T^{-\tau'} g(T/T_{\text{cut}}), \quad (26)$$

where the upper cut-offs S_{cut} and T_{cut} entering the scaling functions f and g will typically depend on system size, e.g. $S_{\text{cut}} \propto L^{d_f}$. In the following, we will pay particular attention to the possible impact of the peculiarities of the quadrupolar stress redistribution in EPM, notably its fluctuating sign, on the avalanche statistics.

B. Avalanches in mean-field models

Shortly after the emergence of the first EPM, mean-field approximations were exploited to determine the statistics of avalanches. Most of these approaches assume a uniform redistribution of the stress released by plastic events, as exposed in Sec. IV.A. An exponent $\tau = 3/2$ is then consistently found in the avalanche size scaling of Eq. 26.

For instance, Sornette (1992) proposed to map the Burridge-Knopoff model for earthquakes (see Sec. III.A) onto a fiber bundle with global load sharing (see Sec. IX.C) and noted that at criticality the extremal load per fiber performs an unbiased random walk, so that the avalanche size is akin to the walker's survival time close to an absorbing boundary, whence an exponent $\tau = 3/2$. If deformation starts further away from the critical point, a larger exponent is then found, $\tau = 5/2$. A posterior, but widely celebrated (Dahmen *et al.*, 1998) model for heterogeneous faults in earthquakes was proposed by Fisher *et al.* (1997), and later applied to the deformation of crystals by Dahmen *et al.* (2009) and more recently to the deformation of granular matter (Dahmen *et al.*, 2011) and amorphous solids (Antonaglia *et al.*, 2014). Here, the problem is directly mapped onto a problem of elastic line depinning (see Sec. IX.B). Once again, above an upper critical dimension that decreases with the interaction range, the model yields the mean-field exponent $\tau = 3/2$. But if a post-yield weakening mechanism is introduced or if stress pulses due to inertial effects are present, the power-law regime only holds for small avalanches, while larger ones trigger runaway events that span the whole system and result in a bump at a characteristic size in the avalanche statistics.

Much more recently, there have been endeavors to extend mean-field approaches in order to account for the non-positiveness of the redistributed stress (which undermines the mean-field reasoning), e.g., via a diffusive term acting on local stresses in the Hébraud-Lequeux model introduced in Sec. IV.B.2. Jagla (2015) studied avalanches in a discrete

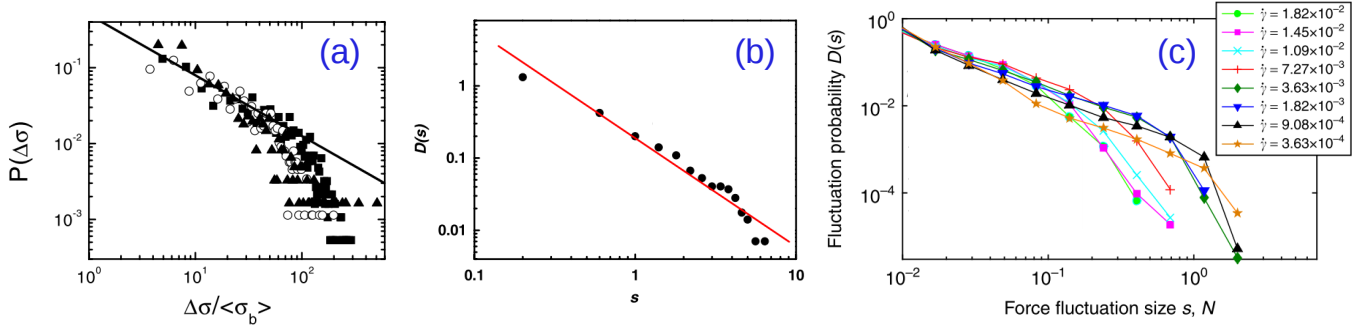


FIG. 17 *Distributions of stress drops in the deformation of amorphous materials.* (a) Distribution of stress drops $\Delta\sigma$ in a foam that is strained in a Couette cell, for three different strain rates. From (Lauridsen *et al.*, 2002) with APS permission. The solid line in this logarithmic plot has a slope of -0.8 . (b) Distribution $D(s)$ of stress drops of normalized magnitude s in a metallic glass ($\text{Cu}_{47.5}\text{Zr}_{47.5}\text{Al}_5$). Adapted from (Sun *et al.*, 2010). The red line represents a power law with exponent $\tau \simeq 1.49$. (c) Distribution $D(s)$ of force fluctuation sizes s in a sheared granular system, for different shear rates and at constant confining pressure $P = 9.6$ kPa. Adapted from (Denisov *et al.*, 2016) with permission. The data suggest truncated power laws $D(s) \sim s^{-\tau} \exp(-s/\gamma^\mu)$, with $\tau = 1.5$ and $\mu = 0.5$.

variant of this model and reported on subtleties that are absent from depinning problems. Indeed, if avalanches are artificially triggered by picking a random block and destabilizing it, the problem can yet again be mapped onto a survival problem for an unbiased random walk, similarly to the fiber bundle, and the mean-field exponent $\tau \simeq 3/2$ is obtained. Incidentally, using a similar random-kick protocol, Lin *et al.* (2014a) had previously found the same result in two EPM variants. If, instead, the physically more relevant protocol of quasistatic loading is used, by uniformly increasing stresses until a block is destabilized, the result still holds in the depinning case. The explanation is that the fairly homogeneous distribution of local stresses σ close to the yield point σ_y , i.e., $p(\sigma)|_{\sigma \approx \sigma_y} \simeq \text{cst}$, is insensitive to the stress shift induced by the uniform loading. By contrast, stress fluctuations in disordered solids deplete local stresses close to σ_y , so much so that $p(\sigma_y^-) = 0$, and thus render $p(\sigma)$ inhomogeneous in stress space. Accordingly, significantly smaller exponents $\tau \simeq 1.1 - 1.2$ are both predicted and observed numerically in that case (Jagla, 2015). Furthermore, the power law is cut off at a value S_{cut} that depends on the distance to criticality and on the system size. An extension of these results to heavy-tailed distributions of stress fluctuations (Lin and Wyart, 2016) is still pending.

C. Experimental observations and atomistic simulations of avalanches

1. Experiments

Various experimental settings have been designed to characterize avalanche statistics in deformed amorphous solids in the last decade, even though experiments are still trailing behind the theoretical predictions and numerical computations in this area. Let us mention examples of such works.

Lauridsen *et al.* (2002) sheared a foam in a Couette cell and investigated its plastic behavior. The distributions $P(S)$ of normalized stress drops S (plotted in Fig. 17a) were shown to follow a power law at three different shear-rates, with an apparent exponent $\tau \simeq 0.8$ in Eq. 26. This value was reported to be consistent with the bubble model of Durian (1997), but contrasts with other theoretical predictions, as we will see. It should however be noted that the power law was fitted over barely a decade in S .

At the other end of the softness spectrum, the compression of millimetric metallic glass rods was studied by Sun *et al.* (2010) and the stress drops were analyzed. Again, $P(S)$ follows a power law regime over one decade of experimental measurements, but this time with exponents in the range $\tau \in [1.37, 1.49]$, as can be seen in Fig. 17b. Among several works that came in the wake of this seminal paper, Antonaglia *et al.* (2014)'s compression experiments of microsamples were argued to be compatible with the mean-field prediction $P(S) \sim S^{-3/2}$. Following the same approach, Tong *et al.* (2016) reported exponents in the range $\tau \in [1.26, 1.6]$ for four different samples of a $\text{Cu}_{50}\text{Zr}_{45}\text{Ti}_5$ alloy.

Quite recently, a granular packing subject to the simultaneous application of pressure and shear was also shown to display stress drops with power-law statistics (Denisov *et al.*, 2016). The power-law exponents, which seem to lie in a relatively broad range in Fig. 17c, were not fitted, but, upon rescaling, were reported to be in good agreement

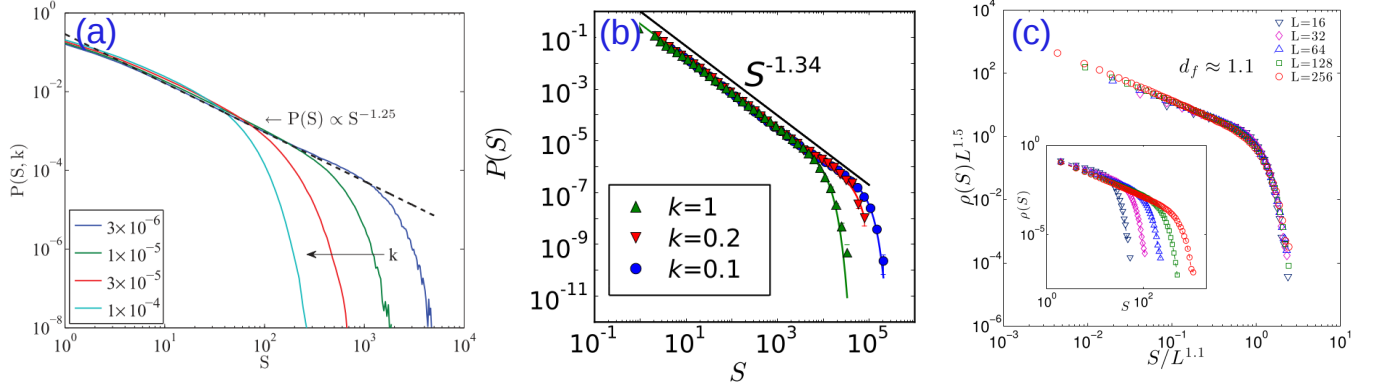


FIG. 18 *2D avalanche size distributions* $P(S)$ obtained with EPM in the quasistatic limit, in diverse settings: **(a)** In a system of linear size $L = 256$ subject to extremal dynamics, with different spring constants k indicated in the legend. The fitted exponent is $\tau \simeq 1.25$. From (Talamali *et al.*, 2011). **(b)** In strain-controlled simulations run with the 'image sum' implementation of the elastic propagator kernel (see Sec.III.C.2). Note the much larger spring constants k , compared to panel (a). From (Budrikis and Zapperi, 2013). **(c)** In systems of different sizes L subject to extremal dynamics. The exponent reported for the unscaled curves (inset) is $\tau \simeq 1.2$, while $\tau \simeq 1.36$ was used to fit the rescaled curve shown in the main plot. From (Lin *et al.*, 2014b).

with the mean-field value $\tau = 3/2$. It remains uncertain to what extent the value reported in this work and in the other ones may have been influenced by the large body of literature claiming that the deformation of (a large variety of) amorphous materials belongs to a unique universality class, the one describing the depinning of an elastic line (Dahmen *et al.*, 2009; Dahmen, 2017). Also note that the data obtained in the granular system of Denisov *et al.* (2016) appear more promising than other experimental examples, insofar as two decades of power law are visible in the raw (non-cumulative) stress drop distribution, at least for the smaller strain rates.

2. Atomistic simulations

In parallel to experiments, stress drops have been analyzed in atomistic simulations of the deformation of glassy materials. In a 2D packing of soft spheres, Maloney and Lemaître (2004) measured power-law distributed energy drops with an exponent $\tau = 0.5 - 0.7$ comparable to that obtained in Durian (1997)'s foam experiments. On the contrary, exponential distributions of stress drops and energy drops were then reported in athermal systems of particles interacting with three distinct potentials in 2D (Maloney and Lemaître, 2006), but also with a more realistic potential for a metallic glass in 3D (Bailey *et al.*, 2007). All these studies were however limited to fairly small system sizes. Using larger systems in 2D and 3D, Salerno and Robbins (2013) found power-law distributed energy drops and stress drops, with distinct values for the exponent τ in the overdamped regime and the underdamped one, and in 2D and 3D. In the overdamped case, the value is identical in 2D and 3D, $\tau = 1.3 \pm 0.1$. We also mention that, opposing the rather consensual view of scale-free avalanches and non-trivial spatiotemporal correlations, Dubey *et al.* (2016) suggested that the characteristics of the stick-slip behavior stemmed from trivial finite-size effects.

D. Avalanche statistics in EPM

The large amount of statistics afforded by EPM can enlighten the debate about the criticality of the yielding transition and the existence (or not) of a unique class of universality by overcoming the uncertainty and limitations of some experimental measurements. In the last years, EPM have tended to challenge the strict amalgamation of the yielding transition with the depinning one.

1. Avalanche sizes in the quasistatic limit

Avalanches are most easily defined in the limit of quasistatic driving, in which the external load is kept fixed during avalanches (Sec. II.C). Applying extremal dynamics to a 2D EPM, Talamali *et al.* (2011) defined an avalanche size S as the number of algorithmic steps Δt during which the external stress Σ remains lower than $\Sigma_{\text{start}} - k\Delta t$, as

though the system were driven by a slowly moving spring of stiffness k . Their numerical simulations displayed a scale-free distribution $P(S) \propto S^{-\tau}$ with $\tau = 1.25 \pm 0.05$ cut by a Gaussian tail (Fig.18a). It was made explicit that this result is at odds with the mean-field exponent $\tau = 3/2$. On the other hand, the measured value is similar to that measured by Durin and Zapperi (2000) ($\tau \simeq 1.27$) for one class of Barkhausen avalanches, due to the motion of ferromagnetic domain walls under an applied magnetic field, and to that predicted for this effect using a model of elastic line depinning with anisotropic (dipolar, but positive) interactions (Zapperi *et al.*, 1998). Of at least equal relevance is the similarity with the avalanche size exponent $\tau \simeq 1.25$ found when simulating differential equations (Bonamy *et al.*, 2008) or cellular automata (Laurson *et al.*, 2010) to describe the interfacial growth of a crack in a heterogeneous medium. Indeed, the alignment of plastic events along the Eshelby ‘easy’ axes was seen as an effective dimensional reduction, leading to avalanches belonging to a quasi 1D problem with positive interactions decaying as r^{-2} , similarly to the interfacial crack growth model of Bonamy *et al.* (2008).

A couple of years later, Budrikis and Zapperi (2013) exploited a closely related EPM, with randomly distributed stress thresholds, to investigate the effect of two distinct implementations of periodicity for the long-range elastic propagator \mathcal{G} defined in Sec. III.C.2. A first series of simulations focuses on the nonstationary plastic activity below the macroscopic yield stress Σ_y , by adiabatically increasing the applied stress Σ . Overloaded blocks yield simultaneously; their strain is increased by $d\gamma = 0.1$ and a new local yield stress is drawn. For $\Sigma \ll \Sigma_y$, avalanche distributions are found to decay as exponentials (or compressed exponentials). For stresses closer to Σ_y , the distribution’s shape can be fitted by Le Doussal and Wiese (2012)’s prediction, viz.,

$$P(S) = c_1 S^{-\tau} \exp(c_2 S - c_3 S^2), \text{ with } \tau \simeq 1.35.$$

The cutoff S_{cut} in $P(S)$ scales with the distance to the critical point, $S_{\text{cut}} \propto (\Sigma_y - \Sigma)^{-1/\sigma}$ with $1/\sigma \simeq 2.3$. In a second series of simulations, apparently inspired by Talamali *et al.* (2011), the system’s strain was pulled by the adiabatic motion of a spring of stiffness $k \in [0.1, 1]$. Hence an external stress $\Sigma = k(\gamma_{\text{tot}} - \gamma)$, where γ_{tot} is the spring position and γ is the plastic strain. Large statistics were collected in the critical steady state and two very close, but not strictly identical, exponents $\tau \simeq 1.35$ were measured for different implementations of the propagator, (see Fig. 18b and Table III for the precise values of τ). These values somewhat differ from Talamali *et al.* (2011)’s measurement, presumably because the distance to criticality ($k \rightarrow 0$) was larger due to the use of larger spring constants; still, they definitely deviate from the mean-field value, too. The authors also considered the avalanche durations T , measured in algorithmic steps, which were fitted by the power law $P(T) \propto T^{-\tau'}$ with $\tau' \simeq 1.5$. Joining these researchers, Sandfeld *et al.* (2015) tested the robustness of these avalanche statistics to variations of the boundaries, implementations of the stress redistribution and to finite-size effects, using an eigenstrain-based finite element method with different types of meshgrids. They found that these variations have no influence on the critical exponents.

Lin *et al.* (2014a) implemented two slightly different automata based on the Hébraud-Lequeux model but embedded in finite dimensions. In stress-controlled simulations with $\Sigma \sim \Sigma_y$, in which sites are randomly ‘kicked’ to trigger an avalanche, they found $\tau \approx 1.42$ in both model variants. Notice the larger value compared to the quasistatic simulations described above, possibly pointing to the influence of the random-kick protocol (Jagla, 2015). Yet, later on, Lin *et al.* (2014b) reported $\tau \simeq 1.36$ in 2D and $\tau \simeq 1.43$ in 3D, for the same protocol. Besides, power-law distributions were reported for the avalanche durations, with exponents $\tau' \simeq 1.6$ in 2D and 1.9 in 3D. In parallel, extremal dynamics were implemented and yielded smaller exponents for the same models, $\tau \simeq 1.2$ in 2D and $\tau \simeq 1.3$ in 3D, closer to previous quasistatic approaches, even though not devoid of finite-size effects.

2. Connection with other critical exponents

A discussion on the density of zones close to yielding and its connection with the critical exponents was opened up by Lin *et al.* (2014a). Denoting $x \equiv \sigma_y - \sigma$ the distance to threshold of local stresses, a stark contrast was emphasized between depinning-like models, with only positive stress increments and $p(x) \sim x^0$ for small x , and EPM, where a pseudo-gap emerges at small x , viz., $p(x) \sim x^\theta$ with $\theta > 0$. In Lin *et al.* (2014a)’s stress-controlled simulations with randomly ‘kicked’ sites, identical values of θ were obtained in two variants of the model embedded in 2D ($\theta \simeq 0.6$) and 3D ($\theta \simeq 0.4$), whereas the stress-strain curves differed (see Table III for the slightly smaller values of θ measured using extremal dynamics).

Shortly afterwards, Lin *et al.* (2014b) proposed to link $p(x)$ with $P(S)$, in a scaling description of the yielding transition. Their scaling argument can be summarized as follows. Starting from Eq. 26, one obtains $\langle \Delta\sigma \rangle \propto L^{d_f(2-\tau)-d}$. Now, in a stationary situation, on average this stress drop must balance the stress increase that is applied to trigger an avalanche. Among the L^d sites, the one with the smallest x , x_{min} , will start the avalanche, so $\Delta\sigma \propto x_{\text{min}}$. If

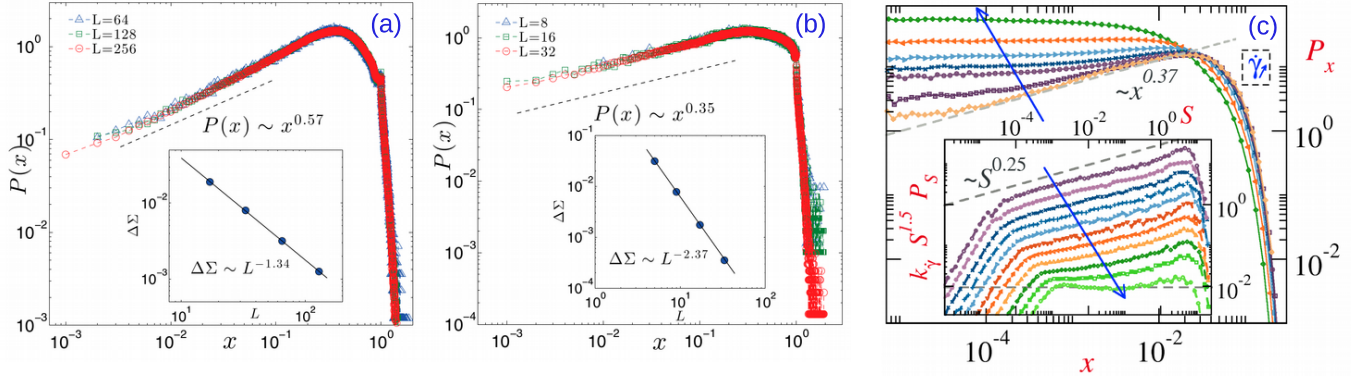


FIG. 19 Probability density of the distance to threshold, $P(x)$, in EPM: (a) In 2D and (b) 3D systems, whose size L is varied. From (Lin *et al.*, 2014b). (c) In a 3D system, as the shear rate is varied. The inset shows the rescaled distribution of avalanche sizes. From (Liu *et al.*, 2016).

$p(x) \sim x^\theta$, then $x_{\min} \propto L^{-\frac{d}{\theta+1}}$. Identifying the two expressions leads to

$$\tau = 2 - \frac{\theta}{\theta + 1} \frac{d}{d_f}, \quad (27)$$

which is supported by their EPM simulations (notably with the random-kick protocol). In this regard, the discrepancy was once again underscored between the depinning transition (with fractal dimensions $d_f \geq d$ typically, due to the compactness of the avalanches, and a velocity-force exponent $\beta \leq 1$) and the yielding transition (with typically $d_f < d$ and a rheological exponent $\beta \geq 1$ in Eq. 24). Generalized scaling relations encompassing both transitions were put forward (see Lin *et al.* (2014b)-Supporting Information).

TABLE III List of values measured for the critical exponents characterizing avalanches in EPM. Only values measured in EPM with *extremal dynamics* (or akin) and a *quadrupolar propagator* are reported. Mean field values are added for comparison.

Exponent	τ	τ'	d_f	θ	γ
Expression	$P(S) \sim S^{-\tau}$	$P(T) \sim T^{-\tau'}$	$S_{\text{cut}} \sim L^{d_f}$	$p(x) = x^\theta$ with $x \equiv \sigma_y - \sigma$	$S \sim T^\gamma$
2D EPM					
(Talamali <i>et al.</i> , 2011) [spring coupling $k \rightarrow 0$]	1.25 ± 0.05	—	~ 1	—	—
(Budrikis and Zapperi, 2013) [spring coupling $k \gtrsim 0.1$]	1.364 ± 0.005	1.5 ± 0.09	$\gtrsim 1^\dagger$	—	~ 1.85
(Lin <i>et al.</i> , 2014b) [extremal]	~ 1.2	~ 1.6	$1.10 \pm 0.04^*$	~ 0.50	—
(Liu <i>et al.</i> , 2016) [$\dot{\gamma} \rightarrow 0$]	1.28 ± 0.05	1.41 ± 0.04	0.90 ± 0.07	0.52 ± 0.03	1.58 ± 0.07
(Budrikis <i>et al.</i> , 2017) [spring coupling, $k \rightarrow 0$]	1.280 ± 0.003	—	—	0.354 ± 0.004	1.8 ± 0.1
3D EPM					
(Lin <i>et al.</i> , 2014b) [extremal]	~ 1.3	~ 1.9	$1.50 \pm 0.05^*$	~ 0.28	—
(Liu <i>et al.</i> , 2016) [$\dot{\gamma} \rightarrow 0$]	1.25 ± 0.05	1.44 ± 0.04	1.3 ± 0.1	0.37 ± 0.05	1.58 ± 0.05
(Budrikis <i>et al.</i> , 2017) [spring coupling, $k \rightarrow 0$]	1.280 ± 0.003	—	—	0.354 ± 0.004	1.8 ± 0.1
MEAN FIELD					
(Fisher <i>et al.</i> , 1997) [depinning]	$3/2$	2	—	0	2
(Jagla, 2015) [Hébraud-Lequeux like]	1.1 – 1.2	—	—	1	—

Legend – : † Estimated from the avalanches shape. * Obtained using the τ exponents from the random-kick protocol.

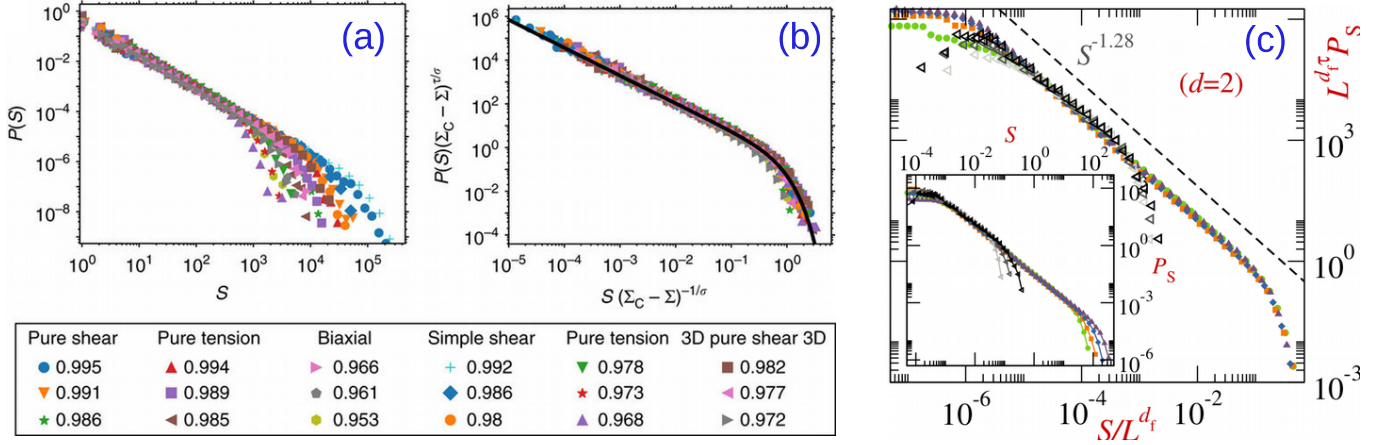


FIG. 20 *Avalanche size distributions $P(S)$ in EPM.* (a) Distributions $P(S)$ obtained in a variety of loading conditions and different external stresses, with an EPM based on a Finite Element routine. The data are collapsed in panel (b) using exponents $\tau \simeq 1.280$ and $1/\sigma \simeq 1.95$. From (Budrikis *et al.*, 2017). The values in the legend refer to the ratio Σ/Σ_y . (c) Main panels show rescaled distributions $L^{d_f \tau} P_S$ vs. S/L^{d_f} in 2D, compared to MD simulations, in the quasistatic limit. The fitted exponents are $\tau \simeq 1.28$ and $d_f \simeq 0.90$. The inset shows the raw data. From (Liu *et al.*, 2016).

3. At finite strain rates

Seeking to narrow the gap between experiments and EPM, Liu *et al.* (2016) analyzed the EPM stress signal with methods mimicking the experimental ones and studied the effect of varying the applied shear rate $\dot{\gamma}$. At very low $\dot{\gamma}$, avalanches are power-law distributed with an exponent $\tau \simeq 1.28$ in 2D and $\tau \simeq 1.25$ in 3D, cut off by finite size effects with $d_f = 0.90$ and 1.3, respectively. These results coincide very well with MD simulations in the quasistatic limit and support the nascent convergence towards an avalanche size exponent $\tau \simeq 1.25$ in 2D or 3D EPM, deviating from the (depinning) mean-field value $3/2$. Much more tentatively, there may be a downward trend of τ with increasing dimensions, which would be compatible with Jagla (2015)'s suggestion $\tau \simeq 1.1 - 1.2$ above the upper critical dimension.

Interestingly, Liu *et al.* (2016) observe a systematic crossover towards higher values of τ when the shear rate is increased, so that τ reaches $\tau \simeq 1.5$ at intermediate $\dot{\gamma}$, before entering the high- $\dot{\gamma}$ regime of pure viscous flow. At the same time, the external driving starts to dominate over the signed stress fluctuations originating from mechanical noise; this nudges the system into a depinning-like scenario, with an exponent θ in $p(x) \sim x^\theta$ decreasing towards zero as $\dot{\gamma}$ reaches finite values both in 2D and 3D. Similarly to pulling the system with a stiff spring (large k), increasing the shear rate generates simultaneous uncorrelated plastic activity in the system, which leads to larger τ , closer to 1.5. Overall, applying a finite shear rate does not destroy the criticality of avalanche statistics; but it affects the critical exponents and eventually produces more trivial effective statistics.

4. Insensitivity to EPM simplifications and settings

At present, technical difficulties hamper a clear discrimination between theoretical predictions on the basis of experimental data. The simplifications used in the models thus need to be carefully examined. Budrikis *et al.* (2017) investigated the effect of the scalar approximation of the stress (see Sec. II.C.3) by comparing the results of a scalar model to those of a finite-element-based fully tensorial model, under different deformation protocols (uniaxial tension, biaxial deformation, pure shear, simple shear) and in both 2D and 3D. Irrespective of the tensoriality of the model, the dimension, and the boundary conditions, a (non mean-field) universal scaling function is observed for the avalanche distribution, shown in Fig. 20 and coinciding with Le Doussal and Wiese (2012)'s proposal

$$P(S) = \frac{A}{2\sqrt{\pi}} S^{-\tau} \exp\left(C\sqrt{u} - \frac{B}{4}u^\delta\right), \quad (28)$$

with an exponent $\tau = 1.280 \pm 0.003$ [note the perfect agreement with Liu *et al.* (2016)'s result], $u \equiv S/S_{\max}$ and $S_{\max} \propto (\Sigma_c - \Sigma)^{-1/\sigma}$ (with $1/\sigma \simeq 1.95$). The constants, A , B , and C are functions of τ , as is $\delta = 2(1 - \tau/3)$.

Heterogeneous deformations, such as bending and indentation, were also considered and yielded similar values for τ , but some range of variation for exponent $\sigma \in [1.53, 2.05]$ and a slightly different form for the cutoff S_{cut} . The

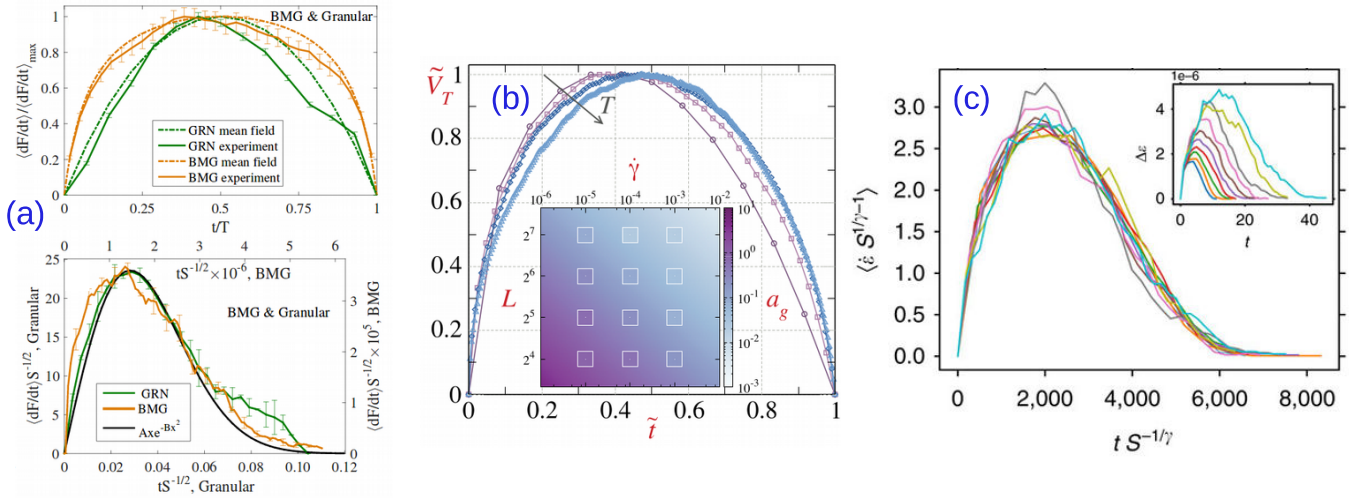


FIG. 21 *Avalanche shapes in experiments and EPM.* (a) Experimental avalanche shapes, for avalanches of fixed duration (top) and fixed sizes (bottom) in a bulk metallic glasses and a granular system. From (Denisov *et al.*, 2017). (b) Avalanche shapes at different fixed durations in a strain-controlled EPM simulation at fixed $\dot{\gamma}$. From (Liu *et al.*, 2016). (c) Avalanche shapes at fixed sizes. From (Budrikis *et al.*, 2017).

latter differences are not unexpected: An independent length scale enters the problem and the yield stress Σ_y used to measure exponent σ is not universal. Finally, the average avalanche size was related to its duration T via $S \sim T^\gamma$ with $\gamma = 1.8 \pm 0.1$.

A possible explanation for the insensitivity of avalanche statistics to the aforementioned aspects may lie with the quasi-1D geometry of the avalanches, resulting from the quadrupolar propagator. Most cooperative phenomena thus appear to be controlled by the stress component along one direction, and a scalar description may be sufficient in this respect. (Scalar models do indeed reproduce the same power-law exponent and evidence a fractal dimension $d_f \approx 1$ in 2D and 3D, as shown in Fig.20c).

5. Effects of inertia

Without the assumption of instantaneous stress redistribution, stress waves are expected to propagate throughout the system (see Sec. III.D and Fig. 9), in a ballistic way or a diffusive one depending on the damping. This is not described by the traditional elastic propagator \mathcal{G} of Eq. 14, but finite-element based EPM have recently made it possible to account for inertial effects (Karimi and Barrat, 2016). Karimi *et al.* (2017) exploited this type of model to study Salerno and Robbins (2013)'s claim, based on extensive atomistic simulations in the quasistatic regime, that inertial effects drive the system into a new (underdamped) class of universality. At odds with this claim, but consistently with results from sandpile models (Khfifi and Loulidi, 2008; Prado and Olami, 1992) and seismic fault models (Carlson and Langer, 1989), they found that inertial effects break down the universal, scale-free avalanche statistics. A characteristic hump (or secondary peak) of large events emerges in the avalanche size distribution $P(S)$, similarly to Fisher *et al.* (1997)'s findings. In Karimi *et al.* (2017)'s work, both the relative weight and the scaling with the system size of this peak are controlled by the damping coefficient Γ , a dimensionless parameter that quantifies the relative impact of dissipation. The effective fractal dimension $d_f'(\Gamma)$ of avalanches also varies with the damping, but was found to satisfy a scaling relation with the exponent $\theta'(\Gamma)$ defined by $p(x) \sim x^{\theta'(\Gamma)}$.

These results are compatible with Papanikolaou (2016)'s phenomenological description of inertial effects, which are accounted for by a temporary vanishment of elasticity after local plastic events (plastic delay): Simulations of the model showed the appearance of a hump of large events in $P(S)$, an increase of the exponent τ , as well as the emergence of dynamical oscillations, accompanied with strain localization.

6. Avalanche shapes

In addition to their duration and size, further insight has been gained into the avalanche dynamics by considering their average temporal signal, i.e., the ‘shape’ of the bursts. This observable can be determined experimentally with higher quality (Antonaglia *et al.*, 2014; Denisov *et al.*, 2017). Avalanche shapes have thus been estimated for various systems displaying crackling noise; examples include earthquakes (Mehta *et al.*, 2006), plastically deforming crystals (Laurson *et al.*, 2013), and the Barkhausen noise (Mehta *et al.*, 2002; Papanikolaou *et al.*, 2011).

In the latter example, the magnetization of a film changes mostly changes via the motion of domain walls (in the central part of the hysteresis loop near the coercive field); its rate of change is recorded as a time series $V(t)$. When the film thickness, which controls the long-range dipolar interactions, is such that mean field is valid, the average shape $V(t|T)$ of avalanches of duration T is well described by an inverted parabola (Papanikolaou *et al.*, 2011), viz.,

$$V(t|T) \propto T\tilde{t}(1 - \tilde{t}) \text{ where } \tilde{t} \equiv t/T. \quad (29)$$

Since oftentimes mean field does not hold, a generalized functional form was proposed by Laurson *et al.* (2013)::

$$V(t|T) \propto T^{\gamma-1} [\tilde{t}(1 - \tilde{t})]^{\gamma-1} \left[1 - a \left(\tilde{t} - \frac{1}{2} \right) \right]. \quad (30)$$

Here, the shape factor γ is also the exponent that controls the scaling between size and duration ($S \sim T^\gamma$), since $S(T)$ is nothing but the integral $\int_0^T V(t|T)dt$. γ and the parameter a controls the asymmetry ($a > 0$ refers to positive skewness); the mean-field formula is recovered for $\gamma = 2$ and $a = 0$. As the interaction range increases from local to infinite, the universality-class parameters evolve from $\gamma \simeq 1.56$, $a \simeq 0.081$ to $\gamma \simeq 2.0$, $a \simeq 0.01$. Dobrinevski *et al.* (2015) provided an analytic formalization for this expression as a one-loop correction around the upper critical dimension; these authors also computed the shape of avalanches of fixed *size* S . The need for this generalization beyond mean field was confirmed by Durin *et al.* (2016).

In the deformation of amorphous solids, the inverted-parabola shape predicted by mean field was shown to provide a satisfactory description in experiments on metallic glasses (Antonaglia *et al.*, 2014) and on granular matter (Denisov *et al.*, 2017). Still, at a more quantitative level, deviations can be seen (see Fig.21a).

On the EPM side, Liu *et al.* (2016) studied the effect of finite shear rates $\dot{\gamma}$ on the avalanche shape. By sorting the avalanches according to their duration T , at fixed $\dot{\gamma}$, they found that short avalanches are noticeably more asymmetric and display faster velocities at earlier times (positive skewness, see Fig.21b). For larger T , it is argued that avalanches most likely result from the superposition of uncorrelated activity, which leads to more mean-field like results. This would explain the gradually more symmetric shapes observed for increasing T (see the evolution of the asymmetry parameter in the inset of Fig. 21b). In the quasistatic limit, asymmetric stress-drop shapes are then expected. Indeed, at low $\dot{\gamma}$, fits with Eq. 30 give a non-mean-field value $\gamma \simeq 1.58$ in both 2D and 3D. This feature gradually disappears at larger $\dot{\gamma}$.

In their EPM studies under different loading conditions, in 2D and 3D, Budrikis *et al.* (2017) measured values for γ in the range [1.74, 1.87]. But, contrary to Liu *et al.* (2016)’s findings, based on a scalar EPM, clearly asymmetric avalanches with positive skewness were only found in the bending and indentation protocols, and not (visibly, at least) in the tension and shear simulations. In addition, the shapes obtained by sorting the avalanches according to their sizes (see Fig.21c) collapsed well with the scaling form proposed by Dobrinevski *et al.* (2015); the shape exponent differed from the mean-field value $\gamma = 2$, being closer to $\gamma \simeq 1.8$.

VIII. RELAXATION, AGING AND CREEP PHENOMENA

So far EPM have mostly been exploited to investigate the macroscopic flow behavior and flow profiles (Sec. V), characterize stationary flow (Sec. VI), or study fluctuations and correlations in the steady flow close to criticality, where one finds scale free dynamics in the plastic avalanches (Sec. VII). Still, some works, however few, are concerned with relaxation, aging, and creep phenomena. This section is dedicated to both the dynamics in the temperature assisted relaxation (aging) of disordered systems and to the transient dynamics under loading (creep), prior to yielding or complete arrest. The latter phenomenon can be either an athermal process, provided that the stress load is above, but close to, the yielding point, or thermally assisted creep, in response to a load below the dynamical yield stress.

A. Relaxation and aging

A striking feature in the theory of viscous (glassforming) liquids is their response to an external perturbation, close to the glass transition: They do not exhibit an exponential structural relaxation, with a simple relaxation time scale, but a stretched exponential relaxation. More specifically this means that the temporal behavior of the response function $R(t)$ (e.g., the response in stress $\Sigma(t)$ to the application of a strain step at time $t = 0$) can often be described by the so called Kohlrausch-Williams-Watt (KWW) function

$$R(t) \propto \exp \left[- \left(\frac{t}{\tau} \right)^b \right] \quad \text{with } R(t) \equiv \frac{\Sigma(t) - \Sigma(\infty)}{\Sigma(0) - \Sigma(\infty)}. \quad (31)$$

In this expression, b typically takes a value between 0 and 1, which stretches the exponential relaxation. This was ascribed to the formation of dynamical heterogeneities close to the glass transition, thus producing separately relaxing domains and leading to a broad distribution of relaxation times (Macedo and Napolitano, 1967), hence a stretched exponential relaxation (Bouchaud, 2008; Campbell *et al.*, 1988).

With this picture in mind, it came as a surprise when a series of dynamical light-scattering measurements on colloidal gels showed the opposite behavior, namely, a compressed exponential structural relaxation, characterized by an exponent $b > 1$ (Cipelletti *et al.*, 2000, 2003; Ramos and Cipelletti, 2001). More recent experiments using X-ray photon correlation spectroscopy have found that this feature is not specific to gels (Orsi *et al.*, 2012), but also arises in supercooled liquids (Caronna *et al.*, 2008), colloidal suspensions (Angelini *et al.*, 2013) and even in hard amorphous materials like metallic glasses (Ruta *et al.*, 2013, 2012). Although this anomalous relaxation was observed ubiquitously in experimental systems, it took more than a decade to reproduce dynamics with compressed exponential decay in molecular-scale simulations, until Bouzid *et al.* (2017) and Chaudhuri and Berthier (2017) eventually reported such dynamics in microscopic models for gels; the main obstacle had been to probe the right parameter range, notably with respect to temperature and also length scales.

From the outset, Cipelletti *et al.* (2000) suggested that the faster than exponential relaxation stems from the elastic deformation fields generated by local relaxation events. Shortly afterwards Bouchaud and Pitard (2001) put forward a mean-field model based on the assumption of elasticity to explain this anomalous relaxation. Should this explanation be correct, EPM should be the ideal tool to test it (Ferrero *et al.*, 2014). In a quiescent system, the driving term vanishes in Eq. 5, which turns into

$$\dot{\sigma}_i(t) = \sum_j 2\mu\mathcal{G}_{ji}\dot{\epsilon}_j^{\text{pl}}(t),$$

where $\epsilon_j^{\text{pl}}(t)$ indicates the field of local plastic deformation and the other notations were defined below Eq. 5. As before, this equation describes the response of the surrounding medium to local relaxation events. Here, only thermally activated processes are relevant, and their modeling is inspired by the the trap model of Denny *et al.* (2003) and Sollich *et al.* (1997)'s Soft Glassy Rheology model (SGR, see Eq. 21), with an Arrhenius-like yielding rate for sites below the threshold, viz.,

$$p_{\pm} \sim \exp \left[- \frac{\sigma_y^2 \mp \text{sgn}(\sigma)\sigma^2}{2\kappa T} \right], \quad (32)$$

while sites with $|\sigma| > \sigma_y$ yield instantaneously. In Eq. 32, the signs correspond to the direction of the yielding event, σ_y is a local yield stress, κ is a dimensional prefactor, and T the ambient temperature.

Such models confirm the dependence of the shape parameter β of structural relaxation on the dimensionality of the system, which Bouchaud and Pitard (2001)'s mean-field arguments predict to be $b = \frac{3}{2}$ in 3D and $b = 2$ in 2D. Moreover, in EPM insight into the microscopic dynamics can be gained by following the motion of tracers advected by the elastic displacement field, as explained in Sec. IV.E. This led Ferrero *et al.* (2014) to distinguish three dynamical regimes in 2D, namely (I) ballistic, (II) subdiffusive and (III) diffusive. In the ballistic regime (see Fig. 22), compressed relaxation was found, with a shape parameter $b \approx 2$. The subdiffusive regime was ascribed to correlations in the relaxation dynamics, a feature that has not been reported in experiments. This disagreement can either be due to oversimplifications of the model or to the fact that experiments are usually performed in 3D, and not 2D. Preliminary EPM studies in 3D observed ballistic motion at short times, with a compressed exponent $b = \frac{3}{2}$, followed by a diffusive regime³.

³ Unpublished data of Ferrero *et al.* (2014).

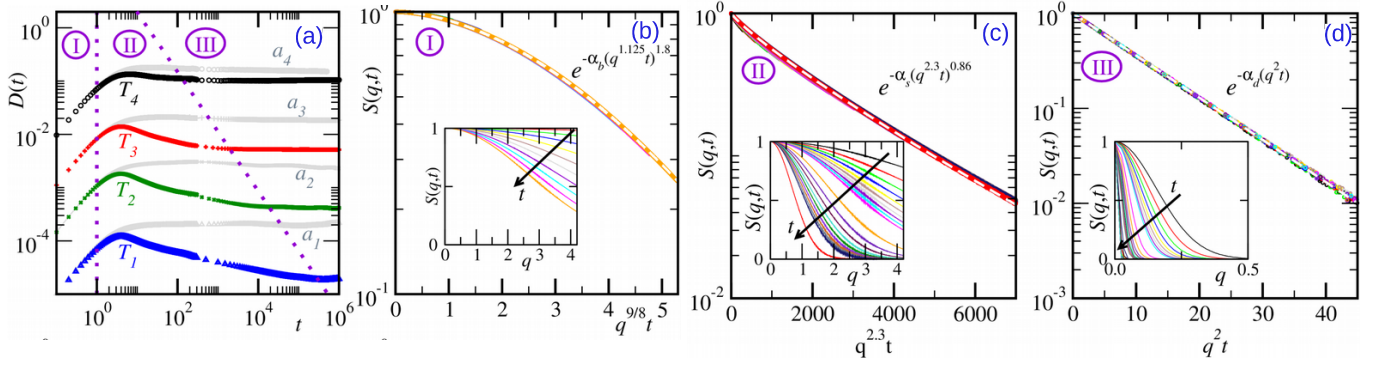


FIG. 22 *Motion of particle tracers and structural relaxation in a quiescent system obtained in an EPM.* (a) Diffusivity $D(t)$ (i.e., mean-square displacement divided by time) of tracer particles (see main text) for four different temperatures, increasing from bottom (blue) to top (black). (b) Rescaled self-part of the intermediate scattering function $S(q, t)$ for t in the first ascending regime of $D(t)$. The motion is close to ballistic (linear in time), with $\tau \approx q^{-1}$ in Eq. 31, and the form factor $\beta \simeq 2$ (defined in the same equation) implies a compressed exponential relaxation. Panels (c) and (d) display the same quantities in dynamical regimes II (subdiffusive) and III (diffusive). All insets show the data from the main graph without rescaling. From (Ferrero *et al.*, 2014).

There remain many other open questions that could be addressed by EPM. For instance the q -dependence of the experimental intermediate scattering functions $S(q, t)$ (Ruta *et al.*, 2012) cannot be captured in EPM at present, but could be included by implementing hybrid models that consider smaller-scale dynamics as well. Besides, the self and the intermediate part of $S(q, t)$ cannot be distinguished in EPM yet, because the tracers do not interact, but may differ in reality. Other questions include the 3D dynamics and the possibility of intermittency in time as well as spatial correlations of the localized relaxation events.

B. Creep

Another field that has stimulated much experimental work in the last years (Bonn *et al.*, 2017) but few rationalization attempts at the mesoscale is creep. The definition of creep is somewhat ambiguous. In some contexts it may refer to stationary motion at a vanishingly small velocity, in particular the creep dynamics of a driven elastic manifold over a disordered landscape at finite temperature (Ferrero *et al.*, 2017), but also the flow of a granular medium subjected to a constant stress $\Sigma \ll \Sigma_y$ supplemented with an additional small cyclic stress modulation (Pons *et al.*, 2016). But here we will restrict our attention to the traditional definition in material science, namely, the slowdown of deformation prior to failure, fluidization or complete arrest, under loading Σ . This load is usually comparable to, or smaller, than the material yield stress Σ_y and creep can in principle be both of thermal and athermal nature.

For $\Sigma > \Sigma_y$ the usual response can be separated into three regimes. Primary creep corresponds to a first slowdown of the dynamics, with a gradual decrease of the (initially high) strain rate $\dot{\gamma}$. The deformation rate is roughly constant in the secondary creep regime but abruptly shoots up in the tertiary regime, which ultimately culminates in macroscopic failure or fluidization. The measured macroscopic quantities are usually the time-dependent $\dot{\gamma}(t)$ and the fluidization or failure time τ_f (Divoux *et al.*, 2011a; Skrzyszewska *et al.*, 2010).

Creep is observed in many experimental systems, from crystalline and amorphous solids to soft materials. In the former materials, a power-law slowing down of the deformation rate with an exponent close to $2/3$ is often reported (Miguel *et al.*, 2002), viz.,

$$\dot{\gamma}(t) \sim t^{-2/3} \text{ or, equivalently, } \gamma(t) \sim t^{1/3}.$$

This law is commonly called Andrade creep and hints at a possible universality of the dynamics. However, experiments and simulations on creep in amorphous systems have found a variety of power-law exponents for the decay of $\dot{\gamma}(t)$ in primary creep, ranging between $1/3$ (Bauer *et al.*, 2006) and 1.0 (the latter value corresponding to logarithmic creep $\dot{\gamma}(t) \sim \ln(t)$), with a multitude of values in-between (Ballesta and Petekidis, 2016; Chaudhuri and Horbach, 2013; Divoux *et al.*, 2011b; Landrum *et al.*, 2016; Leocmach *et al.*, 2014; Sentjabrskaja *et al.*, 2015). Bonn *et al.* (2017) extensively reviewed the literature on the topic. Scaling results for the fluidization (or failure) time τ_f also vary and basically fall in two classes. Among other works, Divoux *et al.* (2011b) found a power-law scaling of τ_f , defined as the time to reach a homogeneous stationary flow, viz., $\tau_f \sim (\Sigma_y - \Sigma)^{-\beta}$, where β varies between 4 and 6. On the other

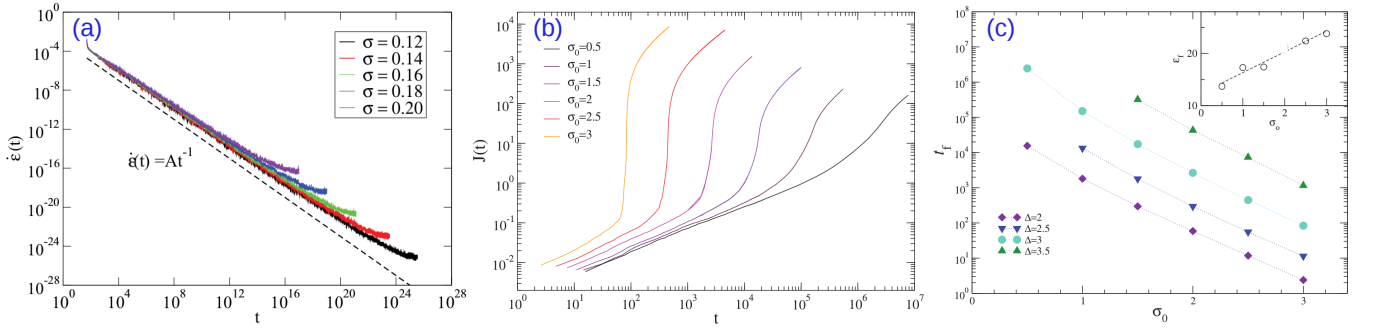


FIG. 23 *EPM characterization of creep.* (a) Strain rate $\dot{\epsilon}$ as a function of time t for different applied stresses Σ in the EPM of Bouttes and Vandembroucq (2013). (b) Non-linear compliance $J \equiv \frac{\epsilon}{\Sigma_0}$ as a function of time for different applied stresses Σ_0 , obtained with Merabia and Detcheverry (2016)'s mesoscopic model. (c) Dependence of the fluidization time t_f on Σ_0 . Panels (b) and (c) are extracted from (Merabia and Detcheverry, 2016).

hand, other works defined τ_f as the duration of the rapid increase of $\dot{\gamma}(t)$ at the end of secondary creep and reported an inverse exponential dependence $\tau_f \sim \exp\left(\frac{\Sigma_0}{\Sigma}\right)$, where a characteristic stress scale Σ_0 has been introduced (Gibaud *et al.*, 2010; Gopalakrishnan and Zukoski, 2007; Lindström *et al.*, 2012).

Thus, rather than a universal behavior, experiments suggest a multitude of dependencies, notably on the preparation protocol prior to the application of the step stress (quench or pre-shear), on temperature, age and also on the dominant physical process at play during creep. In some systems the initial creep regime appears to be completely reversible and one expects the creep to be a result of visco-elasticity. Accordingly, Jaishankar and McKinley (2013) were able to reproduce the experimental power-law creep in Acacia gum solutions using a modified Maxwell model featuring fractional time derivatives. On the other hand, on the basis of molecular dynamics simulations, Shrivastav *et al.* (2016) claim that the power-law creep in a variety of glassy systems can be related to a percolation dynamics of mobile regions, thus plasticity, which would render EPM particularly suitable to tackle the open questions in the field. Among the ‘hot topic’ highlighted by Bonn *et al.* (2017), the detection of precursors that may point to incipient failure stands as the Atlantis in many disciplines from material science to engineering and geology.

Using a lattice-based EPM, Bouttes and Vandembroucq (2013) made a first endeavor to address thermal creep and showed its strong dependence on initial conditions and the impact of aging on the creep behavior. In the model, each site is assigned an energy barrier E_0 (renewed after every plastic event) in the stress-free configuration, with a uniform distribution of E_0 . The elastic stress redistributed by plastic events via usual quadrupolar elastic propagator of Eq. 15 biases this potential. The plastic activation probabilities are analogous to Eq. 32, with an Arrhenius-like law, and are resolved with a kinetic Monte-Carlo algorithm. The resulting creep dynamics $\dot{\gamma}(t)$, studied in pure shear, depend on the applied stress Σ and temperature T and all display an apparent exponent suggestive of logarithmic creep (see Fig. 23a). Besides, the fluidization time τ_f is found to decrease with increasing Σ and T .

Merabia and Detcheverry (2016) explored the transient thermal creep of a viscous liquid under an external stress step prior to steady flow, at relatively high temperatures. They also resorted to a kinetic Monte-Carlo scheme and Arrhenius-type plastic activation rates, but they used a non-uniform distribution of intrinsic trap depths $\rho(E_0)$. With an exponential distribution $\rho(E_0) \sim \exp[-\alpha E_0]$ (leaving aside a lower cutoff), the model is formally similar to the SGR model (see Sec. IV.D.1), but here the temperature parameter is interpreted as the room temperature, instead of an effective noise temperature, and samples are assumed to be thermally equilibrated before stress is applied ($\alpha k_B T > 1$). Contrary to (Bouttes and Vandembroucq, 2013), the simulated creep does not always slow down logarithmically. Instead, a power-law decay $\Sigma \sim \dot{\gamma}^\alpha$ is observed, in agreement with a mean-field analysis. Merabia and Detcheverry (2016) also considered a Gaussian distribution $\rho(E_0)$. In that case the steady-state flow curve grows logarithmically, $\Sigma \sim \ln(\dot{\gamma})$, and the fluidization time τ_f follows the inverse exponential dependence on Σ , $\tau_f \sim \exp\left(\frac{\Sigma_0}{\Sigma}\right)$ (see Fig. 23c), found in experiments on carbopol black gels by Gibaud *et al.* (2010). The latter result is robust to variations of the Gaussian half-peak width.

The authors also tried different stress propagators of short range character, besides the quadrupolar (Eshelby-like) one. It turns out that their mean-field predictions agree best with the simulations with a short-range propagator and an exponential distribution of energy barriers, whereas there is a systematic offset in the creep exponent with respect to the more realistic quadrupolar propagator. This is somewhat counter-intuitive because increasing the interaction range usually leads to a more mean-field-like behavior.

Liu *et al.* (2017) resorted to a mean-field approach based on the Hébraud-Lequeux model (see Sec. IV.B.2) to

study the effect of the loading $\Sigma \equiv \langle \sigma \rangle$ and aging on athermal creep, for $\Sigma > \Sigma_y$. The initial distribution of stresses $\mathcal{P}(\sigma, t = 0)$ was taken as a proxy for the sample age, on the basis that aging results in stress relaxation and thus a narrower distribution $\mathcal{P}(\sigma, t = 0)$. For Σ slightly above the yield stress and long aging, there is first a power-law decay $\dot{\gamma}(t) \sim t^{-\mu}$ ($\mu > 0$) to a minimal value and then an acceleration up to the steady-state value. This evolution is consistent with several experimental measurements in bentonite suspensions and colloidal hard-sphere systems. But, contrary to expectations, the model exhibits a parameter-dependent (thus, non-universal) power-law exponent μ . Within the model, the first creep regime is dominated by the plastic activation of sites that have not yielded yet, which become rarer and rarer, until the memory of the initial configuration is lost and steady-state fluidization is achieved. This occurs at a fluidization time τ_f that decreases as Σ increases, but in a non-universal way.

In conclusion, these few seminal papers proposing a mesoscopic approach to creep leave room for further exploration with EPM, for instance about the universality (or not) of the long-time response in thermal and athermal systems. It would also be interesting to determine if precursors can be defined to predict failure and, once the validity of EPM is established, to upscale the mesoscopic approach into a valid macroscopic description of the creep response.

IX. RELATED TOPICS

Amorphous solids seem to form a specific class of materials. However, the phenomenology exposed above suggests underlying theoretical connections with other problems. And, indeed, EPM are related to a spectrum of other models, with physics-based distinctions, in particular in the interaction kernels. This section reviews, and attempts to compare to EPM, some of these related approaches, from mesoscale models for crystalline plasticity and elastic line depinning to fiber bundles, fuse networks and random spring models.

A. Mesoscale models of crystalline plasticity

1. Crystal plasticity

Like amorphous solids, driven crystalline materials respond elastically to infinitesimal deformations, via an affine deformation of their structure, but undergo plastic deformation under higher loading. To be energetically favorable, plastic deformation increments must somehow preserve the regular stacking of atoms; one must then determine when it is more favorable to jump to the closest regular structure ('switch neighbors') than to keep on with the affine deformation of the current structure. For a perfect crystal, such a criterion would predict an elastic limit of around 5%.

Real crystals actually have a much lower elastic limit because they harbor structural defects, which were created at the stage of their preparation and which play a key role in the deformation. These defects in the regular ordering take the form of dislocations and grain boundaries which separate incompatible crystalline domains. Dislocations are line defects obtained by making a half-plane cut in a perfect crystal and mismatching the cut surfaces before stitching them back together. Similarly to creases on a carpet, which can be pushed across the rug to gradually move it without lifting it as a whole, dislocations can glide across the crystal (and occasionally "climb" when they encounter a defect), thereby generating slip planes. Grain boundaries also promote deformation; in these regions, gliding is facilitated by the mismatch-induced weakness of the local bonds. On the other hand, the presence of impurities, e.g., solute atoms in the crystal, may pin a dislocation at some location in space until it is eventually freed by a moving dislocation, which results in a dent in the stress *vs.* strain curve; this is the so called Portevin-Le Chatelier effect.

The stress field around a dislocation is well known (it decays inversely proportionally to the distance to the line) and the attractive or repulsive interactions between dislocations can also be rigorously computed. As a matter of fact, the elastic propagator used in EPM can be regarded as the stress field induced by four edge dislocations whose Burgers vectors sum to zero (Ben-Zion and Rice, 1993; Ispánovity *et al.*, 2014; Tüzes *et al.*, 2017). However, owing to the vast lengthscales separating the individual dislocation from the macroscopic material, it is beneficial to coarse-grain the description to the mesoscale, by considering the dislocation density field.

2. Models and results

Mesoscale dislocation models, which exist in several variants (Field Dislocation Model, Continuum Dislocation Dynamics), bear formal similarities with EPM.

Noticing that the plastic deformation induced by crystallographic slip generates an elastic stress field $\tau_{int}(\mathbf{r})$ (via the very same elastic propagator as in EPM), Zaiser and Moretti (2005) separated this internal stress $\tau_{int}(\mathbf{r})$ from the aspects more specific to dislocations and crystals and arrived at the following equation in two dimensions:

$$\frac{1}{B}\partial_t\gamma(\mathbf{r}) = \tau_{ext} + \tau_{int}(\mathbf{r}) + \frac{DG}{\rho}\partial_x^2\gamma + \delta\tau(\mathbf{r}, \gamma), \quad (33)$$

where B , D , and G are material constants, τ_{ext} is the externally applied stress, and ρ is the dislocation density. The last two terms on the rhs have no strict counterparts in EPM; they account for the mechanisms generated by interactions between dislocation that alter the stress required to set a dislocation in motion. The third term is a homogenizing term while the fourth one is a (ρ -dependent) fluctuating term; its dependence on the plastic strain γ may be used to effectively describe strain hardening effects due to the multiplication of dislocations. In EPM, such effects would belong to the rules that govern the onset of a plastic event. Armed with this model, the authors then study the slip avalanches in order to explain the experimentally observed deformation patterns consisting of slip lines and bands, echoing the endeavors in this direction on the EPM side. They find scaling exponents for such avalanches that are comparable, but not strictly equal, to the mean-field exponents for the depinning problem; this difference is not unexpected, owing to the fluctuating sign of their elastic propagator, which is identical to the EPM one (see Sec. VII). Also, large avalanches are cut off due to strain hardening, which is one possible explanation for the macroscopic smoothness of the deformation.

Contrasting with this macroscopically smooth situation, the deformation dynamics may feature strong intermittency, which points to collective effects. Power-law-distributed fluctuations have recently been evidenced in the acoustic emissions as well as in the stress *vs.* strain curves of loaded crystals (Weiss *et al.*, 2015; Zhang *et al.*, 2017). These fluctuations may be “mild”, with bursts superimposed on a relatively constant, seemingly uncorrelated fluctuation background, which is the case for many bulk samples, especially those with an *fcc* structure. On the other hand, intermittency becomes dominant in *bcc* crystals and in smaller samples, where large bursts dominate the statistics. Samples with fewer defects also tend to have “wilder” fluctuations. A mean-field rationalization of these phenomena considers the density ρ_m of mobile dislocations and expresses its evolution with the strain γ as

$$\frac{d\rho_m}{d\gamma} = A - C\rho_m + \sqrt{2D}\rho_m\xi(\gamma),$$

where A is a nucleation rate, C is the rate of annihilation of dislocation pairs, and D controls the intensity of the white noise ξ (Weiss *et al.*, 2015). Notice that the latter is multiplied by ρ , owing to the long-ranged interactions between dislocations; the presence of multiplicative mechanical noise makes collective cascade effects possible. Such a model allowed the authors to capture the distinct types of fluctuations in the dynamics, from mild to wild, depending on the noise intensity D . More recently, Valdenaire *et al.* (2016) rigorously coarse-grained a fully discrete 2D dislocation picture into a continuum model centered on a kinetic equation for the dislocation density, with superficial similarities with the EPM equation of motion, Eq. 5.

3. Relation to EPM

Although the microscopic defects and the microscopic deformation mechanisms differ between crystals and disordered solids, the macroscopic phenomenology and, to some extent, the mesoscopic one share many similarities: Microscopic defects interact via long-range interactions and their activity is, in some conditions, controlled by temperature. Globally, the dynamics are highly intermittent at low shear rates and involve scale-invariant avalanches, as indicated, *inter alia*, by acoustic emission measurements on stressed ice crystals (Miguel *et al.*, 2001). This intermittency is generically known as crackling noise (Sethna *et al.*, 2001) and does not connect EPM only to crystal plasticity, but also to the fields of seismology and tribology.

The phenomenological similarity is paralleled by a proximity in the models. In some EPM, the stress redistributed by a shear transformation is actually described as the effect of a combination of dislocations (Ben-Zion and Rice, 1993; Ispánovity *et al.*, 2014; Tüzes *et al.*, 2017). Conversely, quadrupolar interactions may be directly implemented in mesoscale models of crystal plasticity, for instance in Eq. 2 of (Papanikolaou *et al.*, 2012). More generally, the basic equations of evolution in the two fields look very much alike, and models sometimes seem to have bearing on both classes of materials (Shiba and Onuki, 2010). Rottler *et al.* (2014) numerically investigated the transition between the dislocation-mediated plasticity of crystals and the shear-transformation-based deformation of amorphous solids. They found that the directions of the *nonlinear* displacements under strain could be well predicted from the low-frequency

vibrational modes and that polycrystals already behave comparably to glasses, despite their regular structure at the grain scale.

Nevertheless, the connection between crystals and disordered solids should not be overstated. Even though flow defects (“soft spots”) in the latter might to some extent persist over rearrangements (Schoenholz *et al.*, 2014), on no account can they be assimilated to well identified structural defects moving through a crystal. Following from this discrepancy are the facts that, contrary to plastic rearrangements, dislocations are strongly dependent on the preparation of the material (which determines the dislocation density), and may be pinned by defects, annihilate through the merger of partials (“opposite” defects) or multiply.

B. Depinning transition

1. The classical depinning problem

In several systems, an interface is driven through a disordered medium by a uniform external force. This interface can be a magnetic or ferroelectric domain wall, the water front (contact line) in a wetting problem, the fracture front, or even charge density waves and arrays of vortices in superconductors. In all these cases, the interplay between the quenched disorder (e.g., due to impurities) and the elastic interactions along the interface is at the root of a common phenomenology and a universal dynamical response.

If the external force is weak, the interface will advance and soon get pinned and unable to advance any further. If the force is strong enough, instead, the interface will overcome even the largest pinning centers, reaching a steady state of constant velocity. This is the well documented dynamical phase transition known as depinning. Beyond the transition itself, the literature now also describes the equilibrium configuration of the elastic line, several variations of the problem (short/long-range elasticity, different disorder types, etc.), thermally activated dynamical regimes and, in general, tackles the transport problem and its relation with the geometry of the interface. The interested reader is referred to one of the following self-contained works or reviews: (Agoritsas *et al.*, 2012; Chauve *et al.*, 2000; Ferrero *et al.*, 2013; Fisher, 1998; Kolton *et al.*, 2009).

2. Models

The most celebrated model to describe the depinning problem is the quenched Edwards-Wilkinson (QEW) equation. A $d = 1$ -dimensional interface without overhangs is driven by an external pulling force f . In the overdamped limit, its local displacement at time t , described by a single-valued function, $h(x, t)$, obeys

$$\eta \partial_t h(x, t) = c \nabla^2 h(x, t) + f + F_p(x, h) + \xi(x, t) \quad (34)$$

where $c \nabla^2 h(x, t)$ represents the elastic force due to the surface tension, the (quenched) disorder induced by impurities is encoded in the pinning force $F_p(x, h)$ and thermal fluctuations are included as a Langevin thermal noise $\xi(x, t)$. In general, two different kinds of disorder are considered: *random bond* disorder, in which the pinning potential is short-range correlated in the direction of motion ($\langle V(h, i)V(h', j) \rangle = \delta_{ij} \delta_{hh'}$), and *random-field* disorder, where the pinning force is short-range correlated (thus generating correlations of the potential in the direction of motion, $\langle V(h, i)V(h', j) \rangle = \delta_{ij} \min(h, h')$).

Of course, the QEW model just mentioned is minimal. Some of its variants take into account additional ingredients. For example, charge density waves and vortices involve a periodic elastic structure, in fracture and wetting the elastic interactions are long-ranged, and anharmonic corrections to elasticity or anisotropies could also be relevant. These features would call for a rewriting of Eq. 34 into a more general form involving an elastic interaction energy \mathcal{H}^{el}

$$\eta \partial_t h(x, t) = \frac{-\delta \mathcal{H}^{\text{el}}[h]}{\delta h(x)} + f + F_p(x, h) + \xi(x, t) \quad (35)$$

Remarkably, all these different problems, grouped in a few distinct universality classes, share the same basic physics, discussed in the following.

3. Phenomenology

The velocity-force characteristics $\langle \dot{h} \rangle = v(f)$ is well known for the depinning problem (see Fig.24a); the information conveyed by this “equation of state” is enriched by a vast analytical and numerical knowledge of universal

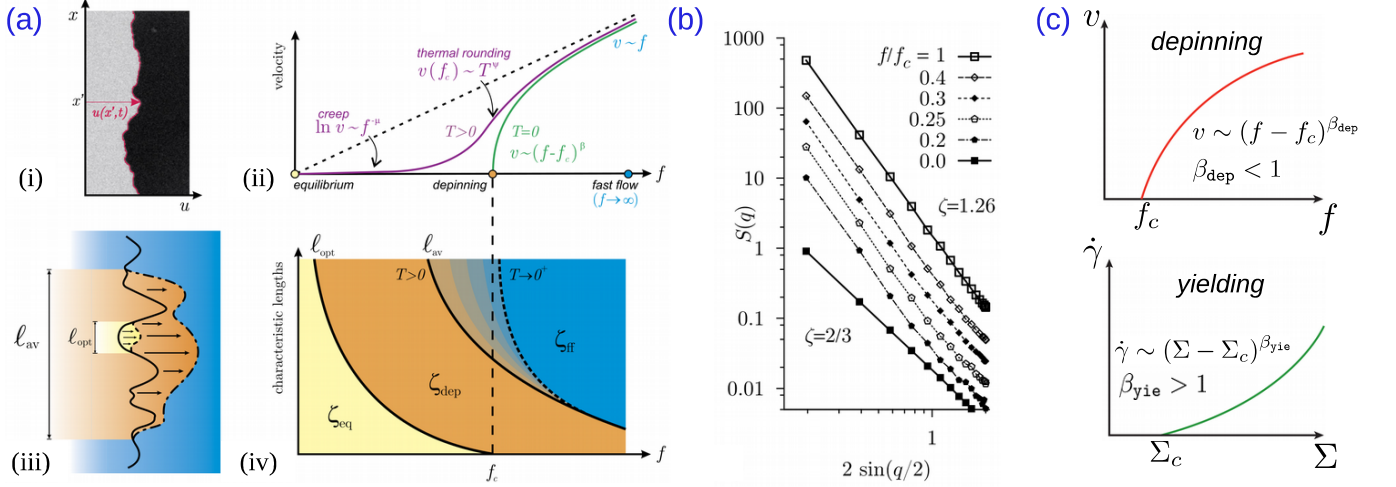


FIG. 24 *The depinning picture.* (a) Connection between transport and geometry in depinning. From (Ferrero *et al.*, 2013). (i) Snapshot of a domain wall in a 2D ferromagnet. (ii) Typical velocity-force characteristics. (iii) Crossover lengths ℓ_{opt} and ℓ_{av} representing the optimal excitation and the deterministic avalanches, respectively. (iv) Geometric crossover diagram. (b) Steady-state structure factor $S(q)$ of the line in the limit of vanishing temperature for different forces (curves are shifted for clarity). Adapted from (Kolton *et al.*, 2006). (c) Comparison of the depinning and yielding critical transitions in a correspondence ($v \leftrightarrow \dot{\gamma}$), ($f \leftrightarrow \Sigma$).

properties at three special points: (i) equilibrium, i.e., $f = 0$; (ii) depinning, i.e., $f = f_c$ at $T = 0$; and (iii) fast-flow $f \gg f_c$. Around these points, at vanishing temperature, the steady-state interface $h(x)$ displays a self-affine geometry (in the sense that it is invariant under dimensional rescaling, viz., $h(ax) \sim a^\zeta h(x)$) above a microscopic length scale, with characteristic roughness exponents: (i) ζ_{eq} , (ii) ζ_{dep} , and (iii) ζ_{ff} .

Turning to transport properties, at *equilibrium*, the mean velocity is zero and the dynamics is glassy. When the applied force approaches zero macroscopic movement can be observed only at finite temperatures and at very long times. Collective rearrangements on a scale of size ℓ_{opt} ($\ell_{opt} \rightarrow \infty$ as $f \rightarrow 0$) are needed to overcome barriers $E_b(\ell_{opt})$ growing as $E_b \sim \ell_{opt}^\theta$, with $\theta > 0$ a universal exponent related to the roughness by $\theta = d - 2 + 2\zeta_{eq}$. This is the *creep* regime. At the zero temperature *depinning transition* the velocity vanishes as $v(f, T = 0) \sim (f - f_c)^\beta$ for $f > f_c$ while $v = 0$ for $f < f_c$. Approaching f_c from above the motion is very jerky and involves collective rearrangements of a typical longitudinal size ℓ_{av} that diverges at f_c . The avalanche size S , defined as the area covered by the moving interface, has power-law statistics, viz.,

$$P(S) \sim S^{-\tau_{dep}}, \text{ with } \tau_{dep} = 2 - \frac{2}{d + \zeta_{dep}}. \quad (36)$$

At finite temperature, the sharp depinning transition is rounded, the velocity behaves as $v(f_c, T) \sim T^\psi$ and the size ℓ_{av} is finite at the transition. In the *fast-flow* regime $f \gg f_c$, the response is linear, viz., $v \sim f$. Here impurities generate an effective thermal noise on the interface. Therefore, the fast-flow roughness corresponds to the Edwards-Wilkinson roughness $\zeta_{ff} = (2 - d)/2$.

One of the remarkable lessons learned from this simple model is the possibility to relate transport and geometry. If the applied force f lies in between two of the above mentioned reference points, the interface geometry (in particular the roughness exponent, see Fig.24b) depends on the observation scale and its relative position compared to the characteristic lengths ($\ell_{opt}, \ell_{av}, \dots$). Granted that one knows the functional dependencies of these characteristic lengths with f and the velocity-force characteristics for a given system, transport properties (which intrinsically pertain to the dynamics) can be deduced from the static interface geometry, and vice-versa.

4. Similarities and differences with EPM

The manifest qualitative similarity between the yielding transition and the depinning one has enticed many researchers to look for a unification of these theories. The analogy has promoted the vision of yielding as a critical phenomenon and has given rise to interesting advances, but, in our opinion, the (spurious) belief in a strict equivalence of the problems has been deceptive in some regards.

To stay on firm ground, a formal approach consists in finding an EPM analog to the depinning equation, Eq. 35. In the stress-controlled situation (with applied stress Σ^{ext}), Weiss *et al.* (2014) (Eq. S3 of the Supplemental Information) and Tyukodi *et al.* (2016b) thus proposed to substitute the EPM equation of motion (Eq. 5) with

$$\eta \partial_t \epsilon^{\text{pl}}(r, t) = \mathcal{P} \left[\frac{-\delta U^{\text{el}}[\epsilon^{\text{pl}}]}{\delta \epsilon^{\text{pl}}(r)} + \Sigma^{\text{ext}} - F_p(r, \epsilon^{\text{pl}}) \right], \quad (37)$$

where $U^{\text{el}}[\epsilon^{\text{pl}}] \equiv -\frac{1}{2} \iint dr dr' \epsilon^{\text{pl}}(r) \mathcal{G}(r - r') \epsilon^{\text{pl}}(r')$, with \mathcal{G} the elastic propagator, and $\mathcal{P}(x)$ denotes the positive part of x (x if $x > 0$, 0 otherwise). In so doing, the deformation of an amorphous solid is mapped onto a problem of motion through an (abstract) disordered space for the ϵ^{pl} -manifold pulled by the ‘force’ Σ^{ext} . The positive part \mathcal{P} in Eq. 37 creates genuine threshold dynamics; it has no direct counterpart in the depinning equation but was argued by Tyukodi *et al.* (2016b) not to be a core dissimilarity between yielding and depinning.

This formal similarity between the two classes of phenomena seems to buttress the application of results from the depinning problem (hence mean field, owing to the long range of the elastic propagator) to the question of, e.g., avalanche statistics in disordered solids (see Sec. VII). However, the following differences must be borne in mind.

First, and perhaps foremost, as often mentioned along the present review, the interaction kernel in depinning problems is positive, whereas the quadrupolar elastic propagator \mathcal{G} used in EPM has positive and negative bits. This has profound consequences on the critical behavior at the yielding transition observed in EPM, in particular with respect to the possibility of strain localization and the avalanche statistics. Furthermore, while in depinning v vanishes at f_c as $v \sim (f - f_c)^\beta$ with $\beta < 1$, the strain rate $\dot{\gamma}$ does so at the yielding transition as $\dot{\gamma} \sim (\Sigma - \Sigma_c)^\beta$ with $\beta > 1$, as schematically shown in Fig.24c. Note that, if the systems were at equilibrium, this difference in the value of β would imply a change in the order of the continuous phase transition. Other consequences can be deduced from the general scaling relations proposed by Lin *et al.* (2014b) [Supplementary Information], which are claimed to encompass the depinning and the yielding cases:

$$\beta = \nu (d - d_f + z) \quad (38)$$

$$\nu = \frac{1}{d - d_f + \alpha_k} \quad (39)$$

$$\tau = 2 - \frac{d_f - d + 1/\nu}{d_f} - \frac{\theta}{\theta + 1} \frac{d}{d_f} \quad (40)$$

Here d_f is the fractal dimension of the avalanches, z is the dynamical exponent, ν is the exponent controlling the divergence of the correlating length at the transition and α_k is the dimension of the elastic interaction kernel. In EPM $\alpha_k = 0$ and $d_f < d$ so that $\beta > 1$. In depinning, $\alpha_k = 2$ for short-ranged elasticity and $\alpha_k = 1$ for long-ranged elasticity, $\theta = 0$ and $d_f \geq d$.

Secondly comes the question of the nature of the disorder in the pinning force F_p . In elastic depinning models, regardless of how realistic the chosen correlations of F_p are, the origin of the disorder is generally extrinsic. More precisely, it reflects the disorder of the substrate on which the elastic manifold advances, hence $F_p = F_p(h)$. On the other hand, in the yielding phenomenon, as stated by Papanikolaou (2016), ‘the pinning disorder for every particle originates in the actual interface that attempts to depin (other nearby particles); a disordered solid pins itself during deformation’. Therefore, it is inaccurate to consider that F_p only depends on the local value of ϵ_{pl} . In particular, a *given* system will *not* encounter the same pinning forces F_p along its deformation between, say, $\epsilon_{pl} = 0$ and $\epsilon_{pl} = 1$ if it is sheared slowly and if it is sheared fast. Typically, at high shear rates, the potential energies of the inherent structures of the material are higher, and the pinning forces (as evidenced by the variations of potential energies of the inherent structures with the shear rate, in atomistic simulations). This dependence should impact the $\dot{\gamma} = f(\Sigma)$ curve.

Last, the EPM equation of motion (Eq. 5) cannot always be reduced to an expression akin to Eq. 37, because of the memory effects contained in the plastic activity variable n .

Let us now mention a subclass of problems that may be more closely related to EPM: the so called ‘‘plastic depinning’’. This phenomenon is observed for example in particle assemblies driven over random substrates whenever irreversible plastic deformations actually occur, or in charge density wave problems. Unfortunately, this comparison has been much less exploited by the amorphous solids community, even though the connection was very recently pointed out in Reichhardt and Reichhardt (2016)’s review.

To conclude on the topic, there undoubtedly remains much to be learned from the more than 30 years of studies on depinning phenomena. Some intriguing open questions left from this comparison are the following: Are the transport properties of driven amorphous solids related to geometrical properties, as they are in elastic manifolds? Is it possible, for example, to infer from a picture at which strain-rate a dense emulsion is being sheared?

C. Fiber bundle, fuse networks and continuum models for the study of cracks and fracture

1. Brief introduction to cracks and fracture

In partial overlap with the scope of EPM, the question of the failure of hard solids under loading, e.g. in tension, has attracted much attention over the last centuries. Pioneering in this respect, as recalled by Alava *et al.* (2006), is Leonardo da Vinci's observation that metal wires of equal cross sections loaded in tension by a weight fail more readily if they are longer, which runs counter to basic continuum mechanics predictions for a uniform medium. In fact, the failure of brittle solids, in particular rocks, is ascribed to the growth and propagation of pre-existing cracks (at the scale of the crystalline grains constituting the material) or, more generally, defects.

If one considers an individual crack in a homogeneous medium, according to Griffith (1921)'s criterion, its growth hinges on a competition between a surface energy term averse to the opening of solid-air interfaces and an elastic energy term favoring its growth and thereby reducing the elastic energy stored in the bulk. For example, for a single elongated elliptic of length a in a 2D medium, the sum of these competing terms reads

$$E_T = \underbrace{\frac{-\pi\Sigma^2 a^2}{2E}}_{\text{elastic energy}} + \underbrace{2\gamma a}_{\text{surface energy}},$$

where E is the Young modulus of the material, γ is the interfacial energy, and Σ is the applied stress. Thus, the evolution of the crack depends on the sign of the derivative $\frac{dE_T}{da}$ (Alava *et al.*, 2006). However, cracks very seldom have so simple a geometric shape. Roughly speaking, owing to the presence of heterogeneities, the crack will zigzag around hard spots. This will result in undulations and protrusions in the *post-mortem* fracture surface, which exhibits a self-similar (fractal) pattern: If the surface height at a point (x, z) is denoted by $h(x, z)$, the root mean square $w(l)$ of the height in a region of size $\Delta x \approx \Delta z \approx l$ obeys

$$w(l) \equiv \sqrt{\langle h(x, z)^2 \rangle - \langle h(x, z) \rangle^2} \sim l^{\zeta_{\perp}},$$

where ζ_{\perp} is the (out-of-plane) Hurst exponent, or roughness exponent. Interestingly, this exponent seems to be weakly sensitive to the material or the loading, with values centered around $\zeta_{\perp} \simeq 0.8$ and early claims of universality (Bouchaud *et al.*, 1990). The fractal dimension d_f of the surface is then related to ζ_{\perp} via $d_f = 3 - \zeta_{\perp}$ for 3D fracture. While the material is being fractured, the crack propagates along a rough, scale-invariant frontline (see Fig. 25a), characterized by the in-plane roughness exponent ζ_{\parallel} . Roughness bears practical importance, since it modifies the scaling of the surface energy term.

Let us mention two subtleties. First, the exponents ζ_{\parallel} and ζ_{\perp} are not independent (Ertaş and Kardar, 1994). Second, ζ_{\perp} might in fact mix two distinct exponents, insofar as Ponson *et al.* (2006)'s fracture experiments on silica and aluminium alloys hint at anisotropic height variations in the fracture plane, with distinct behaviours along the front line and along the crack propagation direction.

In addition to being spatially nontrivial, the propagation of the crack front also displays marked variations in time. The associated dynamics is highly intermittent and involves avalanches of events which span a broad range of energies. Indeed, the crackling noise emitted during these events has a power-law power spectrum, for instance in composite materials (Garcimartin *et al.*, 1997). For instance, the crack produced when tearing apart two sandblasted Plexiglas sheets stuck together through annealing undergoes a stick-slip motion at small scales that is reminiscent of dry solid friction (Måløy and Schmittbuhl, 2001), which in turn may tell us about earthquake dynamics (Svetlizky and Fineberg, 2014).

At this stage, a discrepancy with respect to soft solids ought to be mentioned: In (rock) fracture, the microruptures very generally do not have time to heal on the time scale of the deformation; without recovery process, the material is thus permanently damaged. However, the crack velocity may still have an influence on the dynamics of the process owing to the finite duration of the avalanches.

2. Fiber bundles

Arguably, the simplest way to model fracture is to consider two blocks bound by N aligned fibers. These fibers share the global load and break irreversibly when their elongation x exceeds a randomly distributed threshold; this is the basis of fiber-bundle models (Herrmann and Roux, 2014). In *democratic* fiber bundles, the load of broken fibers is redistributed equally to all survivors. Analytical progress is possible in this intrinsically mean-field model.

In particular, it is easy to show that, *on average*, when the bundle is stretched by x (with $x = 0$ the reference configuration), a fraction $C(x)$ of fibers have broken, where $C(x)$ is the cumulative distribution of thresholds, and the total load (normalized by the initial number of fibers, of stiffness κ each) reads $\bar{f}(x) = \kappa x [1 - C(x)]$. It follows that the maximum strength per fiber of the bundle is, on average,

$$f_c = \max_x \kappa x [1 - C(x)].$$

If one pulls on a *given* bundle, however, the load f will not evolve along the smooth *average* profile $\bar{f}(x)$, but along a rugged profile $\{f(x_k) \equiv \kappa x_k [1 - C(x_k)], k = 1 \dots N\}$ due to the randomness of the thresholds $x_1 \leq x_2 \leq \dots \leq x_N$, sorted according to the order of failure. The $f(x_k)$ thus perform a random walk in “time” k with a time-dependent bias $\langle f(x_{k+1}) - f(x_k) \rangle$ (Sornette, 1992). If, starting from a stable situation, the rupture of the k -th bond leads to S additional failures, viz.,

$$f(x_{k+i}) < f(x_k) \text{ for } i \text{ between } 1 \text{ and } S \text{ (but not for } i = S + 1), \quad (41)$$

an avalanche of size S will occur under fixed load. Noting that (i) this is a problem of first return for the walker $f(x_k)$ [or, equivalently, of survival close to the absorbing boundary $f = f(x_k)^-$], and that (ii) close to global failure $f \approx f_c$ the random walk is unbiased, i.e., $\langle f(x_{k+1}) - f(x_k) \rangle = 0$, Sornette (1992) showed that the distribution of avalanche sizes s obeys

$$p(S) \sim S^{-\tau}, \text{ where } \tau = 3/2. \quad (42)$$

More precisely, for a uniform distribution of thresholds between x_0 and x_m , the distribution reads

$$p(S) \sim S^{-5/2} \left(1 - e^{-\frac{S}{S_{\text{cut}}}} \right),$$

where the cutoff size $S_{\text{cut}} \equiv \frac{x_m^2}{2(x_m - 2x_0)^2}$ diverges at the critical point $x = x_m/2$ (Pradhan *et al.*, 2005). The power law with exponent $\tau = 3/2$ of Eq. 42 is recovered for $S \ll S_{\text{cut}}$, whereas for $S \gg S_{\text{cut}}$ the random walk of the $f(x_k)$ is biased upward and a steeper power law is obtained, with an exponent $5/2$. The scaling $p(S) \sim S^{-5/2}$ is found generically if all avalanches since the start of the deformation ($x = 0$) are taken into account (Hemmer and Hansen, 1992). The gradual shift to an exponent $\tau = 3/2$ then signals imminent failure. Interestingly, the power-law behavior fades out in favor of a much faster decay of $p(s)$ if the load released by broken fibers is redistributed *locally* to the first neighbors only, instead of being shared by all intact fibers (Kloster *et al.*, 1997).

3. Fuse networks

Unfortunately, the picture promoted by mean-field or 1D fiber bundles is incapable of describing the heterogeneous and anisotropic propagation of cracks. Extending the approach to higher dimensions, fuse networks connect lattice nodes (say, nodes i and j) by fuses of conductance K_{ij} that break past a threshold $x \in [0, 1]$, thereby burning the fuse ($K_{ij} \rightarrow 0$). To take an example, the distribution p of the thresholds can be set as a power law, $p(x) \sim x^\theta$ with $\theta > 0$. The voltages V_i are imposed at two opposite edges of the system, as depicted in Fig. 25c. The Hamiltonian of the system reads

$$\mathcal{H}_{nc} = \frac{1}{2} \sum_{\langle i,j \rangle} K_{ij} (V_i - V_j)^2, \quad (43)$$

where the sum runs over neighboring sites. Note that, if the distribution p is uniform, i.e., $\theta = 0$, the model can then be viewed as a discretization of Poisson’s equation in the vacuum, $\nabla^2 V = 0$. Fuse networks are thus closer to EPM than fiber bundles, insofar as the stress redistribution when one fuse burns (in the pristine network) is strongly anisotropic, with a shielding of the current fore and aft and an enhancement sideways (Barthelemy *et al.*, 2002; Rathore, 2016). It can then be understood that failure occurs along a line of burnt fuses, the “crack” line, provided that there is finite disorder ($\theta > 0$) and the network is large (Shekhawat *et al.*, 2013). Besides, in a 2D fuse network, Hansen *et al.* (1991) computed a roughness exponent ζ approximately equal to 0.7 for weak disorder, not far from experimental values for fractured surfaces $\zeta_\perp \approx 0.8$ (note that $\zeta = \zeta_\perp$ in 2D).

The expression of the Hamiltonian in Eq. 44 evokes a random bond Ising model; the equivalence is formally exact if the voltages are restricted to the values ± 1 , and the thresholds are infinite, thus making bonds unbreakable (*perfect*

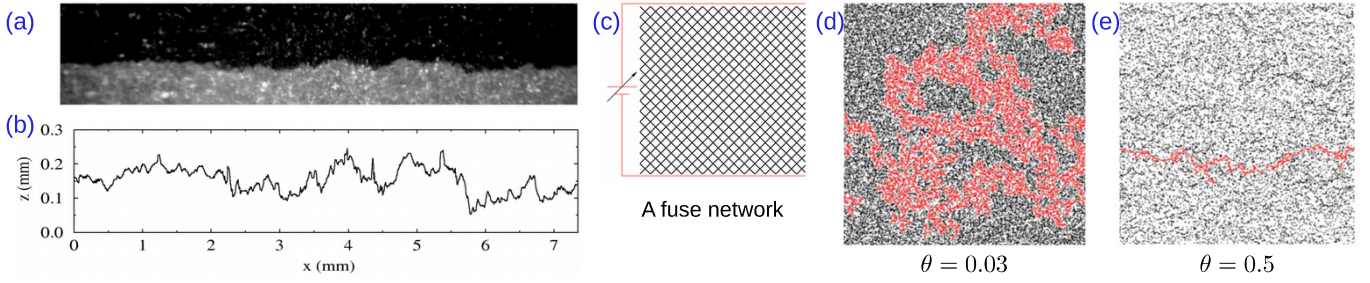


FIG. 25 Observation and modeling of crack propagation. (a) Raw image of the front of an in-plane crack propagating between Plexiglas plates. The intact region appears in black and the image-processed front line is shown in (b). The roughness along the propagation (z) direction has a power-law spectrum characterized by the roughness exponent ζ_{\parallel} . From (Schmittbuhl and Måløy, 1997). (c) Sketch of the random fuse network and (d-e) failure process for distinct probability density functions for the thresholds, $p(x) \sim x^{\theta}$. The crack has been colored in red. From (Shekhawat *et al.*, 2013).

ductility). These differences are not negligible in any way. Indeed, the interactions between nodes are thereby much reduced, in spatial extent and magnitude; by contrast, in random fiber or fuse models, the impact of breaking a bond is magnified close to failure, owing to the small number of intact bonds which will share the load. Nevertheless, the process of fracture can be mimicked in the random Ising models by imposing spin +1 (-1) on the left (right) edges of the sample and monitoring the interface line between the +1 and -1 domains. Rosti *et al.* (2001) studied the probability that this interface passes through an artificial “notch”, i.e., a segment in which the bond strengths K_{ij} have been set to zero, and observed a transition from low to high probabilities as the notch length was increased above a disorder-dependent threshold value. Similar results were obtained in experiments in which sheets of papers with pre-cut notches were torn.

4. Spring models

From a mechanical perspective, should one replace the voltage V_i in Eq. 43 with the displacement \mathbf{u}_i at node i , *viz.*,

$$\mathcal{H}'_{nc} = \frac{1}{2} \sum_{\langle i,j \rangle} K_{ij} (\mathbf{u}_i - \mathbf{u}_j)^2, \quad (44)$$

the interpretation of the Hamiltonian as the energy of a network of random springs of stiffness K_{ij} will become apparent. The x , y , and z components of the displacements in \mathcal{H}'_{nc} decouple, so that model is actually scalar (De Gennes, 1976). However, it features noncentral forces: the force exerted by j on i is not aligned with \mathbf{e}_{ij} . A more consistent description of a network of nodes connected by harmonic springs relies (to leading order) on the Hamiltonian

$$\mathcal{H}_c = \frac{1}{2} \sum_{\langle i,j \rangle} K_{ij} [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{e}_{ij}]^2. \quad (45)$$

On a triangular lattice, with bonds of uniform strength $K_{ij} = 1$, the continuum limit of this Hamiltonian represents an isotropic elastic medium with a Poisson ratio of $1/3$ in 2D and $1/4$ in 3D (Monette and Anderson, 1994). As bonds are gradually removed in a random fashion, the initially rigid system transitions to a non-solid state with vanishing elastic moduli at a critical bond fraction p_c . Such a transition is also observed with the models based on the scalar Hamiltonian \mathcal{H}_{nc} or the noncentral Hamiltonian \mathcal{H}'_{nc} , although at a distinct fraction p_c . Somewhat surprisingly, the scalings of the shear and bulk moduli with the fraction of bonds p around p_c differ between the \mathcal{H}_c and \mathcal{H}_{nc} -based models; the discrepancy stems from the distinct symmetries, in the same way as the Heisenberg model differs from the Ising model (Feng and Sen, 1984). The distinction subtly differs from the scalar *vs.* tensorial dichotomy in EPM, in that the EPM propagator is always derived from the same constitutive model (tensorial continuum elasticity); the scalar description simply discards some tensor components at the end of the day.

Regarding the avalanches of ruptures close to the point of global failure, i.e., under loading $f \approx f_c$, Zapperi *et al.* (1999) claimed that both the random fuse network of Eq. 43 and the central-force spring model of Eq. 45 (supplemented with bond-bending forces) fall in the universality class of spinodal nucleation, in that the avalanche sizes S are distributed according to

$$p(S) \sim S^{-\tau} \Phi [s(f_c - f)], \quad \text{where } \tau = 3/2$$

and Φ is a scaling function. Zaiser *et al.* (2015) also found that fuse networks yielded results similar to spring models regarding the initiation of failure, with localized correlations in the damage patterns.

We conclude this section on spring models with a historical note referring to the fact that such models had in fact been pioneered by De Gennes (1976) to tackle the “converse problem”, namely, gel formation (e.g., through cross-linking): Instead of gradually destroying bonds, he cranked up the fraction p of bonds by randomly connecting pairs of neighbours until bonds percolated throughout the system; this occurred at a critical fraction p_c , supposedly corresponding to gel formation. In any event, the nature of the transition associated with the random depletion (or creation) of bonds, which pertains to percolation, is distinct from what is observed in *random* fuse or spring networks. In the latter models, the disorder in the yield thresholds bestows critical importance to the spatial redistribution of stresses following ruptures. This distinction is at the origin of different scaling relations, e.g., between the failure force and the system size (Hansen *et al.*, 1989).

5. Beyond random spring models

Refinements have been suggested to bring random fuse (or spring) networks closer to models of material deformation and fracture. First, the irreversible breakage of the fuses past a threshold mirrors perfectly brittle fracture. At the opposite end, perfect plasticity is mirrored by the saturation of the fuse intensity past a threshold. But a continuum of possibilities can be explored between these extreme cases, whereby the conductivity of the fuse is decreased to mimic partial weakening, similarly to what can be done in EPM.

Another limitation of the models stems directly from the description of the bonds on a regular lattice: let alone the presence of soft modes in several cases, the (\mathcal{H}_c -based) central-force model, discretized on a triangular lattice, displays an anisotropic tensile failure surface (despite an isotropic linear response), with an anisotropy ratio of 50% (Monette and Anderson, 1994). These deficiencies can be remedied in part by complementing the spring-stretching energies in \mathcal{H}_c with bond-bending energies. This refinement leads to an isotropic elastic medium with adjustable Poisson coefficient and a more isotropic failure surface.

As with EPM, the following step in the endeavor to refine the description led to the introduction of a finite-element approach, which relies on a continuum description down to the scale of one mesh element. The equations of inhomogeneous elasticity are solved and a damage (of magnitude D) is introduced by reducing the local elastic constant $E \rightarrow (1 - D)E$ whenever the local stress exceeds a threshold value. The process can evolve into avalanches, and eventually to a vanishment of the elastic resistance through the propagation of a fracture through the system. Incidentally, this mechanism had first been implemented by Zapperi *et al.* (1997) using a fuse network with damage operating on the fuse resistances; the model displayed scale-free behavior with power-law distribution of event sizes, $P(S) \sim S^{-\tau}$ with $\tau \simeq 1.2$.

Amitrano *et al.* (1999) refined the modeling approach by using a pressure-modified (Mohr-Coulomb) criterion for the onset of plasticity, viz.,

$$C + \sigma_n \tan \phi - \sigma < 0,$$

where C represents the cohesion of the material, σ_n and σ are the normal and shear stresses, respectively. A transition from brittle failure with very localized damage (at low internal friction angle ϕ , i.e., little sensitivity of the yield criterion to pressure) to ductile with diffuse damage (at large ϕ) was observed. At low ϕ the damage around a single event is similar to the stress redistribution considered in EPM, while for large ϕ it becomes much more directional. The transition from ductile to brittle shares qualitatively similarities with the strain localization transition, but the control parameter is different from those discussed in Sec. V.C.

In the case of large ϕ and brittle failure, a description of compressive failure under uniaxial stress as a critical phenomenon analogous to depinning was proposed by Girard *et al.* (2010) and elaborated by Weiss *et al.* (2014). The interpretation in terms of a criticality notably affords a detailed description of size effects on the critical stress (Girard *et al.*, 2012).

X. OUTLOOK

In the last ten years, EPM have become an essential theoretical tool to understand the flow of solids. Starting from elementary models intended to reproduce earthquake dynamics, they have blossomed into more refined approaches that have helped rationalize many experimentally observed features, at least at a qualitative level, and unveil new facets of the rheology of these materials.

Future developments in the field can be expected in a number of directions, following current experimental and theoretical interests. In rheology, considerable attention has recently been devoted to the study of transient regimes, for instance creep under an imposed stress, and oscillatory regimes. In the latter category, the Large Amplitude Oscillatory Strain (LAOS) protocol probes the nonlinear behavior and the frequency dependent one at the same time, and therefore involves a complex interplay between plastic deformation and internal relaxation. Reproducing the complex response of particular systems under such protocols is particularly challenging for simple models. Several issues could be investigated within the framework of EPM, such as the onset of tracer diffusion as the amplitude of the oscillatory strain is increased, or the fatigue behavior leading to failure. Recently, it was suggested that the LAOS protocol could induce strain localization in systems with a monotonic flow curve, based on a study of a spatially resolved version of the soft glassy rheology model, presented in Sec. IV.D.1 (Radhakrishnan and Fielding, 2016, 2017). Creep (see Sec. VIII) is an equally challenging phenomenon, and recently a mean field version of EPM was used to illustrate the very strong sensitivity of this phenomenon to initial conditions (Liu *et al.*, 2017). A more unexpected emerging avenue is the study of systems with internal activity, such as living tissues or dense cell assemblies. The general ideas exploited in the description of amorphous systems can indeed be extended by incorporating different types of events and units, e.g., cell division (local anisotropic dilation) and cell death (local isotropic contraction). A new analysis in that direction was conducted by Matoz-Fernandez *et al.* (2017) based on a mean field description. At present, new experimental tools are providing information on the statistical fluctuations in such systems, which will allow to calibrate these models.

From the viewpoint of statistical physics, the yielding transition described by EPM stands as a new type of dynamical phase transition, with specificities that are still to be understood in detail. Considerable efforts have been devoted to the theoretical study of the related problem of the depinning transition (Sec. IX.B). In the latter case, (mostly) exact exponents, scaling functions, and avalanche shapes were derived using scaling analysis and renormalization techniques. For the yielding transition, the (slow) process of consensus building has not converged yet, but there are reasons to believe that the results on avalanche statistics obtained in the depinning problem cannot be directly transposed to this field, because the propagator controlling stress redistribution is partly negative, which affects the density of sites close to yielding. Nevertheless, scaling relations between critical exponents have been proposed (Lin *et al.*, 2014b) and tested in diverse models (which was possible thanks to the variety of EPM in the literature), but no analytical calculation beyond mean field is available so far. The discussion is even more open regarding flow curve exponents. The situation is somewhat similar on the experimental side: The depinning phenomenon has benefited from a very detailed experimental characterization in various systems (magnetic domain walls, contact lines, vortices), including avalanche statistics and shapes, which has permitted comparison to the theory. Amorphous plasticity is not on quite so good a footing, with only a few attempts to characterize the distribution of stress drops in deformed systems. The situation is however improving, thanks to several recent efforts, e.g. those combining mechanical deformation and confocal microscopy in colloidal glasses.

The foregoing discussion is related to the critical aspects of the yield phenomenon, discussed in Sec. VI and VII. In a number of real systems (Bonn *et al.*, 2017), the transition to flow is in fact discontinuous and implies a coexistence between flowing and immobile states. EPM and other theoretical studies have suggested possible mechanisms that may influence the continuous or discontinuous character of the transition (see Sec. V). Nevertheless, it turns out to be experimentally difficult to control the transition in a systematic way by changing some experimental parameter. Wortel *et al.* (2016)'s work on weakly vibrated granular media represents a notable exception, insofar as the intensity of external shaking could be used to continuously tweak the flow curve towards nonmonotonicity. Similar systems of vibrated grains have also permitted the experimental realization of a Gardner transition (Seguin and Dauchot, 2016), a transition which may be important for the theory of glasses and which has recently been associated with shear yielding (Urbani and Zamponi, 2017).

These prospective lines of research have hardly been explored using EPM. So, for all our efforts to articulate a comprehensive view of the state of the art here, we can only wish that this review will soon need to be updated with insightful results in these new avenues.

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