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A Mathematical Framework for Resilience: Dynamics, Strategies, Shocks and Acceptable Paths

Michel De Lara*

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Abstract

Resilience is a rehashed concept in natural hazard management — resilience of cities to earthquakes, to floods, to fire, etc. In a word, a system is said to be resilient if there exists a strategy that can drive the system state back to “normal” (acceptable states) after a shock. What formal flesh can we put on such malleable notion? We propose to frame the concept of resilience in the mathematical garbs of control theory under uncertainty. Our setting covers dynamical systems both in discrete or continuous time, deterministic or subject to uncertainties. Our definition of resilience extends others, be they “à la Holling” or rooted in viability theory. Indeed, we require that, after a shock, the system returns to an acceptable “regime”, that is, that the state-control path as a whole must return to a set of acceptable paths (and not only the state values must belong to an acceptable subset of the state set). More generally, as state and control paths are contingent on uncertainties, we require that their tails processes must lay within acceptable domains of stochastic processes. We end by pointing out how such domains can be delineated thanks to so called risk measures.

Keywords: resilience; control theory; uncertainty; risk measures

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1 Introduction

Consider a system whose state evolves with time, being subject to a dynamics driven both by controls and by external perturbations. Suddenly, those perturbations jump out of their normal range and induce a shock. The system is said to be resilient if there exists a strategy that can drive the system state back to its normal regime. Basic references are [8, 10, 9, 1].

In Sect. 2, we introduce basic ingredients from the mathematical framework of control theory under uncertainty. Thus equipped, we frame the concept of resilience in mathematical garbs in Sect. 3. Then, in Sect. 4, we provide illustrations of the abstract general framework.

2 Ingredients for an abstract control system with uncertainties

We outline the mathematical formulations of time, uncertainties, controls, states, flow (dynamics) and strategies.

2.1 Time

The *time set* \mathbb{T} is a (nonempty) subset of the real line \mathbb{R} . The set \mathbb{T} holds a minimal element $t_0 \in \mathbb{T}$ and an upper bound T , which is either a maximal element when $T < +\infty$ (that is, $T \in \mathbb{T}$) or not when $T = +\infty$ (that is, $+\infty \notin \mathbb{T}$).

For any couple $(s, t) \in (\mathbb{R} \cup \{+\infty\})^2$, we use the following notation for the *segment* that joins s to t :

$$s : t = \{r \in \mathbb{T} \mid s \leq r \leq t\} . \quad (1)$$

When $s > t$, $s : t = \emptyset$.

Special cases of discrete and continuous time. This setting includes the discrete time case when \mathbb{T} is a discrete set, be it infinite like $\mathbb{T} = \{t_0 + k\Delta t, \quad k \in \mathbb{N}\}$, or finite like $\mathbb{T} = \{t_0 + k\Delta t, \quad k = 0, 1, \dots, K\}$ (with $K \in \mathbb{N}$). Of course, in the continuous time case, \mathbb{T} is an interval of \mathbb{R} , like $[t_0, T]$ when $T < +\infty$ or $[t_0, +\infty[$ when $T = +\infty$. But the setting makes it possible to consider interval of continuous times separated by discrete times corresponding to jumps.

2.2 Uncertainties, controls, states and flow (dynamics)

At each time $t \in \mathbb{T}$,

- the system under consideration can be described by an element x_t of the *state set* \mathbb{X} ,
- the decision-maker (DM) makes a decision u_t , taken within a *control set* \mathbb{U} ,
- an uncertainty w_t affects the system, drawn within an *uncertainty set* \mathbb{W} .

With these three basic sets, we define

- the *set* $\mathbb{U}^{\mathbb{T}}$ of *control paths*, made of sequences from time \mathbb{T} to \mathbb{U} ; *tail control paths* (starting at time t) are elements of $\mathbb{U}^{t:T}$;
- the *set* $\mathbb{W}^{\mathbb{T}}$ of *scenarios*, made of sequences from time \mathbb{T} to \mathbb{W} ; *tail scenarios* (starting at time t) are elements of $\mathbb{W}^{t:T}$;

We now introduce a dynamics under the form of a *flow* $\{\phi_{s:t}\}_{(s,t) \in \mathbb{T}^2}$, that is, a family of mappings

$$\phi_{s:t} : \mathbb{X} \times \mathbb{U}^{s:t} \times \mathbb{W}^{s:t} \rightarrow \mathbb{X}^{s:t} . \quad (2)$$

When $s > t$, all these expressions are void because $s : t = \emptyset$.

The flow maps an initial state x , a tail control path $\{u_s\}_{s \in t:T}$ and a tail scenario $\{w_s\}_{s \in t:T}$ towards a state path $\{x_s\}_{s \in t:T}$:

$$\{x_s\}_{s \in t:T} = \phi_{t:T}(x, \{u_s\}_{s \in t:T}, \{w_s\}_{s \in t:T}) . \quad (3)$$

In general, a state path cannot be determined for *any* scenario or for *any* control path, for analytical reasons (measurability, continuity) or because of bounds on the controls. To circumvent this difficulty, one can add to the state set \mathbb{X} a cemetery point ∂ . The stationary state path with value ∂ will be the image of those scenarios and control paths for which no state path can be determined.

Special case of an iterated dynamics in discrete time. In discrete time, when $\mathbb{T} = \mathbb{N}$, the flow is generally produced by the iterations of a dynamic

$$x_t = x, \quad x_{s+1} = F_s(x_s, u_s, w_s), \quad t \geq s. \quad (4)$$

How do we include control constraints in this setting? Suppose given a family of set-valued mappings $\mathcal{U}_s : \mathbb{X} \rightrightarrows \mathbb{U}$, $s \in \mathbb{T}$. Let us add to the state set \mathbb{X} a cemetery point ∂ . Then, when $u_t \notin \mathcal{U}_t(x)$, we set $\phi_{t:T}(x, \{u_s\}_{s \in t:T}, \{w_s\}_{s \in t:T}) = \{\partial\}_{s \in t:T}$.

Special case of differential systems. In continuous time, the mapping $\phi_{s:t}$ generally cannot be defined over the whole set $\mathbb{U}^{s:t} \times \mathbb{W}^{s:t}$. Tail control paths and scenarios need to be restricted to subsets of $\mathbb{U}^{s:t}$ and $\mathbb{W}^{s:t}$, like the continuous ones.

For instance, when $\mathbb{T} = \mathbb{R}_+$ and the flow is produced by a smooth dynamical system on a Euclidian space $\mathbb{X} = \mathbb{R}^n$

$$x_t = x, \quad \dot{x}_s = f_s(x_s, u_s), \quad s \geq t, \quad (5)$$

control paths $\{u_s\}_{s \in t:T}$ are generally restricted to piecewise continuous ones for a solution to exist.

Special case of stochastic differential equations. Under certain technical assumptions, a stochastic differential equation

$$d\mathbf{X}_s = f_s(\mathbf{X}_s, \mathbf{U}_s, \mathbf{W}_s)ds + g(\mathbf{X}_s, \mathbf{U}_s, \mathbf{W}_s)d\mathbf{W}_s, \quad (6)$$

where $\{\mathbf{W}_s\}_{s \in \mathbb{R}_+}$ is a Brownian motion, gives rise to solutions in the strong sense. In that case, a flow can be defined.

2.3 Adapted and admissible strategies

A control is an element of the control set \mathbb{U} . A strategy is a mapping with image in the control set \mathbb{U} .

2.3.1 Adapted strategies

An *adapted strategy* is a family $\{\lambda_t\}_{t \in t_0:T}$ of mappings where

$$\lambda_t : \mathbb{X} \times \mathbb{W}^{t_0:t} \rightarrow \mathbb{U}. \quad (7)$$

With this definition, we express that a DM can, at time t , use no more than time t , current state value x and past scenario $\{w_s\}_{s \in t_0:t}$ to make his decision $u_t = \lambda_t(x, \{w_s\}_{s \in t_0:t})$.

A *tail adapted strategy* (starting at time t) is a family $\{\lambda_s\}_{s \in t:T}$ of mappings where

$$\lambda_s : \mathbb{X} \times \mathbb{W}^{s:T} \rightarrow \mathbb{U}, \quad \forall s \in t : T. \quad (8)$$

2.3.2 Admissible strategies

Suppose given a family of set-valued mappings $\mathcal{U}_t : \mathbb{X} \times \mathbb{W}^{t_0:t} \rightrightarrows \mathbb{U}$, $t \in \mathbb{T}$. An *admissible strategy* is an adapted strategy $\{\lambda_t\}_{t \in t_0:T}$ such that control constraints are satisfied in the sense that, for all $t \in \mathbb{T}$,

$$\lambda_t(x, \{w_s\}_{s \in t_0:t}) \in \mathcal{U}_t(x, \{w_s\}_{s \in t_0:t}), \quad \forall (x, \{w_s\}_{s \in t_0:t}) \in \mathbb{X} \times \mathbb{W}^{t_0:t}. \quad (9)$$

From now on, when we say “strategy”, we mean “adapted and admissible strategy”. We will restrict the strategies to certain subsets Λ of the set of all strategies, and the same for tail strategies in Λ_t . With this, we are in measure to describe different regimes of information [3].

Special case of Markovian or state feedback strategies. Markovian or state feedback strategies are of the form

$$\lambda_t : \mathbb{X} \rightarrow \mathbb{U}. \quad (10)$$

With this definition, we express that, at time t , the DM can only use time t and current state value x — but not the past scenario $\{w_s\}_{s \in t_0:t}$ — to make his decision $u_t = \lambda_t(x)$. In some cases (when dynamic programming applies for instance), it is enough to restrict to Markovian strategies, much more economical than general strategies.

Special case of partially observed state feedback strategies. Partially observed state feedback strategies form strict subsets of state feedback strategies, depending on which part of the state is observed.

2.4 Closed loop flow

Starting from an initial state, a tail strategy will induce a state path, along any tail scenario, thanks to the flow: this gives the closed loop flow.

Let Λ be a subset of the set of all strategies, together with a family $\{\Lambda_s\}_{s \in \mathbb{T}}$ of tail strategies subsets. Let $t \in \mathbb{T}$. We suppose that, to any initial state x , tail strategy $\lambda = \{\lambda_s\}_{s \in t:T} \in \Lambda_t$ and tail scenario $\{w_s\}_{s \in t:T}$, we can attach a *unique* tail state-control path $(\{x_s\}_{s \in t:T}, \{u_s\}_{s \in t:T})$ solution of the closed loop equations

$$\{x_s\}_{s \in t:T} = \phi_{t:T}(x, \{u_s\}_{s \in t:T}, \{w_s\}_{s \in t:T}) \quad (11a)$$

$$u_s = \lambda_s(x_s, \{w_r\}_{r \in t:s}), \quad \forall s \in t:T. \quad (11b)$$

For flows given by dynamical systems in continuous or discrete time, uniqueness is easily obtained. Quite naturally, we define the *closed loop flow* $\phi_{t:T}^\lambda$ by

$$\phi_{t:T}^\lambda(x, \{w_s\}_{s \in t:T}) = (\{x_s\}_{s \in t:T}, \{u_s\}_{s \in t:T}). \quad (12)$$

3 Resilience: a mathematical framework

Equipped with the material in Sect. 2, we now frame the concept of resilience in mathematical garbs. For this purpose, we introduce the notions of *tail shock scenarios* and of *acceptable tail state-control paths*. Compared to other definitions of resilience [8, 10, 9], our definition requires that, after a shock, the state-control path as a whole must return to a set of acceptable paths (and not only the state values must belong to an acceptable subset of the state set, asymptotically or not).

3.1 Randomness

To speak of shocks, we follow the framework of stochastic processes. From now on, the state set \mathbb{X} is a measurable set equipped with a σ -field \mathcal{X} , the control set \mathbb{U} with a σ -field \mathcal{U} , and the uncertainty set \mathbb{W} with a σ -field \mathcal{W} . We consider a *sample space* Ω , a measurable set equipped with a σ -field \mathcal{F} (but not necessarily equipped with a probability).

For any subset $S \subset \mathbb{T}$ and any measurable set \mathbb{Y} , we denote by $\mathbb{L}^0(S, \mathbb{Y})$ the set of stochastic processes $\{\mathbf{Y}_t\}_{t \in S}$ from Ω to \mathbb{Y} .

3.2 Tail shock scenarios

We suppose given a family $\{\mathcal{S}^t\}_{t \in \mathbb{T}}$ of *tail shock scenarios*, where

$$\mathcal{S}^t \subset \mathbb{L}^0(t:T, \mathbb{W}), \quad \forall t \in \mathbb{T}. \quad (13)$$

Example of tail shock scenarios. Suppose that the sample space Ω is equipped with a probability \mathbb{P} , and that $\mathbb{T} = \mathbb{R}_+$ and $\mathbb{W} = \mathbb{R}$. Let $\Delta t > 0$ represent a small time period, \bar{W} a high value for uncertainties, and $\beta < 1$, $\beta \approx 1$ a probability level. The set

$$\mathcal{S}^t = \{ \{ \mathbf{W}_s \}_{s \geq t} \in \mathbb{L}^0(t : T, \mathbb{W}) \mid \mathbf{W}_s = 0, \forall s > t + \Delta t, \\ \mathbb{P}[\mathbf{W}_s \geq \bar{W}, \forall s \in [t, t + \Delta t]] \geq \beta \} \quad (14)$$

contains stochastic process that, with high probability at least β , display high values at least \bar{W} for times close to t .

3.3 Acceptable tail state-control paths

We suppose given a family $\{ \mathcal{A}^t \}_{t \in \mathbb{T}}$ of *acceptable tail state-control paths*, where

$$\mathcal{A}^t \subset \mathbb{L}^0(t : T, \mathbb{X} \times \mathbb{U}) . \quad (15)$$

When there are no uncertainties, $\mathcal{A}^t \subset \mathbb{X}^{t:T} \times \mathbb{U}^{t:T}$, as in the following examples.

Example of acceptable tail state-control paths converging to an equilibrium. Let $\mathbb{T} = \mathbb{R}_+$ and $\mathbb{X} = \mathbb{R}^n$. Let \bar{x} be an equilibrium point of the dynamical system $\dot{x}_s = f(x_s, \bar{u})$ when the control is stationary equal to \bar{u} , that is, $0 = f(\bar{x}, \bar{u})$. The acceptable tail paths converging to the equilibrium \bar{x} form the set

$$\mathcal{A}^t = \{ (\{ x_s \}_{s \geq t}, \{ u_s \}_{s \geq t}) \in \mathbb{X}^{[t, +\infty[} \times \mathbb{U}^{[t, +\infty[} \mid x_s \rightarrow_{s \rightarrow +\infty} \bar{x} \} . \quad (16)$$

In general, the equilibrium \bar{x} is supposed to be asymptotically stable, locally or globally.

A more general definition would be

$$\mathcal{A}^t = \left\{ (\{ x_s \}_{s \geq t}, \{ u_s \}_{s \geq t}) \mid \lim_{s \rightarrow +\infty} x_s \text{ exists} \right\} , \quad (17)$$

and, to account for constraints on the values taken by the controls, we can consider

$$\mathcal{A}^t = \left\{ (\{ x_s \}_{s \geq t}, \{ u_s \}_{s \geq t}) \mid \lim_{s \rightarrow +\infty} x_s \text{ exists and } u_s \in \mathcal{U}_s(x_s), \forall s \geq t \right\} , \quad (18)$$

where $\mathcal{U}_t : \mathbb{X} \rightrightarrows \mathbb{U}$, $t \in \mathbb{T}$.

Example of bounded acceptable tail state-control paths. Let $\mathbb{T} = \mathbb{R}_+$ and $\mathbb{X} = \mathbb{R}^n$. If B is a bounded region of $\mathbb{X} = \mathbb{R}^n$, we consider

$$\mathcal{A}^t = \left\{ \left(\{x_s\}_{s \geq t}, \{u_s\}_{s \geq t} \right) \mid x_s \in B, \forall s \geq t \right\} . \quad (19)$$

When B is a ball of small radius $\rho > 0$ around the equilibrium \bar{x} , we obtain state paths that remain close to \bar{x} .

3.4 Resilient control paths and resilient states

Let Λ be a subset of the set of all strategies, together with a family $\{\Lambda_s\}_{s \in \mathbb{T}}$ of tail strategies subsets.

Consider a state $x \in \mathbb{X}$ and a time $t \in \mathbb{T}$. We say that the tail strategy $\lambda = \{\lambda_s\}_{s \in t:T} \in \Lambda_t$ is a *resilient strategy* starting from time t if, for any tail shock scenario $\{\mathbf{W}_s\}_{s \in t:T} \in \mathcal{S}^t$, one has that

$$\{\mathbf{X}_s\}_{s \in t:T} = \phi_{t:T}^\lambda(x, \{\mathbf{W}_s\}_{s \in t:T}) \quad (20a)$$

$$\mathbf{U}_s = \lambda_s(\mathbf{X}_s, \{\mathbf{W}_r\}_{r \in t:s}), \quad \forall s \in t:T \quad (20b)$$

$$\left(\{\mathbf{X}_s\}_{s \in t:T}, \{\mathbf{U}_s\}_{s \in t:T} \right) \in \mathcal{A}^t . \quad (20c)$$

With this definition, a resilient strategy is able to drive the state-control path into an acceptable regime, whatever the tail shock scenarios. As a resilient control path is adapted, it can “adapt” to the past values of the current tail shock scenario but no to its future values. Hence, our notion of resilience does not require clairvoyance of the DM.

We denote by $\Lambda^R(t, x) \subset \Lambda$ the set of *resilient strategies* at time t . The set of *resilient states* at time t is

$$\mathbb{X}^R(t) = \{x \in \mathbb{X} \mid \Lambda^R(t, x) \neq \emptyset\} . \quad (21)$$

3.5 Resilience and cost minimization

When the set of resilient strategies at time t is not empty, how can we select one among the many? Here is a possible way.

3.5.1 Tail cost functions

Suppose we have a family $\{\Psi_t\}_{t \in \mathbb{T}}$ of measurable functions

$$\Psi_t : \mathbb{X}^{t:T} \times \mathbb{U}^{t:T} \times \mathbb{W}^{t:T} \rightarrow \mathbb{R}, \quad t \in \mathbb{T}, \quad (22)$$

that attaches a *disutility* or *cost* — conversely, *value*, *utility*, *payoff* — to any tail state, control and uncertainty path starting from time t .

To any tail strategy $\lambda = \{\lambda_s\}_{s \in t:T} \in \Lambda_t$ and tail scenario $\{w_s\}_{s \in t:T}$, we can attach the *tail cost evaluation*

$$\Psi_t^\lambda(\{w_s\}_{s \in t:T}) = \Psi_t(\{x_s\}_{s \in t:T}, \{u_s\}_{s \in t:T}, \{w_s\}_{s \in t:T}) \in \mathbb{R}, \quad (23)$$

where, thanks to (12),

$$\{x_s\}_{s \in t:T} = \phi_{t:T}^\lambda(x, \{w_s\}_{s \in t:T}) \quad (24a)$$

$$u_s = \lambda_s(x_s, \{w_r\}_{r \in t:s}), \quad \forall s \in t:T. \quad (24b)$$

3.5.2 Aggregated costs minimization

Once we plug the random process $\{\mathbf{W}_s\}_{s \in t:T}$ into the tail cost evaluation Ψ_t^λ in (23), we obtain the random variable $\Psi_t^\lambda(\{\mathbf{W}_s\}_{s \in t:T})$. To be able to minimize costs, we have to aggregate these random costs with respect to all possible elements of the sample space Ω . Here are examples.

We can look for resilient strategies $\lambda \in \Lambda$ that minimize *expected costs*

$$\min_{\lambda \in \Lambda^R(t,x)} \mathbb{E}[\Psi_t^\lambda(\{\mathbf{W}_s\}_{s \in t:T})], \quad (25)$$

or that minimize *worst case costs*

$$\min_{\lambda \in \Lambda^R(t,x)} \sup_{\{\mathbf{W}_s\}_{s \in t:T} \in \mathcal{S}^t} \Psi_t^\lambda(\{\mathbf{W}_s\}_{s \in t:T}), \quad (26)$$

or, more generally, that minimize

$$\min_{\lambda \in \Lambda^R(t,x)} \mathbb{F}[\Psi_t^\lambda(\{\mathbf{W}_s\}_{s \in t:T})], \quad (27)$$

where \mathbb{F} is a risk measure that maps random variables towards the real numbers [7].

4 Illustrations

In Sect. 3, we have provided some illustrations in the course of the exposition. Now, we make the connection between the previous setting and two other settings, the resilience “à la Holling” [8] in §4.1 and the resilience-viability framework [10, 9] in §4.2. We also sketch how concepts from risk measures (introduced initially in mathematical finance [7]) could be imported to tackle some resilience issues.

4.1 Deterministic control dynamical system with attractor

As the paper [8] does not contain a single equation, it is bit risky to force the seminal Holling’s contribution into our setting. However, it is likely that it corresponds to $\mathbb{T} = \mathbb{R}_+$ and to acceptable tail state-control paths of the form

$$\mathcal{A}^t = \{ (\{x_s\}_{s \geq t}, \{u_s\}_{s \geq t}) \mid x_s \text{ converges towards an attractor} \} . \quad (28)$$

Note that, as often in the ecological literature on resilience [1], the underlying dynamical system is not controlled.

4.2 Resilience and viability

Some authors [10, 9] propose to frame resilience within the mathematical theory of viability [2].

Let $\mathbb{A} \subset \mathbb{X}$ denote a set made of “acceptable states”. Let $\mathcal{U}_s : \mathbb{X} \rightrightarrows \mathbb{U}$, $s \in \mathbb{T}$ be a family of set-valued mappings that represent control constraints.

4.2.1 Deterministic and robust viability

Consider the acceptable tail state-control paths

$$\mathcal{A}^t = \{ (\{x_s\}_{s \geq t}, \{u_s\}_{s \geq t}) \mid x_s \in \mathbb{A} , u_s \in \mathcal{U}_s(x_s) , \forall s \geq t \} . \quad (29)$$

With this set of tail state-control paths, a resilient control path is able to drive the state towards the set \mathbb{A} of acceptable states, whatever the tail shock scenarios. When there are no uncertainties, the connexion can be made with the viability kernel. But, with uncertainties, as the requirement is “whatever the tail shock scenarios”, we are in the realm of *robust viability* [4].

4.2.2 Time of crisis

We can also consider the acceptable tail state-control paths

$$\mathcal{A}^t = \{ (\{x_s\}_{s \geq t}, \{u_s\}_{s \geq t}) \mid \exists \tau \in \mathbb{T} , \tau \geq t , \\ x_s \in \mathbb{A} , u_s \in \mathcal{U}_s(x_s) , \forall s \geq \tau \} . \quad (30)$$

In this context, the *time of crisis* is [6]

$$\begin{aligned} \tau(t, x) = \inf \{ \tau \in t : T \mid \exists \lambda = \{ \lambda_s \}_{s \in t:T} \in \Lambda, \forall \{ \mathbf{W}_s \}_{s \in t:T} \in \mathcal{S}^t \\ \{ \mathbf{X}_s \}_{s \in t:T} = \phi_{t:T}^\lambda(x, \{ \mathbf{W}_s \}_{s \in t:T}) \\ \mathbf{U}_s = \lambda_s(\mathbf{X}_s, \{ \mathbf{W}_r \}_{r \in t:s}), \forall s \in t : T \\ \mathbf{X}_s \in \mathbb{A}, \mathbf{U}_s \in \mathcal{U}_s(\mathbf{X}_s), \forall s \geq \tau \}, \end{aligned} \quad (31)$$

with the convention that $\inf \emptyset = +\infty$.

4.2.3 Stochastic viability

Suppose that the sample space Ω is equipped with a probability \mathbb{P} , and let $\beta \in [0, 1]$, represent a probability level. With the following acceptable tail state-control paths

$$\begin{aligned} \mathcal{A}^t = \{ (\{ \mathbf{X}_s \}_{s \in t:T}, \{ \mathbf{U}_s \}_{s \in t:T}) \in \mathbb{L}^0(t : T, \mathbb{X} \times \mathbb{U}) \mid \\ \mathbb{P}[\mathbf{X}_s \in \mathbb{A}, \mathbf{U}_s \in \mathcal{U}_s(\mathbf{X}_s), \forall s \geq t] \geq \beta \}, \end{aligned} \quad (32)$$

we express that the probability to satisfy state and control constraints after time t are at least β [5].

4.3 Resilience and risk measures

Consider a family $\{ \mathbb{F}_t \}_{t \in \mathbb{T}}$ of risk measures that map tail state-control random processes towards the real numbers:

$$\mathbb{F}_t : \mathbb{L}^0(s : T, \mathbb{X} \times \mathbb{U}) \rightarrow \mathbb{R}. \quad (33)$$

The quantity $\mathbb{F}_t[\{ \mathbf{X}_s \}_{s \in t:T}, \{ \mathbf{U}_s \}_{s \in t:T}]$ measures the ‘‘risk’’ borne by the tail random process $(\{ \mathbf{X}_s \}_{s \in t:T}, \{ \mathbf{U}_s \}_{s \in t:T})$. With α a level of risk, we introduce

$$\begin{aligned} \mathcal{A}^t = \{ (\{ \mathbf{X}_s \}_{s \in t:T}, \{ \mathbf{U}_s \}_{s \in t:T}) \in \mathbb{L}^0(t : T, \mathbb{X} \times \mathbb{U}) \mid \\ \mathbb{F}_t[\{ \mathbf{X}_s \}_{s \in t:T}, \{ \mathbf{U}_s \}_{s \in t:T}] \leq \alpha \}. \end{aligned} \quad (34)$$

4.3.1 Robust viability

For instance, the robust viability inspired definition of resilience in §4.2.1 corresponds to (34) with $\alpha < 1$ and the risk measure

$$\mathbb{F}_s(\{ \mathbf{X}_s \}_{s \geq t}, \{ \mathbf{U}_s \}_{s \geq t}) = \sup_{s \geq t} \sup_{\omega \in \Omega} \mathbf{1}_{\mathbb{A}^c}(\mathbf{X}_s(\omega)). \quad (35)$$

Indeed, $\mathbb{F}_t[\{\mathbf{X}_s\}_{s \in t:T}, \{\mathbf{U}_s\}_{s \in t:T}] \leq \alpha < 1$ means that $\mathbf{1}_{\mathbb{A}^c}(\mathbf{X}_s(\omega)) \equiv 0$, that is, the state $(\mathbf{X}_s(\omega))$ always belongs to \mathbb{A} (as \mathbb{A}^c is the complementary set of \mathbb{A} in \mathbb{X}).

4.3.2 Stochastic viability and ambiguity

The stochastic viability inspired definition of resilience in §4.2.3 corresponds to (34) with $\alpha = 1 - \beta$ and the risk measure

$$\mathbb{F}_s(\{\mathbf{X}_s\}_{s \geq t}, \{\mathbf{U}_s\}_{s \geq t}) = \mathbb{P}[\exists s \geq t \mid \mathbf{X}_s \notin \mathbb{A} \text{ or } \mathbf{U}_s \notin \mathcal{U}_s(\mathbf{X}_s)]. \quad (36)$$

Now, suppose that different risk-holders do not share the same beliefs and let \mathcal{P} denote a set of probabilities on (Ω, \mathcal{F}) . We can arrive at an *ambiguity* viability inspired definition of resilience using the risk measure

$$\mathbb{F}_s(\{\mathbf{X}_s\}_{s \geq t}, \{\mathbf{U}_s\}_{s \geq t}) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[\exists s \geq t \mid \mathbf{X}_s \notin \mathbb{A} \text{ or } \mathbf{U}_s \notin \mathcal{U}_s(\mathbf{X}_s)]. \quad (37)$$

Here, $\mathbb{F}_t[\{\mathbf{X}_s\}_{s \in t:T}, \{\mathbf{U}_s\}_{s \in t:T}] \leq \alpha = 1 - \beta$ means that $\mathbb{P}[\mathbf{X}_s \in \mathbb{A}, \mathbf{U}_s \in \mathcal{U}_s(\mathbf{X}_s), \forall s \geq t] \geq \beta$, for all $\mathbb{P} \in \mathcal{P}$.

4.3.3 Exit time

Let μ be a measure on the time set \mathbb{T} like, for instance, the counting measure when $\mathbb{T} = \mathbb{N}$ or the Lebesgue measure when $\mathbb{T} = \mathbb{R}_+$. Then, the random quantity

$$\mu\{s \geq t, \mathbf{X}_s \notin \mathbb{A} \text{ or } \mathbf{U}_s \notin \mathcal{U}_s(\mathbf{X}_s)\} \quad (38)$$

measures the number of times that the tail state-control path $(\{\mathbf{X}_s\}_{s \in t:T}, \{\mathbf{U}_s\}_{s \in t:T})$ *exits* from the viability constraints.

Using risk measures [7] or stochastic orders [11, 12], we have different ways to express that this random quantity remains “small”.

5 Conclusion

Resilience is a rehashed concept in natural hazard management. Most of the formalizations of the concept require that, after a shock, the state of a system returns to an acceptable subset of the state set. Equipped with tool from control theory under uncertainty, we have proposed that resilience is the ability for the state-control path as a whole to return to an acceptable

“regime”, including a containment on the residual risk. Our contribution is formal. Much would remain to be done regarding applications and numerical implementation.

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