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# ASYMPTOTIC DISTRIBUTIONS ASSOCIATED TO PIECEWISE QUASI-POLYNOMIALS

PAUL-EMILE PARADAN AND MICHÈLE VERGNE

## 1. INTRODUCTION

Let  $V$  be a finite dimensional real vector space equipped with a lattice  $\Lambda$ . Let  $P \subset V$  be a rational polyhedron. The Euler-Maclaurin formula ([4], [2]) gives an asymptotic estimate, when  $k$  goes to  $\infty$ , for the Riemann sum  $\sum_{\lambda \in kP \cap \Lambda} \varphi(\lambda/k)$  of the values of a test function  $\varphi$  at the sample points  $\frac{1}{k}\Lambda \cap P$  of  $P$ , with leading term  $k^{\dim P} \int_P \varphi$ . Here we consider the slightly more general case of a weighted sum. Let  $q(\lambda, k)$  be a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$ . We consider, for  $k \geq 1$ , the distribution

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} q(\lambda, k) \varphi(\lambda/k)$$

and we show (Proposition 1.2) that the function  $k \mapsto \langle \Theta(P; q)(k), \varphi \rangle$  admits an asymptotic expansion when  $k$  tends to  $\infty$  in powers of  $1/k$  with coefficients periodic functions of  $k$ .

We extend this result to an algebra  $\mathcal{S}(\Lambda)$  of piecewise quasi-polynomial functions on  $\Lambda \oplus \mathbb{Z} \subset V \oplus \mathbb{R}$ . A function  $m(\lambda, k)$  ( $\lambda \in \Lambda, k \in \mathbb{Z}$ ) in  $\mathcal{S}(\Lambda)$  is supported in an union of polyhedral cones in  $V \oplus \mathbb{R}$ . The main feature of a function  $m(\lambda, k)$  in  $\mathcal{S}(\Lambda)$  is that  $m(\lambda, k)$  is entirely determined by its large behavior in  $k$ . We associate to  $m(\lambda, k)$  a formal series  $A(m)$  of distributions on  $V$  encoding the asymptotic behavior of  $m(\lambda, k)$  when  $k$  tends to  $\infty$ .

The motivating example is the case where  $M$  is a projective manifold, and  $\mathcal{L}$  the corresponding ample bundle. If  $T$  is a torus acting on  $M$ , then write, for  $t \in T$ ,

$$\sum_{i=0}^{\dim M} (-1)^i \text{Tr}(t, H^i(M, \mathcal{O}(\mathcal{L}^k))) = \sum_{\lambda} m(\lambda, k) t^\lambda$$

where  $\lambda$  runs over the lattice  $\Lambda$  of characters of  $T$ . The corresponding asymptotic expansion of the distribution  $\sum_{\lambda} m(\lambda, k) \delta_{\lambda/k}$  is an important object associated to  $M$  involving the Duistermaat-Heckmann

measure and the Todd class of  $M$ , see [9] for its determination. The determination of similar asymptotics in the more general case of twisted Dirac operators is the object of a forthcoming article [7].

Thus let  $m \in \mathcal{S}(\Lambda)$ , and consider the sequence

$$\Theta(m)(k) = \sum_{\lambda \in \Lambda} m(\lambda, k) \delta_{\lambda/k}$$

of distributions on  $V$  and its asymptotic expansion  $A(m)$  when  $k$  tends to  $\infty$ . Let  $T$  be the torus with lattice of characters  $\Lambda$ . If  $g \in T$  is an element of finite order, then  $m^g(\lambda, k) := g^\lambda m(\lambda, k)$  is again in  $\mathcal{S}(\Lambda)$ . Our main result (Theorem 1.8) is that the piecewise quasi-polynomial function  $m$  is entirely determined by the collections of asymptotic expansions  $A(m^g)$ , when  $g$  varies over the set of elements of  $T$  of finite order.

We also prove (Proposition 2.1) a functorial property of  $A(m)$  under pushforward.

We use these results to give new proofs of functoriality of the formal quantization of a symplectic manifold [5] or, more generally, of a spinc manifold [6].

For these applications, we also consider the case where  $V$  is a Cartan subalgebra of a compact Lie group, and anti-invariant distributions on  $V$  of a similar nature.

**1.1. Piecewise polynomial functions.** Let  $V$  be a real vector space equipped with a lattice  $\Lambda$ . Usually, an element of  $V$  is denoted by  $\xi$ , and an element of  $\Lambda$  by  $\lambda$ . In this article, a cone  $C$  will always be a closed convex polyhedral cone, and  $0 \in C$ .

Let  $\Lambda^*$  be the dual lattice, and let  $g \in T := V^*/\Lambda^*$ . If  $G \in V^*$  is a representative of  $g$  and  $\lambda \in \Lambda$ , then we denote  $g^\lambda = e^{2i\pi\langle G, \lambda \rangle}$ .

A periodic function  $m$  on  $\Lambda$  is a function such that there exists a positive integer  $D$  (we do not fix  $D$ ) such that  $m(\lambda_0 + D\lambda) = m(\lambda_0)$  for  $\lambda, \lambda_0 \in \Lambda$ . The space of such functions is linearly generated by the functions  $\lambda \mapsto g^\lambda$  for  $g \in T$  of finite order. By definition, the algebra of quasi-polynomial functions on  $\Lambda$  is generated by polynomials and periodic functions on  $\Lambda$ . If  $V_0$  is a rational subspace of  $V$ , the restriction of  $m$  to  $\Lambda_0 := \Lambda \cap V_0$  is a quasi-polynomial function on  $\Lambda_0$ . The space of quasi-polynomial functions is graded: a quasi-polynomial function homogeneous of degree  $d$  is a linear combination of functions  $t^\lambda h(\lambda)$  where  $t \in T$  is of finite order, and  $h$  an homogeneous polynomial on  $V$  of degree  $d$ . Let  $q(\lambda)$  be a quasi-polynomial function on  $\Lambda$ . There is a sublattice  $\Gamma$  of  $\Lambda$  of finite index  $d_\Gamma$  such that for any given  $\gamma \in \Lambda$ , we have  $q(\lambda) = p_\gamma(\lambda)$  for any  $\lambda \in \gamma + \Gamma$  where  $p_\gamma(\xi)$

is a (uniquely determined) polynomial function on  $V$ . Then define  $q_{pol}(\xi) = \frac{1}{d_\Gamma} \sum_{\gamma \in \Lambda/\Gamma} p_\gamma(\xi)$ , a polynomial function on  $V$ . This polynomial function is independent of the choice of the sublattice  $\Gamma$ . Then  $q(\lambda) - q_{pol}(\lambda)$  is a linear combination of functions of the form  $t^\lambda h(\lambda)$  with  $h(\lambda)$  polynomial and  $t \neq 1$ .

Using the Lebesgue measure associated to  $\Lambda$ , we identify generalized functions on  $V$  and distributions on  $V$ . If  $\theta$  is a generalized function on  $V$ , we may write  $\int_V \theta(\xi) \varphi(\xi) d\xi$  for its value on the test function  $\varphi$ . If  $R$  is a rational affine subspace of  $V$ ,  $R$  inherits a canonical translation invariant measure. If  $P$  is a rational polyhedron in  $V$ , it generates a rational affine subspace of  $V$ , and  $\int_P \varphi$  is well defined for  $\varphi$  a smooth function with compact support.

We say that a distribution  $\theta(k)$  depending of an integer  $k$  is periodic in  $k$  if there exists a positive integer  $D$  such that for any test function  $\varphi$  on  $V$ , and  $k_0, k \in \mathbb{Z}$ ,  $\langle \theta(k_0 + Dk), \varphi \rangle = \langle \theta(k_0), \varphi \rangle$ . Then there exists (unique) distributions  $\theta_\zeta$  indexed by  $D$ -th roots of unity such that  $\langle \theta(k), \varphi \rangle = \sum_{\zeta, \zeta^D=1} \zeta^k \langle \theta_\zeta, \varphi \rangle$ .

Let  $(\Theta(k))_{k \geq 1}$  be a sequence of distributions. We say that  $\Theta(k)$  admits an asymptotic expansion (with periodic coefficients) if there exists  $n_0 \in \mathbb{Z}$  and a sequence of distributions  $\theta_n(k), n \geq 0$ , depending periodically of  $k$ , such that for any test function  $\varphi$  and any non negative integer  $N$ , we have

$$\langle \Theta(k), \varphi \rangle = k^{n_0} \sum_{n=0}^N \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle + o(k^{n_0-N}).$$

We write

$$\Theta(k) \equiv k^{n_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \theta_n(k).$$

The distributions  $\theta_n(k)$  are uniquely determined.

Given a sequence  $\theta_n(k)$  of periodic distributions, and  $n_0 \in \mathbb{Z}$ , we write formally  $M(\xi, k)$  for the series of distributions on  $V$  defined by

$$\langle M(\xi, k), \varphi \rangle = k^{n_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \int_V \theta_n(k)(\xi) \varphi(\xi) d\xi.$$

We can multiply  $M(\xi, k)$  by quasi-polynomial functions  $q(k)$  of  $k$  and smooth functions  $h(\xi)$  of  $\xi$  and obtain the formal series  $q(k)h(\xi)M(\xi, k)$  of the same form with  $n_0$  changed to  $n_0 + \text{degree}(q)$ .

Let  $E = V \oplus \mathbb{R}$ , and we consider the lattice  $\tilde{\Lambda} = \Lambda \oplus \mathbb{Z}$  in  $E$ . An element of  $\tilde{\Lambda}$  is written as  $(\lambda, k)$  with  $\lambda \in \Lambda$  and  $k \in \mathbb{Z}$ . We consider quasi-polynomial functions  $q(\lambda, k)$  on  $\tilde{\Lambda}$ . As before, this space

is graded. We call the degree of a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$  the total degree. A quasi-polynomial function  $q(\lambda, k)$  is of total degree  $d$  if it is a linear combination of functions  $(\lambda, k) \mapsto j(k)t^\lambda k^a h(\lambda)$  where  $j(k)$  is a periodic function of  $k$ ,  $t \in T$  of finite order,  $a$  a non negative integer, and  $h$  an homogeneous polynomial on  $V$  of degree  $b$ , with  $b$  such that  $a + b = d$ .

Let  $q(\lambda, k)$  be a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$ . We construct  $q_{pol}(\xi, k)$  on  $V \times \mathbb{Z}$ , and depending polynomially on  $\xi$  as before. We choose a sublattice of finite index  $d_\Gamma$  in  $\Lambda$  and functions  $p_\gamma(\xi, k)$  depending polynomially on  $\xi \in V$  and quasi-polynomial in  $k$  such that  $q(\lambda, k) = p_\gamma(\lambda, k)$  if  $\lambda \in \gamma + \Gamma$ . Then  $q_{pol}(\xi, k) = \frac{1}{d_\Gamma} \sum_{\gamma \in \Lambda/\Gamma} p_\gamma(\xi, k)$ . We say that  $q_{pol}(\xi, k)$  is the polynomial part (relative to  $\Lambda$ ) of  $q$ . If  $q$  is homogeneous of total degree  $d$ , then the function  $(k, \xi) \mapsto q_{pol}(k\xi, k)$  is a linear combination of functions of the form  $j(k)k^d s(\xi)$  where  $j(k)$  is a periodic function of  $k$  and  $s(\xi)$  a polynomial function of  $\xi$ .

**Proposition 1.1.** *Let  $P$  be a rational polyhedron in  $V$  with non empty interior. Let  $q(\lambda, k)$  be a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$  homogeneous of total degree  $d$ . Let  $q_{pol}(\xi, k)$  be its polynomial part. Let  $k \geq 1$ . The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k) \varphi(\lambda/k)$$

admits an asymptotic expansion when  $k \rightarrow \infty$  of the form

$$k^{\dim V} k^d \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

Furthermore, the term  $k^d \langle \theta_0(k), \varphi \rangle$  is given by

$$k^d \langle \theta_0(k), \varphi \rangle = \int_P q_{pol}(k\xi, k) \varphi(\xi) d\xi$$

where  $q_{pol}$  is the polynomial part (with respect to  $\Lambda$ ) of  $q$ .

*Proof.* Let  $q(\lambda, k) = j(k)k^a g^\lambda h(\lambda)$  be a quasi-polynomial function of total degree  $d$ . Let

$$\langle \Theta_0^g(P)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} g^\lambda \varphi(\lambda/k). \quad (1.1)$$

If  $\Theta_0^g(P)(k)$  admits the asymptotic expansion  $M(\xi, k)$ , then  $\Theta(P; q)(k)$  admits the asymptotic expansion  $j(k)k^a h(k\xi)M(\xi, k)$ . So it is sufficient to consider the case where  $q(\lambda, k) = g^\lambda$  and the distribution  $\Theta_0^g(P)(k)$ .

We now proceed as in [2] for the case  $g = 1$  and sketch the proof. By decomposing the characteristic function  $[P]$  of the polyhedron  $P$

in a signed sum of characteristic functions of tangent cones, via the Brianchon Gram formula, then decomposing furthermore each tangent cone in a signed sum of cones  $C_a$  of the form  $\Sigma_a \times R_a$  with  $\Sigma_a$  is a translate of a unimodular cone and  $R_a$  a rational space, we are reduced to study this distribution for the product of the dimension 1 following situations.

$V = \mathbb{R}, \Lambda = \mathbb{Z}$  and one of the following two cases:

- $P = \mathbb{R}$
- $P = s + \mathbb{R}_{\geq 0}$  with  $s$  a rational number.

For example, if  $P = [a, b]$  is an interval in  $\mathbb{R}$  with rational end points  $a, b$ , we write  $[P] = [a, \infty] + [-\infty, b] - [\mathbb{R}]$ .

For  $P = \mathbb{R}$ , and  $\zeta$  a root of unity, it is easy to see that

$$\langle \Theta^\zeta(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}} \zeta^\mu \varphi(\mu/k)$$

is equivalent to  $k \int_{\mathbb{R}} \varphi(\xi) d\xi$  if  $\zeta = 1$  or is equivalent to 0 if  $\zeta \neq 1$ .

We now study the case where  $P = s + \mathbb{R}_{\geq 0}$ . Let

$$\langle \Theta^\zeta(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}, \mu - ks \geq 0} \zeta^\mu \varphi(\mu/k)$$

and let us compute its asymptotic expansion.

For  $r \in \mathbb{R}$ , the fractional part  $\{r\}$  is defined by  $\{r\} \in [0, 1[$ ,  $r - \{r\} \in \mathbb{Z}$ . If  $\mu$  is an integer greater or equal to  $ks$ , then  $\mu = ks + \{-ks\} + u$  with  $u$  a non negative integer.

We consider first the case where  $\zeta = 1$ . This case has been treated for example in [3] (Theorem 9.2.2), and there is an Euler-Maclaurin formula with remainder which leads to the following asymptotic expansion.

The function  $z \mapsto \frac{e^{xz}}{e^z - 1}$  has a simple pole at  $z = 0$ . Its Laurent series at  $z = 0$  is

$$\frac{e^{xz}}{e^z - 1} = \sum_{n=-1}^{\infty} B_{n+1}(x) \frac{z^n}{(n+1)!}$$

where  $B_n(x)$  ( $n \geq 0$ ) are the Bernoulli polynomials.

If  $s$  is rational, and  $n \geq 0$ , the function  $k \mapsto B_n(\{-ks\})$  is a periodic function of  $k$  with period the denominator of  $s$ , and

$$\sum_{\mu \in \mathbb{Z}, \mu \geq ks} \varphi\left(\frac{\mu}{k}\right) \equiv k \left( \int_s^\infty \varphi(\xi) d\xi - \sum_{n=1}^{\infty} \frac{1}{k^n} \frac{B_n(\{-ks\})}{n!} \varphi^{(n-1)}(s) \right).$$

This formula is easily proven by Fourier transform. Indeed, for  $f(\xi) = e^{i\xi z}$ , the series  $\sum_{\mu \geq ks} f(\mu/k)$  is  $\sum_{u \geq 0} e^{isz} e^{i\{-ks\}z/k} e^{iuz/k}$ . It is convergent

if  $z$  is in the upper half plane, and the sum is

$$F(z)(k) = -e^{isz} \frac{e^{i\{-ks\}z/k}}{e^{iz/k} - 1}.$$

So the Fourier transform of the tempered distribution  $\Theta^{\zeta=1}(k)$  is the boundary value of the holomorphic function  $z \mapsto F(z)(k)$  above. We can compute the asymptotic behavior of  $F(z)(k)$  easily when  $k$  tends to  $\infty$ , since  $\{-ks\} \leq 1$ , and  $z/k$  becomes small.

Rewriting  $[P]$  as the signed sum of the characteristic functions of the cones  $C_a$ , we see that the distribution  $\Theta_0^g(P)(k)$  for  $g = 1$  is equivalent to

$$k^{\dim V} \left( \sum_{n=0}^{\infty} \frac{1}{k^n} \theta_n(k) \right)$$

with  $\theta_0$  independent of  $k$ , and given by  $\langle \theta_0, \varphi \rangle = \int_P \varphi(\xi) d\xi$ .

Now consider the case where  $\zeta \neq 1$ . Then

$$\sum_{\mu \in \mathbb{Z}, \mu \geq ks} \zeta^\mu \varphi(\mu/k) = \sum_{u \geq 0} \zeta^{ks + \{-ks\}} \zeta^u \varphi(s + \{-ks\}/k + u/k).$$

The function  $k \mapsto \zeta^{ks + \{-ks\}}$  is a periodic function of  $k$  with period  $ed$  if  $\zeta^e = 1$  and  $ds$  is an integer. If  $\zeta \neq 1$ , the function  $z \mapsto \frac{e^{xz}}{\zeta e^z - 1}$  is holomorphic at  $z = 0$ . Define the polynomials  $B_{n,\zeta}(x)$  via the Taylor series expansion:

$$\frac{e^{xz}}{\zeta e^z - 1} = \sum_{n=0}^{\infty} B_{n+1,\zeta}(x) \frac{z^n}{(n+1)!}.$$

It is easily seen by Fourier transform that  $\sum_{\mu \in \mathbb{Z}, \mu \geq ks} \zeta^\mu \varphi(\mu/k)$  is equivalent to

$$-k \zeta^{ks + \{-ks\}} \sum_{n=1}^{\infty} \frac{1}{k^n} \frac{B_{n,\zeta}(\{-ks\})}{n!} \varphi^{(n-1)}(s).$$

In particular,  $\Theta^\zeta(k)$  admits an asymptotic expansion in non negative powers of  $1/k$  and each coefficient of this asymptotic expansion is a periodic distribution supported at  $s$ .

Rewriting  $[P]$  in terms of the signed cones  $C_a$ , we see that indeed if  $g \in T$  is not 1, one of the corresponding  $\zeta$  in the reduction to a product of one dimensional cones is not 1, and so

$$\Theta_0^g(P)(k) \equiv k^{\dim V - 1} \left( \sum_{n=0}^{\infty} \frac{1}{k^n} \theta_n(k) \right).$$

So we obtain our proposition. □

Consider now  $P$  a rational polyhedron, with possibly empty interior. Let  $C_P$  be the cone of base  $P$  in  $E = V \oplus \mathbb{R}$ ,

$$C_P := \{(t\xi, t), t \geq 0, \xi \in P\}.$$

Let  $q(\lambda, k)$  be a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$ . We consider again

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} q(\lambda, k) \varphi(\lambda/k).$$

Consider the vector space  $E_P$  generated by the cone  $C_P$  in  $E$ . It is clear that  $\Theta(P; q)$  depends only of the restriction  $r$  of  $q$  to  $E_P \cap (\Lambda \oplus \mathbb{Z})$ . This is a quasi-polynomial function on  $E_P$  with respect to the lattice  $E_P \cap (\Lambda \oplus \mathbb{Z})$ . We assume that the quasi-polynomial function  $r$  is homogeneous of degree  $d_0$ . This degree might be smaller than the total degree of  $q$ . Consider the affine space  $R_P$  generated by  $P$  in  $V$ . Let  $E_P^{\mathbb{Z}} = E_P \cap (V \oplus \mathbb{Z})$ . If  $\xi \in R_P, k \in \mathbb{Z}$ , then  $(k\xi, k) \in E_P^{\mathbb{Z}}$ . We will see shortly (Definition 1.3) that we can define a function  $(\xi, k) \mapsto r_{pol}(\xi, k)$  for  $(\xi, k) \in E_P^{\mathbb{Z}}$ , and that the function  $(\xi, k) \mapsto r_{pol}(k\xi, k)$  on  $R_P \times \mathbb{Z}$  is a linear combination of functions of the form  $k^{d_0} j(k) s(\xi)$  where  $j(k)$  is a periodic function of  $k$  and  $s(\xi)$  a polynomial function of  $\xi$ , for  $\xi$  varying on the affine space  $R_P$ .

We now can state the general formula.

**Proposition 1.2.** *Let  $P$  be a rational polyhedron in  $V$ . Let  $q(\lambda, k)$  be a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$ . Let  $r$  be its restriction to  $E_P \cap (\Lambda \oplus \mathbb{Z})$  and  $r_{pol}$  the "polynomial part" of  $r$  on  $E_P \cap (V \oplus \mathbb{Z})$ . Assume that the quasi-polynomial function  $r$  is homogeneous of degree  $d_0$ . Let  $k \geq 1$ . The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k) \varphi(\lambda/k)$$

*admits an asymptotic expansion when  $k \rightarrow \infty$  of the form*

$$k^{\dim P} k^{d_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

*Furthermore, the term  $k^{d_0} \langle \theta_0(k), \varphi \rangle$  is given by*

$$k^{d_0} \langle \theta_0(k), \varphi \rangle = \int_P r_{pol}(k\xi, k) \varphi(\xi) d\xi.$$

*Proof.* We will reduce the proof of this proposition to the case treated before of a polyhedron with interior. Let  $\text{lin}(P)$  be the linear space parallel to  $R_P$ , and  $\Lambda_0 := \Lambda \cap \text{lin}(P)$ . If  $R_P$  contains a point  $\beta \in \Lambda$ , then  $E_P$  is isomorphic to  $\text{lin}(P) \oplus \mathbb{R}$  with lattice  $\Lambda_0 \oplus \mathbb{Z}$ . Otherwise, we will have to dilate  $R_P$ . More precisely, let  $I_P = \{k \in \mathbb{Z}, kR_P \cap \Lambda \neq \emptyset\}$ .



This is an ideal in  $\mathbb{Z}$ . Indeed if  $k_1 \in I_P, k_2 \in I_P, \alpha_1, \alpha_2 \in R_P$  are such that  $k_1\alpha_1 \in \Lambda, k_2\alpha_2 \in \Lambda$ , then  $\alpha_{1,2} = \frac{1}{n_1k_1+n_2k_2}(n_1k_1\alpha_1 + n_2k_2\alpha_2)$  is in  $R_P$ , and  $(n_1k_1 + n_2k_2)(\alpha_{1,2}) \in \Lambda$ . Thus there exists a smallest  $k_0 > 0$  generating the ideal  $I_P$ . We see that our distribution  $\Theta(P; p)(k)$  is identically equal to 0 if  $k$  is not in  $I_P$ . Let  $\delta_{I_P}(k)$  be the function of  $k$  with

$$\delta_{I_P}(k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ 1 & \text{if } k = uk_0 \in I_P. \end{cases}$$

This is a periodic function of  $k$  of period  $k_0$ . We choose  $\alpha \in R_P$  such that  $k_0\alpha \in \Lambda$ . We identify  $E_P$  to  $\text{lin}(P) \oplus \mathbb{R}$  by the map  $T_\alpha(\xi_0, t) = (\xi_0 + tk_0\alpha, tk_0)$ . In this identification, the lattice  $(\Lambda \oplus \mathbb{Z}) \cap E_P$  becomes the lattice  $\Lambda_0 \oplus \mathbb{Z}$ . Consider  $P_0 = k_0(P - \alpha)$ , a polyhedron with interior in  $\text{lin}(P)$ . Let  $q^\alpha(\gamma, u) = r(\gamma + uk_0\alpha, uk_0)$ . This is a quasi-polynomial function on  $\Lambda_0 \oplus \mathbb{Z}$ . Its total degree is  $d_0$ . We have defined its polynomial part  $q_{pol}^\alpha(\xi, u)$  for  $\xi \in \text{lin}(P), u \in \mathbb{Z}$ .

**Definition 1.3.** Let  $(\xi, k) \in E_P^{\mathbb{Z}}$ . Define:

$$r_{pol}(\xi, k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ q_{pol}^\alpha(\xi - uk_0\alpha, u) & \text{if } k = uk_0 \in I_P. \end{cases}$$

The function  $r_{pol}(\xi, k)$  does not depend of the choice of  $\alpha$ . Indeed, if  $\alpha, \beta \in R_P$  are such that  $k_0\alpha, k_0\beta \in \Lambda$ , then  $q^\beta(\gamma, u) = q^\alpha(\gamma + uk_0(\beta - \alpha), u)$ . Then we see that  $q_{pol}^\beta(\xi, u) = q_{pol}^\alpha(\xi + uk_0(\beta - \alpha), u)$ . Furthermore, the function  $(k, \xi) \mapsto r_{pol}(k\xi, k)$  is of the desired form, a linear combination of functions  $\delta_{I_P}(k)j(k)k^{d_0}s(\xi)$  with  $s(\xi)$  polynomial functions on  $R_P$ .

If  $\varphi$  is a test function on  $V$ , we define the test function  $\varphi_0$  on  $\text{lin}(P)$  by  $\varphi_0(\xi_0) = \varphi(\frac{\xi_0}{k_0} + \alpha)$ . We see that

$$\langle \Theta(P; q)(uk_0), \varphi \rangle = \langle \Theta(P_0; q^\alpha)(u), \varphi_0 \rangle. \quad (1.2)$$

Thus we can apply Proposition 1.1. We obtain

$$\langle \Theta(P; q)(uk_0), \varphi \rangle \equiv u^{\dim P} u^{d_0} \sum_{n=0}^{\infty} \frac{1}{u^n} \langle \omega_n(u), \varphi_0 \rangle.$$

We have

$$u^{d_0} \langle \omega_0(u), \varphi_0 \rangle = \int_{P_0} q_{pol}^\alpha(u\xi_0, u) \varphi_0(\xi_0) d\xi_0 = \int_{P_0} q_{pol}^\alpha(u\xi_0, u) \varphi\left(\frac{\xi_0}{k_0} + \alpha\right)$$

When  $\xi_0$  runs in  $P_0 = k_0(P - \alpha)$ ,  $\xi = \frac{\xi_0}{k_0} + \alpha$  runs over  $P$ . Changing variables, we obtain

$$u^{d_0} \langle \omega_0(u), \varphi_0 \rangle = k^{d_0} \int_P r_{pol}(k\xi, k) \varphi(\xi) d\xi.$$

Thus we obtain our proposition.  $\square$

Let  $P$  be a rational polyhedron in  $V$  and  $q$  a quasi-polynomial function on  $\Lambda \oplus \mathbb{Z}$ . We do not assume that  $P$  has interior in  $V$ . We denote by  $[C_P]$  the characteristic function of  $C_P$ . Then the function  $q(\lambda, k)[C_P](\lambda, k)$  is zero if  $(\lambda, k)$  is not in  $C_P$  or equal to  $q(\lambda, k)$  if  $(\lambda, k)$  is in  $C_P$ . We denote it by  $q[C_P]$ . The space of functions on  $\Lambda \oplus \mathbb{Z}$  we will study is the following space.

**Definition 1.4.** *We define the space  $\mathcal{S}(\Lambda)$  to be the space of functions on  $\Lambda \oplus \mathbb{Z}$  linearly generated by the functions  $q[C_P]$  where  $P$  runs over rational polyhedrons in  $V$  and  $q$  over quasi-polynomial functions on  $\Lambda \oplus \mathbb{Z}$ .*

The representation of  $m$  as a sum of functions  $q[C_P]$  is not unique. For example, consider  $V = \mathbb{R}$ ,  $P = \mathbb{R}$ ,  $P_+ := \mathbb{R}_{\geq 0}$ ,  $P_- := \mathbb{R}_{\leq 0}$ ,  $P_0 := \{0\}$ , then  $[C_P] = [C_{P_+}] + [C_{P_-}] - [C_{P_0}]$ .

**Example 1.5.** *An important example of functions  $m \in \mathcal{S}(\Lambda)$  is the following. Assume that we have a closed cone  $C$  in  $V \oplus \mathbb{R}$ , and a covering  $C = \cup_{\alpha} C_{\alpha}$  by closed cones. Let  $m$  be a function on  $C \cap (\Lambda \oplus \mathbb{Z})$ , and assume that the restriction of  $m$  to  $C_{\alpha} \cap (\Lambda \oplus \mathbb{Z})$  is given by a quasipolynomial function  $q_{\alpha}$ . Then, using exclusion-inclusion formulae, we see that  $m \in \mathcal{S}(\Lambda)$ .*

**Definition 1.6.** *If  $m(\lambda, k)$  belongs to  $\mathcal{S}(\Lambda)$ , and  $g \in T$  is an element of finite order, then define*

$$m^g(\lambda, k) = g^{\lambda} m(\lambda, k).$$

The function  $m^g$  belongs to  $\mathcal{S}(\Lambda)$ .

If  $m \in \mathcal{S}(\Lambda)$ , and  $k \geq 1$ , we denote by  $\Theta(m)(k)$  the distribution on  $V$  defined by

$$\langle \Theta(m)(k), \varphi \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) \varphi(\lambda/k),$$

if  $\varphi$  is a test function on  $V$ . The following proposition follows immediately from Proposition 1.2.

**Proposition 1.7.** *If  $m(\lambda, k) \in \mathcal{S}(\Lambda)$ , the distribution  $\Theta(m)(k)$  admits an asymptotic expansion  $A(m)(\xi, k)$ .*

The function  $m(\lambda, k)$  can be non zero, while  $A(m)(\xi, k)$  is zero. For example let  $V = \mathbb{R}$ ,  $P = \mathbb{R}$  and  $m(\lambda, k) = (-1)^\lambda$ . Then  $\Theta(m)(k)$  is the distribution on  $\mathbb{R}$  given by  $T(k) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda \delta_{\lambda/k}$ ,  $k \geq 1$  and this is equivalent to 0. However, here is an unicity theorem.

**Theorem 1.8.** *Assume that  $m \in \mathcal{S}(\Lambda)$  is such that  $A(m^g) = 0$  for all  $g \in T$  of finite order, then  $m = 0$ .*

*Proof.* We start by the case of a function  $m = q[C_P]$  associated to a single polyhedron  $P$  and a quasi-polynomial function  $q$ . Assume first that  $P$  is with non empty interior  $P^0$ . If  $q$  is not identically 0, we write  $q(\lambda, k) = \sum_{g \in T} g^{-\lambda} p_g(\lambda, k)$  where  $p_g(\lambda, k)$  are polynomials in  $\lambda$ . If  $d$  is the total degree of  $q$ , then all the polynomials  $p_g(\lambda, k)$  are of degree less or equal than  $d$ . We choose  $t \in T$  such that  $p_t(\lambda, k)$  is of degree  $d$ . If we consider the quasi-polynomial  $q^t(\lambda, k)$ , then its polynomial part is  $p_t(\lambda, k)$  and the homogeneous component  $p_t^{top}(\lambda, k)$  of degree  $d$  is not zero. We write  $p_t^{top}(\xi, k) = \sum_{\zeta, a} \zeta^k k^a p_{\zeta, a}(\xi)$  where  $p_{\zeta, a}(\xi)$  is a polynomial in  $\xi$  homogeneous of degree  $d - a$ . Testing against a test function  $\varphi$  and computing the term in  $k^{d+\dim V}$  of the asymptotic expansion by Proposition 1.1, we see that  $\sum_{\zeta, a} \zeta^k k^d \int_P p_{\zeta, a}(\xi) \varphi(\xi) d\xi = 0$ . This is true for any test function  $\varphi$ . So, for any  $\zeta$ , we obtain  $\sum_a p_{\zeta, a}(\xi) = 0$ . Each of the  $p_{\zeta, a}$  being homogeneous of degree  $d - a$ , we see that  $p_{\zeta, a} = 0$  for any  $a, \zeta$ . Thus  $p_t^{top} = 0$ , a contradiction. So we obtain that  $q = 0$ , and  $m = q[C_P] = 0$ . Remark that to obtain this conclusion, we may use only test functions  $\varphi$  with support contained in the interior  $P^0$  of  $P$ .

Consider now a general polyhedron  $P$  and the vector space  $\text{lin}(P)$ . Let us prove that  $m(\lambda, k) = q(\lambda, k)[C_P](\lambda, k)$  is identically 0 if  $A(m^g) = 0$  for any  $g \in T$  of finite order. Using the notations of the proof of Proposition 1.2, we see that  $m(\lambda, k) = 0$ , if  $k$  is not of the form  $uk_0$ . Furthermore, if  $q^\alpha(\gamma, u) = q(\gamma + uk_0\alpha, uk_0)$ , it is sufficient to prove that  $q^\alpha = 0$ . Let  $P_0 = k_0(P - \alpha)$ , a polyhedron with interior in  $V_0$ . Consider  $m_0 = q^\alpha[C_{P_0}]$ . We consider  $T$  as the character group of  $\Lambda$ , so  $T$  surjects on  $T_0$ . Let  $g \in T$  of finite order and such that  $g^{k_0\alpha} = 1$ , and let  $g_0$  be the restriction of  $g$  to  $\Lambda_0$ . Using Equation 1.1, we then see that

$$\langle \Theta(m^g)(uk_0), \varphi \rangle = \langle \Theta(m_0^{g_0})(u), \varphi_0 \rangle.$$

Any  $g_0 \in T_0$  of finite order is the restriction to  $\Lambda_0$  of an element  $g \in T$  of finite order and such that  $g^{k_0\alpha} = 1$ . So we conclude that the asymptotic expansion, when  $u$  tends to  $\infty$ , of  $\Theta(m_0^{g_0})(u)$  is equal to 0 for any  $g_0 \in T_0$  of finite order. Remark again that we need only to know that  $\langle \Theta(m^g)(k), \varphi \rangle \equiv 0$  for test functions  $\varphi$  such that the support

$S$  of  $\varphi$  is contained in a very small neighborhood of compact subsets of  $P$  contained in the relative interior of  $P^0$ .

For any integer  $\ell$ , denote by  $\mathcal{S}_\ell(\Lambda)$  the subspace of functions  $m \in \mathcal{S}(\Lambda)$  generated by the functions  $q[C_P]$  with  $\dim P \leq \ell$ .

When  $\ell = 0$ , our polyhedrons are a finite number of rational points  $f \in V$ , the function  $m(\lambda, k)$  is supported on the union of lines  $(ud_f f, ud_f)$  if  $d_f$  is the smallest integer such that  $d_f f$  is in  $\Lambda$ . Choose a test function  $\varphi$  with support near  $f$ . Then  $u \mapsto \langle \Theta(m)(d_f u), \varphi \rangle$  is identical to its asymptotic expansion  $m(ud_f f, ud_f)\varphi(f)$ . Clearly we obtain that  $m = 0$ .

If  $m \in \mathcal{S}_\ell(\Lambda)$  by inclusion-exclusion, we can write

$$m = \sum_{P, \dim(P)=\ell} q_P[C_P] + \sum_{H, \dim H < \ell} q_H[C_H]$$

and we can assume that the intersections of a polyhedron  $P$  occurring in the first sum, with any polyhedron  $P'$  occurring in the decomposition of  $m$  and different from  $P$  is of dimension strictly less than  $\ell$ . Consider  $P$  in the first sum, so  $\dim(P) = \ell$ . We can thus choose test functions  $\varphi$  with support in small neighborhoods of  $K$ , with  $K$  a compact subset contained in the relative interior of  $P$ . Then

$$\langle \Theta(m^g)(k), \varphi \rangle = \langle \Theta(q_P^g[C_P])(k), \varphi \rangle.$$

The preceding argument shows that  $q_P[C_P] = 0$ . So  $m \in \mathcal{S}_{\ell-1}(\Lambda)$ . By induction  $m = 0$ .  $\square$

## 2. COMPOSITION OF PIECEWISE QUASI-POLYNOMIAL FUNCTIONS

Let  $V_0, V_1$  be vector spaces with lattice  $\Lambda_0, \Lambda_1$ .

Let  $C_{0,1}$  be a closed polyhedral rational cone in  $V_0 \oplus V_1$  (containing the origin). Thus for any  $\mu \in \Lambda_1$ , the set of  $\lambda \in V_0$  such that  $(\lambda, \mu) \in C_{0,1}$  is a rational polyhedron  $P(\mu)$  in  $V_0$ . Let  $P$  be a polyhedron in  $V$ . We assume that for any  $\mu \in \Lambda_1$ ,  $P \cap P(\mu)$  is compact. Thus, for  $m = q_P[C_P] \in \mathcal{S}(\Lambda)$ , and  $c(\lambda, \mu)$  a quasi-polynomial function on  $\Lambda_0 \oplus \Lambda_1$ , we can compute

$$m_c(\mu, k) = \sum_{(\lambda, \mu) \in C_{0,1}} m(\lambda, k)c(\lambda, \mu).$$

**Proposition 2.1.** *The function  $m_c$  belongs to  $\mathcal{S}(\Lambda_1)$ .*

Before establishing this result, let us give an example, which occur for example in the problem of computing the multiplicity of a representation  $\chi^\lambda \otimes \chi^\lambda$  of  $SU(2)$  restricted to the maximal torus.

**Example 2.2.** Let  $V_0 = V_1 = \mathbb{R}$ , and  $\Lambda_0 = \Lambda_1 = \mathbb{Z}$ . Let  $P := [0, 2]$ , and let

$$q(\lambda, k) = \begin{cases} \frac{1}{2}(1 - (-1)^\lambda) & \text{if } 0 \leq \lambda \leq 2k \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$C_{0,1} = \{(x, y) \in \mathbb{R}^2; x \geq 0, -x \leq y \leq x\}$$

and

$$c(\lambda, \mu) = \frac{1}{2}(1 - (-1)^{\lambda-\mu}).$$

Let  $\mu \geq 0$ . Then

$$m_c(\mu, k) = \frac{1}{4} \sum_{0 \leq \lambda \leq 2k, \lambda \geq \mu} (1 - (-1)^\lambda)(1 - (-1)^{\lambda-\mu}) = (1 + (-1)^\mu)(k/2 - \mu/4).$$

So if  $P_1 = [0, 2]$ ,  $P_2 := [-2, 0]$ ,  $P_3 := \{0\}$ , we obtain

$$m_c = q_1[C_{P_1}] + q_2[C_{P_2}] + q_3[C_{P_3}]$$

with

$$\begin{cases} q_1(\mu, k) = (1 + (-1)^\mu)(k/2 - \mu/4), \\ q_2(\mu, k) = (1 + (-1)^\mu)(k/2 + \mu/4), \\ q_3(\mu, k) = -k. \end{cases}$$

We now start the proof of Proposition 2.1.

*Proof.* Write  $c(\lambda, \mu)$  as a sum of products of quasi-polynomial functions  $q_j(\lambda)$ ,  $f_j(\mu)$ , and  $q_P(\lambda, k)$  a sum of products of quasi-polynomial functions  $m_\ell(k)$ ,  $h_\ell(\lambda)$ . Then we see that it is thus sufficient to prove that, for  $q(\lambda)$  a quasi-polynomial function of  $\lambda$ , the function

$$S(q)(\mu, k) = \sum_{\lambda \in kP \cap P(\mu)} q(\lambda) \tag{2.1}$$

belongs to  $\mathcal{S}(\Lambda_1)$ . For this, let us recall some results on families of polytopes  $\mathfrak{p}(\mathbf{b}) \subset E$  defined by linear inequations. See for example [1], or [8].

Let  $E$  be a vector space, and  $\omega_i, i = 1, \dots, N$  be a sequence of linear forms on  $E$ . Let  $\mathbf{b} = (b_1, b_2, \dots, b_N)$  be an element of  $\mathbb{R}^N$ . Consider the polyhedron  $\mathfrak{p}(\mathbf{b})$  defined by the inequations

$$\mathfrak{p}(\mathbf{b}) = \{v \in E; \langle \omega_i, v \rangle \leq b_i, i = 1, \dots, N\}.$$

We assume  $E$  equipped with a lattice  $L$ , and inequations  $\omega_i$  defined by elements of  $L^*$ . Then if the parameters  $b_i$  are in  $\mathbb{Z}^N$ , the polytopes  $\mathfrak{p}(\mathbf{b})$  are rational convex polytopes.

Assume that there exists  $\mathbf{b}$  such that  $\mathbf{p}(\mathbf{b})$  is compact (non empty). Then  $\mathbf{p}(\mathbf{b})$  is compact (or empty) for any  $\mathbf{b} \in \mathbb{R}^N$ . Furthermore, there exists a closed cone  $\mathcal{C}$  in  $\mathbb{R}^N$  such that  $\mathbf{p}(\mathbf{b})$  is non empty if and only if  $\mathbf{b} \in \mathcal{C}$ . There is a decomposition  $\mathcal{C} = \cup_{\alpha} \mathcal{C}_{\alpha}$  of  $\mathcal{C}$  in closed polyhedral cones with non empty interiors, where the polytopes  $\mathbf{p}(\mathbf{b})$ , for  $\mathbf{b} \in \mathcal{C}_{\alpha}$ , does not change of shape. More precisely:

- When  $\mathbf{b}$  varies in the interior of  $\mathcal{C}_{\alpha}$ , the polytope  $\mathbf{p}(\mathbf{b})$  remains with the same number of vertices  $\{s_1(\mathbf{b}), s_2(\mathbf{b}), \dots, s_L(\mathbf{b})\}$ .
- for each  $1 \leq i \leq L$ , there exists a cone  $C_i$  in  $E$ , such that the tangent cone to the polytope  $\mathbf{p}(\mathbf{b})$  at the vertex  $s_i(\mathbf{b})$  is the affine cone  $s_i(\mathbf{b}) + C_i$ .
- the map  $\mathbf{b} \rightarrow s_i(\mathbf{b})$  depends of the parameter  $\mathbf{b}$ , via linear maps  $\mathbb{R}^N \rightarrow E$  with rational coefficients.

Furthermore -as proven for example in [1]- the Brianchon-Gram decomposition of  $\mathbf{p}(\mathbf{b})$  is "continuous" in  $\mathbf{b}$  when  $b$  varies on  $C_{\alpha}$ , in a sense discussed in [1].

Before continuing, let us give a very simple example, let  $b_1, b_2, b_3$  be 3 real parameters and consider  $\mathbf{p}(b_1, b_2, b_3) = \{x \in \mathbb{R}, x \leq b_1, -x \leq b_2, -x \leq b_3\}$ . So we are studying the intersection of the interval  $[-b_2, b_1]$  with the half line  $[-b_3, \infty]$ . Then for  $\mathbf{p}(\mathbf{b})$  to be non empty, we need that  $\mathbf{b} \in \mathcal{C}$ , with

$$\mathcal{C} = \{\mathbf{b}; b_1 + b_2 \geq 0, b_1 + b_3 \geq 0\}.$$

Consider  $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ , with

$$\mathcal{C}_1 = \{\mathbf{b} \in \mathcal{C}; b_2 - b_3 \geq 0\},$$

$$\mathcal{C}_2 = \{\mathbf{b} \in \mathcal{C}; b_3 - b_2 \geq 0\}.$$

On  $\mathcal{C}_1$  the vertices of  $\mathbf{p}(\mathbf{b})$  are  $[-b_3, b_1]$ , while on  $\mathcal{C}_2$  the vertices of  $\mathbf{p}(\mathbf{b})$  are  $[-b_2, b_1]$ .

The Brianchon-Gram decomposition of  $\mathbf{p}(\mathbf{b})$  for  $\mathbf{b}$  in the interior of  $\mathcal{C}_1$  is  $[-b_3, \infty] + [-\infty, b_1] - \mathbb{R}$ . If  $\mathbf{b} \in \mathcal{C}_1$  tends to the point  $(b_1, b_2, -b_1)$  in the boundary of  $\mathcal{C}$ , we see the Brianchon-Gram decomposition tends to that  $[b_1, \infty] + [-\infty, b_1] - \mathbb{R}$ , which is indeed the polytope  $\{b_1\}$ .

Let  $q(\gamma)$  be a quasi-polynomial function of  $\gamma \in L$ . Then, when  $\mathbf{b}$  varies in  $\mathcal{C}_{\alpha} \cap \mathbb{Z}^N$ , the function

$$S(q)(\mathbf{b}) = \sum_{\gamma \in \mathbf{p}(\mathbf{b}) \cap L} q(\gamma)$$

is given by a quasi-polynomial function of  $b$ . This is proven in [8], Theorem 3.8. In this theorem, we sum an exponential polynomial function  $q(\gamma)$  on the lattice points of  $\mathbf{p}(\mathbf{b})$  and obtain an exponential polynomial function of the parameter  $b$ . However, the explicit formula shows

that if we sum up a quasi-polynomial function of  $\gamma$ , then we obtain a quasi-polynomial function of  $\mathbf{b} \in \mathbb{Z}^N$ . Another proof follows from [1] (Theorem 54) and the continuity of Brianchon-Gram decomposition. In [1], only the summation of polynomial functions is studied, via a Brianchon-Gram decomposition, but the same proof gives the result for quasi-polynomial functions (it depends only of the fact that the vertices vary via rational linear functions of  $\mathbf{b}$ ). The relations between partition polytopes  $P_{\mathfrak{p}}(\xi)$  (setting used in [8], [1]) and families of polytopes  $\mathfrak{p}(\mathbf{b})$  is standard, and is explained for example in the introduction of [1].

Consider now our situation with  $E = V$  equipped with the lattice  $\Lambda$ . The polytope  $kP \subset V$  is given by a sequence of inequalities  $\omega_i(\xi) \leq ka_i$ ,  $i = 1, \dots, I$ , where we can assume  $\omega_i \in \Lambda^*$  and  $a_i \in \mathbb{Z}$  by eventually multiplying by a large integer the inequality. The polytope  $P(\mu)$  is given by a sequence of inequalities  $\omega_j(\xi) \leq \nu_j(\mu)$ ,  $j = 1, \dots, J$  where  $\nu_j$  depends linearly on  $\mu$ . Similarly we can assume  $\nu_j(\mu) \in \mathbb{Z}$ . Let

$$(\mu, k) \mapsto \mathbf{b}(\mu, k) = [ka_1, \dots, ka_I, \nu_1(\mu), \dots, \nu_J(\mu)]$$

a linear map from  $\Lambda_1 \oplus \mathbb{Z}$  to  $\mathbb{Z}^N$ . Our polytope  $kP \cap P(\mu)$  is the polytope  $\mathfrak{p}(\mathbf{b}(k, \mu))$  and

$$S(q)(\mu, k) = \sum_{\lambda \in \mathfrak{p}(\mathbf{b}(k, \mu)) \cap \Lambda} q(\lambda) = S(q)(\mathbf{b}(\mu, k)).$$

Consider one of the cones  $\mathcal{C}_\alpha$ . Then  $\mathbf{b}(\mu, k) \in \mathcal{C}_\alpha$ , if and only if  $(\mu, k)$  belongs to a rational polyhedral cone  $C_\alpha$  in  $V_1 \oplus \mathbb{R}$ . If  $Q$  is a quasi-polynomial function of  $\mathbf{b}$ , then  $Q(\mathbf{b}(\mu, k))$  is a quasi-polynomial function of  $(\mu, k)$ . Thus on each of the cones  $C_\alpha$ ,  $S(q)(\mu, k)$  is given by a quasi-polynomial function of  $(\mu, k)$ . From Example 1.5, we conclude that  $S(q)$  belongs to  $\mathcal{S}(\Lambda_1)$ .  $\square$

### 3. PIECEWISE QUASI-POLYNOMIAL FUNCTIONS ON THE WEYL CHAMBER

For applications, we have also to consider the following situation.

Let  $G$  be a compact Lie group. Let  $T$  be a maximal torus of  $G$ ,  $\mathfrak{t}$  its Lie algebra,  $W$  be the Weyl group. Let  $\Lambda \subset \mathfrak{t}^*$  be the weight lattice of  $T$ . We choose a system  $\Delta^+ \subset \mathfrak{t}^*$  of positive roots, and let  $\rho \in \mathfrak{t}^*$  be the corresponding element. We consider the positive Weyl chamber  $\mathfrak{t}_{\geq 0}^*$  with interior  $\mathfrak{t}_{> 0}^*$ .

We consider now  $\mathcal{S}_{\geq 0}(\Lambda)$  the space of functions generated by the functions  $q[C_P]$  with polyhedrons  $P$  contained in  $\mathfrak{t}_{\geq 0}^*$ . This is a subspace of  $\mathcal{S}(\Lambda)$ . If  $t \in T$  is an element of finite order, the function  $m^t(\lambda, k) = t^\lambda m(\lambda, k)$  is again in  $\mathcal{S}_{\geq 0}(\Lambda)$ .

If  $m \in \mathcal{S}_{\geq 0}(\Lambda)$ , we define the following anti invariant distribution with value on a test function  $\varphi$  given by

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda} m(\lambda, k) \sum_{w \in W} \epsilon(w) \varphi(w(\lambda + \rho)/k)$$

**Proposition 3.1.** *If for every  $t \in T$  of finite order, we have  $\Theta_a(m^t) \equiv 0$ , then  $m = 0$ .*

*Proof.* Consider  $\varphi$  a test function supported in the interior of the Weyl chamber. Thus, for  $\lambda \geq 0$ ,  $\varphi(w(\lambda + \rho)/k)$  is not zero only if  $w = 1$ . So

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda \geq 0} m(\lambda, k) \varphi((\lambda + \rho)/k)$$

while

$$\langle \Theta(k), \varphi \rangle = \sum_{\lambda \geq 0} m(\lambda, k) \varphi(\lambda/k).$$

Let  $(\partial_\rho \varphi)(\xi) = \frac{d}{d\epsilon} \varphi(\xi + \epsilon \rho)|_{\epsilon=0}$  and consider the series of differential operators with constant coefficients  $e^{\partial_\rho/k} = 1 + \frac{1}{k} \partial_\rho + \dots$ . We then see that, if  $\langle A(\xi, k), \varphi \rangle$  is the asymptotic expansion of  $\langle \Theta(k), \varphi \rangle$ , the asymptotic expansion of  $\langle \Theta_a(k), \varphi \rangle$  is  $\langle A(\xi, k), e^{\partial_\rho/k} \varphi \rangle$ . Proceeding as in the proof of Theorem 1.8, we see that if  $\langle \Theta_a(m^t)(k), \varphi \rangle \equiv 0$  for all  $t \in T$  of finite order, then  $m(\lambda, k)$  is identically 0 when  $\lambda$  is on the interior of the Weyl chamber.

Consider all faces (closed)  $\sigma$  of the closed Weyl chamber. Define  $\mathcal{S}_{\ell, \geq 0} \subset \mathcal{S}(\Lambda)$  to be the space of  $m = \sum_{\sigma, \dim(\sigma) \leq \ell} m_\sigma$ , where  $m_\sigma \in \mathcal{S}_{\geq 0}(\Lambda)$  is such that  $m_\sigma(\lambda, k) = 0$  if  $\lambda$  is not in  $\sigma$ . Let us prove by induction on  $\ell$  that if  $m \in \mathcal{S}_{\ell, \geq 0}$  and  $\Theta_a^t(m^t) \equiv 0$ , for all  $t \in T$  of finite order, then  $m = 0$ .

If  $\ell = 0$ , then  $m(\lambda, k) = 0$  except if  $\lambda = 0$ , and our distribution is

$$m(0, k) \sum_w \epsilon(w) \varphi(w\rho/k).$$

Now, take for example  $\varphi(\xi) = \prod_{\alpha > 0} (\xi, H_\alpha) \chi(\xi)$  where  $\chi$  is invariant with small compact support and identically equal to 1 near 0. Then  $\langle \Theta_a(m), \varphi \rangle$  for  $k$  large is equal to  $c \frac{1}{k^N} m(0, k)$ , where  $N$  is the number of positive roots, and  $c$  a non zero constant. So we conclude that  $m(0, k) = 0$ .

Now consider  $m = \sum_{\dim \sigma = \ell} m_\sigma + \sum_{\dim f < \ell} m_f$ . Choose  $m_\sigma$  in the first sum. Let  $\sigma^0$  be the relative interior of  $\sigma$ . Let  $\Delta_0$  be the set of roots  $\alpha$ , such that  $\langle H_\alpha, \sigma \rangle = 0$ . Then  $\mathfrak{t}^* = \mathfrak{t}_1^* \oplus \mathfrak{t}_0^*$ , where  $\mathfrak{t}_0^* = \sum_{\alpha \in \Delta_0} \mathbb{R} \alpha$  and  $\mathfrak{t}_1^* = \mathbb{R} \sigma$ . We write  $\xi = \xi_0 + \xi_1$  for  $\xi \in \mathfrak{t}^*$ , with  $\xi_0 \in \mathfrak{t}_0^*$ ,  $\xi_1 \in \mathfrak{t}_1^*$ . Then  $\rho = \rho_0 + \rho_1$  with  $\rho_1 \in \mathfrak{t}_1^*$  and  $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$ . Let  $W_0$  be the subgroup



of the Weyl group generated by the reflections  $s_\alpha$  with  $\alpha \in \Delta_0$ . It leaves stable  $\sigma$ .

Consider  $\varphi$  a test function of the form  $\varphi_0(\xi_0)\varphi_1(\xi_1)$  with  $\varphi_0(\xi_0) = \chi_0(\xi_0) \prod_{\alpha \in \Delta_0^+} \langle \xi_0, H_\alpha \rangle$  with  $\chi_0(\xi_0)$  a function on  $\mathfrak{t}_0^*$  with small support near 0 and identically 1 near 0, while  $\varphi_1(\xi_1)$  is supported on a compact subset contained in  $\sigma^0$ .

For  $k$  large,

$$\langle \Theta_a^t, \varphi \rangle = \frac{1}{|W|} m_\sigma(\lambda, k) \sum_{w \in W_0} \phi(w(\lambda + \rho)/k).$$

So

$$\langle \Theta_a^t, \varphi \rangle = c_0 \frac{1}{k^{N_0}} \sum_{\lambda \in \sigma} m_\sigma(\lambda, k) \varphi_1((\lambda + \rho_1)/k).$$

As in the preceding case, this implies that  $m_\sigma(\lambda, k) = 0$  for  $\lambda \in \sigma^0$ . Doing it successively for all  $\sigma$  entering in the first sum, we conclude that  $m \in \mathcal{S}_{\geq 0, \ell-1}(\Lambda)$ . By induction, we conclude that  $m = 0$ .  $\square$

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