HAL
open science

Asymptotic distributions associated to piecewise quasi-polynomials

Paul-Emile Paradan, Michele Vergne

To cite this version:

Paul-Emile Paradan, Michele Vergne. Asymptotic distributions associated to piecewise quasi-polynomials. 2017. hal-01577829

HAL Id: hal-01577829

https://hal.science/hal-01577829v1

Preprint submitted on 28 Aug 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ASYMPTOTIC DISTRIBUTIONS ASSOCIATED TO PIECEWISE QUASI-POLYNOMIALS

PAUL-EMILE PARADAN AND MICHÈLE VERGNE

1. Introduction

Let V be a finite dimensional real vector space equipped with a lattice Λ. Let $P \subset V$ be a rational polyhedron. The Euler-Maclaurin formula ([4], [2]) gives an asymptotic estimate, when k goes to ∞, for the Riemann sum $\sum_{\lambda \in kP \cap \Lambda} \varphi(\lambda/k)$ of the values of a test function φ at the sample points $\frac{1}{k}\Lambda \cap P$ of P, with leading term $k^{\dim P} \int_P \varphi$. Here we consider the slightly more general case of a weighted sum. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We consider, for $k \geq 1$, the distribution

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} q(\lambda, k) \varphi(\lambda/k)$$

and we show (Proposition 1.2) that the function $k \mapsto \langle \Theta(P; q)(k), \varphi \rangle$ admits an asymptotic expansion when k tends to ∞ in powers of $1/k$ with coefficients periodic functions of k.

We extend this result to an algebra $\mathcal{S}(\Lambda)$ of piecewise quasi-polynomial functions on $\Lambda \oplus \mathbb{Z} \subset V \oplus \mathbb{R}$. A function $m(\lambda, k)$ ($\lambda \in \Lambda, k \in \mathbb{Z}$) in $\mathcal{S}(\Lambda)$ is supported in an union of polyhedral cones in $V \oplus \mathbb{R}$. The main feature of a function $m(\lambda, k)$ in $\mathcal{S}(\Lambda)$ is that $m(\lambda, k)$ is entirely determined by its large behavior in k. We associate to $m(\lambda, k)$ a formal series $A(m)$ of distributions on V encoding the asymptotic behavior of $m(\lambda, k)$ when k tends to ∞.

The motivating example is the case where M is a projective manifold, and $\mathcal{L}$ the corresponding ample bundle. If T is a torus acting on M, then write, for $t \in T$,

$$\sum_{i=0}^{\dim M} (-1)^i \mathrm{Tr}(t, H^i(M, \mathcal{O}(\mathcal{L}^k))) = \sum_{\lambda} m(\lambda, k) t^{\lambda}$$

where λ runs over the lattice Λ of characters of T. The corresponding asymptotic expansion of the distribution $\sum_\lambda m(\lambda, k) \delta_{\lambda/k}$ is an important object associated to M involving the Duistermaat-Heckmann

measure and the Todd class of M, see [9] for its determination. The determination of similar asymptotics in the more general case of twisted Dirac operators is the object of a forthcoming article [7].

Thus let $m \in \mathcal{S}(\Lambda)$, and consider the sequence

$$\Theta(m)(k) = \sum_{\lambda \in \Lambda} m(\lambda, k)\delta_{\lambda/k}$$

of distributions on V and its asymptotic expansion $A(m)$ when k tends to ∞. Let T be the torus with lattice of characters Λ. If $g \in T$ is an element of finite order, then $m^g(\lambda, k) := g^\lambda m(\lambda, k)$ is again in $\mathcal{S}(\Lambda)$. Our main result (Theorem 1.8) is that the piecewise quasi-polynomial function m is entirely determined by the collections of asymptotic expansions $A(m^g)$, when g varies over the set of elements of T of finite order.

We also prove (Proposition 2.1) a functorial property of $A(m)$ under pushforward.

We use these results to give new proofs of functoriality of the formal quantization of a symplectic manifold [5] or, more generally, of a spinc manifold [6].

For these applications, we also consider the case where V is a Cartan subalgebra of a compact Lie group, and anti-invariant distributions on V of a similar nature.

1.1. **Piecewise polynomial functions.** Let V be a real vector space equipped with a lattice Λ. Usually, an element of V is denoted by ξ, and an element of Λ by λ. In this article, a cone C will always be a closed convex polyhedral cone, and $0 \in C$.

Let Λ^* be the dual lattice, and let $g \in T := V^*/\Lambda^*$. If $G \in V^*$ is a representative of g and $\lambda \in \Lambda$, then we denote $g^\lambda = e^{2i\pi\langle G, \lambda\rangle}$.

A periodic function m on Λ is a function such that there exists a positive integer D (we do not fix D) such that $m(\lambda_0 + D\lambda) = m(\lambda_0)$ for $\lambda, \lambda_0 \in \Lambda$. The space of such functions is linearly generated by the functions $\lambda \mapsto g^\lambda$ for $g \in T$ of finite order. By definition, the algebra of quasi-polynomial functions on Λ is generated by polynomials and periodic functions on Λ. If V_0 is a rational subspace of V, the restriction of m to $\Lambda_0 := \Lambda \cap V_0$ is a quasi-polynomial function on Λ_0. The space of quasi-polynomial functions is graded: a quasi-polynomial function homogeneous of degree d is a linear combination of functions $t^\lambda h(\lambda)$ where $t \in T$ is of finite order, and h an homogeneous polynomial on V of degree d. Let $q(\lambda)$ be a quasi-polynomial function on Λ. There is a sublattice Γ of Λ of finite index d_Γ such that for any given $\gamma \in \Lambda$, we have $q(\lambda) = p_\gamma(\lambda)$ for any $\lambda \in \gamma + \Gamma$ where $p_\gamma(\xi)$

is a (uniquely determined) polynomial function on V. Then define $q_{pol}(\xi) = \frac{1}{d_\Gamma}\sum_{\gamma\in\Lambda/\Gamma} p_\gamma(\xi)$, a polynomial function on V. This polynomial function is independent of the choice of the sublattice Γ. Then $q(\lambda) - q_{pol}(\lambda)$ is a linear combination of functions of the form $t^\lambda h(\lambda)$ with $h(\lambda)$ polynomial and $t \neq 1$.

Using the Lebesgue measure associated to Λ, we identify generalized functions on V and distributions on V. If θ is a generalized function on V, we may write $\int_V \theta(\xi)\varphi(\xi)d\xi$ for its value on the test function φ. If R is a rational affine subspace of V, R inherits a canonical translation invariant measure. If P is a rational polyhedron in V, it generates a rational affine subspace of V, and $\int_P \varphi$ is well defined for φ a smooth function with compact support.

We say that a distribution $\theta(k)$ depending of an integer k is periodic in k if there exists a positive integer D such that for any test function φ on V, and $k_0, k \in \mathbb{Z}$, $\langle\theta(k_0 + Dk), \varphi\rangle = \langle\theta(k_0), \varphi\rangle$. Then there exists (unique) distributions θ_ζ indexed by D-th roots of unity such that $\langle\theta(k), \varphi\rangle = \sum_{\zeta,\zeta^D=1} \zeta^k \langle\theta_\zeta, \varphi\rangle$.

Let $(\Theta(k))_{k\geq 1}$ be a sequence of distributions. We say that $\Theta(k)$ admits an asymptotic expansion (with periodic coefficients) if there exists $n_0 \in \mathbb{Z}$ and a sequence of distributions $\theta_n(k), n \geq 0$, depending periodically of k, such that for any test function φ and any non negative integer N, we have

$$\langle\Theta(k), \varphi\rangle = k^{n_0} \sum_{n=0}^{N} \frac{1}{k^n} \langle\theta_n(k), \varphi\rangle + o(k^{n_0-N}).$$

We write

$$\Theta(k) \equiv k^{n_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \theta_n(k).$$

The distributions $\theta_n(k)$ are uniquely determined.

Given a sequence $\theta_n(k)$ of periodic distributions, and $n_0 \in \mathbb{Z}$, we write formally $M(\xi, k)$ for the series of distributions on V defined by

$$\langle M(\xi, k), \varphi\rangle = k^{n_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \int_V \theta_n(k)(\xi)\varphi(\xi)d\xi.$$

We can multiply $M(\xi, k)$ by quasi-polynomial functions $q(k)$ of k and smooth functions $h(\xi)$ of ξ and obtain the formal series $q(k)h(\xi)M(\xi, k)$ of the same form with n_0 changed to $n_0 + \text{degree}(q)$.

Let $E = V \oplus \mathbb{R}$, and we consider the lattice $\tilde{\Lambda} = \Lambda \oplus \mathbb{Z}$ in E. An element of $\tilde{\Lambda}$ is written as (λ, k) with $\lambda \in \Lambda$ and $k \in \mathbb{Z}$. We consider quasi-polynomial functions $q(\lambda, k)$ on $\tilde{\Lambda}$. As before, this space

is graded. We call the degree of a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$ the total degree. A quasi-polynomial function $q(\lambda, k)$ is of total degree d if it is a linear combination of functions $(\lambda, k) \mapsto j(k)t^\lambda k^a h(\lambda)$ where $j(k)$ is a periodic function of k, $t \in T$ of finite order, a a non negative integer, and h an homogeneous polynomial on V of degree b, with b such that $a + b = d$.

Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We construct $q_{pol}(\xi, k)$ on $V \times \mathbb{Z}$, and depending polynomially on ξ as before. We choose a sublattice of finite index d_Γ in Λ and functions $p_\gamma(\xi, k)$ depending polynomially on $\xi \in V$ and quasi-polynomial in k such that $q(\lambda, k) = p_\gamma(\lambda, k)$ if $\lambda \in \gamma + \Gamma$. Then $q_{pol}(\xi, k) = \frac{1}{d_\Gamma} \sum_{\gamma \in \Lambda/\Gamma} p_\gamma(\xi, k)$. We say that $q_{pol}(\xi, k)$ is the polynomial part (relative to Λ) of q. If q is homogeneous of total degree d, then the function $(k, \xi) \mapsto q_{pol}(k\xi, k)$ is a linear combination of functions of the form $j(k)k^d s(\xi)$ where $j(k)$ is a periodic function of k and $s(\xi)$ a polynomial function of ξ.

Proposition 1.1. *Let P be a rational polyhedron in V with non empty interior. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$ homogeneous of total degree d. Let $q_{pol}(\xi, k)$ be its polynomial part. Let $k \geq 1$. The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k)\varphi(\lambda/k)$$

admits an asymptotic expansion when $k \to \infty$ of the form

$$k^{\dim V} k^d \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

Furthermore, the term $k^d \langle \theta_0(k), \varphi \rangle$ is given by

$$k^d \langle \theta_0(k), \varphi \rangle = \int_P q_{pol}(k\xi, k)\varphi(\xi) d\xi$$

where q_{pol} is the polynomial part (with respect to Λ) of q.

Proof. Let $q(\lambda, k) = j(k)k^a g^\lambda h(\lambda)$ be a quasi-polynomial function of total degree d. Let

$$\langle \Theta_0^g(P)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} g^\lambda \varphi(\lambda/k). \tag{1.1}$$

If $\Theta_0^g(P)(k)$ admits the asymptotic expansion $M(\xi, k)$, then $\Theta(P; q)(k)$ admits the asymptotic expansion $j(k)k^a h(k\xi)M(\xi, k)$. So it is sufficient to consider the case where $q(\lambda, k) = g^\lambda$ and the distribution $\Theta_0^g(P)(k)$.

We now proceed as in [2] for the case $g = 1$ and sketch the proof. By decomposing the characteristic function $[P]$ of the polyhedron P

in a signed sum of characteristic functions of tangent cones, via the Brianchon Gram formula, then decomposing furthermore each tangent cone in a signed sum of cones C_a of the form $\Sigma_a \times R_a$ with Σ_a is a translate of a unimodular cone and R_a a rational space, we are reduced to study this distribution for the product of the dimension 1 following situations.

$V = \mathbb{R}, \Lambda = \mathbb{Z}$ and one of the following two cases:

- $P = \mathbb{R}$
- $P = s + \mathbb{R}_{\geq 0}$ with s a rational number.

For example, if $P = [a, b]$ is an interval in $\mathbb{R}$ with rational end points a, b, we write $[P] = [a, \infty] + [-\infty, b] - [\mathbb{R}]$.

For $P = \mathbb{R}$, and ζ a root of unity, it is easy to see that

$$\langle \Theta^\zeta(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}} \zeta^\mu \varphi(\mu/k)$$

is equivalent to $k \int_{\mathbb{R}} \varphi(\xi) d\xi$ if $\zeta = 1$ or is equivalent to 0 if $\zeta \neq 1$.

We now study the case where $P = s + \mathbb{R}_{\geq 0}$. Let

$$\langle \Theta^\zeta(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}, \mu - ks \geq 0} \zeta^\mu \varphi(\mu/k)$$

and let us compute its asymptotic expansion.

For $r \in \mathbb{R}$, the fractional part $\{r\}$ is defined by $\{r\} \in [0, 1[, r - \{r\} \in \mathbb{Z}$. If μ is an integer greater or equal to ks, then $\mu = ks + \{-ks\} + u$ with u a non negative integer.

We consider first the case where $\zeta = 1$. This case has been treated for example in [3] (Theorem 9.2.2), and there is an Euler-Maclaurin formula with remainder which leads to the following asymptotic expansion.

The function $z \mapsto \frac{e^{xz}}{e^z - 1}$ has a simple pole at $z = 0$. Its Laurent series at $z = 0$ is

$$\frac{e^{xz}}{e^z - 1} = \sum_{n=-1}^{\infty} B_{n+1}(x) \frac{z^n}{(n+1)!}$$

where $B_n(x)$ $(n \geq 0)$ are the Bernoulli polynomials.

If s is rational, and $n \geq 0$, the function $k \mapsto B_n(\{-ks\})$ is a periodic function of k with period the denominator of s, and

$$\sum_{\mu \in \mathbb{Z}, \mu \geq ks} \varphi(\frac{\mu}{k}) \equiv k(\int_s^\infty \varphi(\xi) d\xi - \sum_{n=1}^{\infty} \frac{1}{k^n} \frac{B_n(\{-ks\})}{n!} \varphi^{(n-1)}(s)).$$

This formula is easily proven by Fourier transform. Indeed, for $f(\xi) = e^{i\xi z}$, the series $\sum_{\mu \geq ks} f(\mu/k)$ is $\sum_{u \geq 0} e^{isz} e^{i\{-ks\}z/k} e^{iuz/k}$. It is convergent

if z is in the upper half plane, and the sum is

$$F(z)(k) = -e^{isz}\frac{e^{i\{-ks\}z/k}}{e^{iz/k}-1}.$$

So the Fourier transform of the tempered distribution $\Theta^{\zeta=1}(k)$ is the boundary value of the holomorphic function $z \mapsto F(z)(k)$ above. We can compute the asymptotic behavior of $F(z)(k)$ easily when k tends to ∞, since $\{-ks\} \leq 1$, and z/k becomes small.

Rewriting $[P]$ as the signed sum of the characteristic functions of the cones C_a, we see that the distribution $\Theta_0^g(P)(k)$ for $g = 1$ is equivalent to

$$k^{\dim V}(\sum_{n=0}^{\infty}\frac{1}{k^n}\theta_n(k))$$

with θ_0 independent of k, and given by $\langle\theta_0, \varphi\rangle = \int_P \varphi(\xi)d\xi$.

Now consider the case where $\zeta \neq 1$. Then

$$\sum_{\mu\in\mathbb{Z},\mu\geq ks}\zeta^{\mu}\varphi(\mu/k) = \sum_{u\geq 0}\zeta^{ks+\{-ks\}}\zeta^{u}\varphi(s+\{-ks\}/k+u/k).$$

The function $k \mapsto \zeta^{ks+\{-ks\}}$ is a periodic function of k with period ed if $\zeta^e = 1$ and ds is an integer. If $\zeta \neq 1$, the function $z \mapsto \frac{e^{xz}}{\zeta e^z-1}$ is holomorphic at $z = 0$. Define the polynomials $B_{n,\zeta}(x)$ via the Taylor series expansion:

$$\frac{e^{xz}}{\zeta e^z-1} = \sum_{n=0}^{\infty}B_{n+1,\zeta}(x)\frac{z^n}{(n+1)!}.$$

It is easily seen by Fourier transform that $\sum_{\mu\in\mathbb{Z},\mu\geq ks}\zeta^{\mu}\varphi(\mu/k)$ is equivalent to

$$-k\zeta^{ks+\{-ks\}}\sum_{n=1}^{\infty}\frac{1}{k^n}\frac{B_{n,\zeta}(\{-ks\})}{n!}\varphi^{(n-1)}(s).$$

In particular, $\Theta^{\zeta}(k)$ admits an asymptotic expansion in non negative powers of $1/k$ and each coefficient of this asymptotic expansion is a periodic distribution supported at s.

Rewriting $[P]$ in terms of the signed cones C_a, we see that indeed if $g \in T$ is not 1, one of the corresponding ζ in the reduction to a product of one dimensional cones is not 1, and so

$$\Theta_0^g(P)(k) \equiv k^{\dim V-1}(\sum_{n=0}^{\infty}\frac{1}{k^n}\theta_n(k)).$$

So we obtain our proposition.

□

Consider now P a rational polyhedron, with possibly empty interior. Let C_P be the cone of base P in $E = V \oplus \mathbb{R}$,

$$C_P := \{(t\xi, t), t \geq 0, \xi \in P\}.$$

Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We consider again

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} q(\lambda, k)\varphi(\lambda/k).$$

Consider the vector space E_P generated by the cone C_P in E. It is clear that $\Theta(P; q)$ depends only of the restriction r of q to $E_P \cap (\Lambda \oplus \mathbb{Z})$. This is a quasi-polynomial function on E_P with respect to the lattice $E_P \cap (\Lambda \oplus \mathbb{Z})$. We assume that the quasi-polynomial function r is homogeneous of degree d_0. This degree might be smaller that the total degree of q. Consider the affine space R_P generated by P in V. Let $E_P^{\mathbb{Z}} = E_P \cap (V \oplus \mathbb{Z})$. If $\xi \in R_P, k \in \mathbb{Z}$, then $(k\xi, k) \in E_P^{\mathbb{Z}}$. We will see shortly (Definition 1.3) that we can define a function $(\xi, k) \mapsto r_{pol}(\xi, k)$ for $(\xi, k) \in E_P^{\mathbb{Z}}$, and that the function $(\xi, k) \mapsto r_{pol}(k\xi, k)$ on $R_P \times \mathbb{Z}$ is a linear combination of functions of the form $k^{d_0} j(k) s(\xi)$ where $j(k)$ is a periodic function of k and $s(\xi)$ a polynomial function of ξ, for ξ varying on the affine space R_P.

We now can state the general formula.

Proposition 1.2. *Let P be a rational polyhedron in V. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. Let r be its restriction to $E_P \cap (\Lambda \oplus \mathbb{Z})$ and r_{pol} the "polynomial part" of r on $E_P \cap (V \oplus \mathbb{Z})$. Assume that the quasi-polynomial function r is homogeneous of degree d_0. Let $k \geq 1$. The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k)\varphi(\lambda/k)$$

admits an asymptotic expansion when $k \to \infty$ of the form

$$k^{\dim P} k^{d_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

Furthermore, the term $k^{d_0} \langle \theta_0(k), \varphi \rangle$ is given by

$$k^{d_0} \langle \theta_0(k), \varphi \rangle = \int_P r_{pol}(k\xi, k)\varphi(\xi) d\xi.$$

Proof. We will reduce the proof of this proposition to the case treated before of a polyhedron with interior. Let $\mathrm{lin}(P)$ be the linear space parallel to R_P, and $\Lambda_0 := \Lambda \cap \mathrm{lin}(P)$. If R_P contains a point $\beta \in \Lambda$, then E_P is isomorphic to $\mathrm{lin}(P) \oplus \mathbb{R}$ with lattice $\Lambda_0 \oplus \mathbb{Z}$. Otherwise, we will have to dilate R_P. More precisely, let $I_P = \{k \in \mathbb{Z}, kR_P \cap \Lambda \neq \emptyset\}$.

This is an ideal in $\mathbb{Z}$. Indeed if $k_1 \in I_P, k_2 \in I_P$, $\alpha_1, \alpha_2 \in R_P$ are such that $k_1\alpha_1 \in \Lambda$, $k_2\alpha_2 \in \Lambda$, then $\alpha_{1,2} = \frac{1}{n_1k_1+n_2k_2}(n_1k_1\alpha_1 + n_2k_2\alpha_2)$ is in R_P, and $(n_1k_1 + n_2k_2)(\alpha_{1,2}) \in \Lambda$. Thus there exists a smallest $k_0 > 0$ generating the ideal I_P. We see that our distribution $\Theta(P;p)(k)$ is identically equal to 0 if k is not in I_P. Let $\delta_{I_P}(k)$ be the function of k with

$$\delta_{I_P}(k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ 1 & \text{if } k = uk_0 \in I_P. \end{cases}$$

This is a periodic function of k of period k_0. We choose $\alpha \in R_P$ such that $k_0\alpha \in \Lambda$. We identify E_P to $\mathrm{lin}(P) \oplus \mathbb{R}$ by the map $T_\alpha(\xi_0, t) = (\xi_0 + tk_0\alpha, tk_0)$. In this identification, the lattice $(\Lambda \oplus \mathbb{Z}) \cap E_P$ becomes the lattice $\Lambda_0 \oplus \mathbb{Z}$. Consider $P_0 = k_0(P - \alpha)$, a polyhedron with interior in $\mathrm{lin}(P)$. Let $q^\alpha(\gamma, u) = r(\gamma + uk_0\alpha, uk_0)$. This is a quasi-polynomial function on $\Lambda_0 \oplus \mathbb{Z}$. Its total degree is d_0. We have defined its polynomial part $q^\alpha_{pol}(\xi, u)$ for $\xi \in \mathrm{lin}(P)$, $u \in \mathbb{Z}$.

Definition 1.3. *Let $(\xi, k) \in E_P^{\mathbb{Z}}$. Define:*

$$r_{pol}(\xi, k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ q^\alpha_{pol}(\xi - uk_0\alpha, u) & \text{if } k = uk_0 \in I_P. \end{cases}$$

The function $r_{pol}(\xi, k)$ does not depend of the choice of α. Indeed, if $\alpha, \beta \in R_P$ are such that $k_0\alpha, k_0\beta \in \Lambda$, then $q^\beta(\gamma, u) = q^\alpha(\gamma + uk_0(\beta - \alpha), u)$. Then we see that $q^\beta_{pol}(\xi, u) = q^\alpha_{pol}(\xi + uk_0(\beta - \alpha), u)$. Furthermore, the function $(k, \xi) \mapsto r_{pol}(k\xi, k)$ is of the desired form, a linear combination of functions $\delta_{I_P}(k)j(k)k^{d_0}s(\xi)$ with $s(\xi)$ polynomial functions on R_P.

If φ is a test function on V, we define the test function φ_0 on $\mathrm{lin}(P)$ by $\varphi_0(\xi_0) = \varphi(\frac{\xi_0}{k_0} + \alpha)$. We see that

$$\langle \Theta(P; q)(uk_0), \varphi\rangle = \langle \Theta(P_0; q^\alpha)(u), \varphi_0\rangle. \tag{1.2}$$

Thus we can apply Proposition 1.1. We obtain

$$\langle \Theta(P; q)(uk_0), \varphi\rangle \equiv u^{\dim P} u^{d_0} \sum_{n=0}^{\infty} \frac{1}{u^n} \langle \omega_n(u), \varphi_0\rangle.$$

We have

$$u^{d_0}\langle \omega_0(u), \varphi_0\rangle = \int_{P_0} q^\alpha_{pol}(u\xi_0, u)\varphi_0(\xi_0)d\xi_0 = \int_{P_0} q^\alpha_{pol}(u\xi_0, u)\varphi(\frac{\xi_0}{k_0} + \alpha)$$

When ξ_0 runs in $P_0 = k_0(P-\alpha)$, $\xi = \frac{\xi_0}{k_0} + \alpha$ runs over P. Changing variables, we obtain

$$u^{d_0}\langle \omega_0(u), \varphi_0\rangle = k^{d_0}\int_P r_{pol}(k\xi, k)\varphi(\xi)d\xi.$$

Thus we obtain our proposition. □

Let P be a rational polyhedron in V and q a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We do not assume that P has interior in V. We denote by $[C_P]$ the characteristic function of C_P. Then the function $q(\lambda, k)[C_P](\lambda, k)$ is zero if (λ, k) is not in C_P or equal to $q(\lambda, k)$ if (λ, k) is in C_P. We denote it by $q[C_P]$. The space of functions on $\Lambda \oplus \mathbb{Z}$ we will study is the following space.

Definition 1.4. *We define the space $\mathcal{S}(\Lambda)$ to be the space of functions on $\Lambda \oplus \mathbb{Z}$ linearly generated by the functions $q[C_P]$ where P runs over rational polyhedrons in V and q over quasi-polynomial functions on $\Lambda \oplus \mathbb{Z}$.*

The representation of m as a sum of functions $q[C_P]$ is not unique. For example, consider $V = \mathbb{R}$, $P = \mathbb{R}, P_+ := \mathbb{R}_{\geq 0}, P_- := \mathbb{R}_{\leq 0}, P_0 := \{0\}$, then $[C_P] = [C_{P_+}] + [C_{P_-}] - [C_{P_0}]$.

Example 1.5. *An important example of functions $m \in \mathcal{S}(\Lambda)$ is the following. Assume that we have a closed cone C in $V \oplus \mathbb{R}$, and a covering $C = \cup_\alpha C_\alpha$ by closed cones. Let m be a function on $C \cap (\Lambda \oplus \mathbb{Z})$, and assume that the restriction of m to $C_\alpha \cap (\Lambda \oplus \mathbb{Z})$ is given by a quasipolynomial function q_α. Then, using exclusion-inclusion formulae, we see that $m \in \mathcal{S}(\Lambda)$.*

Definition 1.6. *If $m(\lambda, k)$ belongs to $\mathcal{S}(\Lambda)$, and $g \in T$ is an element of finite order, then define*

$$m^g(\lambda, k) = g^\lambda m(\lambda, k).$$

The function m^g belongs to $\mathcal{S}(\Lambda)$.

If $m \in \mathcal{S}(\Lambda)$, and $k \geq 1$, we denote by $\Theta(m)(k)$ the distribution on V defined by

$$\langle \Theta(m)(k), \varphi\rangle = \sum_{\lambda \in \Lambda} m(\lambda, k)\varphi(\lambda/k),$$

if φ is a test function on V. The following proposition follows immediately from Proposition 1.2.

Proposition 1.7. *If $m(\lambda, k) \in \mathcal{S}(\Lambda)$, the distribution $\Theta(m)(k)$ admits an asymptotic expansion $A(m)(\xi, k)$.*

The function $m(\lambda, k)$ can be non zero, while $A(m)(\xi, k)$ is zero. For example let $V = \mathbb{R}$, $P = \mathbb{R}$ and $m(\lambda, k) = (-1)^\lambda$. Then $\Theta(m)(k)$ is the distribution on $\mathbb{R}$ given by $\mathrm{T}(k) = \sum_{\lambda=-\infty}^{\infty}(-1)^\lambda \delta_{\lambda/k}, k \geq 1$ and this is equivalent to 0. However, here is an unicity theorem.

Theorem 1.8. *Assume that $m \in \mathcal{S}(\Lambda)$ is such that $A(m^g) = 0$ for all $g \in T$ of finite order, then $m = 0$.*

Proof. We start by the case of a function $m = q[C_P]$ associated to a single polyhedron P and a quasi-polynomial function q. Assume first that P is with non empty interior P^0. If q is not identically 0, we write $q(\lambda, k) = \sum_{g\in T} g^{-\lambda} p_g(\lambda, k)$ where $p_g(\lambda, k)$ are polynomials in λ. If d is the total degree of q, then all the polynomials $p_g(\lambda, k)$ are of degree less or equal than d. We choose $t \in T$ such that $p_t(\lambda, k)$ is of degree d. If we consider the quasi-polynomial $q^t(\lambda, k)$, then its polynomial part is $p_t(\lambda, k)$ and the homogeneous component $p_t^{top}(\lambda, k)$ of degree d is not zero. We write $p_t^{top}(\xi, k) = \sum_{\zeta,a} \zeta^k k^a p_{\zeta,a}(\xi)$ where $p_{\zeta,a}(\xi)$ is a polynomial in ξ homogeneous of degree $d - a$. Testing against a test function φ and computing the term in $k^{d+\dim V}$ of the asymptotic expansion by Proposition 1.1, we see that $\sum_{\zeta,a} \zeta^k k^d \int_P p_{\zeta,a}(\xi)\varphi(\xi)d\xi = 0$. This is true for any test function φ. So, for any ζ, we obtain $\sum_a p_{\zeta,a}(\xi) = 0$. Each of the $p_{\zeta,a}$ being homogeneous of degree $d - a$, we see that $p_{\zeta,a} = 0$ for any a, ζ. Thus $p_t^{top} = 0$, a contradiction. So we obtain that $q = 0$, and $m = q[C_P] = 0$. Remark that to obtain this conclusion, we may use only test functions φ with support contained in the interior P^0 of P.

Consider now a general polyhedron P and the vector space $\mathrm{lin}(P)$. Let us prove that $m(\lambda, k) = q(\lambda, k)[C_P](\lambda, k)$ is identically 0 if $A(m^g) = 0$ for any $g \in T$ of finite order. Using the notations of the proof of Proposition 1.2, we see that $m(\lambda, k) = 0$, if k is not of the form uk_0. Furthermore, if $q^\alpha(\gamma, u) = q(\gamma + uk_0\alpha, uk_0)$, it is sufficient to prove that $q^\alpha = 0$. Let $P_0 = k_0(P - \alpha)$, a polyhedron with interior in V_0. Consider $m_0 = q^\alpha[C_{P_0}]$. We consider T as the character group of Λ, so T surjects on T_0. Let $g \in T$ of finite order and such that $g^{k_0\alpha} = 1$, and let g_0 be the restriction of g to Λ_0. Using Equation 1.1, we then see that

$$\langle \Theta(m^g)(uk_0), \varphi\rangle = \langle \Theta(m_0^{g_0})(u), \varphi_0\rangle.$$

Any $g_0 \in T_0$ of finite order is the restriction to Λ_0 of an element $g \in T$ of finite order and such that $g^{k_0\alpha} = 1$. So we conclude that the asymptotic expansion, when u tends to ∞, of $\Theta(m_0^{g_0})(u)$ is equal to 0 for any $g_0 \in T_0$ of finite order. Remark again that we need only to know that $\langle \Theta(m^g)(k), \varphi\rangle \equiv 0$ for test functions φ such that the support

S of φ is contained in a very small neighborhood of compact subsets of P contained in the relative interior of P^0.

For any integer ℓ, denote by $\mathcal{S}_\ell(\Lambda)$ the subspace of functions $m \in \mathcal{S}(\Lambda)$ generated by the functions $q[C_P]$ with $\dim P \leq \ell$.

When $\ell = 0$, our polyhedrons are a finite number of rational points $f \in V$, the function $m(\lambda, k)$ is supported on the union of lines $(ud_f f, ud_f)$ if d_f is the smallest integer such that $d_f f$ is in Λ. Choose a test function φ with support near f. Then $u \mapsto \langle \Theta(m)(d_f u), \varphi \rangle$ is identical to its asymptotic expansion $m(ud_f f, ud_f)\varphi(f)$. Clearly we obtain that $m = 0$.

If $m \in \mathcal{S}_\ell(\Lambda)$ by inclusion-exclusion, we can write

$$m = \sum_{P;\dim(P)=\ell} q_P[C_P] + \sum_{H,\dim H<\ell} q_H[C_H]$$

and we can assume that the intersections of a polyhedron P occurring in the first sum, with any polyhedron P' occurring in the decomposition of m and different from P is of dimension strictly less than ℓ. Consider P in the first sum, so $\dim(P) = \ell$. We can thus choose test functions φ with support in small neighborhoods of K, with K a compact subset contained in the relative interior of P. Then

$$\langle \Theta(m^g)(k), \varphi \rangle = \langle \Theta(q_P^g[C_P])(k), \varphi \rangle.$$

The preceding argument shows that $q_P[C_P] = 0$. So $m \in \mathcal{S}_{\ell-1}(\Lambda)$. By induction $m = 0$. □

2. Composition of piecewise quasi-polynomial functions

Let V_0, V_1 be vector spaces with lattice Λ_0, Λ_1.

Let $C_{0,1}$ be a closed polyhedral rational cone in $V_0 \oplus V_1$ (containing the origin). Thus for any $\mu \in \Lambda_1$, the set of $\lambda \in V_0$ such that $(\lambda, \mu) \in C_{0,1}$ is a rational polyhedron $P(\mu)$ in V_0. Let P be a polyhedron in V. We assume that for any $\mu \in \Lambda_1$, $P \cap P(\mu)$ is compact. Thus, for $m = q_P[C_P] \in \mathcal{S}(\Lambda)$, and $c(\lambda, \mu)$ a quasi-polynomial function on $\Lambda_0 \oplus \Lambda_1$, we can compute

$$m_c(\mu, k) = \sum_{(\lambda,\mu)\in C_{0,1}} m(\lambda, k)c(\lambda, \mu).$$

Proposition 2.1. *The function m_c belongs to $\mathcal{S}(\Lambda_1)$.*

Before establishing this result, let us give an example, which occur for example in the problem of computing the multiplicity of a representation $\chi^\lambda \otimes \chi^\lambda$ of $SU(2)$ restricted to the maximal torus.

Example 2.2. *Let $V_0 = V_1 = \mathbb{R}$, and $\Lambda_0 = \Lambda_1 = \mathbb{Z}$. Let $P := [0,2]$, and let*

$$q(\lambda, k) = \begin{cases} \frac{1}{2}(1-(-1)^\lambda) & \text{if } 0 \leq \lambda \leq 2k \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$C_{0,1} = \{(x,y) \in \mathbb{R}^2; x \geq 0, -x \leq y \leq x\}$$

and

$$c(\lambda, \mu) = \frac{1}{2}(1-(-1)^{\lambda-\mu}).$$

Let $\mu \geq 0$. Then

$$m_c(\mu, k) = \frac{1}{4} \sum_{0 \leq \lambda \leq 2k, \lambda \geq \mu} (1-(-1)^\lambda)(1-(-1)^{\lambda-\mu}) = (1+(-1)^\mu)\,(k/2 - \mu/4)\,.$$

So if $P_1 = [0,2], P_2 := [-2,0], P_3 := \{0\}$, we obtain

$$m_c = q_1[C_{P_1}] + q_2[C_{P_2}] + q_3[C_{P_3}]$$

with

$$\begin{cases} q_1(\mu, k) = (1+(-1)^\mu)\,(k/2 - \mu/4)\,, \\ q_2(\mu, k) = (1+(-1)^\mu)\,(k/2 + \mu/4)\,, \\ q_3(\mu, k) = -k. \end{cases}$$

We now start the proof of Proposition 2.1.

Proof. Write $c(\lambda, \mu)$ as a sum of products of quasi-polynomial functions $q_j(\lambda)$, $f_j(\mu)$, and $q_P(\lambda, k)$ a sum of products of quasi-polynomial functions $m_\ell(k)$, $h_\ell(\lambda)$. Then we see that it is thus sufficient to prove that, for $q(\lambda)$ a quasi-polynomial function of λ, the function

$$S(q)(\mu, k) = \sum_{\lambda \in kP \cap P(\mu)} q(\lambda) \tag{2.1}$$

belongs to $\mathcal{S}(\Lambda_1)$. For this, let us recall some results on families of polytopes $\mathfrak{p}(\mathbf{b}) \subset E$ defined by linear inequations. See for example [1], or [8].

Let E be a vector space, and $\omega_i, i = 1, \ldots, N$ be a sequence of linear forms on E. Let $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ be an element of $\mathbb{R}^N$. Consider the polyhedron $\mathfrak{p}(\mathbf{b})$ defined by the inequations

$$\mathfrak{p}(\mathbf{b}) = \{v \in E; \langle \omega_i, v \rangle \leq b_i, i = 1, \ldots, N\}.$$

We assume E equipped with a lattice L, and inequations ω_i defined by elements of L^*. Then if the parameters b_i are in $\mathbb{Z}^N$, the polytopes $\mathfrak{p}(\mathbf{b})$ are rational convex polytopes.

Assume that there exists $\mathbf{b}$ such that $\mathfrak{p}(\mathbf{b})$ is compact (non empty). Then $\mathfrak{p}(\mathbf{b})$ is compact (or empty) for any $\mathbf{b} \in \mathbb{R}^N$. Furthermore, there exists a closed cone $\mathcal{C}$ in $\mathbb{R}^N$ such that $\mathfrak{p}(\mathbf{b})$ is non empty if and only if $\mathbf{b} \in \mathcal{C}$. There is a decomposition $\mathcal{C} = \cup_\alpha \mathcal{C}_\alpha$ of $\mathcal{C}$ in closed polyhedral cones with non empty interiors, where the polytopes $\mathfrak{p}(\mathbf{b})$, for $\mathbf{b} \in \mathcal{C}_\alpha$, does not change of shape. More precisely:

• When $\mathbf{b}$ varies in the interior of $\mathcal{C}_\alpha$, the polytope $\mathfrak{p}(\mathbf{b})$ remains with the same number of vertices $\{s_1(\mathbf{b}), s_2(\mathbf{b}), \ldots, s_L(\mathbf{b})\}$.

• for each $1 \leq i \leq L$, there exists a cone C_i in E, such that the tangent cone to the polytope $\mathfrak{p}(\mathbf{b})$ at the vertex $s_i(\mathbf{b})$ is the affine cone $s_i(\mathbf{b}) + C_i$.

• the map $\mathbf{b} \to s_i(\mathbf{b})$ depends of the parameter $\mathbf{b}$, via linear maps $\mathbb{R}^N \to E$ with rational coefficients.

Furthermore -as proven for example in [1]- the Brianchon-Gram decomposition of $\mathfrak{p}(\mathbf{b})$ is "continuous" in $\mathbf{b}$ when b varies on C_α, in a sense discussed in [1].

Before continuing, let us give a very simple example, let b_1, b_2, b_3 be 3 real parameters and consider $\mathfrak{p}(b_1, b_2, b_3) = \{x \in \mathbb{R}, x \leq b_1, -x \leq b_2, -x \leq b_3\}$. So we are studying the intersection of the interval $[-b_2, b_1]$ with the half line $[-b_3, \infty]$. Then for $\mathfrak{p}(\mathbf{b})$ to be non empty, we need that $\mathbf{b} \in \mathcal{C}$, with

$$\mathcal{C} = \{\mathbf{b}; b_1 + b_2 \geq 0, b_1 + b_3 \geq 0\}.$$

Consider $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, with

$$\mathcal{C}_1 = \{\mathbf{b} \in \mathcal{C}; b_2 - b_3 \geq 0\},$$

$$\mathcal{C}_2 = \{\mathbf{b} \in \mathcal{C}; b_3 - b_2 \geq 0\}.$$

On $\mathcal{C}_1$ the vertices of $\mathfrak{p}(\mathbf{b})$ are $[-b_3, b_1]$, while on $\mathcal{C}_2$ the vertices of $\mathfrak{p}(\mathbf{b})$ are $[-b_2, b_1]$.

The Brianchon-Gram decomposition of $\mathfrak{p}(\mathbf{b})$ for $\mathbf{b}$ in the interior of $\mathcal{C}_1$ is $[-b_3, \infty] + [-\infty, b_1] - \mathbb{R}$. If $\mathbf{b} \in \mathcal{C}_1$ tends to the point $(b_1, b_2, -b_1)$ in the boundary of $\mathcal{C}$, we see the Brianchon-Gram decomposition tends to that $[b_1, \infty] + [-\infty, b_1] - \mathbb{R}$, which is indeed the polytope $\{b_1\}$.

Let $q(\gamma)$ be a quasi-polynomial function of $\gamma \in L$. Then, when $\mathbf{b}$ varies in $\mathcal{C}_\alpha \cap \mathbb{Z}^N$, the function

$$S(q)(\mathbf{b}) = \sum_{\gamma \in \mathfrak{p}(\mathbf{b}) \cap L} q(\gamma)$$

is given by a quasi-polynomial function of b. This is proven in [8], Theorem 3.8. In this theorem, we sum an exponential polynomial function $q(\gamma)$ on the lattice points of $\mathfrak{p}(\mathbf{b})$ and obtain an exponential polynomial function of the parameter b. However, the explicit formula shows

that if we sum up a quasi-polynomial function of γ, then we obtain a quasi-polynomial function of $\mathbf{b} \in \mathbb{Z}^N$. Another proof follows from [1] (Theorem 54) and the continuity of Brianchon-Gram decomposition. In [1], only the summation of polynomial functions is studied, via a Brianchon-Gram decomposition, but the same proof gives the result for quasi-polynomial functions (it depends only of the fact that the vertices vary via rational linear functions of $\mathbf{b}$). The relations between partition polytopes $P_\Phi(\xi)$ (setting used in [8], [1]) and families of polytopes $\mathfrak{p}(\mathbf{b})$ is standard, and is explained for example in the introduction of [1].

Consider now our situation with $E = V$ equipped with the lattice Λ. The polytope $kP \subset V$ is given by a sequence of inequalities $\omega_i(\xi) \leq ka_i$, $i = 1, \ldots, I$, where we can assume $\omega_i \in \Lambda^*$ and $a_i \in \mathbb{Z}$ by eventually multiplying by a large integer the inequality. The polytope $P(\mu)$ is given by a sequence of inequalities $\omega_j(\xi) \leq \nu_j(\mu)$, $j = 1, \ldots, J$ where ν_j depends linearly on μ. Similarly we can assume $\nu_j(\mu) \in \mathbb{Z}$. Let

$$(\mu, k) \mapsto \mathbf{b}(\mu, k) = [ka_1, \ldots, ka_I, \nu_1(\mu), \ldots, \nu_J(\mu)]$$

a linear map from $\Lambda_1 \oplus \mathbb{Z}$ to $\mathbb{Z}^N$. Our polytope $kP \cap P(\mu)$ is the polytope $\mathfrak{p}(\mathbf{b}(k, \mu))$ and

$$S(q)(\mu, k) = \sum_{\lambda \in \mathfrak{p}(\mathbf{b}(k,\mu)) \cap \Lambda} q(\lambda) = S(q)(\mathbf{b}(\mu, k)).$$

Consider one of the cones $\mathcal{C}_\alpha$. Then $\mathbf{b}(\mu, k) \in \mathcal{C}_\alpha$, if and only if (μ, k) belongs to a rational polyhedral cone C_α in $V_1 \oplus \mathbb{R}$. If Q is a quasi-polynomial function of $\mathbf{b}$, then $Q(\mathbf{b}(\mu, k))$ is a quasi-polynomial function of (μ, k). Thus on each of the cones C_α, $S(q)(\mu, k)$ is given by a quasi-polynomial function of (μ, k). From Example 1.5, we conclude that $S(q)$ belongs to $\mathcal{S}(\Lambda_1)$. □

3. Piecewise quasi-polynomial functions on the Weyl chamber

For applications, we have also to consider the following situation.

Let G be a compact Lie group. Let T be a maximal torus of G, $\mathfrak{t}$ its Lie algebra, W be the Weyl group. Let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of T. We choose a system $\Delta^+ \subset \mathfrak{t}^*$ of positive roots, and let $\rho \in \mathfrak{t}^*$ be the corresponding element. We consider the positive Weyl chamber $\mathfrak{t}^*_{\geq 0}$ with interior $\mathfrak{t}^*_{>0}$.

We consider now $\mathcal{S}_{\geq 0}(\Lambda)$ the space of functions generated by the functions $q[C_P]$ with polyhedrons P contained in $\mathfrak{t}^*_{\geq 0}$. This is a subspace of $\mathcal{S}(\Lambda)$. If $t \in T$ is an element of finite order, the function $m^t(\lambda, k) = t^\lambda m(\lambda, k)$ is again in $\mathcal{S}_{\geq 0}(\Lambda)$.

If $m \in \mathcal{S}_{\geq 0}(\Lambda)$, we define the following anti invariant distribution with value on a test function φ given by

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda} m(\lambda, k) \sum_{w \in W} \epsilon(w) \varphi(w(\lambda + \rho)/k)$$

Proposition 3.1. *If for every $t \in T$ of finite order, we have $\Theta_a(m^t) \equiv 0$, then $m = 0$.*

Proof. Consider φ a test function supported in the interior of the Weyl chamber. Thus, for $\lambda \geq 0$, $\varphi(w(\lambda + \rho)/k)$ is not zero only if $w = 1$. So

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda \geq 0} m(\lambda, k) \varphi((\lambda + \rho)/k)$$

while

$$\langle \Theta(k), \varphi \rangle = \sum_{\lambda \geq 0} m(\lambda, k) \varphi(\lambda/k).$$

Let $(\partial_\rho \varphi)(\xi) = \frac{d}{d\epsilon}`\varphi(\xi + \epsilon\rho)|_{\epsilon=0}$ and consider the series of differential operators with constant coefficients $e^{\partial_\rho/k} = 1 + \frac{1}{k}\partial_\rho + \cdots$. We then see that, if $\langle A(\xi, k), \varphi \rangle$ is the asymptotic expansion of $\langle \Theta(k), \varphi \rangle$, the asymptotic expansion of $\langle \Theta_a(k), \varphi \rangle$ is $\langle A(\xi, k), e^{\partial_\rho/k} \varphi \rangle$. Proceeding as in the proof of Theorem 1.8, we see that if $\langle \Theta_a(m^t)(k), \varphi \rangle \equiv 0$ for all $t \in T$ of finite order, then $m(\lambda, k)$ is identically 0 when λ is on the interior of the Weyl chamber.

Consider all faces (closed) σ of the closed Weyl chamber. Define $\mathcal{S}_{\ell, \geq 0} \subset \mathcal{S}(\Lambda)$ to be the space of $m = \sum_{\sigma, \dim(\sigma) \leq \ell} m_\sigma$, where $m_\sigma \in \mathcal{S}_{\geq 0}(\Lambda)$ is such that $m_\sigma(\lambda, k) = 0$ if λ is not in σ. Let us prove by induction on ℓ that if $m \in \mathcal{S}_{\ell, \geq 0}$ and $\Theta_a^t(m^t) \equiv 0$, for all $t \in T$ of finite order, then $m = 0$.

If $\ell = 0$, then $m(\lambda, k) = 0$ except if $\lambda = 0$, and our distribution is

$$m(0, k) \sum_{w} \epsilon(w) \varphi(w\rho/k).$$

Now, take for example $\varphi(\xi) = \prod_{\alpha > 0} (\xi, H_\alpha) \chi(\xi)$ where χ is invariant with small compact support and identically equal to 1 near 0. Then $\langle \Theta_a(m), \varphi \rangle$ for k large is equal to $c \frac{1}{k^N} m(0, k)$, where N is the number of positive roots, and c a non zero constant. So we conclude that $m(0, k) = 0$.

Now consider $m = \sum_{\dim \sigma = \ell} m_\sigma + \sum_{\dim f < \ell} m_f$. Choose m_σ in the first sum. Let σ^0 be the relative interior of σ. Let Δ_0 be the set of roots α, such that $\langle H_\alpha, \sigma \rangle = 0$. Then $\mathfrak{t}^* = \mathfrak{t}_1^* \oplus \mathfrak{t}_0^*$, where $\mathfrak{t}_0^* = \sum_{\alpha \in \Delta^0} \mathbb{R}\alpha$ and $\mathfrak{t}_1^* = \mathbb{R}\sigma$. We write $\xi = \xi_0 + \xi_1$ for $\xi \in \mathfrak{t}^*$, with $\xi_0 \in \mathfrak{t}_0^*, \xi_1 \in \mathfrak{t}_1^*$. Then $\rho = \rho_0 + \rho_1$ with $\rho_1 \in \mathfrak{t}_1^*$ and $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$. Let W_0 be the subgroup

of the Weyl group generated by the reflections s_α with $\alpha \in \Delta_0$. It leaves stable σ.

Consider φ a test function of the form $\varphi_0(\xi_0)\varphi_1(\xi_1)$ with $\varphi_0(\xi_0) = \chi_0(\xi_0)\prod_{\alpha\in\Delta_0^+}\langle \xi_0, H_\alpha\rangle$ with $\chi_0(\xi_0)$ a function on $\mathfrak{t}_0^*$ with small support near 0 and identically 1 near 0, while $\varphi_1(\xi_1)$ is supported on a compact subset contained in σ^0.

For k large,

$$\langle \Theta_a^t, \varphi\rangle = \frac{1}{|W|} m_\sigma(\lambda, k) \sum_{w\in W_0} \phi(w(\lambda+\rho)/k).$$

So

$$\langle \Theta_a^t, \varphi\rangle = c_0 \frac{1}{k^{N_0}} \sum_{\lambda\in\sigma} m_\sigma(\lambda, k)\varphi_1((\lambda+\rho_1)/k).$$

As in the preceding case, this implies that $m_\sigma(\lambda, k) = 0$ for $\lambda \in \sigma^0$. Doing it successively for all σ entering in the first sum, we conclude that $m \in \mathcal{S}_{\geq 0,\ell-1}(\Lambda)$. By induction, we conclude that $m = 0$. □

References

[1] Nicole Berline and Michèle Vergne, *Analytic continuation of a parametric polytope and wall-crossing*, Configuration spaces, CRM Series, vol. 14, Ed. Norm., Pisa, 2012, pp. 111–172.

[2] ———, *Local asymptotic Euler-Maclaurin expansion for Riemann sums over a semi-rational polyhedron*, arXiv **1502.01671** (2015).

[3] Henri Cohen, *Number theory. Vol. II. Analytic and modern tools*, Graduate Texts in Mathematics, vol. 240, Springer, New York, 2007.

[4] Victor Guillemin and Shlomo Sternberg, *Riemann sums over polytopes*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 7, 2183–2195 (English, with English and French summaries). Festival Yves Colin de Verdière.

[5] Paul-Émile Paradan, *Formal geometric quantization*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 1, 199–238 (English, with English and French summaries).

[6] ———, *Formal Geometric Quantization III, Functoriality in the spin-c setting*, arxiv **1704.06034** (2017).

[7] Paul-Emile Paradan and Michèle Vergne, *The equivariant index of twisted Dirac operators and semi-classical limits*, to appear (2017).

[8] András Szenes and Michèle Vergne, *Residue formulae for vector partitions and Euler-MacLaurin sums*, Adv. in Appl. Math. **30** (2003), no. 1-2, 295–342. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[9] Michèle Vergne, *The equivariant Riemann-Roch theorem and the graded Todd class*, arXiv **1612.04651** (2016).

Institut Montpelliérain Alexander Grothendieck, CNRS , Université de Montpellier
E-mail address: paul-emile.paradan@umontpellier.fr

Université Denis Diderot, Institut Mathématique de Jussieu, Sophie Germain
E-mail address: michele.vergne@imj-prg.fr