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# Robust Viability Analysis of a Controlled Epidemiological Model

Lilian Sofia Sepulveda Salcedo\*      Michel De Lara†

August 28, 2017

## Abstract

Managing infectious diseases is a world public health issue, plagued by uncertainties. In this paper, we analyze the problem of viable control of a dengue outbreak under uncertainty. For this purpose, we develop a controlled Ross-Macdonald model in discrete time, with mosquito vector control by fumigation and with uncertainties affecting the dynamics. The robust viability kernel is the set of all initial states such that there exists at least a strategy of insecticide spraying which guarantees that the number of infected people remains below a threshold, for all times, and whatever the sequences of uncertainties (scenarios). Having chosen three nested subsets of uncertainties — a deterministic one (without uncertainty), a medium one and a large one — we can measure the incidence of the uncertainties on the size of the kernel, in particular on its reduction with respect to the deterministic case. The numerical results show that the viability kernel without uncertainties is highly sensitive to the variability of parameters — here the biting rate, the probability of infection to mosquitoes and humans, and the proportion of female mosquitoes per person. So the robust viability kernel is a possible tool to reveal the importance of uncertainties regarding epidemics control.

**Keywords:** epidemics control; viability; uncertainty and robustness; Ross-Macdonald model; dengue.

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## 1 Introduction

Managing infectious diseases is a world public health issue. The joint dynamics of infectious agents, vectors and hosts, and their spatial movements make the control or eradication problems difficult. On top of that, uncertainties abound. Despite advances in epidemiological surveillance systems, and data on infected, recovered people, etc., there remain inaccuracies and errors. Factors such as ambient temperature, host age, social customs also contribute to uncertainty. In this paper, we focus on the impact of uncertainty on the viable control of a Ross-Macdonald epidemiological model.

In [8], we used viability theory to analyze the problem of maintaining the number of infected individuals below a threshold, with limited fumigation capacity. The setting was deterministic, without any uncertainty in the model parameters. We said that a state is viable if there exists at least one admissible control trajectory — time-dependent mosquito mortality rates bounded

by control capacity — such that, starting from this state, the resulting proportion of infected individuals remains below a given infection cap for all times. We defined the so-called viability kernel as the set of viable states. We obtained three different expressions of the viability kernel, depending on the couple control capacity-infection cap.

In this paper, we analyze what happens to the viability kernel when additional uncertain factors, that affect the dynamics of the disease, are considered. We make use of the so-called *robust viability* approach [7], looking for vector fumigation policies able to maintain the proportion of infected individuals remains below a given infection cap, for all times and for all scenarios of uncertainties. Several studies have applied the deterministic viable control method to managing natural resources, for example [3], [5] and [4]. In discrete time, different examples can be found in [7]. Yet, few studies have undertaken a robust approach to these issues [2, 12]. Comparison of deterministic and robust viable states can contribute to shed light on the distance between the outcomes of these two extreme approaches: ignoring uncertainty versus hedging against any risk (totally risk averse context [2]). In robust viability, constraints must be satisfied even under uncertainties related to an unlikely pessimistic scenario. By contrast, reducing uncertainties to zero amounts at dressing the problem as deterministic [1].

The paper is organized as follows. In Section 2, we introduce the Ross-Macdonald model in discrete time, discuss uncertainties and set the robust viability problem. Then, we introduce the robust viability kernel and present the dynamic programming equation to compute it. In Section 3, we provide a numerical application in the case of the dengue outbreak in 2013 in Cali, Colombia. We evaluate the impact of different sets of uncertainties on the robust viability kernel. We conclude in Section 4.

## 2 The Robust Viability Problem

In §2.1, we introduce the Ross-Macdonald model in discrete time, we discuss different uncertainties and we set the robust viability problem. Then, in §2.2 we introduce the robust viability kernel. In §2.3, we present the dynamic programming equation to compute the robust viability kernel.

## 2.1 Discrete time Ross-Macdonald model with uncertainties

To address the robust viability problem for the Ross-Macdonald model with uncertainties, we work with a model in discrete time, with time period one day denoted by

$$t = t_0, t_0 + 1, \dots, T - 1, T, \quad (1)$$

where  $t_0 \in \mathbb{N}$  is the initial time (day) and  $T \in \mathbb{N}$ ,  $T \geq t_0 + 1$  is the horizon. Working in continuous time would make the introduction of uncertainties more delicate, as well as the numerical calculation of the robust viability kernel.

### Discrete time Ross-Macdonald model

From the continuous time controlled Ross-Macdonald model (34) used in [8] (see Appendix A.1), we obtain a discrete time model as follows (see details in Appendix A.2). First, we suppose that mosquito mortality rates induced by fumigation remain stationary during a time period (day). Second, we replace time derivatives by increments over a time period. We obtain the following controlled Ross-Macdonald model in discrete time

$$M(t+1) = A_M(t)H(t)(1 - M(t)) + (1 - u(t))M(t) \quad (2a)$$

$$H(t+1) = A_H(t)M(t)(1 - H(t)) + (1 - \gamma)H(t), \quad (2b)$$

where

- time  $t$  runs from initial day  $t_0$  to horizon  $T$ , with time step one day, as in (1),
- the state vector  $(M(t), H(t)) \in [0, 1]^2$  represents the proportion of mosquitoes and the proportion of humans that are infected at the beginning of day  $t$ ,
- the control variable  $u(t)$  represents the mosquito mortality rate applied during all day  $t$ , due to application of chemical control over the mosquitoes,
- the parameter  $\gamma \geq 0$  is the recovery rate for humans ( $\text{day}^{-1}$ ),

- the vector  $(A_M(t), A_H(t)) \in [0, +\infty]^2$  — whose components are infection rates for mosquito and human, respectively — carries all the uncertainties; we discuss them below.

The state space  $[0, 1]^2$  is invariant by the dynamics (2) under mild assumptions. Indeed, as soon as

$$0 \leq A_M(t) \leq 1, \quad 0 \leq A_H(t) \leq 1, \quad 0 \leq \gamma \leq 1, \quad 0 \leq u(t) \leq 1, \quad (3)$$

we easily check that

$$(M(t), H(t)) \in [0, 1]^2 \Rightarrow (M(t+1), H(t+1)) \in [0, 1]^2, \quad (4)$$

as  $M(t+1)$  and  $H(t+1)$  are convex combinations of quantities in  $[0, 1]$  by (2).

## Uncertainties

Uncertain factors in the dynamics of dengue transmission are the number of larval breeding sites, the rapid growth and urbanization of humans population, seasonal variability of mosquitoes population correlated with environmental factors such as rainfall, etc. The interactions between all these factors yield significant differences in parameters values — such as mosquito bite rate, the proportion of mosquitoes females, probabilities of infection in humans and mosquitoes — involved in the formulation of compartmental models of the process of transmission of an infectious diseases as dengue.

In the continuous time Ross-Macdonald model (34) outlined in Appendix A.1, we consider the following parameters as uncertain: bite rate ( $\alpha$ ), probability of mosquito infection ( $p_m$ ), probability of infection of a human ( $p_H$ ) and number of female mosquitoes per person ( $\frac{N_m}{N_h}$ ). As detailed in Appendix A.5, we decided to incorporate uncertainty in the discrete time Ross-Macdonald model (2) by propagating the uncertainties in the parameters of the continuous time Ross-Macdonald model through the aggregate parameters  $A_m = \alpha p_m$  and  $A_h = \alpha p_h \frac{M}{H}$  in (35). Thus, the couples

$$(A_M(t), A_H(t)) \in \mathbb{R}^2, \quad \forall t = t_0, \dots, T-1 \quad (5)$$

represent uncertainties variables that affect the population of mosquitoes and humans population, respectively.

We define a *scenario* of uncertainties as a sequence, of length  $T - t_0$  of uncertainty couples:

$$(A_M(\cdot), A_H(\cdot)) = \left( (A_M(t_0), A_H(t_0)), \dots, (A_M(T-1), A_H(T-1)) \right) \in (\mathbb{R}^2)^{T-t_0}. \quad (6)$$

## 2.2 The robust viability kernel

Viable management in the robust sense evokes a pessimistic context. Here, it means that we want to satisfy the constraint of keeping the number of infected humans below the threshold imposed, *whatever the uncertainties*. We now give precise mathematical statement of this problem.

### Control constraints

Let  $(\underline{u}, \bar{u})$  be a couple such that

$$0 \leq \underline{u} \leq \bar{u} \leq 1 . \quad (7)$$

We impose that the control variable  $u(t)$  — the mosquito mortality rate applied during all day  $t$  in (2) — satisfies the constraints

$$\underline{u} \leq u(t) \leq \bar{u} . \quad (8)$$

We suppose that the upper and lower control bounds satisfy (7) to make it possible that the state space  $[0, 1]^2$  is invariant by the dynamics (2) — see (3) and see (4).

### Uncertainty constraints

The two terms  $A_M(t)$  and  $A_H(t)$  in (2) encapsulate all the uncertainties affecting each population, respectively. We suppose that they can take any value in a known *set*  $\mathbb{S}$  of *uncertainties*:

$$(A_M(t), A_H(t)) \in \mathbb{S} \subset [0, 1]^2 . \quad (9)$$

We impose  $\mathbb{S} \subset [0, 1]^2$  so that, jointly with (7), the state space  $[0, 1]^2$  is invariant by the dynamics (2), by (3) and (4). We could have allowed the set  $\mathbb{S}$  to differ from one time to another (hence being a sequence of  $\mathbb{S}(t)$  for  $t = t_0, \dots, T-1$ ). However, for simplicity reasons, we consider the stationary case.

### State constraints

Let  $\bar{H}$  be a real number such that

$$0 < \bar{H} < 1 , \quad (10)$$

which represents the *maximum tolerated proportion of infected humans*, or *infection cap*. Thinking about public health policies set by governmental entities, we impose the following constraint: the proportion  $H(t)$  of infected humans must always remain below the infection cap  $\bar{H}$ . Therefore, we impose the state constraint

$$H(t) \leq \bar{H}, \quad \forall t = t_0, \dots, T-1, T. \quad (11)$$

### Strategies

To define the robust viability kernel, we need the notion of strategy. A *control strategy*  $\mathbf{u}$  is a sequence of mappings from states  $(M, H)$  towards controls  $u$  as follows:

$$\mathbf{u} = (\mathbf{u}_t)_{t=t_0, \dots, T-1} \text{ with } \mathbf{u}_t : [0, 1]^2 \rightarrow \mathbb{R}. \quad (12)$$

A control strategy  $\mathbf{u}$  is said to be *admissible* if

$$\mathbf{u}_t : [0, 1]^2 \rightarrow [\underline{u}, \bar{u}]. \quad (13)$$

For any scenario  $(A_M(\cdot), A_H(\cdot))$  as in (6), and any initial state

$$(M(0), H(0)) = (M_0, H_0) \in [0, 1]^2, \quad (14)$$

a control strategy  $\mathbf{u}$  as in (12) produces — through the dynamics (2) — a state trajectory  $(M(\cdot), H(\cdot))$  by the closed-loop dynamics

$$M(t+1) = A_M(t)H(t)(1 - M(t)) + \left(1 - \mathbf{u}_t(M(t), H(t))\right)M(t) \quad (15a)$$

$$H(t+1) = A_H(t)M(t)(1 - H(t)) + (1 - \gamma)H(t), \quad (15b)$$

for  $t = t_0, \dots, T-1$ , and a control trajectory  $u(\cdot)$  by

$$u(t) = \mathbf{u}_t(M(t), H(t)), \quad t = t_0, \dots, T-1. \quad (16)$$

### Robust viability kernel

Now, we are ready to lay out the definition of the robust viability kernel.



**Definition 1.** *The robust viability kernel (at initial time  $t_0$ ) is*

$$\mathbb{V}_{\mathbb{S}}(t_0) = \left\{ \begin{array}{l} (M_0, H_0) \\ \in [0, 1]^2 \end{array} \left| \begin{array}{l} \text{there is at least one} \\ \text{admissible strategy } \mathbf{u} \text{ as in (13)} \\ \text{such that, for every scenario} \\ (A_M(\cdot), A_H(\cdot)) \in \mathbb{S}^{T-t_0}, \\ \text{the state trajectory} \\ (M(\cdot), H(\cdot)) \text{ given by (14)–(15)} \\ \text{satisfies the state constraint (11)} \end{array} \right. \right\}. \quad (17)$$

The states belonging to the robust viability kernel are called *viable robust states*.

We use the notation  $\mathbb{V}_{\mathbb{S}}(t_0)$  to mark the dependency of the robust viability kernel of (17) with respect to the set  $\mathbb{S}$  of uncertainties in (9). Indeed, in the sequel, we will study how  $\mathbb{V}_{\mathbb{S}}(t_0)$  varies with  $\mathbb{S}$ .

### 2.3 Dynamic programming equation

In (17), we observe that the set of scenarios — with respect to which the robust viability kernel is defined — is the rectangle  $\mathbb{S}^{T-t_0}$ . This is a strong assumption that corresponds to stagewise independence of uncertainties. Indeed, when the head uncertainty trajectory  $\left( (A_M(t_0), A_H(t_0)), \dots, (A_M(t), A_H(t)) \right)$  is known, the domain where the tail uncertainty trajectory  $\left( (A_M(t+1), A_H(t+1)), \dots, (A_M(T-1), A_H(T-1)) \right)$  takes its value does not depend on the head uncertainty trajectory, as it is the rectangle  $\mathbb{S}^{T-t+1}$ .

This rectangularity property of the set of scenarios makes it possible to compute the robust viability kernel  $\mathbb{V}_{\mathbb{S}}(t_0)$  of (17) by dynamic programming. The proof of the following result is an easy extension of a proof to be found in [7].

Let  $\Phi$  denote the dynamics corresponding to (2):

$$\Phi(M, H, u, A_M, A_H) = \begin{pmatrix} A_M H (1 - M) + (1 - u) M \\ A_H M (1 - H) + (1 - \gamma) H \end{pmatrix}. \quad (18)$$

Let us introduce the so-called *constraints set*

$$\mathbb{A} = \{(M, H) | 0 \leq M \leq 1, 0 \leq H \leq \overline{H}\} = [0, 1] \times [0, \overline{H}] \quad (19)$$

and let  $1_{\mathbb{A}}(M, H)$  denote its *indicator function*

$$1_{\mathbb{A}}(M, H) = \begin{cases} 1 & \text{if } (M, H) \in \mathbb{A} , \\ 0 & \text{if } (M, H) \notin \mathbb{A} . \end{cases} \quad (20)$$

**Proposition 2.** *The robust viability kernel  $\mathbb{V}_{\mathbb{S}}(t_0)$  of (17) is given by*

$$\mathbb{V}_{\mathbb{S}}(t_0) = \{(M, H) \in [0, 1]^2 \mid \mathbf{V}_{t_0}(M, H) = 1\} , \quad (21)$$

where the function  $\mathbf{V}_{t_0}$  is solution of the following backward induction — called dynamic programming equation — that connects the so-called value functions

$$\mathbf{V}_T(M, H) = 1_{\mathbb{A}}(M, H) \quad (22a)$$

$$\mathbf{V}_t(M, H) = 1_{\mathbb{A}}(M, H) \sup_{u \in [\underline{u}, \bar{u}]} \inf_{(A_M, A_H) \in \mathbb{S}} \mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H)) , \quad (22b)$$

where  $t$  runs down from  $T - 1$  to  $t_0$ .

We are now equipped with the concept of robust viability kernel and we dispose of a method to compute it.

### 3 Numerical Results and Viability Analysis

Now, after having set the theory in Section 2, we will provide a numerical application in the case of the dengue outbreak in 2013 in Cali, Colombia. We introduce three nested sets of uncertainties in §3.1, we study their impact on the robust viability kernels computed in §3.2, and finally, in §3.3, we discuss the results thus obtained.

#### 3.1 Three nested sets of uncertainties

With the purpose of evaluating the sensitivity of the size and shape of the robust viability kernel  $\mathbb{V}_{\mathbb{S}}(t_0)$  of (17) with respect to the set  $\mathbb{S}$  of uncertainties in (9), we consider the following three possible cases —  $\mathring{\mathbb{S}}$ ,  $\bar{\mathbb{S}}$  and  $\bar{\bar{\mathbb{S}}}$  — for the set  $\mathbb{S}$ :

L) Low case (deterministic)

$$\mathring{\mathbb{S}} = \left\{ \mathring{A}_M \right\} \times \left\{ \mathring{A}_H \right\}, \quad (23)$$

M) Middle case

$$\bar{\mathbb{S}} = \left[ \underline{A}_M, \bar{A}_M \right] \times \left[ \underline{A}_H, \bar{A}_H \right], \quad (24)$$

H) High case

$$\bar{\bar{\mathbb{S}}} = \left[ \underline{\underline{A}}_M, \bar{\bar{A}}_M \right] \times \left[ \underline{\underline{A}}_H, \bar{\bar{A}}_H \right]. \quad (25)$$

We make the additional assumption that

$$\mathring{\mathbb{S}} \subset \bar{\mathbb{S}} \subset \bar{\bar{\mathbb{S}}} \subset [0, 1]^2, \quad (26)$$

that is,

$$0 \leq \underline{\underline{A}}_M \leq \underline{A}_M \leq \mathring{A}_M \leq \bar{A}_M \leq \bar{\bar{A}}_M \leq 1, \quad (27a)$$

$$0 \leq \underline{\underline{A}}_H \leq \underline{A}_H \leq \mathring{A}_H \leq \bar{A}_H \leq \bar{\bar{A}}_H \leq 1. \quad (27b)$$

By Definition 1, we easily obtain that

$$\mathbb{V}_{\bar{\bar{\mathbb{S}}}}(t_0) \subset \mathbb{V}_{\bar{\mathbb{S}}}(t_0) \subset \mathbb{V}_{\mathring{\mathbb{S}}}(t_0), \quad (28)$$

as less and less initial states can comply with the constraints in (17) as the set  $\mathbb{S}$  of uncertainties increases.

### 3.2 Numerical computation of robust viability kernels

For the numerical calculation of the robust viability kernel, we use the dynamic programming equation (22) with horizon

$$T = 60 \text{ days}. \quad (29)$$

We discretize the state space  $[0, 1]^2$ , the control space (see (47b)-(47c))

$$[\underline{u}, \bar{u}] = [0.0333, 0.05] \text{ day}^{-1} \quad (30)$$

and the sets  $\bar{\mathbb{S}}$  and  $\bar{\bar{\mathbb{S}}}$  of uncertainties in (24)-(25). We take a partition of 70 elements for

- the interval  $[0, 1]$ , where the proportion  $M$  of infected mosquitoes takes its values,
- the interval  $[0, \overline{H}]$ , where the proportion  $H$  of infected humans takes its values,
- the interval  $[\underline{u}, \overline{u}]$ , where the control variable  $u$  takes its values,
- the intervals  $[\underline{A}_M, \overline{A}_M]$  and  $[\underline{A}_H, \overline{A}_H]$  (or the intervals  $[\underline{\underline{A}}_M, \overline{\overline{A}}_M]$  and  $[\underline{\underline{A}}_H, \overline{\overline{A}}_H]$ ) where the uncertainties  $A_M(t)$  and  $A_H(t)$  take their values, respectively, as in (24)-(25).

To compute the robust viability kernel  $\mathbb{V}_S(t_0)$  by (22), we implement the following dynamic programming Algorithm 1.

```

Initialization  $\mathbf{V}_T(M, H) = 1_{\mathbb{A}}(M, H)$ ;
for  $t = T, T - 1, \dots, t_0$  do
  forall  $(M, H) \in [0, 1] \times [0, \overline{H}]$  do
    forall  $u$  do
      forall  $(A_M, A_H)$  do
         $\mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H))$ 
         $\min_{(A_M, A_H)} \mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H))$ 
         $\max_u \min_{(A_M, A_H)} \mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H))$ 
       $\mathbf{V}_t(M, H) = 1_{\mathbb{A}}(M, H) \mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H))$ 

```

**Algorithm 1:** Dynamic programming algorithm to compute the robust viability kernel by (22)

In general, the image  $\Phi(M, H, u, A_M, A_H)$  of the dynamics does not fall exactly on one element of the  $70 \times 70$  grid — corresponding to the cells of the state space  $[0, 1] \times [0, \overline{H}]$  — over which the numerical value function  $\mathbf{V}_{t+1}$  is defined. We have taken a conservative stand: we put  $\mathbf{V}_{t+1}(\Phi(M, H, u, A_M, A_H)) = 1$  if and only if  $\mathbf{V}_{t+1} = 1$  for *all* points in the grid that surround the image  $\Phi(M, H, u, A_M, A_H)$ .

The (discrete) robust viability kernel is defined as the set of points of the  $70 \times 70$  grid where the indicator function  $\mathbf{V}_{t_0}(M, H)$  is equal to 1.

### 3.3 Robust viability analysis

Now, as we are able to compute robust viability kernels as just seen in §3.2, we will compare them under the three nested sets of uncertainties introduced in §3.1.

#### Comparison between deterministic viability kernels in discrete time and in continuous time

We consider the low case (deterministic), for which we take

$$\mathring{\mathbb{S}} = \{0.07660\} \times \{0.00722\} \text{ day}^{-1} \times \text{day}^{-1} . \quad (31)$$

These numbers were obtained in (47a) from the parameters adjusted to the 2013 dengue outbreak in Cali, Colombia, as described in [8]. Details can be found in Appendix A.

In Figure 1, we present both the deterministic viability kernel in discrete time — that is, obtained when  $\mathbb{S} = \mathring{\mathbb{S}}$  in (23) — and the deterministic viability kernel in continuous time — that is, the one obtained in [8]. The solid line corresponds to the discrete time case and the dotted line corresponds to the continuous time case.

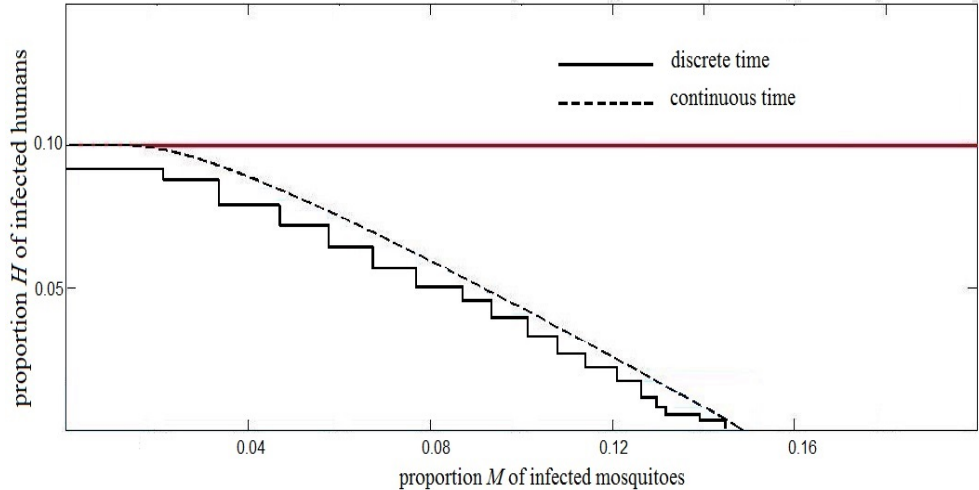


Figure 1: Viability kernels for the deterministic case in discrete time ( $\mathring{\mathbb{S}} = \{0.07660\} \times \{0.00722\}$ ) and in continuous time

We observe that the viability kernel for the deterministic case in discrete time is slightly smaller than the deterministic viability kernel in continuous time. Therefore, the discretization process — described in §3.2 — leads to a moderate and conservative degradation of the theoretical viability kernel.

### Comparison between robust viability kernels as the set $\mathbb{S}$ of uncertainties increases

In Figure 2, we display the three robust viability kernels corresponding to the three cases presented in §3.1: low case (deterministic) with  $\mathbb{S} = \mathring{\mathbb{S}}$ , middle case with  $\mathbb{S} = \bar{\mathbb{S}}$  and high case with  $\mathbb{S} = \bar{\bar{\mathbb{S}}}$ .

For the middle case, we take

$$\bar{\mathbb{S}} = [0, 5] \times [0, 25] \text{ day}^{-1} \times \text{day}^{-1} . \quad (32)$$

These numbers were obtained from the ranges for the parameters adjusted to the 2013 dengue outbreak in Cali, Colombia, as described in [8]. Details can be found in Appendix A.5.

For the high case, we doubled the right ends of each interval in (32),

giving:

$$\bar{\mathbb{S}} = [0, 10] \times [0, 50] \text{ day}^{-1} \times \text{day}^{-1} . \quad (33)$$

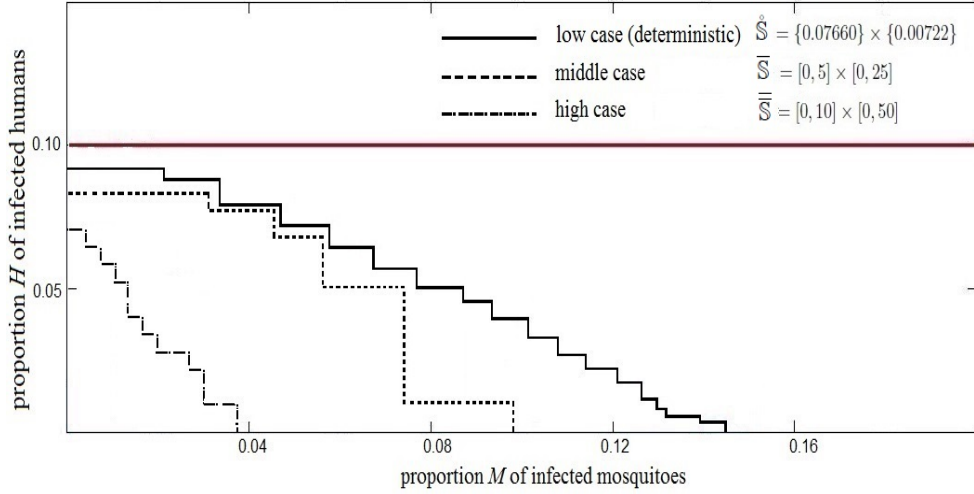


Figure 2: Deterministic viability kernel ( $\mathring{\mathbb{S}} = \{0.0766\} \times \{0.00722\}$ ) and robust viability kernel for the middle case ( $\bar{\mathbb{S}} = [0, 5] \times [0, 25]$ ) and the high case ( $\bar{\bar{\mathbb{S}}} = [0, 10] \times [0, 50]$ )

## Discussion

In Figure 2 — both in the middle case (dashed line) and more markedly in the high case (line made of segments and dots) — we observe the following: a large part of the initial states ( $M_0, H_0$ ) that are identified as viable in the deterministic case (continuous line) are no longer viable when taking into account uncertainties.

In addition, as we expand the set  $\mathbb{S}$  where uncertainties take their values, the gap with the deterministic case is larger and larger. In the high case, when we double the extremities of  $\bar{\mathbb{S}}$ , we observe in Figure 2 that all the initial states that are below the solid broken line (boundary of the deterministic viability kernel) and above the lower broken line (made of segments and dots) no longer belong to the viability kernel. Thus, enlarging the set of uncertainties can have a strong impact on the viability kernel. When we have much variability in the values of the uncertain variables ( $A_m, A_H$ ), the robust viability kernel is greatly reduced.

Therefore, we cannot guarantee compliance with the constraint imposed on the proportion of infected humans for all uncertainties, if we take initial conditions belonging to the deterministic kernel. Picking initial conditions within the deterministic kernel would yield optimistic conclusions with respect to infection control, when uncertainties affect the epidemics dynamics.

These observations depend on the temporal horizon and on the structure of the set of scenarios. Recall that, in §2.3, we stressed the importance of having a rectangular set of scenarios to obtain a dynamic programming equation, making it possible to recursively compute robust viability kernels. This rectangularity property of the set of scenarios corresponds to stage-wise independence of uncertainties: whatever the value of the uncertainty at time  $t$ , the next uncertainties can take *any* values (within the rectangle). This rectangularity property contributes to having small robust viability kernels. Indeed, scenarios can display arbitrary evolutions, switching from one extreme to another between time periods (days). High and low values for the uncertainties alternate, submitting the epidemics dynamics to a strong stress, and thus narrowing the possibility of satisfying the state constraints for all times. Such contrasted scenarios deserve the label of worst-case scenarios (they represent a small fraction of the extreme points of the rectangular set of scenarios). This is why amplifying the distance between extreme uncertainties shrinks the robust viability kernel.

## 4 Conclusion

In [8], the two authors obtained a neat expression of the deterministic viability kernel for a continuous time Ross-Macdonald model with control. However, uncertainties abound and we wanted to assess their importance regarding epidemics control. The numerical results show that the viability kernel without uncertainties is highly sensitive to the variability of parameters — here the biting rate, the probability of infection to mosquitoes and humans, and the proportion of female mosquitoes per person. So the robust viability kernel is a possible tool to reveal the importance of uncertainties regarding epidemics control.



# A Appendix. Fitting an epidemiological model for dengue

Here, we present how, from the Ross-Macdonald model in continuous time, we infer the controlled Ross-Macdonald model (2) in discrete time.

## A.1 Ross-Macdonald model in continuous time

The Ross-Macdonald model in continuous time is the following differential system

$$\frac{dm}{dt} = \alpha p_m h (1 - m) - \delta m , \quad (34a)$$

$$\frac{dh}{dt} = \alpha p_h \frac{N_m}{N_h} m (1 - h) - \gamma h , \quad (34b)$$

where the parameters  $\alpha$ ,  $p_m$ ,  $p_h$ ,  $\xi = \frac{N_m}{N_h}$ ,  $\delta$  and  $\gamma$  are given in Table 1. The state space is the unit square  $[0, 1]^2$ .

Parameter	Description	Unit
$\alpha \geq 0$	biting rate per time unit	time <sup>-1</sup>
$\xi = N_m/N_h \geq 0$	number of female mosquitoes per human	dimensionless
$1 \geq p_h \geq 0$	probability of infection of a susceptible human by infected mosquito biting	dimensionless
$1 \geq p_m \geq 0$	probability of infection of a susceptible mosquito when biting an infected human	dimensionless
$\gamma \geq 0$	recovery rate for humans	time <sup>-1</sup>
$\delta \geq 0$	(natural) mortality rate for mosquitoes	time <sup>-1</sup>

Table 1: Parameters of the continuous time Ross-Macdonald model (34)

For notational simplicity, we put

$$A_m = \alpha p_m , \quad A_h = \alpha p_h \frac{N_m}{N_h} = \alpha p_h \xi . \quad (35)$$

We turn the dynamical system (34) into a controlled system by replacing the natural mortality rate  $\delta$  for mosquitoes in (34) by a piecewise continuous

function, called *control trajectory*

$$u(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad t \mapsto u(t). \quad (36)$$

Therefore, the *controlled Ross-Macdonald model* is

$$\frac{dm}{dt} = A_m h(t)(1 - m(t)) - u(t)m(t), \quad (37a)$$

$$\frac{dh}{dt} = A_h m(t)(1 - h(t)) - \gamma h(t). \quad (37b)$$

## A.2 Ross-Macdonald model in discrete time

From the continuous time controlled Ross-Macdonald model (37), we obtain a discrete time model as follows in three steps. First, we consider a time period  $\Delta t > 0$  and we restrict time  $t$  to

$$t = t_0, t_0 + \Delta t, \dots, T - \Delta t, T \text{ with } \frac{T}{\Delta t} \in \mathbb{N}, \quad (38)$$

where we suppose that the horizon  $T$  is a multiple of the time period  $\Delta t$ . Second, we suppose that mosquito mortality rates  $u(t)$  induced by fumigation remain stationary in (37) during every time period  $[t, t + \Delta t[$ . Third, we replace time derivatives in (37) by increments over a time period  $\Delta t$ . We obtain the following numerical approximations

$$m(t + \Delta t) \approx (A_m \Delta t) h(t)(1 - m(t)) + (1 - u(t)\Delta t)m(t) \quad (39)$$

$$h(t + \Delta t) \approx (A_H \Delta t) m(t)(1 - h(t)) + (1 - \gamma \Delta t)h(t).$$

Introducing

$$\Delta t = 1 \text{ day}, \quad M(t) = m(t), \quad H(t) = h(t), \quad (40)$$

we infer the following controlled Ross-Macdonald model in discrete time:

$$M(t + 1) = A_M H(t)(1 - M(t)) + (1 - u(t))M(t) \quad (41)$$

$$H(t + 1) = A_H M(t)(1 - H(t)) + (1 - \gamma)H(t).$$

### A.3 Parameters and daily data deduced from health reports

We introduce the vector of parameters

$$\theta = (\alpha, p_h, p_m, \xi, \delta) \in \Theta \subset \mathbb{R}_+^5, \quad (42)$$

consisting of the five parameters previously defined in Table 1. The parameter set  $\Theta \subset \mathbb{R}_+^5$  is given by the Cartesian product of the five intervals in the third column of Table 2.

With these notations, the Ross-Macdonald model (34) now writes

$$\begin{aligned} \frac{dm(t; \theta)}{dt} &= \alpha p_m h(t; \theta) (1 - m(t; \theta)) - \delta m, \\ \frac{dh(t; \theta)}{dt} &= \alpha p_h \xi m(t; \theta) (1 - h(t; \theta)) - \gamma h, \\ (m(t_0; \theta), h(t_0; \theta)) &= (m_0, h_0). \end{aligned} \quad (43)$$

Notice that the rate  $\gamma$  of human recovery does not appear in the parameter vector  $\theta$  in (42). Indeed, in the data provided by the Municipal Secretariat of Public Health (Cali, Colombia), we only have new cases of dengue registered per day; there is no information regarding how many individuals recover daily. We choose an infectiousness period of 10 days, that is, a rate of human recovery fixed at

$$\gamma = 0.1 \text{ day}^{-1}. \quad (44)$$

Under this assumption, the *daily incidence data* (i.e., numbers of newly registered cases reported on daily basis) provided by the Municipal Secretariat of Public Health (Cali, Colombia) can be converted into the *daily prevalence data* (i.e., numbers of infected individuals on a given day, be they new or not). With this, we deduce values of daily proportion of infected individuals in the form of the set

$$\mathbb{O} = \left\{ (r, \hat{h}_r), r = r_0, r_0 + 1, \dots, R \right\}, \quad (45)$$

where  $r$  refers to day  $r$  within the observation period of  $R + 1 - r_0$  days, and where  $\hat{h}_r$  stands for the fraction of infected individuals at day  $r$ . Naturally, the first couple in the set  $\mathbb{O}$  in (45) defines the initial condition  $h(r_0; \theta) = \hat{h}_0$ . Unfortunately, there is no available data for the fraction of infected mosquitoes.

As mosquito abundance is strongly correlated with dengue outbreaks [9], we have chosen a relation  $m(r_0; \theta) = 3\hat{h}_0$  at the beginning of an epidemic outburst (other choices gave similar numerical results).

#### A.4 Parameter estimation

To estimate a parameter vector  $\theta \in \Theta$  in (42) that fits with the data provided by the Municipal Secretariat of Public Health (Cali, Colombia), we apply the curve-fitting approach based on least-square method. More precisely, we look for an optimal solution to the problem

$$\min_{\theta \in \Theta} \sum_{r=r_0}^R (h(r; \theta) - \hat{h}_r)^2, \quad R = 60 \text{ days}, \quad (46)$$

subject to the differential constraint (43). Regarding numerics, we have solved this optimization problem with the `lsqcurvefit` routine (MATLAB Optimization Toolbox), starting with an admissible  $\theta^{(0)} \in \Theta$  (the exact value stands in the second column of Table 2). The routine generates a sequence  $\theta^{(1)}, \theta^{(2)} \dots$  that we stop once it is stationary, up to numerical precision. For a better result, we have combined two particular methods (Trust-Region-Reflective Least Squares Algorithm [15] and Levenberg-Marquardt Algorithm [11]) in the implementation of the `lsqcurvefit` MATLAB routine.

Parameter	Initial value	Range	Reference	Estimated value	Unit
$\alpha$	1	[0, 5]	[6], [13]	0.3600	day <sup>-1</sup>
$p_m$	0.5	[0, 1]		0.2128	dimensionless
$p_h$	0.5	[0, 1]		0.1990	dimensionless
$\xi$	1	[1, 5]	[10], [14]	1.0087	dimensionless
$\delta$	0.035	[0.033, 0.066]	[6], [14]	0.0333	day <sup>-1</sup>

Table 2: Initial values, admissible ranges, respective source references, estimated values of parameters (numerical solution of the optimization problem (46)–(43)) and units

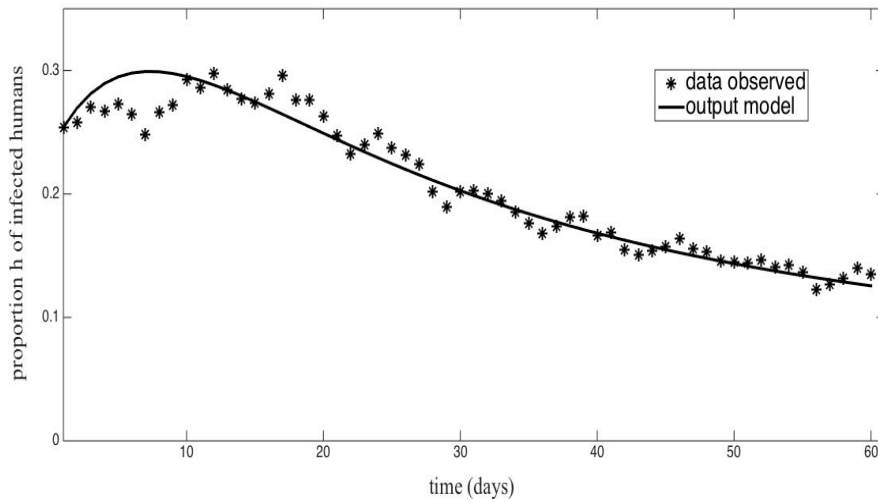


Figure 3: Fraction of individuals infected with dengue, obtained by adjustment of the Ross-Macdonald model (34) (smooth solid curve) versus registered daily prevalence cases (star isolated points) during the 2013 dengue outbreak in Cali, Colombia

The last column of Table 2 provides estimated values for the parameters  $\theta = (\alpha, p_h, p_m, \xi, \delta)$ , and Figure 3 displays the curve-fitting results. Therefore, the aggregate parameters (35) are estimated as

$$A_m = 0.076608 \text{ day}^{-1}, \quad A_h = 0.0722633 \text{ day}^{-1}. \quad (47a)$$

For the natural mortality rate  $\delta = \underline{u}$  of mosquitoes (see Table 1), we obtain

$$\delta = \underline{u} = 0.0333 \text{ day}^{-1} . \quad (47b)$$

For the mosquito mortality maximal rate  $\bar{u}$ , we take

$$\bar{u} = 0.05 \text{ day}^{-1} . \quad (47c)$$

## A.5 Ranges for the uncertain aggregate parameters

We could have identified directly the Ross-Macdonald model (41) in discrete time, that is, identified values for the parameters  $A_M$  and  $A_H$ . However, we wanted to obtain the Ross-Macdonald model (41) in discrete time from the Ross-Macdonald model (34) in continuous time, because the parameters  $(\alpha, p_h, p_m, \xi, \delta)$  of the latter have more sense than the aggregate parameters (35).

In addition, from the ranges for the parameters  $(\alpha, p_h, p_m, \xi, \delta)$  displayed in the third column of Table 2, we obtain ranges for the aggregate parameters (35):

$$A_m = \alpha p_m \in [0, 5] \text{ day}^{-1} , \quad A_h = \alpha p_h \xi \in [0, 25] \text{ day}^{-1} . \quad (48)$$

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