



HAL
open science

On the size of chaos in the mean field dynamics

Thierry Paul, Mario Pulvirenti, Sergio Simonella

► **To cite this version:**

Thierry Paul, Mario Pulvirenti, Sergio Simonella. On the size of chaos in the mean field dynamics. Archive for Rational Mechanics and Analysis, In press. hal-01577284v4

HAL Id: hal-01577284

<https://hal.science/hal-01577284v4>

Submitted on 5 Sep 2017 (v4), last revised 30 Jun 2018 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE SIZE OF CHAOS IN THE MEAN FIELD DYNAMICS

THIERRY PAUL, MARIO PULVIRENTI, AND SERGIO SIMONELLA

ABSTRACT. We consider the error arising from the approximation of an N -particle dynamics with its description in terms of a one-particle kinetic equation. We estimate the distance between the j -marginal of the system and the factorized state, obtained in a mean field limit as $N \rightarrow \infty$. Our analysis relies on the evolution equation for the “correlation error” rather than on the usual BBGKY hierarchy. The rate of convergence is shown to be $O(j^2/N)$ in any bounded interval of time (size of chaos), as expected from heuristic arguments. Our formalism applies to an abstract hierarchical mean field model with bounded collision operator and a large class of initial data, covering (a) stochastic jump processes converging to the homogeneous Boltzmann and the Povzner equation and (b) quantum systems giving rise to the Hartree equation.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 2. Abstract model and main result | 6 |
| 2.1. The model | 6 |
| 2.2. Main result | 13 |
| 2.3. The correlation equations | 14 |
| 3. Proof of Theorem 2.2 | 16 |
| 4. Return to the concrete examples | 24 |
| 4.1. Stochastic models | 24 |
| 4.2. Quantum mean field | 26 |
| Appendix. Derivation of the correlation equations | 27 |
| References | 31 |

1. INTRODUCTION

The kinetic description of particle systems is based on the propagation of chaos. This property allows to substitute the complex dynamics of a huge number of particles by a single nonlinear partial differential equation for the probability density (in quantum systems, the reduced density matrix) of a given particle. More precisely, one applies a statistical description. At time zero, the N -particle system is assumed to be “chaotic” in the sense that each particle is distributed identically and independently from the others, at least up to an error, vanishing when N diverges. The dynamics creates correlations and the statistical independence is lost at any positive time. However, after suitable rescaling of space and time, the statistical

independence of any finite group of particles can be recovered, in the limit $N \rightarrow \infty$. As a consequence a given particle evolves according to an effective equation. The nature of this dynamics is determined by the microscopic details of the system and by the regime of physical parameters. Such a mechanism works in the formal (in a few cases, rigorous) derivation of the most common kinetic equations.

In this paper we consider the following class of mean field models.

- *Kac model.* The N -particle system evolves according to a stochastic process. To each particle, say particle i , we associate a velocity $v_i \in \mathbb{R}^3$. The vector $\mathcal{V}_N = \{v_1, \dots, v_N\}$ changes by means of two-body collisions at random times, with random scattering angle. The probability density $f^N(\mathcal{V}_N, t)$ evolves according to the master equation (forward Kolmogorov equation)

$$\partial_t f^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{f^N(\mathcal{V}_N^{i,j}) - f^N(\mathcal{V}_N)\},$$

where $\mathcal{V}_N^{i,j} = \{v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_N\}$ and the pair v'_i, v'_j gives the outgoing velocities after a collision with scattering (unit) vector ω and incoming velocities v_i, v_j . $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$ is the differential cross-section of the two-body process. The resulting kinetic equation is the homogeneous Boltzmann equation

$$\partial_t f(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{f(v')f(v'_1) - f(v)f(v_1)\}.$$

Such a model has been introduced by Kac [24, 25] and has been largely investigated over recent times, see e.g. [28]. Very similar stochastic systems including space variables and (space-)delocalized collisions are frequently used to justify numerical schemes [32, 33]. We will not mention them explicitly although they could be included in our analysis.

- *'Soft spheres' model.* A slightly more realistic variant, taking into account the positions of particles $X_N = \{x_1, \dots, x_N\} \in \mathbb{R}^{3N}$ and relative transport, was introduced by Cercignani [9] and further investigated in [27]. The probability density $f^N(X_N, \mathcal{V}_N, t)$ evolves according to the equation

$$\begin{aligned} \partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\times \{f^N(X_N, \mathcal{V}_N^{i,j}) - f^N(X_N, \mathcal{V}_N)\}. \end{aligned}$$

Here $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive function with compact support. Now a pair of particles collides at a random distance with rate modulated by h . The associated kinetic equation is the Povzner equation

$$\begin{aligned} \partial_t f(x, v) + v \cdot \nabla_x f(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\quad \times \{f(x, v') f(x_1, v'_1) - f(x, v) f(x_1, v_1)\}, \end{aligned}$$

which can be seen as an h -mollification of the inhomogeneous Boltzmann equation (formally obtained when h converges to a Dirac mass at the origin).

- *Quantum mean field model.* The N -particle quantum system has a mean field Hamiltonian

$$H_N = -\frac{\hbar^2}{2} \sum_{i=1}^N \Delta_{x_i} + \frac{1}{N} \sum_{i < j} V(x_i - x_j),$$

where V is a two-body potential on \mathbf{R}^d and d is the dimension of the physical space. A state of the system is a density matrix ρ^N whose time evolution is given by the von Neumann equation

$$\partial_t \rho^N = \frac{1}{i\hbar} [H_N, \rho^N]$$

(equivalent, modulo a global phase, to the Schrödinger equation). The effective equation for the one-particle density matrix ρ is the Hartree equation

$$\partial_t \rho = \frac{1}{i\hbar} [H_{MF}, \rho]$$

where

$$H_{MF} = -\frac{\hbar^2}{2} \Delta_x + \int dy V(x - y) \rho(y, y).$$

Here $\rho(x, y)$ is the kernel of ρ and hence $\rho(y, y)$ is the spatial density.

In all the above cases we assume symmetry in the particle labels. Moreover, we assume that the initial state factorizes (or at least does so in the limit $N \rightarrow \infty$), namely $f^N(0) = f_0^{\otimes N}$ and $\rho^N(0) = \rho_0^{\otimes N}$. At time $t > 0$, in spite of the correlations created by the dynamics, the system is still factorizing in the limit $N \rightarrow \infty$ through its j -particle marginals f_j^N, ρ_j^N , defined as partial integrations of f^N and partial traces of ρ^N , in the sense that these marginals converge, for any fixed j and in the limit $N \rightarrow \infty$, to $f^{\otimes j}$ and $\rho^{\otimes j}$ respectively, $f = f(t)$ and $\rho = \rho(t)$ being the solutions of the associated effective equations.

This ‘propagation of chaos’ has been proved for the models under consideration and, under certain assumptions, informations on the convergence rate are also available (see Section 4 below for bibliographical references).

A natural question arises (*size of chaos*): how large can be a group of $j = j(N)$ distinct particles, $j(N)$ diverging with N , so that one sees the decorrelation of such systems?

A simple heuristic argument on the Kac model gives an indication on $j(N)$. Let us consider a tagged group of j particles and consider, for any $i = 1, 2, \dots, j$, the set B_i of particles influencing the dynamics of particle i up to the time t . We can assume that the cardinality of B_i is finite to have a correct kinetic behaviour in the limit. If the propagation of chaos takes place, the groups B_i must be disjoint. On the other hand the probability that two fixed particles interact is $O(1/N)$. Therefore the probability that any pair of particles, in the group of j , is dynamically correlated is $O(j^2/N)$ and hence it suffices that $j/\sqrt{N} \rightarrow 0$ to ensure that the correlations are vanishing.

The purpose of the present paper is to prove this property for a class of mean field models including the ones listed above. One can deal with them simultaneously in terms of an abstract formalism which will be introduced in Section 2.1. Indeed $f_j^N(\mathcal{V}_j, t)$, $f_j^N(X_j \mathcal{V}_j, t)$ and $\rho_j^N(t)$ are ruled out by a hierarchy of equations with the same structure (BBGKY hierarchy). Under suitable hypotheses, the operators occurring in these hierarchies satisfy the same bounds. Notice that, physically, these models are very different. In particular, the Kac and the soft spheres models are stochastic processes for interacting particles, the quantum mean field is time reversible.

Inspired from [31], our main tool is a precise notion of decorrelation. Let us present it here, for the sake of concreteness, in the case of Kac model. (For the general definition in the abstract setting, see Definition 2.1 below.) Define

$$(1) \quad E_j(t) := \sum_{K \subset J} (-1)^{|K|} f_{J \setminus K}^N(t) f(t)^{\otimes K}$$

where $J = \{1, 2, \dots, j\}$, K is any subset of J , $J \setminus K$ is the relative complement of K in J and $|K| =$ cardinality of K . $f_A^N(t)$ stands for the $|A|$ -marginal

$f_{|A|}^N(t)$ computed in the configuration $\{v_i\}_{i \in A}$. Similarly, $f(t)^{\otimes K} = f(t)^{\otimes |K|}$ evaluated in $\{v_i\}_{i \in K}$. Here f is the solution to the homogeneous Boltzmann equation.

Eq. (1) has an inverse formula (proven below), that is

$$(2) \quad f_j^N(t) = \sum_{K \subset J} E_{J \setminus K}(t) f(t)^{\otimes K},$$

where $E_{J \setminus K}$ is defined as $F_{J \setminus K}^N$ is.

$E_j(t)$ measures the tendency of f_j^N to factorize and to converge in the mean field limit. For this reason E_j will be called the *correlation error* (of order j) of the mean field system.

These quantities have been already used (under the name “ v -functions”) to deal with kinetic limits of stochastic models [11, 7, 4, 12, 13, 6, 8, 14] and they have been recently investigated in the more singular low density limit of hard spheres [31]. In the latter reference, the correlation error of the N -particle system is given by

$$(3) \quad E'_j(t) := \sum_{K \subset J} (-1)^{|K|} f_{J \setminus K}^N f_1^N(t)^{\otimes K},$$

which is closely related to the cumulant expansion of a probability distribution of particles at a given time. Note that E'_j quantifies the mere deviation of f_j^N from the product of one-particle marginals without any reference to the kinetic equation, in contrast with (1) which measures both factorization and convergence. Unlike in [31], in the context of the present paper (1) and (3) provide equivalent information, and it is convenient to work directly on the functions E_j since they satisfy a simple evolution equation.

The plan of the paper is the following. In Section 2 we introduce the model, state our main results and write the correlation equations satisfied by E_j . The proofs of the results are presented in Section 3, while the derivation of the correlation equations is given in the Appendix. Similar equations for the evolution of the correlation error have been derived in many of the aforementioned references for stochastic systems. Finally, Section 4 collects comments on applications of the results and comparisons with the existing literature.

Let us conclude this introduction with a remark on the fundamental case of classical particles, i.e. the mean field limit of a Newtonian system. This case eludes our abstract setting and strategy. Indeed the presence

of derivatives makes singular the BBGKY operator, which would require the introduction of analytic spaces (see [19]). An efficient approach for the convergence to the Vlasov equation is the direct control of the empirical measures, and the problem is naturally solved in weak topologies [30, 5, 15, 21]. Concerning the size of chaos we refer to [18], where it is shown (avoiding empirical measures) that the j -marginal converges to the factorized state with rate bounded by $(j/N)^{1/p}$, for any p -Wasserstein distance with $p \geq 2$, while the bound is j/\sqrt{N} for $p = 1$.

2. ABSTRACT MODEL AND MAIN RESULT

2.1. The model. In this section we introduce an abstract setting which extends the trace-class operator formalism customary in quantum mechanics. This allows to deal, simultaneously, with classical cases. First, we assume the existence of states admitting a family of marginals (Sec. 2.1.1, 2.1.2), then we state assumptions on the evolution operators and write down the evolution equations (Sec. 2.1.3, 2.1.4) and finally we introduce the correlation error (Sec. 2.1.5).

2.1.1. State of the particle system. Let \mathfrak{H} be a complex separable Hilbert space and let \mathfrak{H}_n , $n = 1, 2, \dots, N$ be the n -fold tensor power of \mathfrak{H}

$$\mathfrak{H}_n := \mathfrak{H}^{\otimes n}, \quad \mathfrak{H}_1 := \mathfrak{H}.$$

For any operator A acting on \mathfrak{H}_n , we denote by A^* its adjoint. As usual $|A| = \sqrt{A^*A}$. To unburden notations, we drop the n -indices here and below, when no confusion arises.

For $1 \leq k, s \leq n$, let $\sigma_{k,s} = \sigma_{s,k} \in \mathcal{B}(\mathfrak{H}_n)$ be the unitary, involutory operator defined by

$$\sigma_{k,s}(e_1 \otimes \dots \otimes e_k \otimes \dots \otimes e_s \otimes \dots \otimes e_n) = e_1 \otimes \dots \otimes e_s \otimes \dots \otimes e_k \otimes \dots \otimes e_n, \quad k \neq s$$

for any orthonormal basis $\{e_j\}_{j \geq 1}$ of \mathfrak{H} . By convention, we define $\sigma_{k,k} = \mathbb{I}_{\mathfrak{H}_n}$. A given $A \in \mathcal{B}(\mathfrak{H}_n)$ is said symmetric if

$$A = \sigma_{k,s} A \sigma_{k,s}^*$$

for all $k, s = 1, \dots, n$.

For any n , we postulate the existence of a $*$ -algebra $\mathfrak{A}_n \subset \mathcal{B}(\mathfrak{H}_n)$, $\mathfrak{A}_n = \mathfrak{A}^{\otimes n}$, $\mathfrak{A}_1 = \mathfrak{A}$ (possibly not containing the identity), stable under the map $A \rightarrow |A|$.

We assume \mathfrak{A}_n to be equipped with a norm $\|\cdot\|_1$ defined by

$$\|A\|_1 := \ell(|A|) \quad A \in \mathfrak{A}_n ,$$

where

$$\ell : \mathfrak{A}_n \rightarrow \mathbf{C}$$

is a positive linear functional satisfying the following properties:

- $|\ell(A)| \leq \ell(|A|)$, $A \in \mathfrak{A}_n$;
- for $A \in \mathfrak{A}_j$ and $B \in \mathfrak{A}_{n-j}$

$$\ell(A \otimes B) = \ell(A)\ell(B);$$

as a consequence

$$\|A \otimes B\|_1 = \ell(|A \otimes B|) = \ell(|A| \otimes |B|) = \|A\|_1 \|B\|_1;$$

- for any $A \in \mathfrak{A}_n$, and any $\sigma_{k,s}$,

$$\|\sigma_{k,s} A \sigma_{k,s}^*\|_1 = \|A\|_1.$$

We consider the completion of the above algebras and keep for it the same notation. Thus \mathfrak{A}_n with norm $\|\cdot\|_1$ is a Banach space and we extend ℓ as a continuous functional by completion.

A state of the N -particle system is, by definition, an element $F^N \in \mathfrak{A}_N$, positive (as operator in $\mathcal{B}(\mathfrak{H}_N)$), symmetric and such that $\|F^N\|_1 = 1$.

2.1.2. *Marginals.* Let $\ell_n : \mathfrak{A}_n \rightarrow \mathfrak{A}_{n-1}$ be positive linear maps such that:

- $\ell_n(A_1 \otimes \cdots \otimes A_n) = \ell(A_n)(A_1 \otimes \cdots \otimes A_{n-1})$, $A_i \in \mathfrak{A}$.

Note that ℓ_n is symmetry preserving.

Moreover we assume that

- $|\ell_n(A)| \leq \ell_n(|A|)$, $A \in \mathfrak{A}_n$.

For $1 \leq j < n$, we indicate by $\ell_{j,n} : \mathfrak{A}_n \rightarrow \mathfrak{A}_j$ the transformation

$$\ell_{j,n} = \ell_{j+1} \cdots \ell_n .$$

This map is a contraction and preserves the norm of positive elements:

$$(4) \quad \|\ell_{j,n}(A)\|_1 = \ell(|\ell_{j,n}(A)|) \leq \ell(\ell_{j,n}(|A|)) = \|A\|_1$$

for $A \in \mathfrak{A}_n$ and the equality holds for A positive.

The j -particle marginal of the N -particle state F^N is given by

$$F_j^N := \ell_{j,N}(F^N) \in \mathfrak{A}_j.$$

By construction, F_j^N is a j -particle state.

2.1.3. *Evolution equations.* The evolution of a state $F^N \rightarrow F^N(t)$ in \mathfrak{A}_N is supposed to be given by the N -particle dynamics associated to a two-body interaction:

$$(5) \quad \frac{d}{dt} F^N = (K_0^N + V^N) F^N,$$

where

$$(6) \quad K_0^N = \sum_{i=1}^N \mathbb{I}_{\mathfrak{A}_{i-1}} \otimes K_0 \otimes \mathbb{I}_{\mathfrak{A}_{N-i}}$$

and

$$(7) \quad V^N = \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{i,j},$$

with

$$(8) \quad V_{i,j}(A) := \sigma_{1,i}^* \sigma_{2,j}^* V \otimes \mathbb{I}_{\mathfrak{A}_{N-2}} (\sigma_{1,i} \sigma_{2,j} A \sigma_{2,j}^* \sigma_{1,i}^*) \sigma_{2,j} \sigma_{1,i}, \quad A \in \mathfrak{A}_N$$

for a (possibly unbounded) linear operator K_0 on \mathfrak{A} and a symmetry preserving, two-body potential $V \in \mathcal{B}(\mathfrak{A}_2)$.

Formula (8) expresses the following simple fact. If $A = A_1 \otimes \cdots \otimes A_N$ and

$$V(A_i \otimes A_j) = \sum_{r,s} \alpha_{r,s} B_r \otimes C_s,$$

then

$$V_{i,j}(A) = \sum_{r,s} \alpha_{r,s} A_1 \otimes \cdots \otimes A_{i-1} \otimes B_r \otimes \cdots \otimes C_s \otimes A_{j+1} \cdots \otimes A_N.$$

In other words, the action of $V_{i,j}$ is the same as $V_{1,2} = V$ on the slots i and j (see the table below for concrete examples).

We assume that both K_0 and $K_0^N + V^N$ generate a strongly continuous, positivity preserving, isometric semigroup (with respect to the norm $\|\cdot\|_1$) and there exists a unique mild solution to (5) with initial datum $F^N(0) \in \mathfrak{A}_N$. Symmetry is automatically preserved by the symmetry of K_0^N and V^N .

Finally, for any $F \in \mathfrak{A}$, $F^N \in \mathfrak{A}_N$ and $i, r > j$, $i \neq r$ we assume

$$(9) \quad \ell(K_0(F)) = 0 \quad \text{and} \quad \ell_{j,N}(V_{i,r}(F^N)) = 0.$$

These last properties are necessary to deduce the forthcoming hierarchy.

The following table summarizes the applications of the above abstract model to the three models presented in the introduction, namely the **K**(ac), **S**(oft spheres) and **Q**(uantum) models. For these models, we have that \mathfrak{H} is $L^2(\mathbf{R}^3, dv)$, $L^2(\mathbf{R}^6, dx dv)$ and $L^2(\mathbf{R}^d, dx)$ respectively, while \mathfrak{A} is given by

$L^1 \cap L^\infty(\mathbf{R}^3), L^1 \cap L^\infty(\mathbf{R}^6)$ (as multiplication operators¹) and $\mathcal{L}^1(L^2(\mathbf{R}^d))$ (the space of the trace-class operators).

| | Kac | Soft spheres | Quantum mean field |
|----------------|---|---|---|
| ℓ | $f \rightarrow \int dv f$ | $f \rightarrow \int \int f dx dv$ | $A \rightarrow \text{Tr} A$ |
| $\ \cdot\ _1$ | $\ \cdot\ _{L^1}$ | $\ \cdot\ _{L^1}$ | $A \rightarrow \text{Tr} A $ |
| K_0 | 0 | $-v \cdot \nabla_x$ | $\frac{1}{i\hbar}[-\frac{\hbar^2}{2}\Delta, \cdot]$ |
| $V_{i,j}(F^N)$ | $\int d\omega$ $B(\omega; v_i - v_j)$ $\{f^N(\mathcal{V}_N^{i,j})$ $-f^N(\mathcal{V}_N)\}$ | $h(x_i - x_j)$ $B\left(\frac{x_i - x_j}{ x_i - x_j }; v_i - v_j\right)$ $\{f^N(X_N, \mathcal{V}_N^{i,j})$ $-f^N(X_N, \mathcal{V}_N)\}$ | $\frac{1}{i\hbar}[V(x_i - x_j), \rho^N]$ |

In both cases **S** and **Q**, K_0 is only densely defined. As we shall see, we will use only the groups generated by K_0 , i.e. respectively

$$e^{K_0 t} f(x, v) = f(x - vt, v), \quad f \in L^1 \cap L^\infty$$

for the case **S**, and

$$e^{K_0 t} A = U_0(-t) A U_0(t), \quad A \in \mathcal{L}^1$$

where $U_0(t) = e^{-i\frac{\hbar}{2}\Delta_x t}$ as unitary operator on L^2 for the case **Q**. The operators $V_{i,j} : \mathfrak{A}_N \rightarrow \mathfrak{A}_N$ have to be understood as bounded operators from $L^\infty \cap L^1 \rightarrow L^\infty \cap L^1$ for both cases **K** and **S**.

2.1.4. *Hierarchies.* Applying subsequently $\ell_N, \ell_{N-1}, \dots$ to (5) and using the symmetry and Eq. (9), we get the BBGKY hierarchy of equations

$$(10) \quad \partial_t F_j^N = \left(K_0^j + \frac{T_j}{N} \right) (F_j^N) + \alpha(j, N) C_{j+1} (F_{j+1}^N)$$

for $j = 1, \dots, N$, where

$$(11) \quad \alpha(j, N) = \frac{(N-j)}{N},$$

K_0^j, T_j are operators on \mathfrak{A}_j ,

$$(12) \quad K_0^j := \sum_{i=1}^j \mathbb{I}_{\mathfrak{A}_{i-1}} \otimes K_0 \otimes \mathbb{I}_{\mathfrak{A}_{j-i}},$$

$$(13) \quad T_j := \sum_{1 \leq i < r \leq j} T_{i,r}$$

¹The restriction to L^∞ is merely due to the abstract formulation. This assumption can be removed by density in the main theorem 2.2 below.

with²

$$(14) \quad T_{i,r}(A) = \sigma_{1,i}^* \sigma_{2,r}^* V \otimes \mathbb{I}_{\mathfrak{A}_{j-2}}(\sigma_{1,i} \sigma_{2,r} A \sigma_{2,r}^* \sigma_{1,i}^*) \sigma_{2,r} \sigma_{1,i}, \quad A \in \mathfrak{A}_j$$

and

$$(15) \quad C_{j+1}(A) := \ell_{j+1} \left(\sum_{i \leq j} T_{i,j+1}(A) \right) = \sum_{i=1}^j C_{i,j+1}(A), \quad A \in \mathfrak{A}_{j+1}$$

with

$$(16) \quad C_{i,j+1}(A) := \ell_{j+1}(T_{i,j+1}(A)),$$

$$(17) \quad C_{i,j+1} : \mathfrak{A}_{j+1} \rightarrow \mathfrak{A}_j.$$

We have

$$(18) \quad \|T_j\| \leq \frac{j(j-1)}{2} \|T_{i,r}\| \leq \frac{j(j-1)}{2} \|V\|, \quad \|C_{j+1}\| \leq j \|C_{i,j+1}\| \leq j \|V\|$$

(meant for $\|T_j\|_{\mathfrak{A}_j \rightarrow \mathfrak{A}_j}$, $\|T_{i,r}\|_{\mathfrak{A}_j \rightarrow \mathfrak{A}_j}$, $\|C_{j+1}\|_{\mathfrak{A}_{j+1} \rightarrow \mathfrak{A}_j}$, $\|C_{i,j+1}\|_{\mathfrak{A}_{j+1} \rightarrow \mathfrak{A}_j}$, $\|V\|_{\mathfrak{A}_2 \rightarrow \mathfrak{A}_2}$).

Associated to V , we introduce the nonlinear mapping $Q : \mathfrak{A}^{\otimes 2} \rightarrow \mathfrak{A}$ by the formula

$$(19) \quad Q(F, G) := \ell_2(V(F \otimes G))$$

and the nonlinear mean field equation on \mathfrak{A}

$$(20) \quad \partial_t F = K_0(F) + Q(F, F), \quad F(0) \geq 0, \quad \|F(0)\|_1 = 1.$$

Eq. (20) is the Boltzmann, Povzner or Hartree equation according to the specifications established in the table above. By assumption, it possesses a (global) unique mild solution, preserving $\|\cdot\|_1$ and positivity.

Observe that, after definitions (19) and (16),

$$(21) \quad C_{i,j+1}(F^{\otimes(j+1)}) = F^{\otimes(i-1)} \otimes Q(F, F) \otimes F^{\otimes(j-i)}$$

and, if F is a solution of (20), then $\{F^{\otimes j}\}_{j \geq 1}$ solves

$$(22) \quad \partial_t F^{\otimes j} = K_0^j(F^{\otimes j}) + C_{j+1}(F^{\otimes(j+1)}).$$

In other words, $\{F^{\otimes j}\}_{j \geq 1}$ is a solution of the formal limit of the hierarchy (10) as $N \rightarrow \infty$.

²According to our definition, it should be specified that $T_{i,r} : \mathfrak{A}_j \rightarrow \mathfrak{A}_j$ depends explicitly on j . We avoid to introduce a further notation, this fact being clear from the context.

2.1.5. *Correlation error.* Let us now fix j and $K \subset \{1, \dots, j\} = J$. Writing $K = \{i_s; s = 1, \dots, k\}$ with $i_1 < i_2 < \dots < i_k$, $|K| = k$, we define the unitary operator

$$(23) \quad \sigma_{K,j} = \sigma_{i_1,1} \sigma_{i_2,2} \cdots \sigma_{i_k,k} ,$$

$\sigma_{\emptyset,j} = \mathbb{I}_{\mathfrak{H}_j}$. We introduce a mapping on $A_k \in \mathfrak{A}_k$ into $\mathcal{B}(\mathfrak{H}_j)$

$$(24) \quad A_k \longrightarrow A_K^J ,$$

by

$$(25) \quad A_K^J = \sigma_{K,j} (A_k \otimes \mathbb{I}_{\mathfrak{H}_{j-k}}) \sigma_{K,j}^* .$$

For instance in the case of marginals, dropping from now on the explicit dependence on N ($F_j^N = F_j$), we get

$$(26) \quad F_{J \setminus K}^J = \sigma_{J \setminus K,j} (F_{j-k} \otimes \mathbb{I}_{\mathfrak{H}_k}) \sigma_{J \setminus K,j}^* .$$

Moreover, for $F \in \mathfrak{A}$ we write

$$(27) \quad F^{\otimes K,J} = \sigma_{J \setminus K,j} (\mathbb{I}_{\mathfrak{H}_{j-k}} \otimes F^{\otimes k}) \sigma_{J \setminus K,j}^* \in \mathcal{B}(\mathfrak{H}_j) .$$

A more explicit equivalent definition is

$$F^{\otimes K,J} = A_1 \otimes A_2 \otimes \cdots \otimes A_j ,$$

$$A_i = F \text{ if } i \in K \text{ and } A_i = \mathbb{I}_{\mathfrak{H}} \text{ otherwise.}$$

Note that

$$(28) \quad F^{\otimes K,J} F^{\otimes R,J} = F^{\otimes R,J} F^{\otimes K,J} = F^{\otimes (K \cup R),J}$$

if $R \cap K = \emptyset$ both in J . Note also that $F^{\otimes K,J}$ and $A_{J \setminus K}^J$ commute.

Even if $\mathbb{I}_{\mathfrak{H}_k} \notin \mathfrak{A}_k$ (and $A_{J \setminus K}^J \notin \mathfrak{A}_j$, $F^{\otimes K,J} \notin \mathfrak{A}_j$), a product of the form $F^{\otimes K,J} A_{J \setminus K}^J$ lies always in \mathfrak{A}_j and

$$(29) \quad \|F^{\otimes K,J} A_{J \setminus K}^J\|_1 = \|F\|_1^k \|A_{j-k}\|_1 .$$

Fixing the convention

$$(30) \quad F^{\otimes \emptyset,J} = A_{\emptyset}^J = \mathbb{I}_{\mathfrak{H}_j}$$

we introduce, in the following definition, a family of symmetric elements in \mathfrak{A}_j characterizing the state of the N -particle system.

Definition 2.1. For any $j = 1, \dots, N$, setting $J = \{1, \dots, j\}$ and $k = |K|$, we define the “correlation error” of order j by

$$(31) \quad E_j = \sum_{K \subset J} (-1)^k F^{\otimes K,J} F_{J \setminus K}^J .$$

Eq. (31) also reads

$$(32) \quad E_j = \sum_{K \subset J} (-1)^k \sigma_{J \setminus K, j} (F_{j-k} \otimes F^{\otimes k}) \sigma_{J \setminus K, j}^* .$$

Note that, by (30), the terms $K = \emptyset$ and $K = J$ have to be interpreted as $F_J^J = F_j$ and $(-1)^j F^{\otimes J, J} = (-1)^j F^{\otimes j}$ respectively.

Formula (31) is inverted by

$$(33) \quad F_j = \sum_{K \subset J} F^{\otimes K, J} E_{J \setminus K}^J$$

where, using (24), we write

$$(34) \quad E_{J \setminus K}^J = \sigma_{J \setminus K, j} (E_{j-k} \otimes \mathbb{I}_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^*$$

$$(35) \quad = \sum_{R \subset J \setminus K} (-1)^{|R|} F^{\otimes R, J} F_{J \setminus (K \cup R)}^J$$

and $E_{\emptyset}^J = \mathbb{I}_{\mathfrak{H}_j}$. To prove (35), denoting $I = \{1, \dots, j-k\}$ and $J \setminus K = \{i_s; s = 1, \dots, j-k\}$ with i_s increasing, we observe that (32) together with the change of variables induced by $\sigma_{J \setminus K, j} = \sigma_{i_1, 1} \cdots \sigma_{i_{j-k}, j-k}$,

$$I \supset R' = \{\ell_1, \dots, \ell_{|R|}\} \rightarrow R = \{i_{\ell_1}, \dots, i_{\ell_{|R|}}\} \subset J \setminus K ,$$

imply

$$E_{J \setminus K}^J = \sum_{R \subset J \setminus K} (-1)^{|R|} \sigma_{J \setminus K, j} (\sigma_{I \setminus R', j-k} (F_{j-k-|R|} \otimes F^{\otimes |R|}) \sigma_{I \setminus R', j-k}^* \otimes \mathbb{I}_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^* .$$

On the other hand

$$\begin{aligned} & \sigma_{J \setminus K, j} (\sigma_{I \setminus R', j-k} (F_{j-k-|R|} \otimes F^{\otimes |R|}) \sigma_{I \setminus R', j-k}^* \otimes \mathbb{I}_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^* \\ &= \sigma_{J \setminus K, j} (\sigma_{I \setminus R', j-k} \otimes I_{\mathfrak{H}_k}) (F_{j-k-|R|} \otimes F^{\otimes |R|} \otimes \mathbb{I}_{\mathfrak{H}_k}) (\sigma_{I \setminus R', j-k}^* \otimes I_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^* \\ &= \sigma_{J \setminus K, j} (\sigma_{I \setminus R', j-k} \otimes I_{\mathfrak{H}_k}) (\mathbb{I}_{\mathfrak{H}_{j-k-|R|}} \otimes F^{\otimes |R|} \otimes \mathbb{I}_{\mathfrak{H}_k}) (\sigma_{I \setminus R', j-k}^* \otimes I_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^* \\ & \quad \cdot \sigma_{J \setminus K, j} (\sigma_{I \setminus R', j-k} \otimes I_{\mathfrak{H}_k}) (F_{j-k-|R|} \otimes \mathbb{I}_{\mathfrak{H}_{k+|R|}}) (\sigma_{I \setminus R', j-k}^* \otimes I_{\mathfrak{H}_k}) \sigma_{J \setminus K, j}^* \\ &= \sigma_{J \setminus R, j} (\mathbb{I}_{\mathfrak{H}_{j-|R|}} \otimes F^{\otimes |R|}) \sigma_{J \setminus R, j}^* \\ & \quad \cdot \sigma_{J \setminus (K \cup R), j} (F_{j-k-|R|} \otimes \mathbb{I}_{\mathfrak{H}_{k+|R|}}) \sigma_{J \setminus (K \cup R), j}^* . \end{aligned}$$

Formula (33) follows then from (35) and (28):

$$\begin{aligned}
 \sum_{K \subset J} F^{\otimes K, J} E_{J \setminus K}^J &= \sum_{K \subset J} \sum_{R \subset J \setminus K} (-1)^{|R|} F^{\otimes K, J} F^{\otimes R, J} F_{J \setminus (K \cup R)}^J \\
 &= \sum_{K \subset J} \sum_{R \subset J \setminus K} (-1)^{|R|} F^{\otimes (K \cup R), J} F_{J \setminus (K \cup R)}^J \\
 &= \sum_{T \subset J} \sum_{R \subset T} (-1)^{|R|} F^{\otimes T, J} F_{J \setminus T}^J \\
 &= \sum_{T \subset J} \delta_{T, \emptyset} F^{\otimes T, J} F_{J \setminus T}^J \\
 &= F_J^J \equiv F_j.
 \end{aligned}$$

2.2. Main result. Let $F^N(t)$ be the time-evolved state of the N -particle system, solution of (5) with initial datum $F^N(0) \in \mathfrak{A}_N$. Let $F(t)$ be the solution of the kinetic equation (20) with initial datum $F \in \mathfrak{A}$ and $E_j(t)$, $j = 1, \dots, N$, the correlation errors associated to the marginals of $F^N(t)$, as given by Definition 2.1.

In the sequel, we will denote for any operator $H : \mathfrak{A}_j \rightarrow \mathfrak{A}_{j'}$, $j, j' = 1, \dots, N$,

$$(36) \quad \|H\| = \|H\|_{\mathfrak{A}_j \rightarrow \mathfrak{A}_{j'}}.$$

Theorem 2.2. *Let us suppose that, for all $j = 1, \dots, N$ and for some $C_0 \geq 1$,*

$$(37) \quad \|E_j(0)\|_1 \leq C_0^j \left(\frac{j}{\sqrt{N}} \right)^j.$$

Then, for all $t > 0$ and all $j = 1, \dots, N$, one has

$$(38) \quad \|E_j(t)\|_1 \leq (C_2 e^{C_1 t \|V\|})^j \left(\frac{j}{\sqrt{N}} \right)^j$$

where $C_1 \geq 0$, $C_2 \geq 1$ are given by formula (78).

Suppose in addition that $\|E_1(0)\|_1 \leq \frac{B_0}{N}$ for some $B_0 > 0$. Then for all $t > 0$

$$\|E_1(t)\|_1 \leq \frac{1}{N} (B_2 e^{B_1 t \|V\|})$$

for $B_1 > 0$, $B_2 \geq 1$ given by formula (81).

Corollary 2.3. *Suppose that (37) holds and that $\|E_1(0)\|_1 \leq \frac{B_0}{N}$ for some $B_0 > 0$. Then for all $t > 0$ and all $j = 1, \dots, N$, the marginals satisfy*

$$\|F_j(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{D_1 t \|V\|} \frac{j^2}{N}$$

where $D_2 = \sup\{B_2, 8(eC_2)^2\}$, $D_1 = \sup\{B_1, 2C_1\}$.

Proof. We have, according to (33) and Theorem 2.2,

$$\begin{aligned}
& \|F_j(t) - F(t)^{\otimes j}\|_1 \\
& \leq \sum_{k=1}^j \binom{j}{k} \|E_k(t)\|_1 \leq \|E_1(t)\|_1 + \sum_{k=2}^j \binom{j}{k} \left(\frac{k^2 C_2^2 e^{2C_1 t \|V\|}}{N}\right)^{k/2} \\
& = \|E_1(t)\|_1 + \sum_{k=2}^j j(j-1)\cdots(j-k+1) \frac{k^k}{k!} \left(\frac{C_2 e^{C_1 t \|V\|}}{\sqrt{N}}\right)^k \\
& \leq \|E_1(t)\|_1 + \frac{1}{\sqrt{2\pi}} \sum_{k=2}^j \left(\frac{e^j C_2 e^{C_1 t \|V\|}}{\sqrt{N}}\right)^k \leq \|E_1(t)\|_1 + \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{e^j C_2 e^{C_1 t \|V\|}}{\sqrt{N}}\right)^2}{1 - \frac{j e^{C_2} e^{C_1 \|V\|} t}{\sqrt{N}}}
\end{aligned}$$

since $\frac{k^k}{k!} \leq \frac{e^k}{\sqrt{2\pi k}}$. Then the result follows if

$$(39) \quad N \geq 4(j e C_2)^2 e^{2C_1 \|V\| t}.$$

On the other hand if (39) is violated

$$8 \frac{(e^j C_2)^2 e^{2C_1 t \|V\|}}{N} \geq \frac{8}{4} = 2 \geq \|F_j(t) - F(t)^{\otimes j}\|_1$$

since $F_j(t)$ and $F(t)^{\otimes j}$ remain normalized for all t , and this concludes the proof. \square

Remark 2.4. *The bound (38) is trivial when $j \geq \sqrt{N}$, by virtue of the obvious inequality $\|E_j(t)\|_1 \leq 2^j$. Therefore we will consider in the sequel only the case $j < \sqrt{N}$. Furthermore note that there is no need in Corollary 2.3 for the initial condition $F^N(0)$ to be factorized.*

Theorem 2.2 will be proven in Section 3.

2.3. The correlation equations. In this section we write the equations satisfied by the errors E_j introduced in Definition 2.1.

We make use of the notation (11) and $J^i = \mathcal{J} \setminus \{i\}$, $J^{i,s} = \mathcal{J} \setminus \{i, s\}$. We introduce four (time-dependent) operators D_j, D_j^1, D_j^{-1} and D_j^{-2} , $j = 1, \dots, N$, by the following formulas.

$$\begin{aligned}
 D_j : \mathfrak{A}_j &\rightarrow \mathfrak{A}_j \\
 A_j &\mapsto \alpha(j, N) \sum_{i \in J} C_{i, j+1} \left(F^{\otimes \{i, J \cup \{j+1\}} A_{J^i \cup \{j+1\}}^{J \cup \{j+1\}} + F^{\otimes \{j+1, J \cup \{j+1\}} A_J^{J \cup \{j+1\}} \right) \\
 &\quad - \frac{1}{N} \sum_{\substack{i, s \in J \\ i \neq s}} C_{i, j+1} \left(F^{\otimes \{s, J \cup \{j+1\}} A_{J^s \cup \{j+1\}}^{J \cup \{j+1\}} \right)
 \end{aligned} \tag{40}$$

where we used (24) and (27) (and the convention $\sum_{i \neq s \in J} = 0$ for $J = \{1\}$). Here $F = F(t)$ is the solution of (20). The meaning of the above operator is transparent. Given A_j we can construct via formula (25) the operators $A_S^{J \cup \{j+1\}}$ with $|S| = j$. The right hand side of the above expression yields an operator in \mathfrak{A}_j . Similar arguments apply in the following formulas.

From now on, to unburden the notation we will drop the upper indices of set, i.e. $D_j, \mathfrak{A}_j \rightarrow \mathfrak{A}_j$, is written as

$$\begin{aligned}
 A_j &\mapsto \alpha(j, N) \sum_{i \in J} C_{i, j+1} \left(F^{\otimes \{i\}} A_{J^i \cup \{j+1\}} + F^{\otimes \{j+1\}} A_J \right) \\
 &\quad - \frac{1}{N} \sum_{\substack{i, s \in J \\ i \neq s}} C_{i, j+1} \left(F^{\otimes \{s\}} A_{J^s \cup \{j+1\}} \right) .
 \end{aligned} \tag{41}$$

Analogously, we define:

$$\begin{aligned}
 D_j^1 : \mathfrak{A}_{j+1} &\rightarrow \mathfrak{A}_j, \quad j = 1, \dots, N-1, \\
 A_{j+1} &\mapsto \alpha(j, N) C_{j+1} (A_{j+1}) ,
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 D_j^{-1} : \mathfrak{A}_{j-1} &\rightarrow \mathfrak{A}_j, \quad j = 2, \dots, N, \\
 A_{j-1} &\mapsto -\frac{j}{N} \sum_{i \in J} Q(F, F)^{\otimes \{i\}} A_{J^i} + \frac{1}{N} \sum_{\substack{i, s \in J \\ i \neq s}} T_{i, s} \left(F^{\otimes \{i\}} A_{J^i} \right) + \\
 &\quad - \frac{1}{N} \sum_{\substack{i, s \in J \\ i \neq s}} C_{i, j+1} \left(F^{\otimes \{i, s\}} A_{J^{i, s} \cup \{j+1\}} + F^{\otimes \{s, j+1\}} A_{J^s} \right) ,
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 D_j^{-2} : \mathfrak{A}_{j-2} &\rightarrow \mathfrak{A}_j, \quad j = 3, \dots, N, \\
 A_{j-2} &\mapsto \frac{1}{N} \sum_{\substack{i, s \in J \\ i \neq s}} \left(\frac{1}{2} T_{i, s} \left(F^{\otimes \{i, s\}} A_{J^{i, s}} \right) - Q(F, F)^{\otimes \{i\}} F^{\otimes \{s\}} A_{J^i, s} \right) .
 \end{aligned} \tag{44}$$

Let us consider the equation:

$$(45) \quad \begin{aligned} \partial_t E_j &= \left(K_0^j + \frac{T_j}{N} \right) (E_j) + D_j(E_j) \\ &+ D_j^1(E_{j+1}) + D_j^{-1}(E_{j-1}) + D_j^{-2}(E_{j-2}) \end{aligned}$$

where, by convention,

$$(46) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N}Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N}(T_{1,2}F \otimes F - Q(F, F)^{\otimes\{1\}}F^{\otimes\{2\}} - Q(F, F)^{\otimes\{2\}}F^{\otimes\{1\}}). \end{cases}$$

Note that the first line contains operators which do not change the particle number. D_j^1 is an operator increasing by one the number of particles considered. D_j^{-1} and D_j^{-2} are operators decreasing the number of particles by one and two respectively.

Eq. (45) is inhomogeneous so that it has nontrivial solutions even for initial data $E_j(0) = 0$, $j > 0$ (namely when the initial state is chaotic).

Notice that, by (18), (29) and the normalization of F ,

$$(47) \quad \|D_j\| \leq \sum_{i \in J} 2\|C_{i,j+1}\| + \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} \|C_{i,j+1}\| \leq 2j\|V\| + \frac{j^2}{N}\|V\| \leq 3j\|V\|.$$

Similarly,

$$(48) \quad \|D_j^1\| \leq j\|V\|, \quad \|D_j^{-1}\| \leq 4\frac{j^2}{N}\|V\|, \quad \|D_j^{-2}\| \leq \frac{3j^2}{2N}\|V\|.$$

The following result will be proven in the Appendix.

Proposition 2.5. *Let F satisfy (20). Then the hierarchy of equations (10) is equivalent to the hierarchy (45) in the sense that*

$$\{F_j\}_{j=1,\dots,N} \text{ solves (10)} \implies \{E_k\}_{k=1,\dots,N} \text{ solves (45)}.$$

3. PROOF OF THEOREM 2.2

We start by focusing on the evolution generated by the operator

$$K_0^j + \frac{T_j}{N} + D_j$$

preserving the particle number. We construct the (two-parameters) semi-group $U_j(t, s)$ for $s \leq t$, satisfying

$$(49) \quad \begin{aligned} \partial_t U_j(t, s) &= \left(K_0^j + \frac{T_j}{N} + D_j(t) \right) U_j(t, s), \quad j = 1, \dots, N, \\ U_j(s, s) &= \mathbb{I}_{\mathfrak{X}_j}. \end{aligned}$$

Recall that

$$(50) \quad \left\| e^{\left(K_0^j + \frac{T_j}{N}\right)t} \right\| = 1$$

by assumption.

Moreover, we assume preliminarily that

$$(51) \quad \|V\| = \frac{1}{4}.$$

From (18) and (47) we get then $\|T_j\| \leq j^2/8$ and

$$(52) \quad \|D_j\| \leq j.$$

We deduce, by Gronwall Lemma,

$$(53) \quad \|U_j(t, s)\| \leq e^{j(t-s)}.$$

We turn next to the contribution of the operators changing the particle number and notice that (cf. (48))

$$(54) \quad \|D_j^1\| \leq j, \quad \|D_j^{-1}\| \leq \frac{j^2}{N}, \quad \|D_j^{-2}\| \leq \frac{j^2}{N}.$$

In order to estimate E_j , we express the solution of Eq. (45) in terms of the Dyson series

$$(55) \quad E_j(t) = \sum_{n \geq 0} \sum_{k_1 \dots k_n} \int_s^t dt_1 \int_s^{t_1} dt_2 \dots \int_s^{t_{n-1}} dt_n U_{j_1}(t, t_1) D_{j_1}^{k_1}(t_1) U_{j_2}(t_1, t_2) \dots D_{j_n}^{k_n}(t_n) U_{j_0}(t_n, s) (E_{j_0}(s)),$$

where:

- the term $n = 0$ is $U_j(t, s) (E_j(s))$;
- $\sum_{k_1 \dots k_n}$ denotes the sum over all possible choices for the sequence $k_1 \dots k_n$ with $k_i \in \{1, -1, -2\}$;
- $j_1 = j, j_2 = j + k_1, \dots, j_{m+1} = j_m + k_m$ and j_0 is the number of particles at time s , namely $j_0 = j + \sum_{\ell=1}^n k_\ell = j_n + k_n$;
- we use the convention expressed by (46).
- the Dyson series (55) follows by iterating (45) in integral form via the Duhamel formula, and since the terms $D_N^1(E_N) = 0$ and $D_1^{-1}(E_0), D_1^{-2}(E_{-1}), D_2^{-2}(E_0)$ are explicitly known (see (46)), the iteration stops when these terms appear; namely the sums are constrained to $N \geq j_{s+1} = j + \sum_{\ell=1}^s k_\ell > 0$ for $s < n$ (but it can be $j_0 = j + \sum_{\ell=1}^n k_\ell = 0$).

When convergent, the sum in the r.h.s. of (55) defines a solution of (45) with initial condition $E_j(s)$.

For any sequence $k_1 \dots k_n = \underline{k}$, we denote by $n_{\underline{k}}^+ = n^+$ (resp. $n^-, n^\bar{}$) the number of times where 1 (resp. $-1, -2$) appears in $\{k_1, \dots, k_n\}$, namely $n_{\underline{k}}^+ = \sum_{1 \leq i \leq n} \delta_{1, k_i}$. It is the number of operators D_j^1 (resp. D_j^{-1}, D_j^{-2}) appearing in the string $U_{j_1}(t, t_2) D_{j_1}^{k_1} \dots D_{j_n}^{k_n} U_{j_0}(t_n, s)$. We have that

$$(56) \quad j_0 = j + n^+ - n^- - 2n^\bar{=} = j + n^+ - m^-, \quad n = n^+ + n^- + n^\bar{=}$$

where $m^- = n^- + 2n^\bar{=}$ is the number of negative steps performed by the process.

Note that the r.h.s. of (55) is a (finite dimensional) functional integral over the space of all random walks with single positive and single and double negative jumps. where we assume that k_1, \dots, k_n satisfy the following constraint. For all integers s s.t. $1 \leq s < n$,

$$j + \sum_{i=1}^s k_i > 0; \quad j + \sum_{i=1}^n k_i \geq 0,$$

and both quantities are not larger than N .

Proposition 3.1. *Suppose that, for some $A_0 \geq 1$,*

$$(57) \quad \|E_j(s)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} A_0^j, \quad \text{for all } j = 1, \dots, N.$$

Then there exists τ_0 sufficiently small such that for any $\tau = t - s \leq \tau_0$,

$$(58) \quad \|E_j(t)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} (A_1)^j, \quad \text{for all } j = 1, \dots, N,$$

where $A_1 = C(\tau_0)A_0$, for some explicitly computable constant $C(\tau_0) \geq 1$, depending only on τ_0 .

This Proposition is the heart of the paper. The proof of Theorem 2.2 at the end of the present section will consist in iterating this result, together with a scaling argument in order to remove the simplifying assumption (51).

Proof. The strategy of the proof of Proposition 3.1 will be to split the sum over k_1, k_2, \dots, k_n in the Dyson expansion (55) in several parts, and to use alternatively the three estimates at disposal

- $\|E_j\|_1 \leq \text{card}\{K \subset J = \{1, \dots, j\}\} = 2^j$;
- $\|D_j^{-1, -2}\| \leq \frac{j^2}{N}$;

$$\bullet \|D_j^{1,-1,-2}\| \leq j .$$

We have, for $j_0 \neq 0$,

$$\begin{aligned} & \|U_{j_1}(t, t_1) D_{j_1}^{k_1} U_{j_2}(t_1, t_2) \dots D_{j_n}^{k_n} U_{j_0}(t_n, s) (E_{j_0}(s)) \|_1 \\ & \leq \prod_{i=0}^n e^{(j+k_i+\dots+k_1)(t_i-t_{i+1})} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1 \\ (59) \quad & \leq \prod_{i=0}^n e^{(j+i)(t_i-t_{i+1})} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1 \end{aligned}$$

($t_0 = t$ and $t_{n+1} = s$). For $j_0 = 0$, as already mentioned, one has to replace in (55) the corresponding quantities defined in (46).

Using that

$$(60) \quad \prod_{i=0}^n e^{(j+i)(t_i-t_{i+1})} \leq e^{j(t-s)} e^{n(t-s)} .$$

and $\int_s^{t=s+\tau} dt_1 \int_s^{t_1} dt_2 \dots \int_s^{t_{n-1}} dt_n = \frac{\tau^n}{n!}$, (59) leads to

$$(61) \quad \|E_j(t)\|_1 \leq e^{j\tau} \sum_{n \geq 0} \sum_{k_1 \dots k_n} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1 \frac{\tau^n}{n!} e^{n\tau} .$$

We split now

$$(62) \quad \sum_{k_1 \dots k_n} = \sum_{k_1 \dots k_n} \chi(j_0 < j) + \sum_{k_1 \dots k_n} \chi(j_0 \geq j) ,$$

namely, using (56),

$$(63) \quad \sum_{k_1 \dots k_n} = \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- < 0}} + \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- \geq 0}} := \sum_{k_1 \dots k_n}^{<} + \sum_{k_1 \dots k_n}^{\geq} .$$

The corresponding terms in the r.h.s. of (61) will be denoted by $\mathcal{E}_j^{<}$ and \mathcal{E}_j^{\geq} respectively.

Let us first bound

$$\mathcal{E}_j^{<} = e^{j\tau} \sum_{n > 0} \sum_{k_1 \dots k_n} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1 \frac{\tau^n}{n!} e^{n\tau} .$$

In this case $n^- + 2n^- - n^+ = m^- - n^+ = j - j_0 > 0$ where m^- is the number of negative jumps.

Therefore there must be n_0 such that $\sum_{i=1}^{n_0} k_i = 0$, $\sum_{i=n_0+1}^s k_i < 0$ for all $n_0 + 1 \leq s < n$ and $\sum_{i=n_0+1}^n k_i = j_0 - j$. This means that the random walk is definitively below j from n_0 on.

The sequence $k_{n_0+1} \dots k_n$ has the associated numbers $\tilde{n}^+, \tilde{n}^-, \tilde{n}^-$ satisfying

$$(64) \quad \tilde{n}^+ + \tilde{n}^- + \tilde{n}^- = n - n_0 \quad \text{and} \quad \tilde{n}^+ - \tilde{n}^- - 2\tilde{n}^- = j_0 - j .$$

This implies that

$$(65) \quad \tilde{n}^- + \tilde{n}^= \geq \frac{j - j_0}{2} \quad \text{and} \quad \tilde{n}^+ \leq n - n_0 - \frac{j - j_0}{2}.$$

Therefore, using (54), we get

$$\|D_{j_{n_0+1}}^{k_{n_0+1}}\| \dots \|D_{j_n}^{k_n}\| \leq \left(\frac{j^2}{N}\right)^{(j-j_0)/2} j^{(n-n_0-(j-j_0)/2)}.$$

Clearly the same estimate holds true also when $j_0 = 0$.

On the other hand, observe that, for all $\underline{k} = k_1 \dots k_n$ and $i = 1, \dots, n$, $\|D_{j_i}^{k_i}\| \leq j_i \leq (j + n^+)$, so that

$$(66) \quad \|D_{j_1}^{k_1}\| \dots \|D_{j_{n_0}}^{k_{n_0}}\| \leq (j + n^+)^{n_0}.$$

Thus we get, using (57), and the obvious inequality

$$j^{n-n_0} (j + n^+)^{n_0} \leq (j + n^+)^n,$$

$$(67) \quad \mathcal{E}_j^< \leq e^{j\tau} \sum_{n>0} \sum_{k_1 \dots k_n} \frac{\tau_1^n}{n!} (j + n^+)^n \left(\frac{j_0^2}{N}\right)^{j_0/2} \left(\frac{j}{N}\right)^{\frac{(j-j_0)}{2}} A_0^{j_0},$$

where $\tau_1 = \tau e^\tau$.

By using that $\frac{k^k}{k!} \leq \sum_{r \geq 0} \frac{k^r}{r!} = e^k$, we get

$$(68) \quad \frac{\tau_1^n}{n!} (j + n^+)^n \leq (e\tau_1)^n \left(\frac{j + n}{n}\right)^n \leq (e\tau_1)^n e^j.$$

Note that $e\tau_1 = e\tau e^\tau$ is arbitrarily small provided that $\tau_0 \geq \tau = t - s$ is sufficiently small.

In conclusion, since $A_0 > 1$ and $(\sum_{k_1 \dots k_n} 1) \leq 3^n$, setting $\lambda = \lambda(\tau_0) = 3e\tau_0 e^{\tau_0}$, we have

$$(69) \quad \begin{aligned} \mathcal{E}_j^< &\leq e^{j(\tau+1)} A_0^j \sum_{n>0} \sum_{k_1 \dots k_n} (e\tau e^\tau)^n \left(\frac{j_0^2}{N}\right)^{\frac{j_0}{2}} \left(\frac{j}{N}\right)^{\frac{(j-j_0)}{2}} \\ &\leq (A_0 e^{\tau+1})^j \frac{\lambda}{1-\lambda} \left(\frac{j^2}{N}\right)^{j/2}. \end{aligned}$$

As a consequence, if

$$\lambda = 3e\tau_0 e^{\tau_0} \leq \frac{1}{4},$$

we have

$$(70) \quad \frac{\lambda}{1-\lambda} \leq 1$$

and

$$(71) \quad \mathcal{E}_j^< \leq (A_0 e^{\tau+1})^j \left(\frac{j^2}{N} \right)^{j/2}.$$

Next we estimate

$$(72) \quad \mathcal{E}_j^{\geq} = e^{j\tau} \sum_{n \geq 0} \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- \geq 0}} e^{n\tau} \frac{\tau^n}{n!} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1.$$

To do this, we introduce a parameter $\mu \in (0, 1)$, to be determined later, and we split the above sum into two terms, namely

$$T_1 = e^{j\tau} \sum_{n \geq 0} \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- \geq 0 \\ j_0 \leq \mu\sqrt{N}}} e^{n\tau} \frac{\tau^n}{n!} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1$$

and

$$T_2 = e^{j\tau} \sum_{n \geq 0} \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- \geq 0 \\ j_0 > \mu\sqrt{N}}} e^{n\tau} \frac{\tau^n}{n!} \prod_{i=1}^n \|D_{j_i}^{k_i}\| \|E_{j_0}\|_1.$$

Note that, when $j > \mu\sqrt{N}$, $T_1 = 0$.

By (66), the inductive hypothesis (57) and estimate (68) we deduce

$$T_1 \leq e^{j(\tau+1)} \sum_{n \geq 0} \sum_{n \geq 0} \sum_{\substack{k_1 \dots k_n \\ n^+ - n^- - 2n^- \geq 0 \\ j_0 \leq \mu\sqrt{N}}} (\tau e^{\tau+1})^n \left(\frac{(j+\ell)^2}{N} \right)^{\frac{j+\ell}{2}} A_0^{(j+\ell)}$$

where $\ell = n^+ - m^- = j_0 - j$. But

$$(73) \quad \begin{aligned} \left(\frac{(j+\ell)^2}{N} \right)^{\frac{j+\ell}{2}} A_0^\ell &= \left(\frac{j^2}{N} \right)^{j/2} \left(\frac{(j+\ell)^2}{N} \right)^{\frac{\ell}{2}} \left(\frac{j+\ell}{j} \right)^j A_0^\ell \\ &\leq \left(\frac{j^2}{N} \right)^{j/2} \mu^\ell e^\ell A_0^\ell. \end{aligned}$$

Here we used that $j+\ell = j_0 \leq \mu\sqrt{N}$ and that $\left(\frac{j+\ell}{j} \right)^j \leq e^\ell$. Therefore we conclude that

$$(74) \quad T_1 \leq \frac{1}{1-\lambda} (A_0 e^{\tau+1})^j \left(\frac{j^2}{N} \right)^{j/2},$$

after having fixed $\mu = \frac{1}{A_0 e}$.

For the term T_2 we make use of the estimate

$$(75) \quad \|E_j\|_1 \leq \text{card}\{K \subset J = \{1, \dots, j\}\} = 2^j$$

which is valid for any $j = 1, \dots, N$. Now we can assume $j < \frac{\mu\sqrt{N}}{2}$ because otherwise, reminding that $\mu = \frac{1}{eA_0}$,

$$(76) \quad (4A_0e)^j \left(\frac{j}{\sqrt{N}}\right)^j > (4A_0e)^j \left(\frac{\mu}{2}\right)^j = 2^j \geq \|E_j\|_1.$$

Thus, if $j < \frac{\mu\sqrt{N}}{2}$ and $j_0 > \mu\sqrt{N}$ then

$$\mu\sqrt{N} \leq j_0 = j + \sum_{i=1}^n k_i \leq j + n < \frac{\mu\sqrt{N}}{2} + n,$$

therefore

$$n \geq \frac{\mu\sqrt{N}}{2}.$$

As a consequence

$$T_2 \leq 2^j e^{j(\tau+1)} \sum_{n \geq \frac{\mu\sqrt{N}}{2}} (2\lambda)^n \leq \frac{(2e)^{j(\tau+1)}}{1-2\lambda} (2\lambda)^{\frac{\mu\sqrt{N}}{2}}.$$

Setting $\beta = |\log(2\lambda)|$ we obtain ($\lambda \leq \frac{1}{4}$)

$$(2\lambda)^{\frac{\mu\sqrt{N}}{2}} \leq \left(\frac{1}{N}\right)^{j/2} \sup e^{-\beta \frac{\mu\sqrt{N}}{2}} N^{j/2} \leq \left(\frac{j}{\sqrt{N}}\right)^j \left(\frac{2}{\beta\mu}\right)^j$$

so that

$$(77) \quad T_2 \leq 2 \left(\frac{j^2}{N}\right)^{j/2} (2e)^{j(\tau+1)} \left(\frac{2}{\beta\mu}\right)^j.$$

Collecting (71), (74), (76) and (77), we conclude that there exists an explicitly computable constant $C(\tau_0) \geq 1$ so that, taking τ_0 small enough the Proposition holds with $A_1 = C(\tau_0)A_0$. \square

To prove Theorem 2.2, we first fix τ_0 small enough. Then, for $t < \tau_0$, Proposition 3.1 gives the desired bound with $C_1 = 0$, $C_2 = C(\tau_0)C_0$. For $t \geq \tau_0$, we set $t = k\tau$ with $\frac{\tau_0}{2} \leq \tau \leq \tau_0$ and $k \geq 1$ is a positive integer. Then we iterate Proposition 3.1 to get

$$\|E_j(t)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} (C(\tau_0)^k C_0)^j.$$

The first part of Theorem 2.2, under hypothesis (51), follows by setting $C_1 = 4 \frac{\log C(\tau_0)}{\tau_0}$ and $C_2 = C(\tau_0)C_0$.

In order to evaluate C_1, C_2 we first fix $\lambda(\tau_0) = 3\tau_0 e e^{\tau_0} = \frac{1}{4}$. This easily implies that $|\log(1 - \frac{1}{24e})| \leq \tau_0 \leq \frac{1}{12e}(1 - \frac{1}{24e})$. Moreover tracing the dependence in τ of (71), (74), (76) and (77), we easily show that, when $\lambda(\tau_0) = \frac{1}{4}$,

$C(\tau_0)$ can be taken as $C(\tau_0) = 16(2e)^{\tau_0+1} \leq 16(2e)^{\frac{1}{12e}+1}$ so that one can take

$$(78) \quad C_1 = 4 \frac{16(2e)^{\frac{1}{12e}+1}}{\left| \log\left(1 - \frac{1}{24e}\right) \right|} \text{ and } C_2 = 16(2e)^{1+1/12e} C_0 .$$

What is left in order to finish the proof of the first part of Theorem 2.2, is to release the hypothesis (51). This is easily done by means of a scaling argument. Note that rescaling V as $V/(4\|V\|)$ in (10) and in (20) is equivalent to speed up time by $4\|V\|$ and rescale K_0 as $K_0/(4\|V\|)$. But all the estimates in this section depend only on K_0 through the hypothesis (50) and we supposed that $\|e^{sK_0^j}\| = \|e^{t(K_0^j + \frac{T_j}{N})}\| = 1$ for all $s, t \in \mathbf{R}$. This allows to conclude.

In particular, for any K_0, V , the corresponding correlation error is

$$E_j^{(K_0, V)}(t) = E_j^{\left(\frac{K_0}{4\|V\|}, \frac{V}{4\|V\|}\right)}(4t\|V\|)$$

and we obtain

$$(79) \quad \|E_j^{(K_0, V)}(t)\|_1 \leq (C_2 e^{C_1 t \|V\|})^j \left(\frac{j}{\sqrt{N}} \right)^j .$$

The second statement in Theorem 2.2 is proven directly from the equation (45) which reads for $j = 1$

$$(80) \quad \partial_t E_1 = (K_0 + D_1)(E_1) + D_1^1(E_2) + D_1^{-1}(E_0) .$$

Using the semigroup $U_1(t, 0)$, estimated by (53) as $\|U_1(t, s)\| \leq e^{(t-s)}$, reminding that, under assumption (51),

$$\|D_1^{-1}(E_0)(s)\|_1 = \frac{1}{N} \|Q(F, F)(s)\|_1 \leq \frac{1}{4N}$$

and, by Theorem 2.2,

$$\|D_1^1(E_2)(s)\|_1 \leq \|E_2(s)\|_1 \leq \frac{4}{N} C_2^2 e^{2C_1 s/4} ,$$

we get

$$\begin{aligned} \|E_1(t)\|_1 &\leq e^t \|E_1(0)\|_1 + \int_0^t e^{(t-s)} \|D_1^1(E_2)(s) + D_1^{-1}(E_0)(s)\|_1 ds \\ &\leq e^t \frac{B_0}{N} + \int_0^t e^{t-s} \left(\frac{1}{4N} + \frac{4}{N} C_2^2 e^{2C_1 s/4} \right) ds \leq \frac{B_2 e^{B_1 t/4}}{N} \end{aligned}$$

$$(81) \quad \text{with } B_1 = 2C_1 \text{ and } B_2 = B_0 + \frac{1}{2} + \frac{16C_2^2}{C_1 - 2} .$$

The same argument as before ($t \rightarrow 4t\|V\|$) allows to place the $\|V\|$ in the exponential, so Theorem 2.2 is proven.

□

Remark 3.2. *Let us point out that the proof of Proposition 3.1 out of the equation (45) uses only the properties (50), (52), (54) and the “convention” (46). No particular explicit incarnation for the operators D_j s in (45) is needed. Therefore Theorem 2.2 and Corollary 2.3 remain true under the validity of Proposition 2.5 involving an equation of type (45-46) endowed with the following assumptions*

$$\|e^{(K_0^j + \frac{T_j}{N})t}\| = 1, \quad \|D_j\|, \|D_j^1\| \leq j \quad \text{and} \quad \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

Remark 3.3. *According to the heuristic argument in the Introduction, we believe that our estimates are optimal as regards the size of chaos and the rate of convergence. They are certainly not optimal as regards the time dependence. However in the above proof we did not try to optimize the numerical constants. It is easy to realize that such constants affect only the growth of the error as a function of time, but not the dependence on j and N whose analysis is the main purpose of this paper.*

4. RETURN TO THE CONCRETE EXAMPLES

In this section we comment on our abstract results in relation to several specific cases.

The identification between the abstract structure analyzed so far and the concrete models we have in mind is given by the table. Let us analyze more precisely our results in these concrete contexts.

4.1. Stochastic models. We recall the evolution equation for the probability measure $f^N(\mathcal{V}_N, t)$ describing the Kac model:

$$\partial_t f^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{f^N(\mathcal{V}_N^{i,j}) - f^N(\mathcal{V}_N)\},$$

where $\mathcal{V}_N = \{v_1, \dots, v_N\}$ and $\mathcal{V}_N^{i,j} = \{v_1, \dots, v_{i-1}, v'_i, \dots, v_{j-1}, v'_j, \dots, v_N\}$ is the vector of the velocities after a collision between particle i and j and $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$ is the differential cross-section of the two-body process, which we assume here to be bounded. The resulting kinetic equation reads

$$\partial_t f(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{f(v')f(v'_1) - f(v)f(v_1)\}.$$

For the soft spheres model the probability density $f^N(X_N, \mathcal{V}_N, t)$ evolves according to

$$\begin{aligned} \partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\times \{f^N(X_N, \mathcal{V}_N^{i,j}) - f^N(X_N, \mathcal{V}_N)\}. \end{aligned}$$

Here $X_N = \{x_1, \dots, x_N\}$ and $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive function with compact support. The associated kinetic equation is

$$\begin{aligned} \partial_t f(v, t) + v \cdot \nabla_x f(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\times \{f(x, v') f(x_1, v'_1) - f(x, v) f(x_1, v_1)\}. \end{aligned}$$

For both models the correlation error is defined as

$$E_j(t) := \sum_{K \subset J} (-1)^{|K|} f_{J \setminus K}^N(t) f(t)^{\otimes K}$$

where $J = \{1, 2, \dots, j\}$, K is any subset of J and $|K| = \text{cardinality of } K$. $f_A^N(t)$ stands for the $|A|$ -marginal $f_{|A|}^N(t)$ computed in the configuration $\{z_i\}_{i \in A}$. Similarly, $f(t)^{\otimes K} = f(t)^{\otimes |K|}$ evaluated in $\{z_i\}_{i \in K}$. Moreover either $z_i = v_i$ or $z_i = (x_i, v_i)$ for the Kac and soft spheres model respectively and f is the solution to the kinetic equations above.

According to Corollary 2.3 we have, whenever $\|E_1(0)\|_{L^1(dz)} \leq B_0/N$,

$$(82) \quad \|f_j^N(t) - f^{\otimes j}(t)\|_{L^1(dz_1 \dots dz_j)} \leq D_2 e^{D_1 t \|V\|} \frac{j^2}{N},$$

where D_2 is a geometrical constant and $\|V\| \leq 2\|B\|_\infty (2\pi + \|h\|_\infty)$.

The Kac model has been extensively studied. We refer to the recent papers [28, 29, 22] and to the references therein. Here uniform in time estimates have been derived for models including unbounded kernels. Typically the error in the propagation of chaos is controlled in terms of a Wasserstein distance, which however is sensitive on the metric chosen for the configuration space.

Estimates in L^1 similar to (82) for models with bounded cross-section were obtained first in [20]. The technique uses an explicit representation of the underlying stochastic process, making rigorous the heuristic argument described in our introduction.

In contrast, the method of the present paper focuses on the errors E_j in an abstract setting, thus using only the hierarchy of equations. This

allows us to apply our results also to different cases such as the soft sphere model where the impact parameter is not random. We also remark that collisional mean field models are potentially useful for applications in population dynamics involving a large number of agents.

4.2. Quantum mean field. In this case, $\mathfrak{A} = \mathcal{L}^1(L^2(\mathbf{R}^d))$ is the space of trace-class operators on $L^2(\mathbf{R}^d)$ and

- $K = \frac{1}{i\hbar}[-\frac{\hbar^2}{2}\Delta_{\mathbf{R}^d}, \cdot]$, $K^N = \frac{1}{i\hbar}[-\frac{\hbar^2}{2}\Delta_{\mathbf{R}^{Nd}}, \cdot]$;
- $V_{r,j} = \frac{1}{i\hbar}[V(x_r - x_j), \cdot]$, $V^N = \frac{1}{i\hbar}[\frac{1}{N} \sum_{1 \leq i \neq j \leq N} V(x_i - x_j), \cdot]$.

For any operator $B \in \mathfrak{A}_j$, $j = 1, \dots, N$, we denote its integral kernel by $B(X_j, X'_j) = B(Z_j)$, $X_j = (x_1, \dots, x_j) \in \mathbf{R}^{jd}$ where we denote $Z_j = (z_1, \dots, z_j) \in \mathbf{R}^{2jd}$, $z_k = (x_k, x'_k) \in \mathbf{R}^{2d}$. We also denote $Z_{J \setminus K}$ the vector Z_j after removing the components z_{i_1}, \dots, z_{i_k} , where J stands for $\{1, \dots, j\}$ and $K = \{i_1, \dots, i_k\}$. The formulas defining the error and its inverse read

$$E_j(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1, \dots, i_k \leq j} \frac{1}{k!} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_{J \setminus K}),$$

$$F_j^N(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1, \dots, i_k \leq j} \frac{1}{k!} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_{J \setminus K}).$$

Note that all operators commute since they act on different variables. The quantum N -body dynamics is defined by the equation

$$i\hbar \partial_t F^N(t) = [-\frac{\hbar^2}{2}\Delta_{\mathbf{R}^{Nd}} + \frac{1}{N} \sum_{1 \leq i < j \leq N} V(x_i - x_j), F^N(t)]$$

$$F^N(0) \geq 0, \quad \text{Tr} F^N(0) = 1.$$

Moreover the mean field Hartree equation reads

$$\partial_t F = \frac{1}{i\hbar}[-\frac{\hbar^2}{2}\Delta + V_F, F],$$

where

$$V_F(x) = \int_{\mathbf{R}^d} V(x - y) F(y, y) dy.$$

Corollary 2.3 is reformulated as follows:

$$(83) \quad \text{Tr}|F_j^N(t) - F^{\otimes j}(t)| \leq D_2 e^{2D_1 t \frac{\|V\|_\infty}{\hbar}} \frac{j^2}{N}.$$

A bound like the one (83) is, to our knowledge, new, in view of its explicit dependence in j , t and \hbar and validity for “all” (pure and mixed) initial data.

Introduced in 1927 (one year after the Schrödinger equation), the Hartree equation has received an enormous interest in physics since then. The first derivation from the quantum N -body dynamics of observables goes back to Hepp in [23], using coherent states, and to Spohn in [34] for pure states, using hierarchies. A proof of the mean field limit for bounded potentials and mixed states including rates of convergence can be found in [2], and for Coulomb singularity and pure states in [16], after [3], using hierarchies. At $\hbar = 1$, the rate of convergence in $\frac{1}{\sqrt{N}}$ has been discussed in [26] together with an explicit dependence in j , using heavily a pure states hypothesis on the initial data, and in [1], improved to a (optimal) rate in $\frac{1}{N}$ in [10] without tracing the dependence in j and \hbar . These two papers use the method of second quantization and “coherent” states in Fock space, initiated in [23, 17]. A rate of convergence with explicit (exponential) dependence in j and \hbar can be found in [19], Theorem 7.1. Eq. (83) realizes an improvement of this last result.

APPENDIX. DERIVATION OF THE CORRELATION EQUATIONS

We prove here Proposition 2.5.

We want to compute the time-derivative of (31), which we recall:

$$(84) \quad E_j = \sum_{K \subset J} (-1)^k F^{\otimes K, J} F_{J \setminus K}^J$$

with $J = \{1, 2, \dots, j\}$, $k = |K|$.

We first notice that, using the mapping (24) and the hierarchy (10),

$$(85) \quad \begin{aligned} F^{\otimes K, J} \partial_t F_{J \setminus K}^J &= \left(K_0^{J \setminus K} + \frac{T_{J \setminus K}}{N} \right) \left(F^{\otimes K, J} F_{J \setminus K}^J \right) \\ &+ \alpha(j - k, N) C_{J \setminus K, j+1} \left(F^{\otimes K, J \cup \{j+1\}} F_{(J \setminus K) \cup \{j+1\}}^{J \cup \{j+1\}} \right), \end{aligned}$$

where, for $S \subset J$,

$$(86) \quad K_0^S = \sum_{i \in S} \mathbb{I}_{\mathfrak{a}_{i-1}} \otimes K_0 \otimes \mathbb{I}_{\mathfrak{a}_{j-i}}, \quad T_S = \sum_{\substack{i, r \in S \\ i < r}} T_{i, r}$$

and

$$(87) \quad C_{S, j+1} = \sum_{i \in S} C_{i, j+1}.$$

Moreover, (20) and (27) imply

$$(88) \quad \partial_t F^{\otimes K, J} = K_0^K \left(F^{\otimes K, J} \right) + \sum_{i \in K} F^{\otimes K^i, J} Q(F, F)^{\otimes \{i, J\}},$$

with the notation $K^i = K \setminus \{i\}$.

Therefore we have

$$\begin{aligned}
\partial_t E_j = & K_0^J(E_j) + \sum_{K \subset J} (-1)^k \sum_{i \in K} F^{\otimes K^i, J} Q(F, F)^{\otimes \{i\}, J} F_{J \setminus K}^J \\
& + \sum_{K \subset J} (-1)^k \alpha(j-k, N) \sum_{i \in J \setminus K} C_{i, j+1} \left(F^{\otimes K, J \cup \{j+1\}} F_{(J \setminus K) \cup \{j+1\}}^{J \cup \{j+1\}} \right) \\
(89) \quad & + \frac{1}{2N} \sum_{K \subset J} (-1)^k \sum_{\substack{i, r \in J \setminus K \\ i \neq r}} T_{i, r} \left(F^{\otimes K, J} F_{J \setminus K}^J \right).
\end{aligned}$$

In the following computation we shall simplify the notation by skipping the upper indices of sets J and $J \cup \{j+1\}$, now clear from the context.

We compute next the terms in the three lines on the r.h.s. of (89) separately. They are denoted by \mathcal{T}_i , $i = 1, 2, 3$ respectively.

Using $K^i = K \setminus \{i\}$ and the change $K \rightarrow K^i$,

$$\begin{aligned}
\mathcal{T}_1 = & K_0^J(E_j) + \sum_{i \in J} \sum_{K \subset J^i} (-1)^{(k-1)} F^{\otimes K} Q(F, F)^{\otimes \{i\}} F_{J^i \setminus K} \\
& = K_0^J(E_j) - \sum_{i \in J} Q(F, F)^{\otimes \{i\}} \sum_{K \subset J^i} (-1)^k F^{\otimes K} F_{J^i \setminus K} \\
& = K_0^J(E_j) - \sum_{i \in J} Q(F, F)^{\otimes \{i\}} E_{J^i},
\end{aligned}$$

where in the last step we applied (35) (for $E_{J^i}^J$).

To compute the term \mathcal{T}_2 , we will make use of the combinatorial identity

$$(90) \quad \sum_{K \subset R} (-1)^k \alpha(j-k, N) = \alpha(j, N) \delta_{R, \emptyset} - \frac{1}{N} \delta_{|R|, 1} \quad R \subset J.$$

We postpone the elementary proof of (90) to the end of the section.

Applying again (35) into the second line of (89) we obtain

$$\begin{aligned}
\mathcal{T}_2 = & \sum_{K \subset J} (-1)^k \alpha(j-k, N) \sum_{i \in J \setminus K} C_{i, j+1} \left(F^{\otimes K} \sum_{L \subset J \cup \{j+1\} \setminus K} F^{\otimes L} E_{J \cup \{j+1\} \setminus (K \cup L)} \right) \\
& = \sum_{i \in J} \sum_{K \subset J^i} (-1)^k \alpha(j-k, N) C_{i, j+1} \left(F^{\otimes K} \sum_{L \subset J \cup \{j+1\} \setminus K} F^{\otimes L} E_{J \cup \{j+1\} \setminus (K \cup L)} \right). \\
(91) \quad &
\end{aligned}$$

Now we distinguish the following cases:

- $r = 1$: $i, j+1 \in L$
- $r = 2$: $i, j+1 \notin L$
- $r = 3$: $i \in L, j+1 \notin L$
- $r = 4$: $i \notin L, j+1 \in L$

and set

$$\mathcal{T}_2^1 = \sum_{r=1}^4 \mathcal{T}_2^r$$

with \mathcal{T}_2^i given by (91) with the additional constraint $r = i$ above. Setting $L' = L^{i,j+1} = L \setminus \{i, j+1\}$ and $R = K \cup L'$ and recalling (28),

$$\mathcal{T}_2^1 = \sum_{i \in J} \sum_{R \subset J^i} Q(F, F)^{\otimes \{i\}} F^{\otimes R} E_{J^i \setminus R} \sum_{K \subset R} (-1)^k \alpha(j-k, N)$$

so that (90) leads to

$$\mathcal{T}_2^1 = \alpha(j, N) \sum_{i \in J} Q(F, F)^{\otimes \{i\}} E_{J^i} - \frac{1}{N} \sum_{\substack{i, r \in J \\ i \neq r}} Q(F, F)^{\otimes \{i\}} F^{\otimes \{r\}} E_{J^{i,r}} .$$

Observe that

$$\mathcal{T}_1 + \mathcal{T}_2^1 = K_0^J(E_j) - \frac{j}{N} \sum_{i \in J} Q(F, F)^{\otimes \{i\}} E_{J^i} - \frac{1}{N} \sum_{\substack{i, r \in J \\ i \neq r}} Q(F, F)^{\otimes \{i\}} F^{\otimes \{r\}} E_{J^{i,r}} ,$$

namely there is a crucial compensation for which all the operators, but K_0^J , are $O(\frac{j^2}{N})$.

Furthermore, setting $R = K \cup L$,

$$\begin{aligned} \mathcal{T}_2^2 &= \sum_{i \in J} \sum_{L \subset J^i} \sum_{K \subset J^i / L} (-1)^k \alpha(j-k, N) C_{i,j+1} (F^{\otimes (K \cup L)} E_{J \cup \{j+1\} \setminus (K \cup L)}) \\ &= \sum_{i \in J} \sum_{R \subset J^i} \sum_{K \subset R} (-1)^k \alpha(j-k, N) C_{i,j+1} (F^{\otimes R} E_{J \cup \{j+1\} \setminus R}) \\ &= \alpha(j, N) \sum_{i \in J} C_{i,j+1} (E_{j+1}) - \frac{1}{N} \sum_{\substack{i, r \in J \\ i \neq r}} C_{i,j+1} (F^{\otimes \{r\}} E_{J^r \cup \{j+1\}}) . \end{aligned}$$

To compute \mathcal{T}_2^3 we set $L' = L^i$ and $R = L' \cup K$ so that

$$\begin{aligned} \mathcal{T}_2^3 &= \sum_{i \in J} \sum_{R \subset J^i} \sum_{K \subset R} (-1)^k \alpha(j-k, N) C_{i,j+1} (F^{\otimes R} F^{\otimes \{i\}} E_{J^i \cup \{j+1\} \setminus R}) \\ &= \alpha(j, N) \sum_{i \in J} C_{i,j+1} (F^{\otimes \{i\}} E_{J^i \cup \{j+1\}}) - \frac{1}{N} \sum_{\substack{i, r \in J \\ i \neq r}} C_{i,j+1} (F^{\otimes \{r,i\}} E_{J^{i,r} \cup \{j+1\}}) . \end{aligned}$$

Finally, setting $L' = L^{j+1}$ and $R = K \cup L'$ we obtain

$$\begin{aligned} \mathcal{T}_2^4 &= \sum_{i \in J} \sum_{R \subset J^i} \sum_{K \subset R} (-1)^k \alpha(j-k, N) C_{i,j+1} (F^{\otimes R} F^{\otimes \{j+1\}} E_{J \setminus R}) \\ &= \alpha(j, N) \sum_{i \in J} C_{i,j+1} (F^{\otimes \{j+1\}} E_J) - \frac{1}{N} \sum_{\substack{i, r \in J \\ i \neq r}} C_{i,j+1} (F^{\otimes \{r,j+1\}} E_{J^r}) . \end{aligned}$$

Similarly we compute the term

$$\begin{aligned}\mathcal{T}_3 &= \frac{1}{2N} \sum_{K \subset J} (-1)^k \sum_{i,s \in J \setminus K} \sum_{L \subset J \setminus K} T_{i,s} (F^{\otimes K, J} F^{\otimes L} E_{J \setminus (K \cup L)}) \\ &= \sum_{r=1}^3 \mathcal{T}_3^r\end{aligned}$$

where each term \mathcal{T}_3^r corresponds to the constraints

- $r = 1$: $i, s \in L$
- $r = 2$: $i, s \notin L$
- $r = 3$: $i \in L, s \notin L$.

Setting $L' = L^{i,s}$ and $K \cup L' = R$ we have

$$\begin{aligned}\mathcal{T}_3^1 &= \frac{1}{2N} \sum_{i,s \in J} \sum_{R \subset J^{i,s}} \sum_{K \subset R} (-1)^k T_{i,s} (F^{\otimes \{i,s\}} F^{\otimes R} E_{J^{i,s} \setminus R}) \\ &= \frac{1}{2N} \sum_{i,s \in J} T_{i,s} (F^{\otimes \{i,s\}} E_{J^{i,s}})\end{aligned}$$

and an analogous computation gives

$$\mathcal{T}_3^2 = \frac{1}{2N} \sum_{\substack{i,s \in J \\ i \neq s}} T_{i,s} (E_j) = \frac{T_j}{N} (E_j)$$

and

$$\mathcal{T}_3^3 = \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} T_{i,s} (F^{\otimes \{i\}} E_{J^i}).$$

In conclusion:

$$\begin{aligned}\partial_t E_j &= K_0^j (E_j) + \frac{T_j}{N} (E_j) + \alpha(j, N) \sum_{i \in J} C_{i,j+1} (F^{\otimes \{i\}} E_{J^{i \cup \{j+1\}}} + F^{\otimes \{j+1\}} E_J) \\ &\quad - \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} C_{i,j+1} (F^{\otimes \{s\}} E_{J^{s \cup \{j+1\}}}) \\ &\quad + \alpha(j, N) C_{j+1} (E_{j+1}) \\ &\quad - \frac{j}{N} \sum_{i \in J} Q(F, F)^{\otimes \{i\}} E_{J^i} + \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} T_{i,s} (F^{\otimes \{i\}} E_{J^i}) \\ &\quad - \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} C_{i,j+1} (F^{\otimes \{i,s\}} E_{J^{i,s \cup \{j+1\}}} + F^{\otimes \{s,j+1\}} E_{J^s}) \\ (92) \quad &\frac{1}{2N} \sum_{\substack{i,s \in J \\ i \neq s}} T_{i,s} (F^{\otimes \{i,s\}} E_{J^{i,s}}) - \frac{1}{N} \sum_{\substack{i,s \in J \\ i \neq s}} Q(F, F)^{\otimes \{i\}} F^{\otimes \{s\}} E_{J^{i,s}}.\end{aligned}$$

We organized the terms in the above equation, according to the following rule. The first two lines contain operators which do not change the particle

number. The third line increases the number of particles by one. The 4th and 5th lines decrease the particle number by one. Finally the last line decreases it by two.

Using the definition of D_j in (40)-(41) and of $D_j^1, D_j^{-1}, D_j^{-2}$ in (42), (43) and (44), Eq. (92) reads

$$(93) \quad \begin{aligned} \partial_t E_j &= \left(K_0^j + \frac{T_j}{N} \right) (E_j) + D_j (E_j) \\ &+ D_j^1 (E_{j+1}) + D_j^{-1} (E_{j-1}) + D_j^{-2} (E_{j-2}) . \end{aligned}$$

We end this section with the proof of (90). Denoting $|R| = r$ one has

$$\begin{aligned} \sum_{k=0}^r (-1)^k \binom{r}{k} \alpha(j-k, N) &= \alpha(j, N) \sum_{k=0}^r (-1)^k \binom{r}{k} + \frac{1}{N} \sum_{k=0}^r (-1)^k \binom{r}{k} k \\ &= \alpha(j, N) \delta_{r,0} - \frac{1}{N} r \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \\ &= \alpha(j, N) \delta_{r,0} - \frac{1}{N} r (1-1)^{r-1} \\ &= \alpha(j, N) \delta_{r,0} - \frac{1}{N} r \delta_{r,1} . \end{aligned}$$

Acknowledgements. The authors are grateful to Joaquin Fontbona and Stéphane Mischler for helpful discussions. This work has been partially carried out thanks to the supports of the LIA AMU-CNRS-ECM-INDAM Laboratoire Ypatie des Sciences Mathématiques (LYSM) and the A*MIDEX project (n° ANR-11-IDEX-0001-02) funded by the “Investissements d’Avenir” French Government program, managed by the French National Research Agency (ANR). T.P. thanks also the Dipartimento di Matematica, Sapienza Università di Roma, for its kind hospitality during the elaboration of this work. S.S. acknowledges support of the German Research Foundation (DFG n° 269134396).

REFERENCES

- [1] N. Benedikter, M. Porta and B. Schlein: *01040000 Effective Evolution Equations from Quantum Dynamics*, Springer-Briefs in Mathematical Physics (2016).
- [2] C. Bardos, F. Golse, N. Mauser: *Weak coupling limit of the N particles Schrödinger equation*, Methods Appl. Anal. **7** (2000), no.2, 275–293.
- [3] C. Bardos, L. Erdős, F. Golse, N. Mauser, H.-T. Yau: *Derivation of the Schrödinger-Poisson equation from the quantum N -body problem*, C. R. Acad. Sci. Paris, Sér. I **334** (2002), 515–520.
- [4] C. Boldrighini, A. De Masi, A. Pellegrinotti : *Non equilibrium fluctuations in particle systems modelling Reaction-Diffusion equations*. Stochastic Processes and Appl. **42** , 1-30 (1992).
- [5] W. Braun, K. Hepp: *The Vlasov Dynamics and Its Fluctuations in the $1/N$ Limit of Interacting Classical Particles*, Commun. Math. Phys. **56** (1977), 101–113.
- [6] S. Caprino, M. Pulvirenti: *A cluster expansion approach to a one-dimensional Boltzmann equation: a validity result* Comm. Math. Phys. . **166**, 3 (1995), 603-631.
- [7] S. Caprino, A. De Masi, E. Presutti, M. Pulvirenti: *A derivation of the Broadwell equation*. Comm. Math. Phys. **135** (1991) 3, 443–465.

- [8] S. Caprino, M. Pulvirenti and W. Wagner: *A particle systems approximating stationary solutions to the Boltzmann equation* SIAM J. Math. Anal. **4** (1998), 913-934.
- [9] C. Cercignani: *The Grad limit for a system of soft spheres*, Comm. Pure Appl. Math. **36** (1983), 479-494.
- [10] L. Chen, J. Oon Lee, B. Schlein: *Rate of Convergence Towards Hartree Dynamics*, J. Stat. Phys. **144** (2011), 872–903.
- [11] A.De Masi, E. Presutti: *Mathematical methods for hydrodynamical limits*. Lecture Notes in Mathematics 1501, Springer-Verlag, (1991).
- [12] A. De Masi, E. Orlandi, E. Presutti, L. Triolo: *Glauber evolution with Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics*. Nonlinearity **7**, 633-696, (1994).
- [13] A.De Masi, E. Orlandi, E. Presutti, L. Triolo: *Glauber evolution with Kac potentials. II. Fluctuations*. Nonlinearity **9**, 27–51, (1996).
- [14] A.De Masi, E. Presutti, D. Tsagkarogiannis, M.E. Vares: *Truncated correlations in the stirring process with births and deaths*. Electronic Journal of Probability, **17**, 1-35, (2012).
- [15] R. Dobrushin: *Vlasov equations*, Funct. Anal. Appl. **13** (1979), 115–123.
- [16] L. Erdős, H.-T. Yau: *Derivation of the nonlinear Schrödinger equation from a many body Coulomb system*, Adv. Theor. Math. Phys. **5**(6) (2001), 1169–1205.
- [17] J. Ginibre, G. Velo: *The classical field limit of scattering theory for nonrelativistic many-boson systems. I.-II.*, Comm. Math. Phys. **66** (1979), 37-76 and **68** (1979), 451768.
- [18] F. Golse, C. Mouhot, T. Paul: *On the Mean Field and Classical Limits of Quantum Mechanics*, Commun. Math. Phys. **343** (2016), 165-205.
- [19] F. Golse, T. Paul, M. Pulvirenti: *On the Derivation of the Hartree Equation from the N-Body Schrödinger Equation: Uniformity in the Planck Constant*, arXiv:1606.06436.
- [20] C. Graham, S. Méléard: *Stochastic particle approximations for generalized Boltzmann models and convergence estimates*, Annals of Probability **25** (1997), 11517132.
- [21] M. Hauray, P.-E. Jabin: *Particles approximations of Vlasov equations with singular forces: Propagation of chaos*, Annales scientifiques de l'ENS **48**, 4 (2015), 891-940.
- [22] M. Hauray, S. Mischler: *On Kacs chaos and related problems*, Journal of Functional Analysis **266** (2014) 6055176157.
- [23] K. Hepp: *The classical limit for quantum mechanical correlation functions*, Commun. Math. Phys. **35** (1974).
- [24] M. Kac: *Foundations of kinetic theory*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley and Los Angeles, 1956.
- [25] M. Kac: *Probability and related topics in physical sciences*, Interscience, London-New York, 1959.
- [26] A. Knowles, P. Pickl: *Mean-Field Dynamics: Singular Potentials and Rate of Convergence*, Com. Math. Physics **298** (2010), 101-138.
- [27] M Lachowicz, M Pulvirenti: *A stochastic system of particles modelling the Euler equation*, Arch. Ration. Mech. Anal. **109** (1990), 81-93.
- [28] S. Mischler, C. Mouhot: *Kac's program in kinetic theory*, Inventiones mathematicae **193** (2013), 1-147.
- [29] S. Mischler, C. Mouhot, B. Wennberg: *A new approach to quantitative propagation of chaos for drift, diffusion and jump processes*, Probability Theory and Related Fields **161** (2015), 1-2, p. 1-59.
- [30] H. Neunzert, J. Wick: *Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmen-gen*, Lecture Notes in Math. **395** (1974), 275–290, Springer, Berlin.
- [31] M. Pulvirenti, S. Simonella: *The Boltzmann-Grad limit of a hard sphere system: analysis of the correlation error* Inventiones mathematicae, **207**(3) (2017) 1135-1237.
- [32] M. Pulvirenti, W. Wagner, M.B. Zavelani Rossi: *Convergence of particle schemes for the Boltzmann equation*, Eur. J. Mech. B/Fluids, **13** 3 (1994), 339–351.
- [33] S.Rjasanow, W. Wagner: *Stochastic Numerics for the Boltzmann Equation*, Springer Series in Computational Mathematics **37** (2005).
- [34] H. Spohn: *Kinetic equations from Hamiltonian dynamics*, Rev. Mod. Phys. **52** (1980), 3, 600–640.

(T.P.) CMLS, ECOLE POLYTECHNIQUE, CNRS, UNIVERSITÉ PARIS-SACLAY, 91128 PALAISEAU CEDEX, FRANCE
E-mail address: thierry.paul@polytechnique.edu

(M.P.) INTERNATIONAL RESEARCH CENTER ON THE MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS, MEMOCS,
 UNIVERSITY OF L'AQUILA, ITALY
E-mail address: pulviren@mat.uniroma1.it

(S.S.) ZENTRUM MATHEMATIK, TU MÜNCHEN, BOLTZMANNSTRASSE 3, 85748 GARCHING – GERMANY
E-mail address: s.simonella@tum.de