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PROBABILISTIC CONSTRUCTION AND PROPERTIES OF GAMMA PROCESSES AND EXTENSIONS

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A standard gamma process is widely used for cumulative deterioration modeling purpose and to make prediction over the system future behavior. However, this process may be restrictive within an applicative context since its variance-to-mean ratio is constant over time. Extended gamma process, which was introduced by Cinlar (1980), seems to be a good candidate to overcome the latter restriction. The aim of this paper is to investigate benefits of using an extended gamma process for modeling the system evolution instead of a standard gamma process, from a reliability point of view. With that goal, we propose a condition-based dynamic maintenance policy and evaluate its performance on a finite planning horizon. Numerical experiments are illustrated based on simulated data.

In the oldest reliability literature, lifetimes of industrial systems or components were usually modeled through random variables, e.g. see [3] for a pioneer work on the subject. Based on the development of on-line monitoring which allows for the effective measurement of a system deterioration, numerous papers nowadays model the degradation in itself, which is often considered to be accumulating over time. In such a context, one can find shock models, where the deterioration is suddenly increased at each shock, in contrast to wear models, where the degradation level appears as mostly continuous. Deterioration induced by isolated shocks is typically modeled through compound Poisson or shot noise processes [19] whereas the most classical wear models are Wiener process (with trend) [10,19,25] and gamma process, the use of which seems to go back to the middle of the 80's [1,8]. Inverse gaussian process also appeared as a possible model more recently [24,26], as well as inverse gamma process, see [15,16]. This talk is devoted to the seemingly most common model, that is gamma process (and extensions). We refer to [23] for a comprehensive presentation and

overview of applications of the gamma process to reliability theory, and to [17] for a deep account on its probabilistic construction and jump behavior (together with statistical inference procedures).

Given an increasing and continuous function $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $A(0) = 0$ and $b > 0$, let us first recall that a càdlàg process $\mathbf{X} = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be a (non-homogeneous) gamma process with shape function $A(\cdot)$ and scale parameter b (written $\mathbf{X} \sim \mathcal{G}(A(\cdot), b)$) if

- $X_0 = 0$ almost surely,
- \mathbf{X} has independent increments,
- for all $0 \leq s < t$, the random variable $X_t - X_s$ is gamma distributed $\mathcal{G}(A(t) - A(s), b)$ with the following probability density function with respect to Lebesgue measure:

$$f(x) = \frac{b^{A(t)-A(s)}}{\Gamma(A(t) - A(s))} x^{A(t)-A(s)-1} e^{-bx} \mathbf{1}_{\mathbb{R}_+}(x), \text{ for all } x \in \mathbb{R}.$$

In the specific case where $A(t) = at$ with $a > 0$, the process \mathbf{X} is said to be a homogeneous gamma process. In that case, it is a Lévy process [5]. In the most general case, it is an additive (or non homogeneous Lévy) process [22].

As a first step, probabilistic constructions of a gamma process will be reviewed through series representations of the shape

$$X_t = \sum_{n \geq 1} V_n \mathbf{1}_{[0,t]}(U_n),$$

based on the works of Bondesson [6] and Rosiński [21]. These constructions allow to give a good account of the jump structure of a gamma process $(X_t)_{t \geq 0}$, which can be seen to have an infinite activity [12], namely to jump infinitely many on any finite time interval. This is illustrated in Figure 1, where we can observe that there is an accumulation of jumps of small size. Based on this, the gamma process is well adapted to model deterioration that is accumulating over time as the result of many tiny increments [23].

From an applicative point of view, a system with non-decreasing deterioration $(X_t)_{t \geq 0}$ is typically considered as failed (or too degraded) as soon the deterioration level is beyond a given failure threshold (say h). The time-to-failure of the system hence is the hitting time τ_h of the Borel set $[h, +\infty)$ by the process $(X_t)_{t \geq 0}$, with

$$\tau_h = \inf (t \geq 0 : X_t \geq h).$$

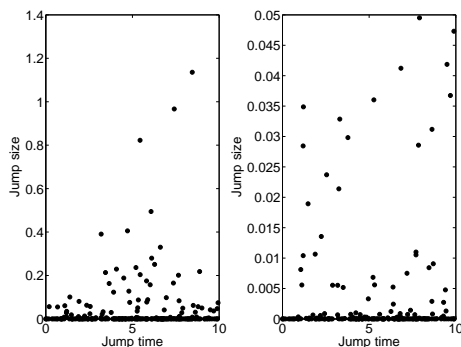


Figure 1. An observation of the jump times (U_n) and jump sizes (V_n) of a gamma process $\mathcal{G}(t, 1)$, restricted to the jumps with size greater than 10^{-10} (left) and zoomed observation of the left plot, restricted to the jumps with size in $[10^{-10}, 0.05]$ (right)

Based on the monotony of a gamma process, it is immediate to get the survival function of τ_h :

$$\bar{F}_{\tau_h}(t) = \mathbb{P}(\tau_h > t) = \mathbb{P}(X_t \leq h) = F_{X_t}(h),$$

where F_{X_t} is the cumulative distribution function of X_t .

However, contrary to what happens with a continuous process such as a Wiener process, the level h is not exactly reached at time τ_h : it is crossed by a jump, so that the level X_{τ_h} at time τ_h is a.s. larger than h , leading to an a.s. positive overshoot ($X_{\tau_h} - h$), see Figure 2. In an applicative context, the failure may be all the more severe as the overshoot is higher, which may entail safety problems. Whence the interest of studying the after-jump level X_{τ_h} together with the jump time τ_h , as well as the size of the jump $X_{\tau_h} - X_{\tau_h^-}$. Several results will be reviewed on these hitting times, together with the before/after jump levels, based on results from [5]. Aging properties of τ_h will also be mentioned, which justify the use of preventive maintenance actions for a gamma deteriorating process.

A few properties of a gamma process will next be pointed out, which can be restrictive in an applicative context. Several extensions from the literature will be reviewed, which allow to overcome such limitations. As an example, the variance-to-mean ratio of a (standard) gamma process is known to be constant over time. This has lead to the development of Extended gamma process, as defined by [9], where the scale parameter may vary over time. See [14] for a practical use in reliability in a discrete time context, and [2] for approximate simulation methods and probabilistic computations. This process is also called Weighted gamma process by [11]

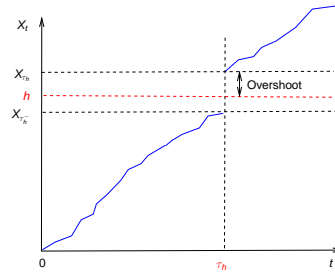


Figure 2. Overshoot of the gamma process

(in a Bayesian context).

Based on the independence of its increments, another possible restriction of the gamma process is that a deterioration increment $X_t - X_s$ (with $0 < s < t$) does not depend on the previous history of the process at time s , whatever severe the deterioration could have been before s . A second extension called Transformed gamma process was recently proposed by [13], which overcomes this restriction.

A third extension called Perturbed gamma process is studied by [7], which allows to model non strictly monotonous deterioration (but with monotonous trend). This can, e.g., allow to deal with measurement errors.

Finally, the development of online monitoring often allows to measure several deterioration indicators at the same time, see, e.g., [20] for a practical example for railway studies. Apart from the previously mentioned univariate extensions, there hence is a crucial need for multivariate deterioration model. A possibility is to consider multivariate (non homogeneous) Lévy processes with gamma marginal processes. A few possible constructions from the literature will be reviewed, such as, e.g., superposition or subordination of independent gamma processes [4], or construction through Lévy copulas [18], which allows for more flexibility in the dependence of the marginal gamma processes.

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