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A continuous probability model allowing occurrence of zero values: Polynomial-exponential

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ABSTRACT

This paper deals with a new two-parameter lifetime distribution with increasing, decreasing and constant hazard rate. This distribution allows the occurrence of zero values and involves the exponential, linear exponential and other combinations of Weibull distributions as submodels. Many statistical properties of the distribution are derived. Maximum likelihood estimation of the parameters is investigated with a simulation study for performance of the estimators. Two real data sets are analyzed for illustrative purposes and it is noted that the distribution is a highly alternative to the gamma, Weibull, Lognormal and exponentiated exponential distributions.

KEYWORDS

Hazard rate; Two-parameter distributions; Reliability and statistical measures; Maximum Likelihood Estimation; Data applications.

1. Introduction with motivations

In analysis of the lifetime data, monotone hazard rates are common. Such data can be modelled using the log-normal, Weibull and gamma distributions. The Weibull distribution is more popular than log-normal and gamma because the survival and hazard rate functions of the last two distributions have not a closed form and hence numerical integrations are required. Gupta and Kundu [6] introduced the exponentiated exponential (EE) distribution as an extension to the exponential distribution and also as an alternative to the gamma distribution. Further developments on the exponentiated exponential distribution can be seen in [7].

In many practical applications, continuous probability models that allow occurrence of zero values have vast importance, for example in forecast models when we observe the monthly rainfall precipitation, it is common in dry periods the non occurrence of precipitation, therefore the occurrence of zero values can be observed in different measures, such as the average, maximum and minimum. In survival analysis, we may

observe data with instantaneous failure due construction problem. Another example of zero occurrence is the hydrologic data in arid and semiarid regions, like annual peak flow discharges. Moreover, zero occurrence can be met in many areas, e.g. manufacturing defects, medical consultations, hydrology, ecology and econometrics.

The above four distributions (log-normal, Weibull, gamma and exponentiated exponential) do not provide the characteristic of zero occurrence. Therefore, we introduce a new continuous probability model that allows occurrence of zero values and it shall be called Polynomial-exponential (PE) distribution. This model represents a powerful alternative to the mentioned distributions above and also an extension to exponential, linear exponential and other combinations of Weibull distributions as it shall be seen later. The model is specified in terms of the cumulative distribution function (cdf) as:

$$F(x) = 1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}}, \quad x \in (0, +\infty) \setminus \{1\},$$

where $\alpha > 0$, $\lambda > 0$ (and $F(x) = 0$ if $x < 0$) and Figure 1 presents plots of the cdf for different values of α and $\lambda = 2$. Note that, there is a discontinuity when $x = 1$, to avoid that we can considered the continuous extension for $x = 1$ given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}}, & x \in (0, +\infty) \setminus \{1\}, \\ 1 - e^{-\lambda \alpha}, & x = 1. \end{cases} \quad (1)$$

Moreover, a significant account of mathematical properties for the new distribution are provided and the hazard rate function has constant, increasing or decreasing shape, which making the PE distribution an alternative to the mentioned distributions above. Another attractive feature of the PE distribution is that it has closed form expressions for its cdf and hazard rate function, which is not the case, for the log-normal and gamma distributions. Further, the distribution has several particular sub-models. For $\alpha = 1$, the PE distribution gives the exponential and when $\alpha = 2$, it reduces to the one parameter linear exponential distribution. Also, when α is an integer, we can express $F(x)$ as $F(x) = 1 - \exp\left(-\lambda \sum_{k=1}^{\alpha} x^k\right)$, $x > 0$, and hence the survival function $S(x) = 1 - F(x)$ of the PE distribution is the product of the survival functions of Weibull distributions with parameters $(\lambda, 1)$, $(\lambda, 2)$, ..., and (λ, α) with respect to value of k , which means the distribution has an exponential general polynomial.

Also, we can note $F(x)$ given by (1) has the form:

$$F(x) = G(H(x)),$$

where $G(x)$ is the cdf associated to the exponential distribution with parameter $\lambda > 0$ and $H(x) = x \frac{x^\alpha - 1}{x-1}$ (with $H(1) = \alpha$) is a positive increasing function with $H(0) = 0$ and $\lim_{x \rightarrow +\infty} H(x) = +\infty$.

The inferential procedure for the parameters of PE distribution is presented using the maximum likelihood estimation. It is shown that one of the estimators can be obtained in closed-form and this allows us to obtain the estimates solving a simple one non-linear equation. The performance of the MLEs are compared using extensive numerical simulations.

The paper is organized as follows. In Section 2 we introduce the PE distribution. Section 3 is devoted to some of its mathematical properties. Estimations of the parameters via the maximum likelihood method are investigated in Section 4. A simulation

analysis is given in Section 5. In Section 6 we apply our proposed model in two real data sets. Finally in Section 7 we conclude the paper.

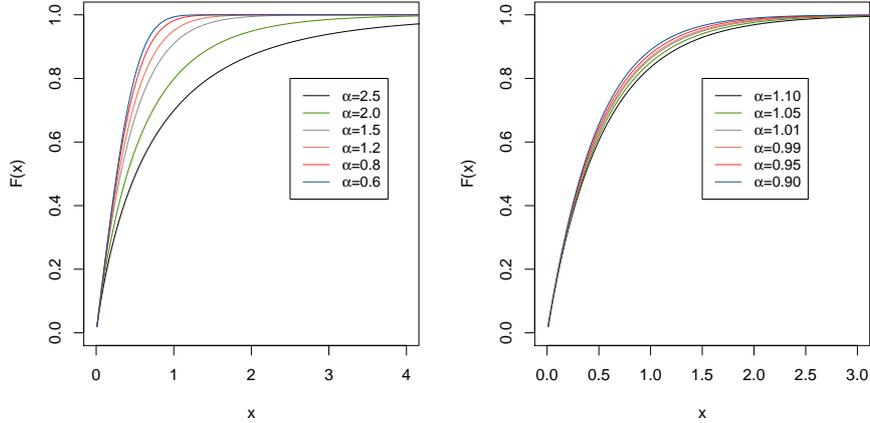


Figure 1. Cumulative density function shapes for PE distribution considering different values of α and fixed $\lambda = 2$.

2. Polynomial-exponential distribution

The associated probability density function (pdf) of the cdf given by the equation (1) is

$$f(x) = \lambda \frac{\alpha x^{\alpha+1} - (\alpha + 1)x^\alpha + 1}{(x - 1)^2} e^{-\lambda x \frac{x^\alpha - 1}{x - 1}}, \quad x \in (0, +\infty) \setminus \{1\}, \quad (2)$$

with the continuous extension for $x = 1$: $f(1) = \frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda}$. Figure 2 presents some plot of the PE distribution for different values of α and λ , and showing various shapes of the density function with left skewness.

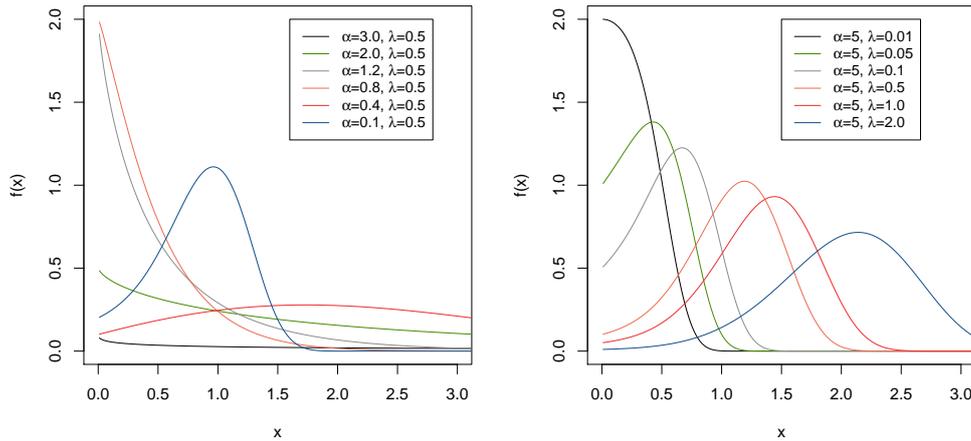


Figure 2. Density function shapes for PE distribution considering different values of α and λ .

As we know, many distributions such as log-normal, Weibull, gamma and exponentiated exponential, to list a few, do not allow occurrence of zero values. In this regard,

the following remark shows that the PE distribution can be used as a model with occurrence of zero values.

Let us observe that $f(0) = \lambda > 0$ for all $\alpha > 0$ and $\lambda > 0$. Further, it follows from equation (3) that $f(x) \sim \lambda \alpha x^{\alpha-1} e^{-\lambda x^\alpha}$ as $x \rightarrow \infty$. Therefore, the upper tail behavior of the pdf is a product of a polynomial power and an exponential polynomial power decay, both of them depend only on α . Obviously, larger values of α lead to faster decay of the upper tail, which interprets α as a shape parameter.

The hazard rate function (hrf) is given by

$$h(x) = \lambda \frac{\alpha x^{\alpha+1} - (\alpha + 1)x^\alpha + 1}{(x-1)^2}, \quad x \in (0, +\infty) \setminus \{1\},$$

with the continuous extension for $x = 1$ given by $h(1) = \frac{\lambda \alpha (\alpha + 1)}{2}$.

The study of the behavior of the hazard function is not an easy task. Glaser's [5] lemma is difficult to be implemented since $\eta(t) = -\frac{d}{dt} \log(f(t))$ does not has a simple form. However, from graphical analysis we observed that the hazard function presents an decreasing hazard rate for $\alpha > 1$ and $\lambda > 0$, increasing hazard rate for $\alpha < 1$ and $\lambda > 0$ and constant rate for $\alpha = 1$, and for this purpose some plots of the hazard function with various values for the parameters α and λ are presented in Figure 3.

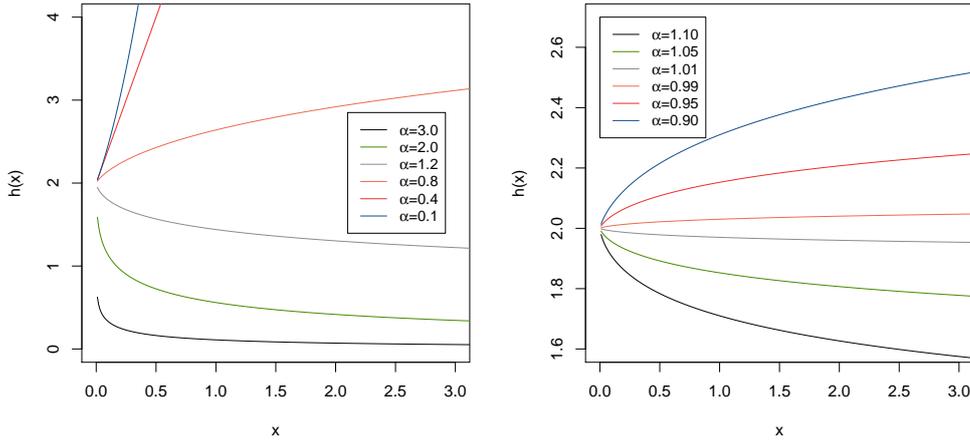


Figure 3. Hazard function shapes for PF distribution for $\lambda = 2$ and considering different values of α .

Moreover, note that, $h(x) \sim \lambda$ as $x \rightarrow 0$, and $h(x) \sim \lambda \alpha x^{\alpha-1}$ as $x \rightarrow \infty$. Hence, we conclude that, the lower tail of the hazard rate function is a constant, while its upper tail is a polynomial which allows for increasing, decreasing and constant hazard rate shapes.

Remark 1. Using the indicator function: $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 elsewhere, we have following analytic expressions for $F(x)$, $f(x)$ and $h(x)$:

$$F(x) = \left(1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}}\right) \mathbf{1}_{(0, +\infty) \setminus \{1\}}(x) \left(1 - e^{-\lambda}\right) \mathbf{1}_{\{1\}}(x) \mathbf{1}_{(0, +\infty)}(x),$$

$$\begin{aligned}
f(x) &= \left(\lambda \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} e^{-\lambda x \frac{x^\alpha-1}{x-1}} \right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)} \times \\
&\times \left(\frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha} \right)^{\mathbf{1}_{\{1\}}(x)} \mathbf{1}_{(0,+\infty)}(x),
\end{aligned} \tag{3}$$

and

$$h(x) = \left(\lambda \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} \right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)} \left(\frac{\lambda\alpha(\alpha+1)}{2} \right)^{\mathbf{1}_{\{1\}}(x)} \mathbf{1}_{(0,+\infty)}(x).$$

These analytic expressions will be useful in the next.

3. Mathematical properties

3.1. Some useful expansions

The result below presents a polynomial expansion of the cdf $F(x)$ given by (1).

Proposition 3.1. *We have the following expansion, for $x \in (0, +\infty) \setminus \{1\}$,*

$$F(x) = 1 - \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} A_{k,\ell,j} \left[(-1)^k x^{k+\alpha\ell+j} \mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j} \mathbf{1}_{(1,+\infty)}(x) \right],$$

where

$$A_{k,\ell,j} = \binom{k}{\ell} \binom{-k}{j} \frac{1}{k!} (-1)^{\ell+j} \lambda^k.$$

Proof of Proposition 3.1. First of all, let us now investigate an expansion for $e^{-\lambda x \frac{x^\alpha-1}{x-1}}$ by distinguishing the case $x \in [0, 1)$ and the case $x > 1$.

- If $x \in [0, 1)$, note that $\frac{x^\alpha-1}{x-1} = (1-x^\alpha) \frac{1}{1-x}$. The exponential and binomial series give

$$\begin{aligned}
e^{-\lambda x \frac{x^\alpha-1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\lambda x \frac{x^\alpha-1}{x-1} \right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} (-\lambda)^k x^k (1-x^\alpha)^k \frac{1}{(1-x)^k} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} \binom{k}{\ell} \binom{-k}{j} \frac{1}{k!} (-1)^{\ell+j} (-\lambda)^k x^{k+\alpha\ell+j}.
\end{aligned}$$

- If $x > 1$, note that $x \frac{x^\alpha-1}{x-1} = -(1-x^\alpha) \frac{1}{1-\frac{1}{x}}$. It follows from the exponential and binomial series that

$$\begin{aligned}
e^{-\lambda x \frac{x^\alpha-1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\lambda x \frac{x^\alpha-1}{x-1} \right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k (1-x^\alpha)^k \frac{1}{(1-\frac{1}{x})^k} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} \binom{k}{\ell} \binom{-k}{j} \frac{1}{k!} (-1)^{\ell+j} \lambda^k x^{\alpha\ell-j}.
\end{aligned}$$

Hence

$$e^{-\lambda x \frac{x^\alpha - 1}{x - 1}} = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} A_{k,\ell,j} \left((-1)^k x^{k+\alpha\ell+j} \mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j} \mathbf{1}_{(1,+\infty)}(x) \right).$$

Therefore

$$F(x) = 1 - e^{-\lambda x \frac{x^\alpha - 1}{x - 1}} = 1 - \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} A_{k,\ell,j} \left((-1)^k x^{k+\alpha\ell+j} \mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j} \mathbf{1}_{(1,+\infty)}(x) \right).$$

This complete the proof of Proposition 3.1. \square

The result below presents an expansion of the pdf $f(x)$ given by (2) via polynomial and the exponential function $e^{-\lambda x}$, which will be important to ensure the permutation of sum and integral in several probabilistic quantities.

Proposition 3.2. *We have the following expansion, for $x \in (0, +\infty) \setminus \{1\}$,*

$$f(x) = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left[(-1)^k R_{k,\ell,j}(x) \mathbf{1}_{(0,1)}(x) + S_{k,\ell,j}(x) \mathbf{1}_{(1,+\infty)}(x) \right] e^{-\lambda x},$$

where

$$B_{k,\ell,j} = \binom{k}{\ell} \binom{-k-2}{j} \frac{1}{k!} (-1)^{\ell+j} \lambda^{k+1}, \quad (4)$$

$$R_{k,\ell,j}(x) = \alpha x^{2k+(\alpha-1)\ell+j+\alpha+1} - (\alpha+1) x^{2k+(\alpha-1)\ell+j+\alpha} + x^{2k+(\alpha-1)\ell+j}$$

and

$$S_{k,\ell,j}(x) = \alpha x^{k+(\alpha-1)\ell-j+\alpha-1} - (\alpha+1) x^{k+(\alpha-1)\ell-j-2+\alpha} + x^{k+(\alpha-1)\ell-j-2}.$$

Proof of Proposition 3.2. Let us observe that

$$\lambda x \frac{x^\alpha - 1}{x - 1} = \lambda x \left(\frac{x^\alpha - 1}{x - 1} - 1 \right) + \lambda x = \lambda x^2 \frac{x^{\alpha-1} - 1}{x - 1} + \lambda x.$$

Therefore, we can express the pdf $f(x)$ as

$$f(x) = \lambda (\alpha x^{\alpha+1} - (\alpha+1) x^\alpha + 1) e^{-\lambda x} \times \frac{1}{(x-1)^2} e^{-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1}}.$$

Let us now investigate an expansion for $\frac{1}{(x-1)^2} e^{-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1}}$ by distinguishing $x \in [0, 1)$ and $x > 1$.

- If $x \in [0, 1)$, it follows from the exponential and binomial series that

$$\begin{aligned}
\frac{1}{(x-1)^2} e^{-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(x-1)^2} \left(-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (-\lambda)^k x^{2k} (1-x^{\alpha-1})^k \frac{1}{(1-x)^{k+2}} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} \binom{k}{\ell} \binom{-k-2}{j} \frac{1}{k!} (-1)^{\ell+j} (-\lambda)^k x^{2k+(\alpha-1)\ell+j}.
\end{aligned}$$

- If $x > 1$, exponential and binomial series give

$$\begin{aligned}
\frac{1}{(x-1)^2} e^{-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(x-1)^2} \left(-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k x^{2k} (1-x^{\alpha-1})^k x^{-2} \frac{1}{(1-\frac{1}{x})^{k+2}} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} \binom{k}{\ell} \binom{-k-2}{j} \frac{1}{k!} (-1)^{\ell+j} \lambda^k x^{k+(\alpha-1)\ell-j-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{(x-1)^2} e^{-\lambda x^2 \frac{x^{\alpha-1}-1}{x-1}} &= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} \binom{k}{\ell} \binom{-k-2}{j} \frac{1}{k!} (-1)^{\ell+j} \lambda^k \\
&\quad \times \left[(-1)^k x^{2k+(\alpha-1)\ell+j} \mathbf{1}_{(0,1)}(x) + x^{k+(\alpha-1)\ell-j-2} \mathbf{1}_{(1,+\infty)}(x) \right].
\end{aligned}$$

Owing to this equality, we obtain the desired expansion:

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left[(-1)^k x^{2k+(\alpha-1)\ell+j} \mathbf{1}_{(0,1)}(x) + x^{k+(\alpha-1)\ell-j-2} \mathbf{1}_{(1,+\infty)}(x) \right] \\
&\quad \times (\alpha x^{\alpha+1} - (\alpha+1)x^{\alpha} + 1) e^{-\lambda x} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left((-1)^k R_{k,\ell,j}(x) \mathbf{1}_{(0,1)}(x) + S_{k,\ell,j}(x) \mathbf{1}_{(1,+\infty)}(x) \right) e^{-\lambda x}.
\end{aligned}$$

This ends the proof of Proposition 3.2. \square

3.2. Moments and moment generating function

Here and after, we consider a random variable X following the $\text{PE}(\alpha, \lambda)$ distribution with $\alpha > 0$ and $\lambda > 0$.

We define the upper incomplete gamma function as $\Gamma(s, x) = \int_x^{+\infty} t^{s-1} e^{-t} dt$ and the lower incomplete gamma function as $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$, $s > 0$, $x \geq 0$.

Let $r \geq 0$. Using the notations and the result of Proposition 3.2, the r -moments of X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left((-1)^k U_{k,\ell,j,r} + V_{k,\ell,j,r} \right),$$

where $U_{k,\ell,j,r} = \int_0^1 x^r R_{k,\ell,j}(x) e^{-\lambda x} dx$ and $V_{k,\ell,j,r} = \int_1^{+\infty} x^r S_{k,\ell,j}(x) e^{-\lambda x} dx$. We have

$$\begin{aligned} U_{k,\ell,j,r} &= \alpha \int_0^1 x^{r+2k+(\alpha-1)\ell+j+\alpha+1} e^{-\lambda x} dx - (\alpha+1) \int_0^1 x^{r+2k+(\alpha-1)\ell+j+\alpha} e^{-\lambda x} dx \\ &+ \int_0^1 x^{r+2k+(\alpha-1)\ell+j} e^{-\lambda x} dx \\ &= \alpha \frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+2, \lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+2}} - (\alpha+1) \frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+1, \lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+1}} \\ &+ \frac{\gamma(r+2k+(\alpha-1)\ell+j+1, \lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+1}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} V_{k,\ell,j,r} &= \alpha \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j+\alpha-1} e^{-\lambda x} dx - (\alpha+1) \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j-2+\alpha} e^{-\lambda x} dx \\ &+ \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j-2} e^{-\lambda x} dx \\ &= \alpha \frac{\Gamma(r+k+(\alpha-1)\ell-j+\alpha, \lambda)}{\lambda^{r+k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1) \frac{\Gamma(r+k+(\alpha-1)\ell-j-1+\alpha, \lambda)}{\lambda^{r+k+(\alpha-1)\ell-j-1+\alpha}} \\ &+ \frac{\Gamma(r+k+(\alpha-1)\ell-j-1, \lambda)}{\lambda^{r+k+(\alpha-1)\ell-j-1}}. \end{aligned}$$

The moment generating function of X is given by, for $t < \lambda$,

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left((-1)^k U_{k,\ell,j}^*(t) + V_{k,\ell,j}^*(t) \right),$$

where $U_{k,\ell,j}^*(t) = \int_0^1 e^{tx} R_{k,\ell,j}(x) e^{-\lambda x} dx$ and $V_{k,\ell,j}^*(t) = \int_1^{+\infty} e^{tx} S_{k,\ell,j}(x) e^{-\lambda x} dx$. We have

$$\begin{aligned} U_{k,\ell,j}^*(t) &= \alpha \int_0^1 x^{2k+(\alpha-1)\ell+j+\alpha+1} e^{-(\lambda-t)x} dx - (\alpha+1) \int_0^1 x^{2k+(\alpha-1)\ell+j+\alpha} e^{-(\lambda-t)x} dx \\ &+ \int_0^1 x^{2k+(\alpha-1)\ell+j} e^{-(\lambda-t)x} dx \\ &= \alpha \frac{\gamma(2k+(\alpha-1)\ell+j+\alpha+2, \lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+\alpha+2}} - (\alpha+1) \frac{\gamma(2k+(\alpha-1)\ell+j+\alpha+1, \lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+\alpha+1}} \\ &+ \frac{\gamma(2k+(\alpha-1)\ell+j+1, \lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+1}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
V_{k,\ell,j}^*(t) &= \alpha \int_1^{+\infty} x^{k+(\alpha-1)\ell-j+\alpha-1} e^{-(\lambda-t)x} dx - (\alpha+1) \int_1^{+\infty} x^{k+(\alpha-1)\ell-j-2+\alpha} e^{-(t-\lambda)x} dx \\
&+ \int_1^{+\infty} x^{k+(\alpha-1)\ell-j-2} e^{-(\lambda-t)x} dx \\
&= \alpha \frac{\Gamma(k+(\alpha-1)\ell-j+\alpha, \lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1) \frac{\Gamma(k+(\alpha-1)\ell-j-1+\alpha, \lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j-1+\alpha}} \\
&+ \frac{\Gamma(k+(\alpha-1)\ell-j-1, \lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j-1}}.
\end{aligned}$$

3.3. On other means and moments

The following result proposes an expansion of the primitive $\int_0^t x^r f(x) dx$, with $t > 0$. It will be useful in the next.

Proposition 3.3. *For any $r \geq 0$ and $t > 0$, we have*

$$\int_0^t x^r f(x) dx = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{j=0}^{\infty} B_{k,\ell,j} \left((-1)^k U_{k,\ell,j,r}^{\circ}(t) + V_{k,\ell,j,r}^{\circ}(t) \mathbf{1}_{[1,+\infty)}(t) \right), \quad (5)$$

where $B_{k,\ell,j}$ is defined by (4),

$$\begin{aligned}
U_{k,\ell,j,r}^{\circ}(t) &= \alpha \frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+2, \lambda \min(t, 1))}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+2}} \\
&- (\alpha+1) \frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+1, \lambda \min(t, 1))}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+1}} \\
&+ \frac{\gamma(r+2k+(\alpha-1)\ell+j+1, \lambda \min(t, 1))}{\lambda^{r+2k+(\alpha-1)\ell+j+1}}
\end{aligned}$$

and

$$\begin{aligned}
V_{k,\ell,j,r}^{\circ}(t) &= \alpha \frac{\Gamma(r+k+(\alpha-1)\ell-j+\alpha, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1) \frac{\Gamma(r+k+(\alpha-1)\ell-j-1+\alpha, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j-1+\alpha}} \\
&+ \frac{\Gamma(r+k+(\alpha-1)\ell-j-1, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j-1}}.
\end{aligned}$$

The proof of Proposition 3.3 follows from Proposition 3.2 with $U_{k,\ell,j,r}^{\circ}(t) = \int_0^{\min(t,1)} x^r R_{k,\ell,j}(x) e^{-\lambda x} dx$ and $V_{k,\ell,j,r}^{\circ}(t) = \int_1^t x^r S_{k,\ell,j}(x) e^{-\lambda x} dx$. The expressions of these integrals in terms of upper incomplete gamma function and the lower incomplete gamma function is obtained proceeding as Subsection 3.2.

Several crucial conditional moments use the integral $\int_0^t x^r f(x) dx$ for various values of r . The most useful of them are presented below. For any $t > 0$,

- The r -th conditional moments of X is given by,

$$E(X^r | X > t) = \frac{1}{1-F(t)} \int_t^{+\infty} x^r f(x) dx = \frac{1}{1-F(t)} \left(E(X^r) - \int_0^t x^r f(x) dx \right).$$

- The r -th reversed moments of X is given by

$$E(X^r | X \leq t) = \frac{1}{F(t)} \int_0^t x^r f(x) dx.$$

Let $\mu = E(X)$.

- The mean deviations of X about μ is given by

$$\delta = E(|X - \mu|) = 2\mu F(\mu) - 2 \int_0^\mu x f(x) dx$$

- The mean deviations of X about the median M is given by

$$\eta = E(|X - M|) = \mu - 2 \int_0^M x f(x) dx.$$

Residual life parameters can be also determined using $E(X^r)$ and $\int_0^t x^r f(x) dx$ for several values of r . In particular,

- The mean residual life is defined as

$$K(t) = E(X - t | X > t) = \frac{1}{S(t)} \left(E(X) - \int_0^t x f(x) dx \right) - t$$

and the variance residual life is given by

$$V(t) = Var(X - t | X > t) = \frac{1}{S(t)} \left(E(X^2) - \int_0^t x^2 f(x) dx \right) - t^2 - 2tK(t) - [K(t)]^2.$$

- The mean reversed residual life is defined as

$$L(t) = E(t - X | X \leq t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx$$

and the variance reversed residual life is given by

$$W(t) = Var(t - X | X \leq t) = 2tL(t) - [L(t)]^2 - t^2 + \frac{1}{F(t)} \int_0^t x^2 f(x) dx.$$

3.4. Stress-strength reliability

Let X be a random variable following the $PE(\alpha, \lambda_1)$ distribution with pdf denoted by $f_X(x)$ and Y be a random variable following the $PE(\alpha, \lambda_2)$ distribution with cdf denoted by $F_Y(x)$, with $\alpha > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Then the stress-strength reliability is defined by $R = P(X > Y)$. Since the integral on $(0, +\infty)$ of the pdf (2) with

parameters $(\alpha, \lambda_1 + \lambda_2)$ denoted by $f_*(x)$ is equal to one, we have

$$\begin{aligned} R &= P(X > Y) = \int_0^{+\infty} f_X(x)F_Y(x)dx \\ &= 1 - \int_0^{+\infty} \lambda_1 \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} e^{-(\lambda_1+\lambda_2)x\frac{x^\alpha-1}{x-1}} dx \\ &= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_0^{+\infty} f_*(x)dx = \frac{\lambda_2}{\lambda_1 + \lambda_2}. \end{aligned}$$

This result is of interest in a parametric estimation context; only λ_1 and λ_2 need to be estimated to have an estimation of R (the maximum likelihood estimators for λ_1 and λ_2 yield the maximum likelihood estimator for R by the plug-in method).

3.5. Order statistics distributions

We now introduce order statistics and present some of their properties in our mathematical framework (general results can be found, for instance, in [4]). Let X_1, X_2, \dots, X_n be n i.i.d. random variables following the PE(α, λ) distribution with $\alpha > 0$ and $\lambda > 0$. Let us consider its order statistics as $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, *i.e.*, for any $i \in \{1, \dots, n\}$, $X_{i:n} \in \{X_1, \dots, X_n\}$ with $X_{1:n} \leq \dots \leq X_{n:n}$ (so $X_{1:n} = X_{(1)} = \inf(X_1, \dots, X_n)$ and $X_{n:n} = X_{(n)} = \sup(X_1, \dots, X_n)$). Let us now present some important distributions related to $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Some important of them involving our distribution are presented below. The general expression of the cdf of $X_{i:n}$ is given by

$$F_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} [F(x)]^{i+k}, \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$F_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[1 - e^{-\lambda x \frac{x^\alpha-1}{x-1}}\right]^{i+k}.$$

For the case $x = 1$, we have

$$F_{X_{i:n}}(1) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[1 - e^{-\lambda\alpha}\right]^{i+k}.$$

The general expression of the pdf of $X_{i:n}$ is given by

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x), \quad x \in \mathbb{R}.$$

Thus, for any $x \in (0, +\infty)/\{1\}$, we have

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!} \lambda \left[1 - e^{-\lambda x \frac{x^\alpha-1}{x-1}}\right]^{i-1} \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} e^{-\lambda(n-i+1)x\frac{x^\alpha-1}{x-1}}.$$

For the case $x = 1$, we have

$$f_{X_{i:n}}(1) = \frac{n!}{(i-1)!(n-i)!} \left[1 - e^{-\lambda\alpha}\right]^{i-1} \frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha(n-i+1)}.$$

As in Remark 1, one can express $f_{X_{i:n}}(x)$ in a one form using $\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)$ and $\mathbf{1}_{\{1\}}(x)$.

For $i < j$ and $x_i < x_j$, the general expression of the joint pdf of $(X_{i:n}, X_{j:n})$ is given by

$$f_{(X_{i:n}, X_{j:n})}(x_i, x_j) = \frac{n!}{(i-1)!(n-j)!(j-i-1)!} [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} \times \\ \times [1 - F(x_j)]^{n-j} f(x_i) f(x_j).$$

For the case $(x_i, x_j) \in (0, +\infty)^2 / \{(1, 1)\}$, we have

$$f_{(X_{i:n}, X_{j:n})}(x_i, x_j) = \frac{n!}{(i-1)!(n-j)!(j-i-1)!} \lambda^2 \left[1 - e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1}}\right]^{i-1} \left[e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1}} - e^{-\lambda x_j \frac{x_j^\alpha - 1}{x_j - 1}}\right]^{j-i-1} \\ \times \frac{(\alpha x_i^{\alpha+1} - (\alpha+1)x_i^\alpha + 1)(\alpha x_j^{\alpha+1} - (\alpha+1)x_j^\alpha + 1)}{(x_i - 1)^2 (x_j - 1)^2} e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1} - \lambda(n-j+1)x_j \frac{x_j^\alpha - 1}{x_j - 1}}.$$

The expression of $f_{(X_{i:n}, X_{j:n})}(x_i, x_j)$ for $(x_i, x_j) \notin (0, +\infty)^2 / \{(1, 1)\}$ can be set in a similar manner, using the values of $F(1)$ and $f(1)$.

In the following proposition, we provide the asymptotic distributions of the extreme values $X_{1:n}$ and $X_{n:n}$, and show that they are exponential and Gumbel distributions, respectively, which adapt the standards of the asymptotic distribution of extremes.

Proposition 3.4. *Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables following the $PE(\alpha, \lambda)$ distribution, then*

- $(nX_{(1)})_{n \geq 1}$ converges in distribution to a random variable X having the exponential distribution of parameter λ .
- $\left(X_{(n)}^\alpha - \frac{\log(n)}{\lambda}\right)_{n \geq 1}$ converges in distribution to a random variable X having the Gumbel distribution of parameters 0 and $\frac{1}{\lambda}$.

Proof of Proposition 3.4. Let us prove the two points in turn.

- Since X_1, \dots, X_n are i.i.d., using standard mathematical arguments, for $x \in (0, +\infty) / \{n\}$, the cdf of $nX_{(1)}$ is given by

$$F_{nX_{(1)}}(x) = 1 - \left(1 - F\left(\frac{x}{n}\right)\right)^n = 1 - e^{-\lambda x \frac{(\frac{x}{n})^\alpha - 1}{\frac{x}{n} - 1}}.$$

So $\lim_{n \rightarrow +\infty} F_{nX_{(1)}}(x) = 1 - e^{-\lambda x} = F_X(x)$. This ends the proof of the first point.

- Again, since X_1, \dots, X_n are i.i.d., using standard mathematical arguments, for

$x \in \left(-\frac{\log(n)}{\lambda}, +\infty\right) / \left\{1 - \frac{\log(n)}{\lambda}\right\}$, the cdf of $X_{(n)}^\alpha - \frac{\log(n)}{\lambda}$ is given by

$$F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) = \left(F \left(\left(x + \frac{\log(n)}{\lambda} \right)^{\frac{1}{\alpha}} \right) \right)^n = \left(1 - e^{-\lambda \left(x + \frac{\log(n)}{\lambda} \right)^{\frac{1}{\alpha}} \frac{x + \frac{\log(n)}{\lambda} - 1}{\left(x + \frac{\log(n)}{\lambda} \right)^{\frac{1}{\alpha} - 1}} \right)^n.$$

Therefore, when $n \rightarrow +\infty$, several equivalences give

$$F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) \sim e^{n \log \left(1 - \frac{e^{-\lambda x}}{n} \right)} \sim e^{-e^{-\lambda x}}.$$

Hence $\lim_{n \rightarrow +\infty} F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) = e^{-e^{-\lambda x}} = F_X(x)$. The second point is proved. \square

3.6. Stochastic ordering

The ordering mechanism in life time distributions can be illustrate by the concept of stochastic ordering. See, for instance, [10]. This subsection presents the basic of this concept, with a result using the proposed distribution. A random variable X is said to be stochastically smaller than a random variable Y in the

- stochastic order ($X \leq_{st} Y$) if the associated cdfs satisfy: $F_X(x) \geq F_Y(x)$ for all x .
- hazard rate order ($X \leq_{hr} Y$) if the associated hrfs satisfy: $h_X(x) \geq h_Y(x)$ for all x .
- likelihood ratio order ($X \leq_{lr} Y$) if the ratio of the associated pdfs given by $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

Important equivalences exist; when the supports of X and Y have a common finite left end-point, then we have: $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$.

Proposition 3.5. *Let X be a random variable following the $PE(\alpha, \lambda_1)$ distribution with pdf denoted by $f_X(x)$ and Y be a random variable following the $PE(\alpha, \lambda_2)$ distribution with pdf denoted by $f_Y(x)$, with $\alpha > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. If $\lambda_1 > \lambda_2$, then we have $X \leq_{lr} Y$.*

Proof of Proposition 3.5. For any $x \in (0, +\infty) / \{1\}$, we have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1)x \frac{x^\alpha - 1}{x - 1}}$$

and $\frac{f_X(1)}{f_Y(1)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1)\alpha}$. As mentioned in Introduction, the function $H(x) = x \frac{x^\alpha - 1}{x - 1}$ is increasing function of x . Indeed, we have $H'(x) = \frac{1 + \alpha x^{\alpha+1} - (1 + \alpha)x^\alpha}{(x - 1)^2}$ and a study of function shows that $1 + \alpha x^{\alpha+1} - (1 + \alpha)x^\alpha \geq 0$ for any $\alpha > 0$. Hence, if $\lambda_1 > \lambda_2$, $\frac{f_X(x)}{f_Y(x)}$ decreases in x and $X \leq_{lr} Y$. Proposition 3.5 is proved. \square

Remark 2. One can prove that $\lambda_1 > \lambda_2$ implies that $X \leq_{hr} Y$, which follows immediately from the definition (2): for any $x \in (0, +\infty) / \{1\}$, we have $h_X(x) =$

$\lambda_1 \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} \geq \lambda_2 \frac{\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1}{(x-1)^2} = h_Y(x)$, and for $x = 1$, we have $h_X(1) = \frac{\lambda_1 \alpha(\alpha+1)}{2} \geq \frac{\lambda_2 \alpha(\alpha+1)}{2} = h_Y(1)$.

3.7. Record values distributions

Let us now focus our attention on record values and present some of their properties in our mathematical context (general results can be found, for instance, in [2]). Let X_1, X_2, \dots, X_n be n i.i.d. random variables following the $\text{PE}(\alpha, \lambda)$ distribution with $\alpha > 0$ and $\lambda > 0$. We define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min\{j; j > U(n-1), X_j > X_{U(n-1)}\}$ for $n \geq 2$. We define the i -th upper record value by $R_i = X_{U(i)}$, with $R_1 = X_1$. The general expression of the cdf of R_i is given by

$$F_{R_i}(x) = 1 - (1 - F(x)) \sum_{k=0}^{i-1} \frac{[-\log(1 - F(x))]^k}{k!}, \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$F_{R_i}(x) = 1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}} \sum_{k=0}^{i-1} \frac{1}{k!} \left[\lambda x \frac{x^\alpha - 1}{x-1} \right]^k.$$

For the case $x = 1$, we have

$$F_{R_i}(1) = 1 - e^{-\lambda \alpha} \sum_{k=0}^{i-1} \frac{(\lambda \alpha)^k}{k!}.$$

The general expression of the pdf of R_i is given by

$$f_{R_i}(x) = \frac{[-\log(1 - F(x))]^{i-1}}{(i-1)!} f(x), \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$f_{R_i}(x) = \frac{1}{(i-1)!} \lambda^i \frac{x^{i-1} (x^\alpha - 1)^{i-1} (\alpha x^{\alpha+1} - (\alpha+1)x^\alpha + 1)}{(x-1)^{i+1}} e^{-\lambda x \frac{x^\alpha - 1}{x-1}}.$$

Note that, for $x = 1$, we have

$$f_{R_i}(1) = \frac{1}{(i-1)!} (\lambda \alpha)^i \frac{\alpha + 1}{2} e^{-\lambda \alpha}.$$

The general expression of the joint pdf of (R_1, \dots, R_n) is given by, for $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $x_1 < \dots < x_n$,

$$f_{(R_1, \dots, R_n)}(x_1, \dots, x_n) = f(x_n) \prod_{k=1}^{n-1} h(x_k).$$

For the case $(x_1, \dots, x_n) \in (0, +\infty)^n / \{(1, \dots, 1)\}$, we have

$$f_{(R_1, \dots, R_n)}(x_1, \dots, x_n) = \lambda^{n+1} \frac{\alpha x_n^{\alpha+1} - (\alpha + 1)x_n^\alpha + 1}{(x_n - 1)^2} e^{-\lambda x_n \frac{x_n^\alpha - 1}{x_n - 1}} \prod_{k=1}^{n-1} \frac{\alpha x_k^{\alpha+1} - (\alpha + 1)x_k^\alpha + 1}{(x_k - 1)^2}.$$

The expression of $f_{(R_1, \dots, R_n)}(x_1, \dots, x_n)$ for $(x_1, \dots, x_n) \notin (0, +\infty)^n / \{(1, \dots, 1)\}$ can be set in a similar manner, using the values of $f(1)$ and $h(1)$.

For $i < j$ and $x_i < x_j$, the general expression of the joint pdf of (R_i, R_j) is given by

$$f_{(R_i, R_j)}(x_i, x_j) = \frac{[-\log(1 - F(x_i))]^{i-1}}{(i-1)!} \frac{\left[\log\left(\frac{1-F(x_i)}{1-F(x_j)}\right)\right]^{j-i-1}}{(j-i-1)!} h(x_i) f(x_j).$$

For the case $(x_i, x_j) \in (0, +\infty)^2 / \{(1, 1)\}$, we have

$$\begin{aligned} f_{(R_i, R_j)}(x_i, x_j) &= \frac{[-\log(1 - F(x_i))]^{i-1}}{(i-1)!} \frac{\left[\log\left(\frac{1-F(x_i)}{1-F(x_j)}\right)\right]^{j-i-1}}{(j-i-1)!} h(x_i) f(x_j) \\ &= \frac{1}{(i-1)!} \lambda^j \left[\frac{x_i^\alpha - 1}{x_i - 1} \right]^{i-1} \frac{1}{(j-i-1)!} \left[\frac{x_j^\alpha - 1}{x_j - 1} - x_i \frac{x_i^\alpha - 1}{x_i - 1} \right]^{j-i-1} \\ &\quad \times \frac{(\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1)(\alpha x_j^{\alpha+1} - (\alpha + 1)x_j^\alpha + 1)}{(x_i - 1)^2 (x_j - 1)^2} e^{-\lambda x_j \frac{x_j^\alpha - 1}{x_j - 1}}. \end{aligned}$$

The expression of $f_{(R_i, R_j)}(x_i, x_j)$ for $(x_i, x_j) \notin (0, +\infty)^2 / \{(1, 1)\}$ can be set in a similar manner, using the values of $F(1)$, $f(1)$ and $h(1)$.

4. Maximum likelihood estimation

Let X_1, X_2, \dots, X_n be a random sample with common distribution the PE(α, λ) distribution with $\alpha > 0$ and $\lambda > 0$. Let $\boldsymbol{\theta} = (\alpha, \lambda)$ be the parameter vector and x_1, x_2, \dots, x_n be the observed values. Then the likelihood function associated to x_1, \dots, x_n is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^n \left(\lambda \frac{\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1}{(x_i - 1)^2} e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1}} \right)^{\mathbf{1}_{(0, +\infty) \setminus \{1\}}(x_i)} \left(\frac{\lambda \alpha (\alpha + 1)}{2} e^{-\lambda \alpha} \right)^{\mathbf{1}_{\{1\}}(x_i)}.$$

For the set of simplicity, let us set $u_i = \mathbf{1}_{(0, +\infty) \setminus \{1\}}(x_i)$, $v_i = \mathbf{1}_{\{1\}}(x_i)$, and $\sum_{i=1}^n u_i + \sum_{i=1}^n v_i = n$. The log-likelihood function can be expressed as

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= \log(\lambda) \sum_{i=1}^n u_i + \sum_{i=1}^n u_i \log \left(\frac{\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1}{(x_i - 1)^2} \right) \\ &\quad - \lambda \sum_{i=1}^n u_i x_i \frac{x_i^\alpha - 1}{x_i - 1} + \sum_{i=1}^n v_i (\log(\lambda) + \log(\alpha) + \log(\alpha + 1) - \log(2)). \end{aligned}$$

The nonlinear log-likelihood equations $\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0$ are given by

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} &= \sum_{i=1}^n u_i x_i^\alpha \frac{(x_i - 1)[1 + \alpha \log(x_i)] - \log(x_i)}{\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1} - \lambda \sum_{i=1}^n u_i x_i \frac{x_i^\alpha}{x_i - 1} \log(x_i) \\ &+ \left(\frac{1}{\alpha} + \frac{1}{\alpha + 1} \right) \sum_{i=1}^n v_i = 0, \end{aligned} \quad (6)$$

and

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n u_i x_i \frac{x_i^\alpha - 1}{x_i - 1} = 0. \quad (7)$$

Note that after some algebraic manipulations in (7) we have that

$$\lambda = \frac{n}{\sum_{i=1}^n u_i x_i \frac{x_i^\alpha - 1}{x_i - 1}}. \quad (8)$$

Replacing (8) in (6) the maximum likelihood estimates of α and λ are determined by solving the one linear equation (6). Since it does not admit any explicit solution, numerical procedures can be used. Under mild conditions the maximum likelihood estimators are asymptotically normal, with an asymptotic variance-covariance matrix depending on the Fisher information matrix. Crucial quantities to determine the entries of this matrix are the second partial derivatives of the log-likelihood function given by

$$\begin{aligned} \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \alpha^2} &= \sum_{i=1}^n u_i x_i^\alpha \frac{\log(x_i)((x_i - 1)[2 + \alpha \log(x_i)] - \log(x_i))(\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1)}{(\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1)^2} \\ &- \sum_{i=1}^n u_i x_i^\alpha \frac{x_i^\alpha [(x_i - 1)[1 + \alpha \log(x_i)] - \log(x_i)]^2}{(\alpha x_i^{\alpha+1} - (\alpha + 1)x_i^\alpha + 1)^2} \\ &- \lambda \sum_{i=1}^n u_i x_i \frac{x_i^\alpha}{x_i - 1} (\log(x_i))^2 - \left(\frac{1}{\alpha^2} + \frac{1}{(\alpha + 1)^2} \right) \sum_{i=1}^n v_i, \end{aligned}$$

and

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda^2} = -\frac{n}{\lambda^2}, \quad \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \alpha} = -\sum_{i=1}^n u_i x_i \frac{x_i^\alpha}{x_i - 1} \log(x_i).$$

5. Simulation analysis

In this section a simulation study is presented to compare the efficiency of the maximum likelihood method. The comparison is performed by computing the Bias and the

mean square errors (MSE) given by

$$\text{Bias}(\alpha_i) = \frac{1}{N} \sum_{i=1}^N \hat{\alpha}_i - \alpha, \quad \text{MSE}(\alpha_i) = \frac{1}{N} \sum_{j=1}^N (\hat{\alpha}_j - \alpha)^2,$$

$$\text{Bias}(\lambda_i) = \frac{1}{N} \sum_{i=1}^N \hat{\lambda}_i - \lambda, \quad \text{MSE}(\lambda_i) = \frac{1}{N} \sum_{j=1}^N (\hat{\lambda}_j - \lambda)^2,$$

where N is the number of estimates obtained through the MLE. The 95% coverage probability of the asymptotic confidence intervals are also evaluated. Here we expect that the most efficient estimation method returns both Bias and MSE closer to zero. Additionally, for a large number of experiments, using a 95% confidence level, the frequencies of intervals that covered the true values of α and λ should be closer to 95%. The programs can be obtained, upon request. The values of the PE were generated considering the following algorithm:

- (1) Generate $U_i \sim \text{Uniform}(0, 1), i = 1, \dots, n$;
- (2) Find x_i from the solution of $F(x_i) - u_i = 0, i = 1, \dots, n$;

The simulation study is performed under the assumption (0.5, 2) and (4, 2), $N = 100,000$ and $n = (20, 35, \dots, 460)$. The chosen values allow us to obtain data with both increasing ($\alpha < 1$) and decreasing ($\alpha > 1$) hazard rate. It is important to point out that, the results of this simulation study were similar for different choices of α and λ . Figures 4 and 5 present the Bias, the MSE and the coverage probability with a 95% confidence level of the estimates obtained through the MLE for different samples of size.

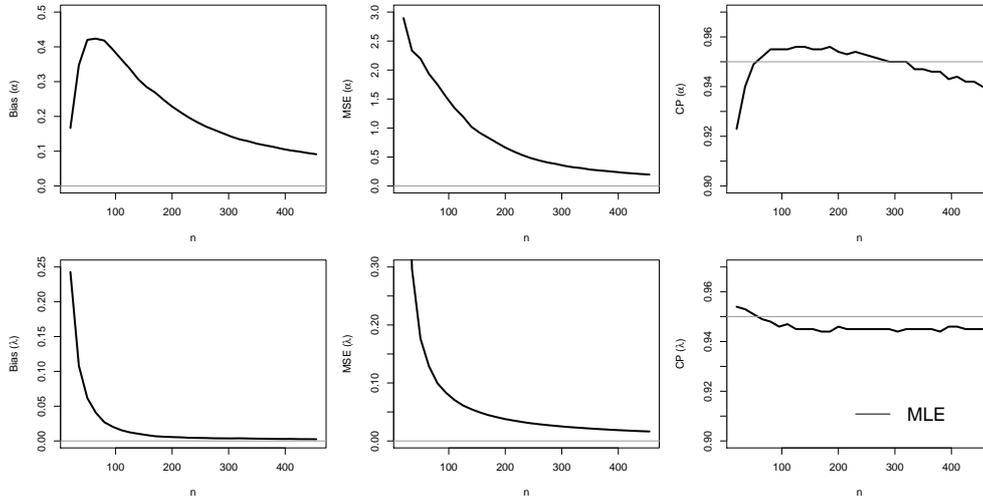


Figure 4. Bias, MSEs related from the estimates of $\alpha = 3$ and $\lambda = 2$ for N simulated samples under the MLE.

From the obtained results, we can conclude that as there is an increase of n both Bias and MSE tend to zero, i.e., the estimator are asymptotic efficiency. Moreover, the coverage probability of the confidence levels tend to the nominal value assumed 0.95. Therefore, the MLE showed to be a good estimator for the parameters of the PE distribution.

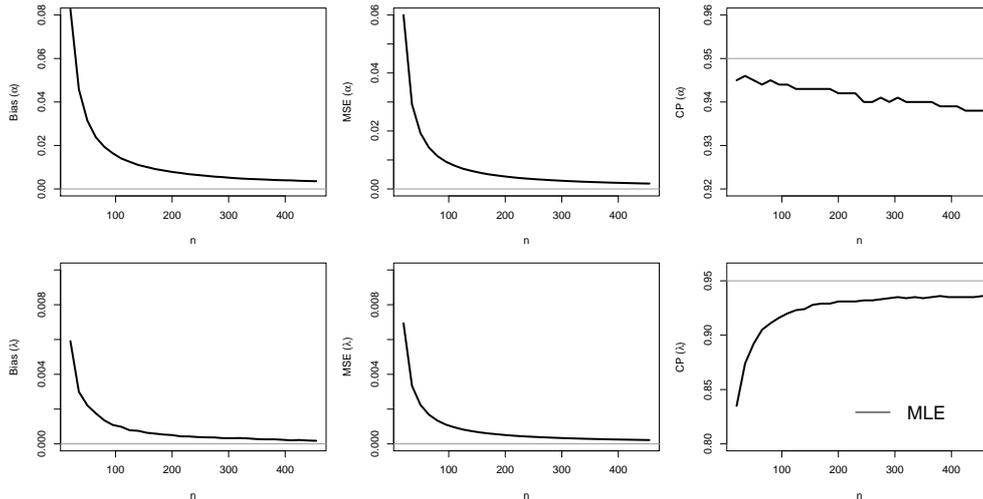


Figure 5. Bias, MSEs related from the estimates of $\alpha = 0.9$ and $\lambda = 0.1$ for N simulated samples under the MLE.

6. Application to real data

In this section, we illustrate the flexibility of our proposed distribution by considering two real data sets. The results obtained from the PE distribution are compared with ones of the Weibull, Gamma, Lognormal and the EE distributions, and nonparametric survival function

Here, different discrimination criterion are considered based on log likelihood function. Let k be the number of parameters to be fitted and $\hat{\theta}$ the MLEs of θ , the discrimination criterion methods are respectively:

- Akaike information criterion $AIC = -2l(\hat{\theta}; \mathbf{x}) + 2k$;
- Corrected Akaike information criterion $AICC = AIC + (2k(k+1))/(n-k-1)$;
- Hannan-Quinn information criterion $HQIC = -2l(\hat{\theta}; \mathbf{x}) + 2k \log(\log(n))$;
- Consistent Akaike information criterion $CAIC = -2l(\hat{\theta}; \mathbf{x}) + k(\log(n) + 1)$.

The best model is the one which provides the minimum values of these criteria. The Kolmogorov-Smirnov (KS) test is also considered aiming to check the goodness of the fit for the models. This procedure is widely known and based on the KS statistic $D_n = \sup_x |F_n(x) - F(x; \theta)|$, where \sup_x is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function and $F(x; \theta)$ is the cdf of the fitted distribution. Under a significance level of 5% if the data comes from $F(x; \theta)$ (null hypothesis), the hypothesis is rejected if the P-value is smaller than 0.05.

The next subsections give a description of the used data and their analysis under the mentioned distributions above.

6.1. Air conditioning system data

The data have been presented by Proschan [9] and further analyzed by Adamidis and Loukas [1]. Table 1 consists of the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

Table 2 displays the MLEs, standard-error and 95% confidence intervals for α and λ . Table 3 presents the results of AIC, AICC, HQIC, CAIC criteria, for the compared

Table 1. Data set related to the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

194	15	41	29	33	181	413	14	58	37	100	65
9	169	447	184	36	201	118	34	31	18	18	67
57	62	7	22	34	90	10	60	186	61	49	14
24	56	20	79	84	44	59	29	118	25	156	310
76	26	44	23	62	130	208	70	101	208	74	57
48	29	502	12	70	21	29	386	59	27	153	26
326	55	320	56	104	220	239	47	246	176	182	33
15	104	35	23	261	87	7	120	14	62	47	225
71	246	21	42	20	5	12	120	11	3	14	71
11	14	11	16	90	1	16	52	95	97	51	11
4	141	18	142	68	77	80	1	16	106	206	82
54	31	216	46	111	39	63	18	191	18	163	24
50	44	102	72	22	39	3	15	197	188	79	88
46	5	5	36	22	139	210	97	30	23	13	14
359	9	12	270	603	3	104	2	438	50	254	5
283	35	12	130	493	487	18	100	7	98	5	85
91	43	230	3	130	102	209	14	57	54	32	67
59	134	152	27	14	230	66	61	34			

distributions.

Table 2. MLE, Standard-error and 95% confidence intervals for α and λ .

θ	MLE	S. error	$CI_{95\%}(\theta)$
α	0.9010	0.01791	(0.7906; 1.0113)
λ	0.0032	0.00003	(0.0076; 0.0282)

Table 3. Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for the compared distributions considering the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

Test	PE	Weibull	Gamma	Lognormal	EE
AIC	2358.61	2359.17	2360.58	2361.76	2360.81
AICc	2354.67	2355.23	2356.64	2357.82	2356.86
CAIC	2367.34	2367.89	2369.30	2370.48	2369.52
HQIC	2361.33	2361.89	2363.30	2364.470	2363.52
P-value	0.63435	0.61336	0.37719	0.56520	0.33913

In Figure 6, we have the TTT-plot, the survival function adjusted by the compared distributions and the non-parametric survival function.

Comparing the empirical survival function with the adjusted models we observe a goodness of the fit for the PE distribution, which is confirmed from different discrimination criterion methods as the PE distribution has the minimum value for all statistics and the largest for the P-value. Consequently, we conclude that the data related to the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes can be described by the PE distribution.

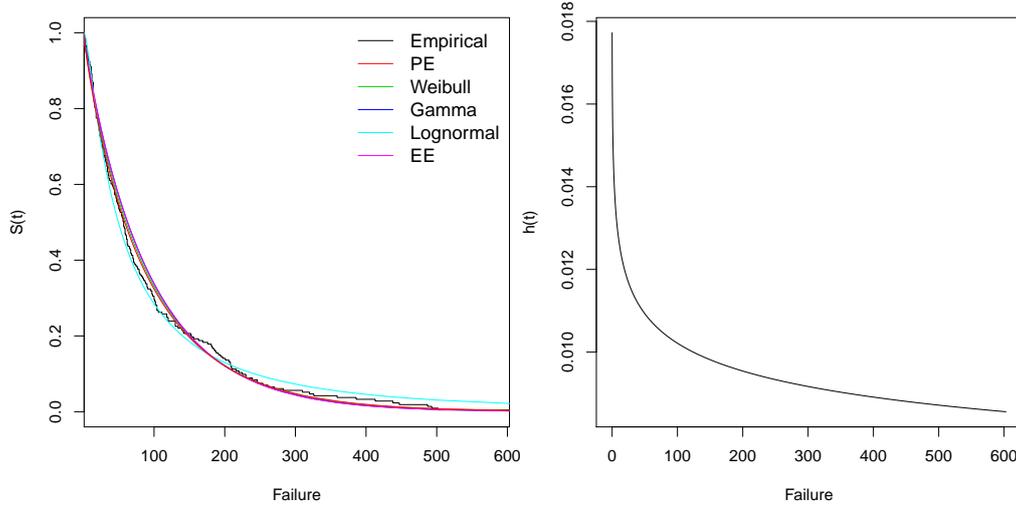


Figure 6. Survival function adjusted by the compared distributions and a non-parametric method considering the data sets related to the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

6.2. Monthly rainfall data

In this subsection, we considered the data set firstly presented in Bakouch et al. [3]. The data set is related to the total monthly rainfall during September at São Carlos located in southeastern Brazil. Such city has an active industrial profile and high agricultural importance where the study of the behavior of dry and wet periods has proved to be strategic and economically significant its development. Table 4 presents the data related to the total monthly rainfall (mm) during September at São Carlos.

Table 4. The data set related to the total monthly (mm) rainfall during September at São Carlos.

26.40	12.50	1.00	44.80	0.00	74.20	179.50	76.70	269.50
49.00	306.80	102.70	73.50	35.20	72.70	28.80	49.30	132.00
151.50	39.70	136.20	112.00	17.70	11.60	225.20	102.60	27.10
17.50	6.70	82.20	40.70	54.60	115.50	89.50	0.00	17.00
127.40	41.70	43.10	84.70	102.50	120.90	80.10	18.10	5.30
59.50	26.80	0.00	34.30	101.10	60.30	31.50	60.40	45.30
49.50	70.44							

Nadarajah and Haghghi [8] observed that maximum likelihood estimate of the shape parameter is non-unique for the Gamma, Weibull and Generalized exponential distributions if data set consists of zeros and therefore none of these three distributions can fit this kind of data set. On the other hand the PE distribution is defined as $x \geq 0$, which allow us to use the original values in the presence of zero. Table 5 displays the MLE, standard-error and 95% confidence intervals for α and λ . Table 6 presents the results of the P-value for the KS test for the compared distributions.

Table 5. MLE, Standard-error and 95% confidence intervals for α and λ .

θ	MLE	S. error	$CI_{95\%}(\theta)$
α	0.9934	0.0119	(0.7799; 1.2069)
λ	0.0144	0.0001	(0.0000; 0.0292)

Table 6. Results of KS test for the compared distributions considering the data set related to the total monthly rainfall during September at São Carlos.

Test	PE	Weibull	Gamma	Lognormal	EE
P-value	0.66711	0.0000	0.0000	0.0000	0.0000

In the Figure 7, the survival function adjusted by the compared distributions and the Kaplan-Meier estimator.

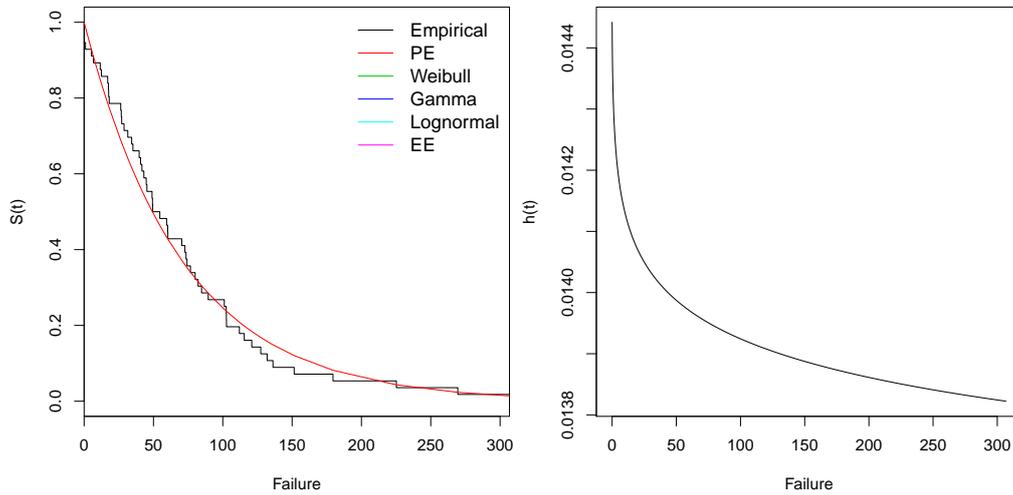


Figure 7. Survival function adjusted by the compared distributions and a non-parametric method considering the data set related to the total monthly rainfall during September at São Carlos.

The adjusted models when compared to the empirical survival show a goodness of the fit for the PE distribution. Additionally, this result is corroborated by the P-value of the KS test. Therefore, our proposed distribution can be used to describe the data related to the total monthly rainfall during September at São Carlos.

7. Concluding remarks

In this paper, we introduced a new two-parameter distribution called polynomial exponential distribution, which generalizes the ordinary exponential, linear exponential and other combinations of Weibull distributions, because survival function of the PE distribution represents the product of the survival functions of Weibull distributions with parameters $(\lambda, 1)$, $(\lambda, 2)$, ..., and (λ, α) . The new distribution could be an alternative model for lifetime data, specially for the presence of instantaneous failures (inliers), since standard distributions such as Gamma, Weibull, Lognormal and exponentiated exponential may not be suitable. We provided a mathematical treatment of the new distribution. The estimation of parameters was discussed by the maximum likelihood approach. Simulation studies were performed to assess the performance of the maximum likelihood estimators. We fitted the proposed distribution to two real data sets and compared its fit to those of commonly known lifetime distributions, establishing that the new model can be a good competitor for the latter. We hope that the proposed distribution may be used in wide applications as well as lifetime modeling. Future studies can be investigated by using other baseline functions $G(x)$, see Introduction section.

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