



HAL
open science

A mathematical comment on gravitational waves

Jean-François Pommaret

► **To cite this version:**

| Jean-François Pommaret. A mathematical comment on gravitational waves. 2017. hal-01576131v1

HAL Id: hal-01576131

<https://hal.science/hal-01576131v1>

Preprint submitted on 22 Aug 2017 (v1), last revised 4 Sep 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A MATHEMATICAL COMMENT ON GRAVITATIONAL WAVES

J.-F. Pommaret
CERMICS, Ecole des Ponts ParisTech,
6/8 Av. Blaise Pascal, 77455 Marne-la-Vallée Cedex 02, France
E-mail: jean-francois.pommaret@wanadoo.fr
URL: <http://cermics.enpc.fr/~pommaret/home.html>

ABSTRACT

In classical General Relativity, the way to exhibit the equations for the gravitational waves is based on two "tricks" allowing to transform the Einstein equations after linearizing them over the Minkowski metric. With specific notations used in the study of *Lie pseudogroups* of transformations of an n -dimensional manifold, let $\Omega = (\Omega_{ij} = \Omega_{ji})$ be a perturbation of the non-degenerate metric $\omega = (\omega_{ij} = \omega_{ji})$ with $\det(\omega) \neq 0$ and call $\omega^{-1} = (\omega^{ij} = \omega^{ji})$ the inverse matrix appearing in the Dalemberertian operator $\square = \omega^{ij} d_{ij}$. The first idea is to introduce the linear transformation $\bar{\Omega}_{ij} = \Omega_{ij} - \frac{1}{2}\omega_{ij} tr(\Omega)$ where $tr(\Omega) = \omega^{ij} \Omega_{ij}$ is the *trace* of Ω , which is invertible when $n \geq 3$. The second important idea is to notice that the composite second order linearized *Einstein* operator $\bar{\Omega} \rightarrow \Omega \rightarrow E = (E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij} tr(R))$ where $\Omega \rightarrow R = (R_{ij} = R_{ji})$ is the linearized *Ricci* operator with trace $tr(R) = \omega^{ij} R_{ij}$ is reduced to $\square \bar{\Omega}_{ij}$ when $\omega^{rs} d_{ri} \bar{\Omega}_{sj} = 0$. The purpose of this short but striking paper is to revisit these two results in the light of the *differential duality* existing in *Algebraic Analysis*, namely a mixture of differential geometry and homological algebra, providing therefore a totally different interpretation. In particular, we prove that the above operator $\bar{\Omega} \rightarrow E$ is nothing else than the formal adjoint of the *Ricci* operator $\Omega \rightarrow R$ and that the map $\Omega \rightarrow \bar{\Omega}$ is just the formal adjoint (transposed) of the defining tensor map $R \rightarrow E$. Accordingly, the *Cauchy* operator (stress equations) can be directly parametrized by the formal adjoint of the *Ricci* operator and the *Einstein* operator is no longer needed.

KEY WORDS

General Relativity, Killing equations, Ricci tensor, Einstein tensor, Conformal Killing equations, Weyl tensor, Lie group, Lie pseudogroup, Algebraic Analysis, Homological algebra, Differential duality, Adjoint operator.

1) INTRODUCTION

In order to make the paper rather self-contained, we recall a few notations and definitions on linear systems of partial differential (PD) equations [8-12,21,22,27]. If E is a vector bundle over the base manifold X with projection π and local coordinates $(x, y) = (x^i, y^k)$ projecting onto $x = (x^i)$ for $i = 1, \dots, n$ and $k = 1, \dots, m$, identifying a map with its graph, a (local) section $f : U \subset X \rightarrow E$ is such that $\pi \circ f = id$ on U and we write $y^k = f^k(x)$ or simply $y = f(x)$. For any change of local coordinates $(x, y) \rightarrow (\bar{x} = \varphi(x), \bar{y} = A(x)y)$ on E , the change of section is $y = f(x) \rightarrow \bar{y} = \bar{f}(\bar{x})$ such that $\bar{f}^l(\varphi(x)) \equiv A_k^l(x)f^k(x)$. The new vector bundle E^* obtained by changing the *transition matrix* A to its inverse A^{-1} is called the *dual vector bundle* of E . We may introduce the tangent bundle T , the cotangent bundle T^* , the vector bundle $S_q T^*$ of q -symmetric covariant tensors and the vector bundle $\wedge^r T^*$ of r -skewsymmetric covariant tensors or r -forms. Differentiating with respect to x^i and using new coordinates y_i^k in place of $\partial_i f^k(x)$, we obtain $\bar{y}_r^l \partial_i \varphi^r(x) = A_k^l(x)y_i^k + \partial_i A_k^l(x)y^k$. Introducing a multi-index $\mu = (\mu_1, \dots, \mu_n)$ with length $|\mu| = \mu_1 + \dots + \mu_n$ and prolonging the procedure up to order q , we may construct in this way a vector bundle $J_q(E)$ over X , called the *jet bundle of order q* with local coordinates $(x, y_q) = (x^i, y_\mu^k)$ with $0 \leq |\mu| \leq q$ and $y_0^k = y^k$. For a later use, we shall set $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$ and define the operator $j_q : E \rightarrow J_q(E) : f \rightarrow j_q(f)$ on sections by the local formula $j_q(f) : (x) \rightarrow (\partial_\mu f^k(x) \mid 0 \leq |\mu| \leq q, k = 1, \dots, m)$. Finally, as the background will always be clear enough, we shall use the same notation for a vector bundle and its set of sections.

DEFINITION 1.1: A *system* of PD equations of order q on E is a vector subbundle $R_q \subset J_q(E)$ locally defined by a constant rank system of linear equations for the jets of order q of the form $a_k^{\tau\mu}(x)y_\mu^k = 0$. Its *first prolongation* $R_{q+1} \subset J_{q+1}(E)$ will be defined by the equations $a_k^{\tau\mu}(x)y_\mu^k = 0, a_k^{\tau\mu}(x)y_{\mu+1_i}^k + \partial_i a_k^{\tau\mu}(x)y_\mu^k = 0$ which may not provide a system of constant rank. A system R_q is said to be *formally integrable* if the R_{q+r} are vector bundles $\forall r \geq 0$ (*regularity condition*) and no new equation of order $q+r$ can be obtained by prolonging the given PD equations more than r times, $\forall r \geq 0$. The *symbols* $g_{q+r} = R_{q+r} \cap S_{q+r} T^* \otimes E$ only depend on g_q [8-12,27].

DEFINITION 1.2: Considering the short exact sequence $0 \rightarrow R_q \rightarrow J_q(E) \xrightarrow{\Phi} F_0 \rightarrow 0$ where $\Phi : j_q(E) \rightarrow J_q(E)/R_q$ is the canonical projection, we may thus introduce the linear operator $\mathcal{D} = \Phi \circ j_q : E \rightarrow F_0$. However, as F_0 is only defined up to an isomorphism, things may not be so simple when $q = 1$ and there is no zero order PD equations. We have the commutative and exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & g_1 & \rightarrow & T^* \otimes E & \xrightarrow{\sigma(\Phi)} & F_0 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \parallel & \\
 0 \rightarrow & R_1 & \rightarrow & J_1(E) & \xrightarrow{\Phi} & F_0 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & E & = & E & \rightarrow & 0 & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
 \end{array}$$

where $\sigma(\Phi)$ is the induced symbol epimorphism.

EXAMPLE 1.3: The infinitesimal isometries of the non-degenerate metric $\omega \in S_2 T^*$ with $\det(\omega) \neq 0$ are defined by the kernel Θ of the linear first order *Killing operator* $T \rightarrow S_2 T^* : \xi \rightarrow \mathcal{D}\xi = \mathcal{L}(\xi)\omega = \Omega$, which involves the Lie derivative \mathcal{L} and provides twice the so-called infinitesimal deformation tensor of continuum mechanics when ω is the Euclidean metric. We may consider the linear first order system of *general infinitesimal Lie equations* in *Medolaghi form*, also called *system of Killing equations* [8,11,28]:

$$\Omega_{ij} \equiv (\mathcal{L}(\xi)\omega)_{ij} \equiv \omega_{rj}(x)\partial_i \xi^r + \omega_{ir}(x)\partial_j \xi^r + \xi^r \partial_r \omega_{ij}(x) = 0$$

which is in fact a family of systems only depending on the *geometric object* ω and its derivatives. Introducing the Christoffel symbols γ , we may differentiate once and add the operator

$\mathcal{L}(\xi)\gamma = \Gamma \in S_2T^* \otimes T$ with the well known Levi-Civita isomorphism $j_1(\omega) = (\omega, \partial_x\omega) \simeq (\omega, \gamma)$ in order to obtain the linear second order system of *general infinitesimal Lie equations* in *Medolaghi form*:

$$\Gamma_{ij}^k \equiv (\mathcal{L}(\xi)\gamma)_{ij}^k \equiv \partial_{ij}\xi^k + \gamma_{rj}^k(x)\partial_i\xi^r + \gamma_{ir}^k(x)\partial_j\xi^r - \gamma_{ij}^r(x)\partial_r\xi^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

This system is formally integrable if and only if ω has a *constant Riemannian curvatures* [2,8-11]. In the diagram, $E = T, F_0 = S_2T^*$ and $\sigma(\Phi) : T^* \otimes T \rightarrow S_2T^* : \xi_i^k \rightarrow \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r$. Similarly, introducing the Jacobian determinant $\Delta(x) = \det(\partial_i f^k(x))$ and the *metric density* $\hat{\omega}_{ij} = |\det(\omega)|^{-\frac{1}{n}}\omega_{ij} \Rightarrow \det(\hat{\omega}) = \pm 1$ as a new *geometric object*, rather than by eliminating a *conformal factor* as usual, the infinitesimal *conformal isometries* are defined by the kernel $\hat{\Theta}$ of the *conformal Killing operator* $\xi \rightarrow \hat{D}\xi = \mathcal{L}(\xi)\hat{\omega} = \hat{\Omega}$. We may consider the first order system of *general infinitesimal Lie equations* in *Medolaghi form*, also called *system of conformal Killing equations* [16,17]:

$$\hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}(x)\partial_i\xi^r + \hat{\omega}_{ir}(x)\partial_j\xi^r - \frac{2}{n}\hat{\omega}_{ij}(x)\partial_r\xi^r + \xi^r\partial_r\hat{\omega}_{ij}(x) = 0$$

We may introduce the *trace* $tr(\Omega) = \omega^{ij}\Omega_{ij}$ with standard notations and obtain therefore $tr(\hat{\Omega}) = 0$ because $\hat{\Omega}_{ij} = |\det(\omega)|^{-\frac{1}{n}}(\Omega_{ij} - \frac{1}{n}\omega_{ij}tr(\Omega))$ by linearization. This system is formally integrable if and only if the corresponding *Weyl tensor* vanishes [8,9,10]. In the diagram $E = T, \hat{F}_0 = \{\hat{\Omega} \in S_2T^* \mid tr(\hat{\Omega}) = 0\}$ and $\sigma(\hat{\Phi}) : T^* \otimes T \rightarrow \hat{F}_0 : \xi_i^k \rightarrow \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r - \frac{2}{n}\omega_{ij}\xi_r^r$. The inclusions $R_1 \subset \hat{R}_1 \Rightarrow g_1 \subset \hat{g}_1$ induces an epimorphism $F_0 \rightarrow \hat{F}_0$ described by $\Omega_{ij} \rightarrow \hat{\Omega}_{ij} = \Omega_{ij} - \frac{1}{n}\omega_{ij}tr(\Omega)$. Contrary to the Abstract, *this is the only combination having a purely mathematical meaning related to group theory but never invertible*. It is only in the next Section that we shall understand the origin of this confusing fact.

We have explained in many books [8-13] or papers [15-22] the way to construct a *differential sequence*:

$$0 \rightarrow \Theta \rightarrow T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} F_2 \rightarrow \dots$$

where each operator generates the CC of the previous one, \mathcal{D} is first order, \mathcal{D}_1 is second order and is the linearization ($R_{l,ij}^k$) of the Riemann tensor over a given flat metric like the Minkowski metric while \mathcal{D}_2 is again first order and is the linearization of the Bianchi identities with:

$$\dim(T) = n, \dim(F_0) = n(n+1)/2, \dim(F_1) = n^2(n^2-1)/2, \dim(F_2) = n^2(n^2-1)(n-2)/24$$

The conformal situation is drastically different but not acknowledged today, because $\hat{g}_3 = 0, \forall n \geq 3$ and we have to study *separately* the cases $n = 3, n = 4, n \geq 5$ even though $\hat{\mathcal{D}}$ is still first order, because $\hat{\mathcal{D}}_1$ is third order when $n = 3$ but still second order and is the linearization $\Sigma_{l,ij}^k$ of the Weyl tensor when $n \geq 4$ while \mathcal{D}_2 is first order when $n = 3$, second order when $n = 4$ but again first order when $n \geq 5$ [19-22]. For $n \geq 4$, we have the commutative and exact diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & S_2T^* & = & S_2T^* & \rightarrow 0 \\ & & 0 & \rightarrow & S_2T^* & \rightarrow & F_1 & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \uparrow & \\ 0 \rightarrow & S_3T^* \otimes T & \rightarrow & S_2T^* \otimes F_0 & \rightarrow & F_1 & \rightarrow 0 \\ & \parallel & & \downarrow & & \downarrow \uparrow & \\ 0 \rightarrow & S_3T^* \otimes T & \rightarrow & S_2T^* \otimes \hat{F}_0 & \rightarrow & \hat{F}_1 & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

providing an epimorphism $F_1 \rightarrow \hat{F}_1$ with kernel S_2T^* , induced by the epimorphism $F_0 \rightarrow \hat{F}_0$ and the relation $\dim(\hat{F}_1) - \dim(F_1) = n(n+1)/2$. The central and right columns split with the usual contraction map $F_1 \rightarrow S_2T^* : R_{l,ij}^k \rightarrow R_{i,rj}^r = R_{ij} = R_{ji}$ and the tensorial lift $\hat{F}_1 \rightarrow F_1 : \Sigma_{l,ij}^k \rightarrow R_{l,ij}^k$ because $\Sigma_{l,rj}^r = 0$.

Using capital letters for the linearized objects as above but now keeping R for the linearized Ricci tensor, we now recall with these new notations a few formulas that can be found in most textbooks [3,5,15,21]. We have thus successively (*care* to the factor 2):

$$\begin{aligned}
2\Gamma_{ij}^k &= \omega^{kr}(d_i\Omega_{rj} + d_j\Omega_{ir} - d_r\Omega_{ij}) = 2\Gamma_{ji}^k \\
2R_{ij} &= \omega^{rs}(d_{ij}\Omega_{rs} + d_{rs}\Omega_{ij} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) = 2R_{ji} \\
tr(R) &= \omega^{ij}R_{ij} = \omega^{ij}d_{ij}tr(\Omega) - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv} \\
E_{ij} &= R_{ij} - \frac{1}{2}\omega_{ij}tr(R) = E_{ji} \Rightarrow tr(E) = \omega^{ij}E_{ij} = -\frac{(n-2)}{2}tr(R) \\
2E_{ij} &\equiv \omega^{rs}(d_{ij}\Omega_{rs} + d_{rs}\Omega_{ij} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}) - \omega_{ij}(\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}) = 0
\end{aligned}$$

and recall the classical computations described in the Abstract:

$$\bar{\Omega}_{ij} = \Omega_{ij} - \frac{1}{2}\omega_{ij}tr(\Omega) \Leftrightarrow \Omega_{ij} = \bar{\Omega}_{ij} - \frac{1}{(n-2)}\omega_{ij}tr(\bar{\Omega})$$

Substituting, we obtain:

$$\begin{aligned}
2E_{ij} &= \square\bar{\Omega}_{ij} - \omega^{rs}d_{ri}\bar{\Omega}_{sj} - \omega^{rs}d_{sj}\bar{\Omega}_{ri} + \omega_{ij}\omega^{ru}\omega^{sv}d_{rs}\bar{\Omega}_{uv} \\
&= \square\bar{\Omega}_{ij} - d_{ri}\bar{\Omega}_j^r - d_{rj}\bar{\Omega}_i^r + \omega_{ij}d_{rs}\bar{\Omega}^{rs}
\end{aligned}$$

We notice at once that, apart from the first Dalembertian term, the other terms factor through $d_r\bar{\Omega}_i^r$ a result leading to add the *differential constraints* $d_r\bar{\Omega}_i^r = 0$ in a coherent way with the identities:

$$2\omega^{ti}d_tE_{ij} = (\omega^{ti}\omega^{rs}d_{rst}\bar{\Omega}_{ij} - \omega^{ti}d_{irt}\bar{\Omega}_j^r) - (\omega^{ti}d_{jrt}\bar{\Omega}_i^r - \omega^{ti}\omega_{ij}d_{rst}\bar{\Omega}^{rs}) = 0 - 0 = 0$$

We shall now revisit these computations in a quite different framework and explain the resulting confusion done between the *div* operator induced by the *Bianchi* operator and the *Cauchy* operator which is the formal adjoint of the *Killing* operator [21,22].

2) DIFFERENTIAL DUALITY

First of all, we describe the initial part of the differential sequence introduced in the preceding Section, calling the successive operators by using the historical names *Killing*, *Riemann*, *Ricci*, *Bianchi*, *Beltrami*. In particular, lowering the indices by means of the constant metric ω , we obtain:

$$\Omega_{ij} = d_i \xi_j + d_j \xi_i \Rightarrow \text{tr}(\Omega) = 2d_r \xi^r \Rightarrow R_{l,ij}^k = 0 \Leftrightarrow R_{ij} = 0 \oplus \Sigma_{l,ij}^k = 0$$

and we have exhibited the last splitting allowing to get a direct sum. As a byproduct, we have thus $E_{ij} = 0$ and it is well known that the so-called *divergence* condition $\omega^{ti} d_t E_{ij} = d_t E_j^t = 0$ is implied by the Bianchi identities $\sum_{(ijr)} d_r R_{l,ij}^k = 0$ where the sum is over the cyclic permutation. Now we recall that the above differential sequence where *Riemann* generates the CC of *Killing*, *Bianchi* generates the CC of *Riemann* and so on, is locally isomorphic to the tensor product of the *Poincaré sequence* by a Lie algebra with $n(n+1)/2$ infinitesimal generators ([11], p 186,224)([21], Section 5). It has therefore the *very special property* that $\text{ad}(\text{Riemann}) = \text{Beltrami}$ generates the CC of $\text{ad}(\text{Bianchi})$ while $\text{ad}(\text{Killing}) = \text{Cauchy}$ generates the CC of $\text{ad}(\text{Riemann})$, a quite difficult result of *homological algebra* saying that the *extension modules* of a differential module M do not depend on the resolution of M [1,4,6,7,12,13,14,25,26]. Of course, the same property is also valid for the corresponding conformal sequence with now $(n+1)(n+2)/2$ infinitesimal generators whenever $n \geq 3$. The key contradicting results are provided by the following Theorem and Corollary [12,13,15,19,21]:

THEOREM 2.1: Contrary to the Ricci operator, the Einstein operator is self-adjoint and we have the following diagram when $n = 4$:

$$\begin{array}{ccccccc}
 4 & \xrightarrow{\text{Killing}} & 10 & \xrightarrow{\text{Riemann}} & 20 & \xrightarrow{\text{Bianchi}} & 20 \longrightarrow 6 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 & & 10 & \xrightarrow{\text{Einstein}} & 10 & \xrightarrow{\text{div}} & 4 \longrightarrow 0 \\
 \\
 0 \leftarrow 4 & \xleftarrow{\text{Cauchy}} & 10 & \xleftarrow{\text{Beltrami}} & 20 & \xleftarrow{\quad} & 20 \\
 & & \parallel & & \uparrow & & \\
 & & 10 & \xleftarrow{\text{Einstein}} & 10 & &
 \end{array}$$

Proof: The 6 terms (4 for R_{ij} and 2 for $\text{tr}(R)$) are exchanged between themselves by *ad*.
Q.E.D.

COROLLARY 2.2: The Einstein equations in vacuum cannot be parametrized and it is thus not possible to express any generic solution by means of the derivatives of a certain number of arbitrary functions or *potentials* like Maxwell equations.

$$\begin{array}{cccc}
 & & \text{Riemann} & 20 \\
 & & \nearrow & \\
 4 & \xrightarrow{\text{Killing}} & 10 & \xrightarrow{\text{Einstein}} & 10 \\
 \\
 4 & \xleftarrow{\text{Cauchy}} & 10 & \xleftarrow{\text{Einstein}} & 10
 \end{array}$$

Proof: According to crucial results of Algebraic Analysis, the test for knowing if a given operator \mathcal{D}_1 can be parametrized by an operator \mathcal{D} , that is if we can find a differential sequence:

$$\xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta$$

where \mathcal{D}_1 generates the CC of an operator \mathcal{D} , has 5 steps:

$$\mathcal{D}_1 \Rightarrow \text{ad}(\mathcal{D}_1) \Rightarrow \text{ad}(\mathcal{D}) \Rightarrow \text{ad}(\text{ad}(\mathcal{D})) = \mathcal{D} \Rightarrow \mathcal{D}_1'$$

where $\text{ad}(\mathcal{D})$ generates the CC of $\text{ad}(\mathcal{D}_1)$ and \mathcal{D}_1' generates the CC of \mathcal{D} , the parametrization being obtained if and only if $\mathcal{D}_1' = \mathcal{D}_1$. We obtain therefore the adjoint differential sequence between

convenient test functions used in order to construct the various adjoint operators:

$$\nu \xleftarrow{ad(\mathcal{D})} \mu \xleftarrow{ad(\mathcal{D}_1)} \lambda$$

Q.E.D.

We are now ready to explain the results presented in the Introduction. Indeed, with arbitrary test functions λ , we have:

$$\lambda^{ij} E_{ij} = \lambda^{ij} (R_{ij} - \frac{1}{2} \omega_{ij} tr(R)) = (\lambda^{ij} - \frac{1}{2} \omega^{ij} tr(\lambda)) R_{ij} = \bar{\lambda}^{ij} R_{ij}$$

Accordingly, as we just saw that $ad(Einstein) = Einstein$ is parametrizing $ad(Killing) = Cauchy$, then $ad(Ricci)$ is thus also parametrizing $Cauchy$ and we obtain through an integration by parts (*care* to the following dumb summations):

$$\begin{aligned} 2\bar{\lambda}^{ij} R_{ij} &= \bar{\lambda}^{ij} \omega^{rs} (d_{ij} \Omega_{rs} + d_{rs} \Omega_{ij} - d_{ri} \Omega_{sj} - d_{sj} \Omega_{ri}) \\ &\equiv (\square \bar{\lambda}^{rs} + \omega^{rs} d_{ij} \bar{\lambda}^{ij} - \omega^{sj} d_{ij} \bar{\lambda}^{ri} - \omega^{ri} d_{ij} \bar{\lambda}^{sj}) \Omega_{rs} \quad mod(div) \\ &= \sigma^{rs} \Omega_{rs} \end{aligned}$$

Surprisingly, all the terms after the Dalemberertian have already been obtained in the preceding Section and factorize through the divergence operator $d_i \bar{\lambda}^{ri}$. Therefore, suppressing the bar for simplicity, we may add the *differential constraints* $d_i \lambda^{ri} = 0$ in a coherent way with the identities:

$$d_r \sigma^{rs} = \omega^{ij} d_{rij} \lambda^{rs} + \omega^{rs} d_{rij} \lambda^{ij} - \omega^{sj} d_{rij} \lambda^{ri} - \omega^{ri} d_{rij} \lambda^{sj} = 0$$

However, it must be noticed that the potential test functions are arbitrary by definition and can be restricted by such differential constraints as will be shown in the last example of this paper thereafter. With more details, we have the identity:

$$Ricci \circ Killing \equiv 0 \Leftrightarrow ad(Killing) \circ ad(Ricci) \equiv 0$$

Now, we recall that if \mathcal{D} has coefficients in a differential field K and defines a differential module over the ring $D = K[d]$ of differential operators, we may define the *differential transcendence degree* $diff\ trd(\mathcal{D}) = m - rk_D(M)$. We obtain thus [11,12,22]:

$$diff\ trd(Killing) = 0 \Rightarrow diff\ trd(ad(Ricci)) = diff\ trd(Ricci) = n(n+1)/2 - n = n(n-1)/2$$

Taking into account the preceding constraints, we obtain a minimum *relative parametrization* that cannot be reduced (see [18,19,20,23,24] for more details and the use of Computer Algebra). Finally, it is important to notice that the *div* operator induced by the *Bianchi* operator in the upper part of the preceding diagram generates the CC of the *Einstein* operator. It follows that the *Cauchy* operator does generate the CC of the $ad(Einstein)=Einstein$ operator in the lower part of the same diagram, though there is no relation at all between these two operators. It is therefore possible to avoid totally the *Einstein* operator which has no mathematical meaning as *no specific diagram chasing can produce it* and to keep only the *Ricci* operator which has indeed a mathematical meaning only depending on the second order jets (*elations*) of the conformal group described by the symbol \hat{g}_2 through a delicate diagram chasing [9,10,11,15,21].

EXAMPLE 2.3: We finally provide an elementary but non-trivial example of the methods used previously and ask the reader to compare the various situations. If $x = (x^1, x^2)$ are the independent variables and $\eta = (\eta^1, \eta^2)$ are the unknowns, let us consider the first order operator \mathcal{D}_1 with coefficients in the differential field $K = \mathbb{Q}(x^1, x^2)$ and the same formal notations as before:

$$d_1 \eta^1 + d_2 \eta^2 - x^2 \eta^1 = \zeta$$

Multiplying by a test function λ and integrating by parts, we get $ad(\mathcal{D}_1)$ in the form:

$$\begin{cases} -d_1 \lambda = \mu^1 \\ -d_2 \lambda = \mu^2 \end{cases} \Rightarrow \lambda = d_1 \mu^2 - d_2 \mu^1 + x^2 \mu^2$$

The generating CC $ad(\mathcal{D})$ are:

$$\begin{cases} -d_{11}\mu^2 + d_{12}\mu^1 - 2x^2d_1\mu^2 + x^2d_2\mu^1 - (x^2)^2\mu^2 - \mu^1 & = \nu^1 \\ -d_{12}\mu^2 + d_{22}\mu^1 - x^2d_2\mu^2 - 2\mu^2 & = \nu^2 \end{cases} \Rightarrow d_1\nu^2 - d_2\nu^1 + x^2\nu^2 = \theta$$

Multiplying by the test functions $\xi = (\xi^1, \xi^2)$, then adding and integrating by parts, we get the *second order* parametrization:

$$\begin{cases} d_{12}\xi^1 + d_{22}\xi^2 - x^2d_2\xi^1 - 2\xi^1 & = \eta^1 \\ -d_{11}\xi^1 - d_{12}\xi^2 + 2x^2d_1\xi^1 + x^2d_2\xi^2 - (x^2)^2\xi^1 - \xi^2 & = \eta^2 \end{cases}$$

and the two differential sequences:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \phi & \xrightarrow{\mathcal{D}_{-1}} & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & \rightarrow & 0 \\ 0 & \leftarrow & \theta & \xleftarrow{ad(\mathcal{D}_{-1})} & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & \leftarrow & 0 \end{array}$$

showing that the differential module over $D = K[d_1, d_2]$ defined by \mathcal{D}_1 is projective (Exercise). Choosing $(\xi^1 = \xi, \xi^2 = 0)$ or $(\xi^1 = 0, \xi^2 = \xi')$, we obtain two minimal parametrizations but we can also suppose that we add the differential constraint $d_1\xi^1 + d_2\xi^2 = 0$ in order to obtain the following *first order* parametrization, a result not evident at first sight:

$$\begin{cases} -x^2d_2\xi^1 - 2\xi^1 & = \eta^1 \\ x^2d_1\xi^1 - (x^2)^2\xi^1 - \xi^2 & = \eta^2 \end{cases}$$

We may also set $\xi^1 = d_2\phi, \xi^2 = -d_1\phi$ and obtain the new second order parametrization:

$$\begin{cases} -x^2d_{22}\phi - 2d_2\phi & = \eta^1 \\ +x^2d_{12}\phi - (x^2)^2d_2\phi + d_1\phi & = \eta^2 \end{cases}$$

We finally notice that the choice $\xi = \mathcal{D}_{-1}\phi$, namely $\xi^1 = d_2\phi, \xi^2 = -d_1\phi + x^2\phi$ is not allowed as it only provides the trivial solution $\eta = 0$.

3) CONCLUSION

As we have seen, only *homological algebra* allows to prove that a differential sequence $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2$ starting with a Lie operator determined by the action of a lie group on a manifold of dimension n and where each operator generates the CC of the previous one is such that, in the adjoint sequence $ad(\mathcal{D}_2), ad(\mathcal{D}_1), ad(\mathcal{D})$, each operator generates the CC of the preceding one. This is in particular the case for the first order *Killing* operator \mathcal{D} , followed by the second order *Riemann* operator \mathcal{D}_1 and the first order *Bianchi* operator \mathcal{D}_2 . The corresponding part of the adjoint sequence is therefore successively made by $ad(\text{Bianchi}), ad(\text{Riemann}) = \text{Beltrami}$ and $ad(\text{Killing}) = \text{Cauchy}$. Accordingly, the classical *div* operator, induced by the *Bianchi* operator and describing the CC of the *Einstein* operator, *has nothing to do* with the *Cauchy* operator. Such a confusion has been produced by the fact that $ad(\text{Einstein}) = \text{Einstein}$ is thus parametrizing the *Cauchy* operator but it is not evident that the transformation of the *Einstein* operator described in the Abstract and in the Introduction, just amounts to parametrize the *Cauchy* operator / *stress* equations by means of the operator $ad(\text{Ricci})$. This result is showing that the *Einstein* operator is no longer needed and must therefore be taken into account in any future work on gravitational waves.

REFERENCES

- [1] Assem, I.: Algèbres et Modules, Masson, Paris (1997).
- [2] Eisenhart, L.P.: Riemannian Geometry, Princeton University Press, Princeton (1926).
- [3] Foster, J., Nightingale, J.D.: A Short Course in General relativity, Longman (1979).
- [4] Hu,S.-T.: Introduction to Homological Algebra, Holden-Day (1968).
- [5] Hughston, L.P., Tod, K.P.: An Introduction to General Relativity, London Math. Soc. Students Texts 5, Cambridge University Press (1990).
- [6] Kashiwara, M.: Algebraic Study of Systems of Partial Differential Equations, Mémoires de la Société Mathématique de France, 63 (1995) (Transl. from Japanese of his 1970 Masters Thesis).
- [7] Northcott, D.G.: An Introduction to Homological Algebra, Cambridge university Press (1966).
- [8] Pommaret, J.-F.: Systems of Partial Differential Equations and Lie Pseudogroups, Gordon and Breach, New York (1978); Russian translation: MIR, Moscow (1983).
- [9] Pommaret, J.-F.: Differential Galois Theory, Gordon and Breach, New York (1983).
- [10] Pommaret, J.-F.: Lie Pseudogroups and Mechanics, Gordon and Breach, New York (1988).
- [11] Pommaret, J.-F.: Partial Differential Equations and Group Theory, Kluwer (1994).
<http://dx.doi.org/10.1007/978-94-017-2539-2>
- [12] Pommaret, J.-F.: Partial Differential Control Theory, Kluwer, Dordrecht (2001).
- [13] Pommaret, J.-F.: Algebraic Analysis of Control Systems Defined by Partial Differential Equations, in "Advanced Topics in Control Systems Theory", Springer, Lecture Notes in Control and Information Sciences 311 (2005) Chapter 5, 155-223.
- [14] Pommaret, J.-F.: Parametrization of Cosserat Equations, Acta Mechanica, 215 (2010) 43-55.
<http://dx.doi.org/10.1007/s00707-010-0292-y>
- [15] Pommaret, J.-F.: The Mathematical Foundations of General Relativity Revisited, Journal of Modern Physics, 4 (2013) 223-239.
<http://dx.doi.org/10.4236/jmp.2013.48A022>
- [16] Pommaret, J.-F.: The Mathematical Foundations of Gauge Theory Revisited, Journal of Modern Physics, 5 (2014) 157-170.
<http://dx.doi.org/10.4236/jmp.2014.55026>
- [17] Pommaret, J.-F.: From Thermodynamics to Gauge Theory: the Virial Theorem Revisited, pp. 1-46 in "Gauge Theories and Differential geometry," NOVA Science Publisher (2015).
- [18] Pommaret, J.-F.: Relative parametrization of Linear Multidimensional Systems, Multidim. Syst. Sign. Process., 26 (2015) 405-437.
DOI 10.1007/s11045-013-0265-0
- [19] Pommaret, J.-F.: Airy, Beltrami, Maxwell, Einstein and Lanczos Potentials revisited, Journal of Modern Physics, 7, 699-728 (2016).
<http://dx.doi.org/10.4236/jmp.2016.77068>
- [20] Pommaret, J.-F.: Deformation Theory of Algebraic and Geometric Structures, Lambert Academic Publisher (LAP), Saarbrücken, Germany (2016).
<http://arxiv.org/abs/1207.1964>
- [21] Pommaret, J.-F.: Algebraic Analysis and Mathematical Physics (2017).
<https://arxiv.org/abs/1706.04105>
- [22] Pommaret, J.-F.: Differential Algebra and Mathematical Physics (2017).
<https://arxiv.org/abs/1707.09763>
- [23] Pommaret, J.-F. and Quadrat, A.: Localization and Parametrization of Linear Multidimensional Control Systems, Systems & Control Letters, 37 (1999) 247-260.
- [24] Quadrat, A.: An Introduction to Constructive Algebraic Analysis and its Applications, Les cours du CIRM, Journées Nationales de Calcul Formel, 1(2), 281-471 (2010).
- [25] Rotman, J.J.: An Introduction to Homological Algebra, Pure and Applied Mathematics, Academic Press (1979).
- [26] Schneiders, J.-P.: An Introduction to D-Modules, Bull. Soc. Roy. Sci. Liège, 63 (1994) 223-295.
- [27] Spencer, D.C.: Overdetermined Systems of Partial Differential Equations, Bull. Am. Math. Soc., 75 (1965) 1-114.
- [28] Vessiot, E.: Sur la Théorie des Groupes Infinis, Ann. Ec. Norm. Sup., 20 (1903) 411-451.
(Can be obtained from <http://numdam.org>).