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**CAUCHY PROBLEM FOR THE SPATIALLY HOMOGENEOUS
LANDAU EQUATION WITH SHUBIN CLASS INITIAL DATUM
AND GELFAND-SHILOV SMOOTHING EFFECT**

HAO-GUANG LI AND CHAO-JIANG XU

ABSTRACT. In this work, we study the nonlinear spatially homogeneous Landau equation with Maxwellian molecules, by using the spectral analysis, we show that the non linear Landau operators is almost linear, and we prove the existence of weak solution for the Cauchy problem with the initial datum belonging to Shubin space of negative index which contains the probability measures. Based on this spectral decomposition, we prove also that the Cauchy problem enjoys $S^{\frac{1}{2}}$ -Gelfand-Shilov smoothing effect, meaning that the weak solution of the Cauchy problem with Shubin class initial datum is ultra-analytic and exponential decay for any positive time.

1. INTRODUCTION

In this work, we study the spatially homogeneous Landau equation

$$\begin{cases} \partial_t f = Q_L(f, f), \\ f|_{t=0} = f_0, \end{cases} \quad (1.1)$$

where $f = f(t, v) \geq 0$ is the density distribution function depending on the variables $v \in \mathbb{R}^3$ and the time $t \geq 0$. The Landau bilinear collision operator is given by

$$Q_L(g, f)(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v_*) (g(v_*) (\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)) dv_* \right),$$

where $a(v) = (a_{i,j}(v))_{1 \leq i,j \leq 3}$ stands for the non-negative symmetric matrix

$$a(v) = (|v|^2 \mathbf{I} - v \otimes v) |v|^\gamma \in M_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

In this work, we only consider the Cauchy problem (1.1) with the Maxwellian molecules, that means $\gamma = 0$. For the non-negative initial datum f_0 , we suppose

$$\int_{\mathbb{R}^3} f_0(v) dv = 1, \quad \int_{\mathbb{R}^3} v_j f_0(v) dv = 0, \quad j = 1, 2, 3, \quad \int_{\mathbb{R}^3} |v|^2 f_0(v) dv = 3. \quad (1.2)$$

We shall study the linearization of the Landau equation (1.1) near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Considering the fluctuation of density distribution function

$$f(t, v) = \mu(v) + \sqrt{\mu}(v)g(t, v),$$

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since $Q_L(\mu, \mu) = 0$, the Cauchy problem (1.1) is reduced to the Cauchy problem

$$\begin{cases} \partial_t g + \mathcal{L}(g) = \mathbf{L}(g, g), & t > 0, v \in \mathbb{R}^3, \\ g|_{t=0} = g_0, \end{cases} \quad (1.3)$$

with $g_0(v) = \mu^{-\frac{1}{2}} f_0(v) - \sqrt{\mu}$, where

$$\mathcal{L}(g) = -\mu^{-\frac{1}{2}} \left(Q_L(\sqrt{\mu}g, \mu) + Q_L(\mu, \sqrt{\mu}g) \right), \quad \mathbf{L}(g, g) = \mu^{-\frac{1}{2}} Q_L(\sqrt{\mu}g, \sqrt{\mu}g).$$

The linear operator \mathcal{L} is non-negative (see [6]) with the null space

$$\mathcal{N} = \text{span} \{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \}.$$

Then the assumption (1.2) on the initial datum f_0 reduces to

$$\begin{cases} \int_{\mathbb{R}^3} \sqrt{\mu}(v) g_0(v) dv = 0, \\ \int_{\mathbb{R}^3} v_j \sqrt{\mu}(v) g_0(v) dv = 0, \quad j = 1, 2, 3, \\ \int_{\mathbb{R}^3} |v|^2 \sqrt{\mu}(v) g_0(v) dv = 0. \end{cases}$$

This shows that $g_0 \in \mathcal{N}^\perp$. We recall the spectral decomposition of the linear Landau operator (see Appendix 6 and [1], [6]).

$$\mathcal{L}(\varphi_{n,l,m}) = \lambda_{n,l} \varphi_{n,l,m}, \quad n, l \in \mathbb{N}, \quad -l \leq m \leq l \quad (1.4)$$

where $\{\varphi_{n,l,m}\}_{n,l \in \mathbb{N}, |m| \leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$ and the Laplace-Beltrami operator on the unit sphere \mathbb{S}^2 ,

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2}) \varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2}(\varphi_{n,l,m}) = l(l+1) \varphi_{n,l,m}.$$

The eigenvalues of (1.4) satisfies : $\lambda_{0,0} = \lambda_{0,1} = \lambda_{1,0} = 0$, $\lambda_{0,2} = 12$ and for $2n + l > 2$,

$$\lambda_{n,l} = 2(2n + l) + l(l + 1). \quad (1.5)$$

Using this spectral decomposition, the definition of the operators $e^{c\mathcal{H}^s}$ and \mathcal{H}^α are then classical.

We introduce the following function spaces: Gelfand-Shilov spaces, for $0 < s \leq 1$,

$$S^{\frac{1}{2s}}(\mathbb{R}^3) = \left\{ u \in \mathcal{S}'(\mathbb{R}^3); \exists c > 0, e^{c\mathcal{H}^s} u \in L^2(\mathbb{R}^3) \right\};$$

and the Shubin spaces, for $\beta \in \mathbb{R}$, (see [15], Ch. IV, 25.3),

$$Q^\beta(\mathbb{R}^3) = \left\{ u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{Q^\beta(\mathbb{R}^3)} = \|\mathcal{H}^{\frac{\beta}{2}} u\|_{L^2(\mathbb{R}^3)} < +\infty \right\}.$$

We have

$$\begin{aligned} Q^\beta(\mathbb{R}^3) &\subset H^\beta(\mathbb{R}^3), \quad \forall \beta \geq 0, \\ H^\beta(\mathbb{R}^3) &\subset Q^\beta(\mathbb{R}^3), \quad \forall \beta < 0, \end{aligned}$$

where $H^\beta(\mathbb{R}^3)$ is the usual Sobolev spaces. In particular, for $\beta < -\frac{3}{2}$ the Shubin space $Q^\beta(\mathbb{R}^3)$ contains the probability measures (see [2, 9, 12, 17] and [8]). See Appendix 6 for more properties of Gelfand-Shilov spaces and the Shubin spaces.

It is showed in [10] that, for $g_0 \in L^2(\mathbb{R}^3)$ with $f_0 = \mu + \sqrt{\mu}g_0 \geq 0$, the solution of the Cauchy problem (1.3) obtained in [18] belongs to $S^{\frac{1}{2}}(\mathbb{R}^3)$ for any $t > 0$. In this work, we consider the initial datum which belongs to the Shubin spaces of negative index. The main theorem of this paper is in the following.

Theorem 1.1. *Let $\alpha \leq 0$, there exists $c_0 > 0$ such that for any initial datum $g_0 \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with*

$$\|\mathbb{S}_2 g_0\|_{L^2(\mathbb{R}^3)} \leq c_0, \quad (1.6)$$

the Cauchy problem (1.3) admits a global weak solution

$$g \in L^{+\infty}([0, +\infty[; Q^\alpha(\mathbb{R}^3)).$$

Moreover, we have the Gelfand-Shilov smoothing effect of Cauchy problem, and there exists $c_1 > 0$ such that for any $t > 0$,

$$\|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g(t)\|_{L^2(\mathbb{R}^3)} \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}.$$

Remark 1.2. .

1) The orthogonal projectors $\{\mathbb{S}_N, N \in \mathbb{N}\}$ is defined, for $g \in \mathcal{S}'(\mathbb{R}^3)$,

$$\mathbb{S}_N g = \sum_{k=0}^N \sum_{2n+l=k} \sum_{|m| \leq l} \langle g, \varphi_{n,l,m} \rangle \varphi_{n,l,m} \in \mathcal{S}(\mathbb{R}^3). \quad (1.7)$$

2) The constant c_0 in (1.6) is not asked to be very small, see the Remark 4.2. On the other hand, the condition (1.6) is a restriction for the initial datum on $\mathbb{S}_2 g_0$, but not a smallness hypothesis for the initial datum g_0 .

3) For the Landau equation (also Boltzmann equation), a physics condition on the initial datum is $f_0 = \mu + \sqrt{\mu}g_0 \geq 0$ which implies the non-negativity of solution $f = \mu + \sqrt{\mu}g$. On the other hand, from the partial differential equations point of view, for the Cauchy problem (1.1) (also (1.3)), we don't need to impose this non-negative condition. So that, in the Theorem 1.1, we do not ask for the initial datum $f_0 = \mu + \sqrt{\mu}g_0$ to be non-negative.

4) Combining this Theorem with the results of [18] and [10] (see also [11, 19]), we get a complete result for the Cauchy problem (1.3) with initial datum $g_0 \in Q^\beta(\mathbb{R}^3) \cap \mathcal{N}^\perp, \beta \in \mathbb{R}$: The existence of global (weak) solution and $S^{\frac{1}{2}}$ -Gelfand-Shilov smoothing effect of Cauchy problem.

5) It is well known that the single Dirac mass on the origin is a stationary solution of the Cauchy problem (1.1). The following example is somehow surprise.

Example 1.1. *Let*

$$f_0 = \delta_0 - \left(\frac{3}{2} - \frac{|v|^2}{2} \right) \mu$$

be the initial datum of the Cauchy problem (1.1), then $f_0 = \mu + \sqrt{\mu}g_0$ with

$$g_0 = \frac{1}{\sqrt{\mu}} \delta_0 - \left(\frac{5}{2} - \frac{|v|^2}{2} \right) \sqrt{\mu} \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp, \quad \alpha < -\frac{3}{2}, \quad (1.8)$$

and $\|\mathbb{S}_2 g_0\|_{L^2(\mathbb{R}^3)} = 0$. Then Theorem 1.1 imply that the Cauchy problem (1.1) admits a global solution

$$f = \mu + \sqrt{\mu}g \in L^{+\infty}([0, +\infty[; Q^\alpha(\mathbb{R}^3)) \cap C^0([0, +\infty[; S^{\frac{1}{2}}(\mathbb{R}^3)).$$

This paper is arranged as follows : In the Section 2, we introduce the spectral analysis of the Landau operators and prove that the nonlinear Landau operator is almost diagonal. By using this decomposition, we can present explicitly the formal solutions to the Cauchy problem (1.3) by transforming it into an infinite system of ordinary differential equations. In the Section 3, we establish an upper bounded estimates for the nonlinear operators. We prove the main theorem 1.1 in the Section 4, and collect the main technical computations in the Section 5. In the Section 6, we give the proof of the Example 1.1 and the characterization of the Gelfand-Shilov spaces and the Shubin spaces.

2. SPECTRAL ANALYSIS AND FORMAL SOLUTIONS

In this section, we study the algebra property of the nonlinear Landau operators on the orthonormal basis $\{\varphi_{n,l,m}\}$ of $L^2(\mathbb{R}^3)$,

$$\mathbf{L}(\varphi_{\bar{n},\bar{l},\bar{m}}, \varphi_{n,l,m}).$$

Recall, for $n, l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l$,

$$\varphi_{n,l,m}(v) = \left(\frac{n!}{\sqrt{2}\Gamma(n+l+3/2)} \right)^{1/2} \left(\frac{|v|}{\sqrt{2}} \right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)} \left(\frac{|v|^2}{2} \right) Y_l^m \left(\frac{v}{|v|} \right),$$

where $\Gamma(\cdot)$ is the standard Gamma function, and

– $L_n^{(\alpha)}$ is the Laguerre polynomial of order α and degree n ,

$$L_n^{(\alpha)}(x) = \sum_{r=0}^n (-1)^{n-r} \frac{\Gamma(\alpha+n+1)}{r!(n-r)!\Gamma(\alpha+n-r+1)} x^{n-r};$$

– $Y_l^m(\sigma)$ is the orthonormal basis of spherical harmonics

$$Y_l^m(\sigma) = N_{l,m} P_l^{|m|}(\cos \theta) e^{im\phi}, \quad |m| \leq l,$$

where $\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$ and $N_{l,m}$ is the normalisation factor. It is obviously that, the conjugate of $Y_l^m(\sigma)$ satisfies

$$\overline{Y_l^m(\sigma)} = Y_l^{-m}(\sigma).$$

– $P_l^{|m|}$ is the Legendre functions of the first kind of order l and degree $|m|$

$$P_l^{|m|}(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx} \left(\frac{1}{2^{|l|} |l|!} \frac{d^{|l|}}{dx^{|l|}} (x^2-1)^{|l|} \right).$$

Then, $\{\varphi_{n,l,m}\} \subset \mathcal{S}(\mathbb{R}^3)$ the Schwartz function space, and

$$\begin{aligned} \varphi_{0,0,0}(v) &= \sqrt{\mu}, & \varphi_{0,1,0}(v) &= v_1 \sqrt{\mu}, \\ \varphi_{0,1,1}(v) &= \frac{v_2 + iv_3}{\sqrt{2}} \sqrt{\mu}, & \varphi_{0,1,-1}(v) &= \frac{v_2 - iv_3}{\sqrt{2}} \sqrt{\mu}, \\ \varphi_{1,0,0}(v) &= \sqrt{\frac{2}{3}} \left(\frac{3}{2} - \frac{|v|^2}{2} \right) \sqrt{\mu}, \end{aligned}$$

and

$$\mathcal{N} = \text{span} \{ \varphi_{0,0,0}, \varphi_{0,1,0}, \varphi_{0,1,1}, \varphi_{0,1,-1}, \varphi_{1,0,0} \}.$$

We have also the explicit form of the eigenfunctions $\{\varphi_{0,2,m_2}, |m_2| \leq 2\}$:

$$\begin{aligned}\varphi_{0,2,0}(v) &= \sqrt{\frac{1}{3}} \left(\frac{3}{2}v_1^2 - \frac{1}{2}|v|^2 \right) \sqrt{\mu}, & \varphi_{0,2,1}(v) &= \frac{v_1v_2 + iv_1v_3}{\sqrt{2}} \sqrt{\mu}, \\ \varphi_{0,2,-1}(v) &= \frac{v_1v_2 - iv_1v_3}{\sqrt{2}} \sqrt{\mu}, & \varphi_{0,2,2}(v) &= \left(\frac{v_2^2 - v_3^2}{2\sqrt{2}} + i \frac{v_2v_3}{\sqrt{2}} \right) \sqrt{\mu}, \\ \varphi_{0,2,-2}(v) &= \left(\frac{v_2^2 - v_3^2}{2\sqrt{2}} - i \frac{v_2v_3}{\sqrt{2}} \right) \sqrt{\mu}.\end{aligned}\quad (2.1)$$

We have the following algebraic identities :

Proposition 2.1. *For $n, l \in \mathbb{N}$, $|m| \leq l$, we have*

$$\begin{aligned}(i) \quad & \mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = -(2(2n+l) + l(l+1)) \varphi_{n,l,m}; \\ (ii) \quad & \mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) \\ &= A_{n,l,m,m_1}^- \varphi_{n+1,l-1,m_1+m} + A_{n,l,m,m_1}^+ \varphi_{n,l+1,m_1+m}, \quad \forall |m_1| \leq 1; \\ (iii) \quad & \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m}) = \frac{4\sqrt{3(n+1)(2n+2l+3)}}{3} \varphi_{n+1,l,m}; \\ (iv) \quad & \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) = A_{n,l,m,m_2}^1 \varphi_{n+2,l-2,m+m_2} \\ &+ A_{n,l,m,m_2}^2 \varphi_{n+1,l,m+m_2} + A_{n,l,m,m_2}^3 \varphi_{n,l+2,m+m_2}, \quad \forall |m_2| \leq 2; \\ (v) \quad & \mathbf{L}(\varphi_{\tilde{n},\tilde{l},\tilde{m}}, \varphi_{n,l,m}) = 0, \quad \forall 2\tilde{n} + \tilde{l} > 2, |\tilde{m}| \leq \tilde{l}.\end{aligned}$$

where the coefficients will be precisely defined in Section 5.

The proof of this Proposition and the estimates of $A_{n,l,m,m_2}^1, A_{n,l,m,m_2}^2$ and A_{n,l,m,m_2}^3 are the main technic parts of this paper, we will give it in the Section 5.

Now we come back to the Cauchy problem (1.3), we search a solution of the form

$$g(t) = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=-l}^l g_{n,l,m}(t) \varphi_{n,l,m}, \quad g_{n,l,m}(t) = \langle g(t), \varphi_{n,l,m} \rangle \quad (2.2)$$

with initial data

$$g|_{t=0} = g_0 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=-l}^l g_{n,l,m}^0 \varphi_{n,l,m}, \quad g_{n,l,m}^0 = \langle g_0, \varphi_{n,l,m} \rangle.$$

The hypothesis $g_0 \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ is equivalent to,

$$g_{0,0,0}^0 = g_{0,1,1}^0 = g_{0,1,0}^0 = g_{0,1,-1}^0 = g_{1,0,0}^0 = 0,$$

and

$$\|g_0\|_{Q^\alpha}^2 = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=-l}^l \left(2n+l+\frac{3}{2}\right)^\alpha |g_{n,l,m}^0|^2 < \infty.$$

See Appendix 6 for the norm of Shubin space.

It follows from Proposition 2.1 that, we have the almost diagonalization of non linear Landau operators, meaning that for the function f, g define by the series

(2.2), for $n, l \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \leq l$

$$\begin{aligned}
& \left(\mathbf{L}(f, g), \varphi_{n,l,m} \right)_{L^2} \\
&= - (2(2n+l) + l(l+1)) f_{0,0,0}(t) g_{n,l,m}(t) \\
&+ \sum_{\substack{|m^*| \leq l+1, |m_1| \leq 1 \\ m_1 + m^* = m}} A_{n-1, l+1, m^*, m_1}^- f_{0,1, m_1}(t) g_{n-1, l+1, m^*}(t) \\
&+ \sum_{\substack{|m^*| \leq l-1, |m_1| \leq 1 \\ m_1 + m^* = m}} A_{n, l-1, m^*, m_1}^+ f_{0,1, m_1}(t) g_{n, l-1, m^*}(t) \\
&+ \frac{4\sqrt{3n(2n+2l+1)}}{3} f_{1,0,0}(t) g_{n-1, l, m}(t) \\
&+ \sum_{\substack{|m^*| \leq l+2, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n-2, l+2, m^*, m_2}^1 f_{0,2, m_2}(t) g_{n-2, l+2, m^*}(t) \\
&+ \sum_{\substack{|m^*| \leq l, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n-1, l, m^*, m_2}^2 f_{0,2, m_2}(t) g_{n-1, l, m^*}(t) \\
&+ \sum_{\substack{|m^*| \leq l-2, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n, l-2, m^*, m_2}^3 f_{0,2, m_2}(t) g_{n, l-2, m^*}(t),
\end{aligned} \tag{2.3}$$

with the conventions

$$g_{n,l,m} \equiv 0, \text{ if } n < 0 \text{ or } l < 0,$$

and

$$\left(\mathcal{L}(g), \varphi_{n,l,m} \right)_{L^2} = \lambda_{n,l} g_{n,l,m}(t), \quad n, l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l.$$

We remark from (2.3) that,

$$\forall f, g \in \mathcal{N}^\perp \Rightarrow \mathbf{L}(f, g) \in \mathcal{N}^\perp. \tag{2.4}$$

So that, formally, if g is a solution of the Cauchy problem (1.3), we find that the family of functions $\{g_{n,l,m}(t); n, l \in \mathbb{N}, |m| \leq l\}$, satisfy the following infinite system of the differential equations, $n, l \in \mathbb{N}, |m| \leq l$,

$$\begin{cases} \partial_t g_{n,l,m}(t) + \lambda_{n,l} g_{n,l,m}(t) = \left(\mathbf{L}(g, g), \varphi_{n,l,m} \right)_{L^2}, & t > 0; \\ g_{n,l,m}|_{t=0} = \langle g_0, \varphi_{n,l,m} \rangle = g_{n,l,m}^0 \end{cases} \tag{2.5}$$

where $\left(\mathbf{L}(g, g), \varphi_{n,l,m} \right)_{L^2}$ was precisely defined in (2.3). We have firstly,

Proposition 2.2. *Let $g_0 \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$, assume that g is a solution of the Cauchy problem (1.3) of the form (2.2), then we have*

$$g_{0,0,0}(t) = g_{0,1,0}(t) = g_{0,1,1}(t) = g_{0,1,-1}(t) = g_{1,0,0}(t) = 0, \quad \forall t \geq 0, \tag{2.6}$$

and

$$g_{0,2,m}(t) = e^{-12t} g_{0,2,m}^0, \quad t \geq 0, |m| \leq 2. \tag{2.7}$$

Proof. (1) Substituting $n = 0, l = 0, m = 0$ into the above infinite ODE system (2.5), one has

$$\partial_t g_{0,0,0}(t) + \lambda_{0,0} g_{0,0,0}(t) = 0.$$

We remind that $\lambda_{0,0} = 0$, then

$$g_{0,0,0}(t) = g_{0,0,0}^0 = 0.$$

(2) Now we set $n = 0, l = 1$, and $|m| \leq 1$, the ODE system (2.5) turn out to be

$$\begin{aligned} & \partial_t g_{0,1,m}(t) + \lambda_{0,1} g_{0,1,m}(t) \\ &= -4g_{0,0,0}(t)g_{0,1,m}(t) + A_{0,0,0,m}^+ g_{0,1,m}(t)g_{0,0,0}(t) \end{aligned}$$

By using the known results

$$\lambda_{0,1} = 0, \quad g_{0,0,0}(t) = 0,$$

one can verify that

$$g_{0,1,m}(t) = g_{0,1,m}^0 = 0, \quad \forall |m| \leq 1.$$

(3) Take now $n = 1, l = 0, m = 0$ in (2.5), we have

$$\begin{aligned} & \partial_t g_{1,0,0}(t) + \lambda_{1,0} g_{1,0,0}(t) \\ &= -4g_{0,0,0}(t)g_{1,0,0}(t) + \sum_{\substack{|m^*| \leq 1, |m_1| \leq 1 \\ m_1 + m^* = 0}} A_{0,1,m^*,m_1}^- g_{0,1,m_1}(t)g_{0,1,m^*}(t) \\ &+ 4g_{1,0,0}(t)g_{0,0,0}(t) + A_{0,0,0,0}^2 g_{0,2,0}(t)g_{0,0,0}(t). \end{aligned}$$

Then

$$\lambda_{1,0} = 0, \quad g_{0,0,0}(t) = 0, \quad g_{0,1,m'}(t) = 0, \quad \forall |m'| \leq 1,$$

imply

$$g_{1,0,0}(t) = g_{1,0,0}^0 = 0.$$

(4) Furthermore, for $n = 0, l = 2$ and $|m| \leq 2$ in (2.5), we have that

$$g_{0,0,0}(t) = 0, \quad g_{0,1,m'}(t) = 0, \quad \forall |m'| \leq 1,$$

imply

$$\partial_t g_{0,2,m}(t) + \lambda_{0,2} g_{0,2,m}(t) = 0.$$

Recalled that $\lambda_{0,2} = 12$ in (1.5), we obtain,

$$g_{0,2,m}(t) = e^{-12t} g_{0,2,m}^0, \quad \forall |m| \leq 2.$$

This ends the proof of Proposition 2.2. \square

Substituting (2.6) and (2.7) into the infinite system of the differential equations (2.5), we have, for all $2n + l > 2, |m| \leq l$,

$$\left\{ \begin{aligned} & \partial_t g_{n,l,m}(t) + \lambda_{n,l} g_{n,l,m}(t) = \\ & + \sum_{\substack{|m^*| \leq l+2, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n-2,l+2,m^*,m_2}^1 e^{-12t} g_{0,2,m_2}^0 g_{n-2,l+2,m^*}(t) \\ & + \sum_{\substack{|m^*| \leq l, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n-1,l,m^*,m_2}^2 e^{-12t} g_{0,2,m_2}^0 g_{n-1,l,m^*}(t) \\ & + \sum_{\substack{|m^*| \leq l-2, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{n,l-2,m^*,m_2}^3 e^{-12t} g_{0,2,m_2}^0 g_{n,l-2,m^*}(t), \\ & g_{n,l,m}|_{t=0} = g_{n,l,m}^0, \end{aligned} \right. \quad (2.8)$$

with the convention

$$A_{n-2,l+2,m^*,m_2}^1 = 0, \text{ if } n-2 < 0; \quad A_{n-1,l,m^*,m_2}^2 = 0, \text{ if } n-1 < 0. \quad (2.9)$$

We can solve this infinite differential equation by induction.

In fact, for $n = 0, l \geq 3, |m| \leq l$, the following system

$$\begin{cases} \partial_t g_{0,l,m}(t) + \lambda_{0,l} g_{0,l,m}(t) \\ = \sum_{\substack{|m^*| \leq l-2, |m_2| \leq 2 \\ m^* + m_2 = m}} A_{0,l-2,m^*,m_2}^3 e^{-12t} g_{0,2,m_2}^0 g_{0,l-2,m^*}(t), \\ g_{0,l,m}(0) = g_{0,l,m}^0. \end{cases}$$

can be solved by induction on l start from $l = 3$ since $g_{0,1,m^*}(t) \equiv 0$ for all $|m^*| \leq 1$. For the general case of l , the index of the right hand side are $l-2$, which have been already known by induction.

Then we solve the differential equations (2.8) for all $n \geq 1, l \geq 0$ and $|m| \leq l$. We also prove by induction on n and for fixed n induction on l . Since for the first two terms on the right hand side of (2.8), the first index are less than $n-1$, and for the last terms on the right hand side, the second index are less than $l-2$, which have been already known by induction. So that in each steps of the induction, the right hand side of (2.8) is already known by induction hypothesis. Then the differential equations are linear differential equations, and can be solved explicitly with any initial detum $g_{n,l,m}^0$. We get then the formal solution of Cauchy problem (1.3) by solve the differential system (2.8), and we have :

Theorem 2.3. *Let $\{g_{n,l,m}^0; n, l \in \mathbb{N}, |m| \leq l\}$ be a complex sequence with*

$$g_{0,0,0}^0 = g_{0,1,1}^0 = g_{0,1,0}^0 = g_{0,1,-1}^0 = g_{1,0,0}^0 = 0.$$

Then the system (2.8) admits a sequence of solutions $\{g_{n,l,m}(t); 2n+l > 2, |m| \leq l\}$.

For all $N \geq 2$, we note that

$$g_N(t) = \sum_{k=2}^N \sum_{\substack{2n+l=k \\ n+l \geq 2}} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m} \quad (2.10)$$

with

$$g_{0,2,m}(t) = e^{-12t} g_{0,2,m}^0, \quad |m| \leq 2, \quad t > 0.$$

Then g_N satisfies the following Cauchy problem

$$\begin{cases} \partial_t g_N + \mathcal{L}(g_N) = \mathbb{S}_N \mathbf{L}(g_N, g_N), \\ g_N|_{t=0} = \sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} g_{n,l,m}^0 \varphi_{n,l,m}. \end{cases} \quad (2.11)$$

The proof of the existence of weak solution of Theorem 1.1 is reduced to prove the convergence of the sequences $\{g_N; N \in \mathbb{N}\}$ in the function space $Q^\alpha(\mathbb{R}^3)$. Namely,

$$g_N \rightarrow g(t) = \sum_{k=2}^{+\infty} \sum_{\substack{2n+l=k \\ n+l \geq 2}} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m} \in Q^\alpha(\mathbb{R}^3), \quad \text{as } N \rightarrow +\infty.$$

The Gelfand-Shilov regularity is reduced to prove: there exists a constant $c_1 > 0$, such that

$$\forall t > 0, \quad \|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum e^{c_1 t(2n+l+\frac{3}{2})} (2n+l+\frac{3}{2})^\alpha |g_{n,l,m}(t)|^2 < \infty.$$

This will be the main jobs of the Section 3 and Section 4.

3. THE TRILINEAR ESTIMATES FOR NON LINEAR OPERATOR

To prove the convergence of the formal solution obtained in Theorem 2.3, we need to estimate the following trilinear terms

$$(\mathbf{L}(f, g), h)_{L^2(\mathbb{R}^3)}, \quad f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp.$$

We need firstly the following estimates for the coefficients A^1, A^2 and A^3 (see their definition (5.10) in Section 5) of the Proposition 2.1.

Proposition 3.1. *For the coefficients of the Proposition 2.1 defined in (5.10), we have the following estimates:*

1) For $n, l \in \mathbb{N}$, $n \geq 2$,

$$\max_{|m^*| \leq l} \sum_{\substack{|m| \leq l+2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n-2, l+2, m, m_2}^1|^2 \leq \frac{16n(n-1)}{3}. \quad (3.1)$$

2) For $n, l \in \mathbb{N}$, $n \geq 1$,

$$A_{n-1, 0, 0, 0}^2 = 0; \quad \max_{|m^*| \leq l} \sum_{\substack{|m| \leq l, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n-1, l, m, m_2}^2|^2 \leq \frac{4n(2n+2l+1)}{3}, \quad \forall l \geq 1. \quad (3.2)$$

3) For $n, l \in \mathbb{N}$, $l \geq 2$,

$$\max_{|m^*| \leq l} \sum_{\substack{|m| \leq l-2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n, l-2, m, m_2}^3|^2 \leq \frac{(2n+2l+1)(2n+2l-1)}{2}. \quad (3.3)$$

We will give the proof of this Proposition in the Section 5.

We now present the trilinear estimation for the nonlinear Landau operator \mathbf{L} , for $g \in \mathcal{S}'(\mathbb{R}^3) \cap \mathcal{N}^\perp$, $N > 2$, we note

$$\tilde{\mathcal{S}}_N g = \sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} g_{n,l,m} \varphi_{n,l,m}, \quad g_{n,l,m} = \langle g, \varphi_{n,l,m} \rangle. \quad (3.4)$$

Then we have the following trilinear estimates:

Proposition 3.2. *Let $f, g, h \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha \leq 0$, then for any $N \geq 2$,*

$$\begin{aligned} & |(\mathbf{L}(\tilde{\mathcal{S}}_N f, \tilde{\mathcal{S}}_N g), \mathcal{H}^\alpha \tilde{\mathcal{S}}_N h)_{L^2}| \\ & \leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|\tilde{\mathcal{S}}_2 f\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathcal{S}}_{N-2} g\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathcal{S}}_N h\|_{L^2}, \end{aligned}$$

and also for any $c > 0$, $t \geq 0$,

$$|(\mathbf{L}(\tilde{\mathcal{S}}_N f, \tilde{\mathcal{S}}_N g), e^{2ct\mathcal{H}} \mathcal{H}^\alpha \tilde{\mathcal{S}}_N h)_{L^2}|$$

$$\leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) e^{2ct} \|\tilde{\mathbb{S}}_2 f\|_{L^2} \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_{N-2} g\|_{L^2} \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_N h\|_{L^2}.$$

The proof of this Proposition is similar to Lemma 3.5 in [7], Proposition 3.2 in [8] and Section 3 in [3].

Proof. Let $f, g, h \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha \leq 0$. For $N \geq 2$, by using the orthogonal property of $\{\varphi_{n,l,m}; n, l \in \mathbb{N}, |m| \leq l\}$, we can deduce from Proposition 2.1 and (2.9) that

$$\begin{aligned} & (\mathbf{L}(\tilde{\mathbb{S}}_N f, \tilde{\mathbb{S}}_N g), \mathcal{H}^\alpha \tilde{\mathbb{S}}_N h) \\ &= \sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 2}} \sum_{\substack{|m| \leq l+2, |m_2| \leq 2 \\ |m+m_2| \leq l}} A_{n-2, l+2, m, m_2}^1 (2n+l+\frac{3}{2})^\alpha f_{0,2,m_2} g_{n-2, l+2, m} h_{n, l, m+m_2} \\ &+ \sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 1, n+l \geq 2}} \sum_{\substack{|m| \leq l, |m_2| \leq 2 \\ |m+m_2| \leq l}} A_{n-1, l, m, m_2}^2 (2n+l+\frac{3}{2})^\alpha f_{0,2,m_2} g_{n-1, l, m} h_{n, l, m+m_2} \\ &+ \sum_{\substack{2 \leq 2n+l \leq N \\ l \geq 2}} \sum_{\substack{|m| \leq l-2, |m_2| \leq 2 \\ |m+m_2| \leq l}} A_{n, l-2, m, m_2}^3 (2n+l+\frac{3}{2})^\alpha f_{0,2,m_2} g_{n, l-2, m} h_{n, l, m+m_2} \\ &\leq \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3. \end{aligned}$$

For the first term \mathbf{B}_1 , we have

$$\begin{aligned} \mathbf{B}_1 &\leq \sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 2}} (2n+l+\frac{3}{2})^\alpha \\ &\quad \times \sum_{|m_2| \leq 2} \sum_{\substack{|m| \leq l+2 \\ |m+m_2| \leq l}} |f_{0,2,m_2}| \left| A_{n-2, l+2, m, m_2}^1 g_{n-2, l+2, m} h_{n, l, m+m_2} \right|, \end{aligned}$$

by using the Cauchy-Schwarz inequality

$$\begin{aligned} & \sum_{|m_2| \leq 2} \left(|f_{0,2,m_2}| \sum_{\substack{|m| \leq l+2 \\ |m+m_2| \leq l}} \left| A_{n-2, l+2, m, m_2}^1 g_{n-2, l+2, m} h_{n, l, m+m_2} \right| \right) \\ &\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m_2| \leq 2} \left(\sum_{\substack{|m| \leq l+2 \\ |m+m_2| \leq l}} \left| A_{n-2, l+2, m, m_2}^1 g_{n-2, l+2, m} h_{n, l, m+m_2} \right|^2 \right)^{\frac{1}{2}} \right) \\ &\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m| \leq l+2} |g_{n-2, l+2, m}|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{|m_2| \leq 2} \sum_{\substack{|m| \leq l+2 \\ |m+m_2| \leq l}} \left| A_{n-2, l+2, m, m_2}^1 h_{n, l, m+m_2} \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By changing the order of summation

$$\sum_{\substack{|m_2| \leq 2, |m| \leq l+2 \\ |m+m_2| \leq l}} = \sum_{|m^*| \leq l} \sum_{\substack{|m_2| \leq 2, |m| \leq l+2 \\ m+m_2=m^*}}$$

and using (3.1) in Proposition 3.1, we have

$$\begin{aligned}
& \sum_{|m_2| \leq 2} \sum_{\substack{|m| \leq l+2 \\ |m+m_2| \leq l}} \left| A_{n-2, l+2, m, m_2}^1 h_{n, l, m+m_2} \right|^2 \\
&= \sum_{|m^*| \leq l} \left| h_{n, l, m^*} \right|^2 \left(\sum_{\substack{|m| \leq l+2, |m_2| \leq 2 \\ m+m_2=m^*}} \left| A_{n-2, l+2, m, m_2}^1 \right|^2 \right) \\
&\leq \frac{16n(n-1)}{3} \sum_{|m^*| \leq l} \left| h_{n, l, m^*} \right|^2
\end{aligned}$$

Substituting back to the estimation of \mathbf{B}_1 , one can verify that

$$\begin{aligned}
\mathbf{B}_1 &\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 2}} (2n+l + \frac{3}{2})^\alpha \sqrt{\frac{16n(n-1)}{3}} \\
&\quad \times \left(\sum_{|m| \leq l+2} |g_{n-2, l+2, m}|^2 \right)^{\frac{1}{2}} \left(\sum_{|m^*| \leq l} |h_{n, l, m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{2\sqrt{3}}{3} \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 2}} (2n+l - \frac{1}{2})^{\alpha+1} \sum_{|m| \leq l+2} |g_{n-2, l+2, m}|^2 \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 2}} (2n+l + \frac{3}{2})^{\alpha+1} \sum_{|m^*| \leq l} |h_{n, l, m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{2\sqrt{3}}{3} \|\tilde{\mathbb{S}}_2 f\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_{N-2} g\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_N h\|_{L^2},
\end{aligned}$$

where we use the estimation $(2n+l + \frac{3}{2})^\alpha (2n-2) \leq (2n+l - \frac{1}{2})^{\alpha+1}$ when $\alpha \leq 0$.

Now we turn back to estimate $\mathbf{B}_2, \mathbf{B}_3$. By using the Cauchy-Schwarz inequality

$$\begin{aligned}
& \sum_{|m_2| \leq 2} \left(|f_{0, 2, m_2}| \sum_{\substack{|m| \leq l \\ |m+m_2| \leq l}} \left| A_{n-1, l, m, m_2}^2 g_{n-1, l, m} h_{n, l, m+m_2} \right| \right) \\
&\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m| \leq l} |g_{n-1, l, m}|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{|m| \leq l, |m_2| \leq 2 \\ |m+m_2| \leq l}} \left| A_{n-1, l, m, m_2}^2 \right|^2 \left| h_{n, l, m+m_2} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m| \leq l} |g_{n-1, l, m}|^2 \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{|m^*| \leq l} \left| h_{n, l, m^*} \right|^2 \left(\sum_{\substack{|m| \leq l, |m_2| \leq 2 \\ m+m_2=m^*}} \left| A_{n-1, l, m, m_2}^2 \right|^2 \right) \right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\sum_{|m_2| \leq 2} \left(|f_{0, 2, m_2}| \sum_{\substack{|m| \leq l-2 \\ |m+m_2| \leq l}} \left| A_{n, l-2, m, m_2}^3 g_{n, l-2, m} h_{n, l, m+m_2} \right| \right)$$

$$\begin{aligned}
&\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m| \leq l-2} |g_{n,l-2,m}|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{|m| \leq l-2, |m_2| \leq 2 \\ |m+m_2| \leq l}} |A_{n,l-2,m,m_2}^3 h_{n,l,m+m_2}|^2 \right)^{\frac{1}{2}} \\
&\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \left(\sum_{|m| \leq l-2} |g_{n,l-2,m}|^2 \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{|m^*| \leq l} |h_{n,l,m^*}|^2 \left(\sum_{\substack{|m| \leq l-2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n,l-2,m,m_2}^3|^2 \right) \right)^{\frac{1}{2}}.
\end{aligned}$$

Substituting the estimations (3.2) and (3.3) in \mathbf{B}_2 , \mathbf{B}_3 , it follows that

$$\begin{aligned}
\mathbf{B}_2 &\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \sum_{\substack{2 \leq 2n+l \leq N \\ n \geq 1, l \geq 1}} (2n+l+\frac{3}{2})^\alpha \sqrt{\frac{4n(2n+2l+1)}{3}} \\
&\quad \times \left(\sum_{|m| \leq l} |g_{n-1,l,m}|^2 \right)^{\frac{1}{2}} \left(\sum_{|m^*| \leq l} |h_{n,l,m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{2\sqrt{3}}{3} \|\tilde{\mathbb{S}}_2 f\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_{N-2} g\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_N h\|_{L^2},
\end{aligned}$$

here for $n \geq 1$, we use

$$\begin{aligned}
(2n+l+\frac{3}{2})^\alpha (n+l+\frac{1}{2}) &\leq (2n+l+\frac{3}{2})^{\alpha+1}; \\
2n(2n+l+\frac{3}{2})^\alpha &\leq (2n+l-\frac{1}{2})^{\alpha+1}, \quad \text{for } l \geq 1, \alpha \leq 0.
\end{aligned}$$

And

$$\begin{aligned}
\mathbf{B}_3 &\leq \|\tilde{\mathbb{S}}_2 f\|_{L^2} \sum_{\substack{2 \leq 2n+l \leq N \\ l \geq 2}} (2n+l+\frac{3}{2})^\alpha \sqrt{\frac{(2n+2l+1)(2n+2l-1)}{2}} \\
&\quad \times \left(\sum_{|m| \leq l-2} |g_{n,l-2,m}|^2 \right)^{\frac{1}{2}} \left(\sum_{|m^*| \leq l} |h_{n,l,m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{2} \|\tilde{\mathbb{S}}_2 f\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_{N-2} g\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_N h\|_{L^2},
\end{aligned}$$

here for $l \geq 2$ and $n \in \mathbb{N}$, we use

$$\begin{aligned}
(2n+l+\frac{3}{2})^\alpha (n+l+\frac{1}{2}) &\leq (2n+l+\frac{3}{2})^{\alpha+1}; \\
(2n+l+\frac{3}{2})^\alpha (n+l-\frac{1}{2}) &\leq (2n+l-\frac{1}{2})^{\alpha+1}, \quad \text{for } \alpha \leq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|(\mathbf{L}(\tilde{\mathbb{S}}_N f, \tilde{\mathbb{S}}_N g), \mathcal{H}^\alpha \tilde{\mathbb{S}}_N h)_{L^2}| \\
&\leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|\tilde{\mathbb{S}}_2 f\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_{N-2} g\|_{L^2} \|\mathcal{H}^{\frac{\alpha+1}{2}} \tilde{\mathbb{S}}_N h\|_{L^2}.
\end{aligned}$$

This is the first result of Proposition 3.2.

For the second inequality of the Proposition 3.2, we just to use,

$$e^{ct(2n+l+\frac{3}{2})} = e^{ct(2(n-2)+(l+2)+\frac{3}{2})} e^{2ct} = e^{ct(2(n-1)+l+\frac{3}{2})} e^{2ct} = e^{ct(2n+(l-2)+\frac{3}{2})} e^{2ct}.$$

This ends the proof of Proposition 3.2. \square

4. THE CONVERGENCE OF THE FORMAL SOLUTION

In this section, we study the convergence of the solutions $\{g_N; N \in \mathbb{N}\}$ defined by (2.10) in Theorem 2.3 where the initial data is the sequence $\{g_{n,l,m}^0 = \langle g_0, \varphi_{n,l,m} \rangle; n, l \in \mathbb{N}, |m| \leq l\}$ with $g_0 \in Q^\alpha \cap \mathcal{N}^\perp$. Note that $g_N \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$, from the definition of (1.7) and (3.4), we have

$$\mathbb{S}_N g_N(t) = \tilde{\mathbb{S}}_N g_N(t) = g_N(t) \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp. \quad (4.1)$$

In particular, for $N = 2$, we have

$$\|g_2(t)\|_{L^2(\mathbb{R}^3)} = \|\tilde{\mathbb{S}}_2 g_2(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-12t} \|\tilde{\mathbb{S}}_2 g_0\|_{L^2(\mathbb{R}^3)}. \quad (4.2)$$

Moreover, we recall the result (2.4) that

$$\mathbb{S}_N \mathbf{L}(g_N(t), g_N(t)) = \tilde{\mathbb{S}}_N \mathbf{L}(g_N(t), g_N(t)).$$

Therefore, we can rewrite the Cauchy problem (2.11) as follows:

$$\begin{cases} \partial_t g_N + \mathcal{L}(g_N) = \tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N), \\ g_N|_{t=0} = \sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} \langle g_0, \varphi_{n,l,m} \rangle \varphi_{n,l,m}. \end{cases} \quad (4.3)$$

Now for $N > 2$, $c > 0$, taking the inner product of $e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t)$ in $L^2(\mathbb{R}^3)$ on both sides of (4.3), we have

$$\begin{aligned} & (\partial_t g_N(t), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t))_{L^2(\mathbb{R}^3)} + (\mathcal{L}g_N(t), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t))_{L^2(\mathbb{R}^3)} \\ &= \left(\tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t) \right)_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where g_N is defined in (2.10). Since

$$\begin{aligned} \lambda_{0,2} &= 12 > \frac{16}{11} \left(2 + \frac{3}{2} \right), \\ \lambda_{n,l} &= 2(2n+l) + l(l+1) \geq \frac{16}{11} \left(2n+l + \frac{3}{2} \right), \quad \forall 2n+l > 2. \end{aligned}$$

The orthogonality of the basis $\{\varphi_{n,l,m}\}_{\{n,l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l\}}$ imply that

$$\left(\mathcal{L}g_N(t), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t) \right)_{L^2(\mathbb{R}^3)} \geq \frac{16}{11} \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha+1}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2.$$

On the other hand

$$\begin{aligned} & 2(\partial_t g_N(t), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t))_{L^2(\mathbb{R}^3)} + 2c(g_N(t), \mathcal{H} e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t))_{L^2(\mathbb{R}^3)} \\ &= \frac{d}{dt} (g_N(t), e^{2ct\mathcal{H}} \mathcal{H}^\alpha g_N(t))_{L^2(\mathbb{R}^3)} = \frac{d}{dt} \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

Therefore, we have

$$\frac{1}{2} \frac{d}{dt} \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2 + \left(\frac{16}{11} - c \right) \|e^{ct\mathcal{H}} \mathcal{H}^{\frac{\alpha+1}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2$$

$$= \left(\mathbf{L}(\tilde{\mathbb{S}}_N g_N, \tilde{\mathbb{S}}_N g_N), e^{2ct\mathcal{H}} \mathcal{H}^\alpha \tilde{\mathbb{S}}_N g_N(t) \right)_{L^2}.$$

It follows from Proposition 3.2 and the inequality (4.2) that, for any $N > 2$, $t > 0$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e^{ct\mathcal{H}} g_N(t)\|_{Q^\alpha(\mathbb{R}^3)}^2 + \left(\frac{16}{11} - c \right) \|e^{ct\mathcal{H}} g_N(t)\|_{Q^{\alpha+1}(\mathbb{R}^3)}^2 \\ & \leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) e^{-(12-c)t} \|\tilde{\mathbb{S}}_2 g_0\|_{L^2(\mathbb{R}^3)} \|e^{ct\mathcal{H}} g_{N-2}\|_{Q^{\alpha+1}(\mathbb{R}^3)} \|e^{ct\mathcal{H}} g_N\|_{Q^{\alpha+1}(\mathbb{R}^3)} \\ & \leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) e^{-(12-c)t} \|\tilde{\mathbb{S}}_2 g_0\|_{L^2(\mathbb{R}^3)} \|e^{ct\mathcal{H}} g_N\|_{Q^{\alpha+1}(\mathbb{R}^3)}^2 \end{aligned} \quad (4.4)$$

where we used the definition of the shubin spaces $Q^{\alpha+1}(\mathbb{R}^3)$ that

$$\|e^{ct\mathcal{H}} g_N\|_{Q^{\alpha+1}(\mathbb{R}^3)}^2 = \sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} e^{2ct(2n+l+\frac{3}{2})} (2n+l+\frac{3}{2})^{\alpha+1} |g_{n,l,m}|^2.$$

Proposition 4.1. *There exists $c_0 > 0$, $c_1 > 0$ such that for all $g_0 \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha \leq 0$, and*

$$\|\tilde{\mathbb{S}}_2 g_0\|_{L^2(\mathbb{R}^3)} = \left(\sum_{|m_2| \leq 2} |\langle g_0, \varphi_{0,2,m_2} \rangle|^2 \right)^{\frac{1}{2}} \leq c_0,$$

if $\{g_{n,l,m}(t); n, l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l\}$ is the solution of (2.8) with initial datum $\{g_{n,l,m}^0 = \langle g_0, \varphi_{n,l,m} \rangle; n, l \in \mathbb{N}, |m| \leq l\}$, then, for any $N \geq 2, t > 0$,

$$\|e^{c_1 t \mathcal{H}} g_N(t)\|_{Q^\alpha(\mathbb{R}^3)}^2 + c_1 \int_0^t \|e^{c_1 \tau \mathcal{H}} g_N(\tau)\|_{Q^{\alpha+1}(\mathbb{R}^3)}^2 d\tau \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}^2. \quad (4.5)$$

We have also, for any $t \geq 0$ and any $N \geq 2$,

$$\|\tilde{\mathbb{S}}_N \mathbf{L}(g_N(t), g_N(t))\|_{Q^{\alpha-2}(\mathbb{R}^3)} \leq 2 \|g_0\|_{Q^\alpha(\mathbb{R}^3)}. \quad (4.6)$$

Remark 4.2. *It is enough to take $0 < c_1 \ll 1$ very small such that*

$$0 < c_0 = \frac{\frac{16}{11} - \frac{3}{2}c_1}{\frac{4\sqrt{3}}{3} + \sqrt{2}} < \frac{\frac{16}{11}}{\frac{4\sqrt{3}}{3} + \sqrt{2}} \approx 0.39.$$

Proof. For $N = 2$, it follows from Proposition 2.2 and $\alpha \leq 0$ that

$$\begin{aligned} & \|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g_2(t)\|_{L^2(\mathbb{R}^3)}^2 + c_1 \int_0^t \|e^{c_1 \tau \mathcal{H}} \mathcal{H}^{\frac{\alpha+1}{2}} g_2(\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau \\ & = \frac{48 - 21c_1}{48 - 14c_1} e^{7c_1 t - 24t} \sum_{|m| \leq 2} \left(\frac{7}{2} \right)^\alpha |g_{0,2,m}^0|^2 \\ & \leq \sum_{|m| \leq 2} \left(\frac{7}{2} \right)^\alpha |g_{0,2,m}^0|^2 \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}^2. \end{aligned}$$

Then for $N > 2$, we can deduce from (4.4) with $c = c_1$ and the hypothesis $\|\tilde{\mathbb{S}}_2 g_0\|_{L^2(\mathbb{R}^3)} \leq c_0$ that

$$\frac{d}{dt} \|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2 + c_1 \|e^{c_1 t \mathcal{H}} g_N(t)\|_{Q^{\alpha+1}(\mathbb{R}^3)}^2 \leq 0.$$

This ends the proof of (4.5) by integration over $[0, t]$.

Now we prove the estimate (4.6). For $h \in Q^{-\alpha+2}(\mathbb{R}^3)$ with $\alpha \leq 0$, by using the first inequality in the Proposition 3.2 with α replacing by $\alpha-1$ where $\alpha-1 \leq -1 < 0$, and the notation of (4.1)

$$\begin{aligned} & |\langle \tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N), h \rangle| \\ &= |(\mathbf{L}(\tilde{\mathbb{S}}_N g_N, \tilde{\mathbb{S}}_N g_N), \mathcal{H}^{\alpha-1} \tilde{\mathbb{S}}_N \mathcal{H}^{1-\alpha} h)_{L^2(\mathbb{R}^3)}| \\ &\leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|\tilde{\mathbb{S}}_2 g_N\|_{L^2(\mathbb{R}^3)} \|\mathcal{H}^{\frac{\alpha}{2}} \tilde{\mathbb{S}}_{N-2} g_N\|_{L^2(\mathbb{R}^3)} \|\tilde{\mathbb{S}}_N h\|_{Q^{-\alpha+2}(\mathbb{R}^3)} \\ &\leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|g_2\|_{L^2(\mathbb{R}^3)} \|\mathcal{H}^{\frac{\alpha}{2}} g_{N-2}\|_{L^2(\mathbb{R}^3)} \|h\|_{Q^{-\alpha+2}(\mathbb{R}^3)}. \end{aligned}$$

Then by the definition of the norm and (4.5), we have

$$\begin{aligned} & \|\tilde{\mathbb{S}}_N \mathbf{L}(g_N(t), g_N(t))\|_{Q^{\alpha-2}(\mathbb{R}^3)} \\ &= \sup_{\|h\|_{Q^{2-\alpha}(\mathbb{R}^3)}=1} |\langle \tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N), h \rangle| \\ &\leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|\tilde{\mathbb{S}}_2 g_0\|_{L^2} \|\mathcal{H}^{\frac{\alpha}{2}} g_{N-2}\|_{L^2(\mathbb{R}^3)} \leq \frac{16}{11} \|\mathcal{H}^{\frac{\alpha}{2}} g_0\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

which ends the proof of the Proposition 4.1. \square

In particular, we get the following surprise results

Corollary 4.3. *For any $f, g \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha \leq 0$, we have*

$$\mathbf{L}(f, g) \in Q^{\alpha-2}(\mathbb{R}^3) \cap \mathcal{N}^\perp$$

and

$$\|\mathbf{L}(f, g)\|_{Q^{\alpha-2}(\mathbb{R}^3)} \leq \left(\frac{4\sqrt{3}}{3} + \sqrt{2} \right) \|\tilde{\mathbb{S}}_2 f\|_{L^2(\mathbb{R}^3)} \|g\|_{Q^\alpha(\mathbb{R}^3)}.$$

Convergence in Shubin space. We prove now the convergence of the sequence

$$g_N(t) \rightarrow g(t) = \sum_{k=2}^{+\infty} \sum_{\substack{2n+l=k \\ n+l \geq 2}} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m}$$

where for all $N \geq 2$, g_N was defined in (2.10) with the coefficients $\{g_{n,l,m}(t)\}$ defined in (2.8), the initial datum is $\{g_{n,l,m}^0 = \langle g_0, \varphi_{n,l,m} \rangle; n, l \in \mathbb{N}, |m| \leq l\}$ with $g_0 \in Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$. By Proposition 4.1 and the orthogonality of the basis $\{\varphi_{n,l,m}\}$,

$$\sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} e^{2c_1 t(2n+l+\frac{3}{2})} (2n+l+\frac{3}{2})^\alpha |g_{n,l,m}(t)|^2 \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}^2.$$

It follows that for all $t \geq 0$,

$$\begin{aligned} \|g_N(t)\|_{Q^\alpha(\mathbb{R}^3)}^2 &= \|\mathcal{H}^{\frac{\alpha}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)}^2 \\ &= \sum_{\substack{2 \leq 2n+l \leq N \\ n+l \geq 2}} \sum_{|m| \leq l} (2n+l+\frac{3}{2})^\alpha |g_{n,l,m}(t)|^2 \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}^2. \end{aligned}$$

By using the monotone convergence theorem, we have

$$g_N \rightarrow g(t) \in Q^\alpha(\mathbb{R}^3).$$

Moreover, for any $T > 0$,

$$\lim_{N \rightarrow \infty} \|g_N - g\|_{L^\infty([0, T]; Q^\alpha(\mathbb{R}^3))} = 0.$$

On the other hand, using (4.6) and Corollary 4.3, we have also

$$\tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N) \rightarrow \mathbf{L}(g, g),$$

in $Q^{\alpha-2}(\mathbb{R}^3)$.

We recall the definition of weak solution of (1.3):

Definition 4.4. Let $g_0 \in \mathcal{S}'(\mathbb{R}^3)$, $g(t, v)$ is called a weak solution of the Cauchy problem (1.3) if it satisfies the following conditions:

$$\begin{aligned} g &\in C^0([0, +\infty[; \mathcal{S}'(\mathbb{R}^3)), \quad g(0, v) = g_0(v), \\ \mathcal{L}(g) &\in L^2([0, T[; \mathcal{S}'(\mathbb{R}^3)), \quad \mathbf{L}(g, g) \in L^2([0, T[; \mathcal{S}'(\mathbb{R}^3)), \quad \forall T > 0, \\ \langle g(t), \phi(t) \rangle - \langle g_0, \phi(0) \rangle &+ \int_0^t \langle \mathcal{L}g(\tau), \phi(\tau) \rangle d\tau \\ &= \int_0^t \langle g(\tau), \partial_\tau \phi(\tau) \rangle d\tau + \int_0^t \langle \mathbf{L}(g(\tau), g(\tau)), \phi(\tau) \rangle d\tau, \quad \forall t \geq 0, \end{aligned}$$

For any $\phi(t) \in C^1([0, +\infty[; \mathcal{S}(\mathbb{R}^3))$.

We prove now the main Theorem 1.1.

Existence of weak solution.

Let $\{g_{n,l,m}, n, l \in \mathbb{N}, n+l \geq 2, |m| \leq l\}$ be the solution of the infinite system (2.8) in Theorem 2.3 with the initial datum give in the Proposition 4.1, then for any $N \geq 2$, g_N satisfy the equation (4.3).

We have, firstly, from the Proposition 4.1, there exists positive constant $C > 0$, for any $N \geq 2$ and any $T > 0$,

$$\begin{aligned} \|g_N\|_{L^\infty([0, T]; Q^\alpha(\mathbb{R}^3))} &\leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}, \\ \|\mathcal{L}(g_N)\|_{L^2([0, T]; Q^{\alpha-3}(\mathbb{R}^3))} &\leq C \|g_0\|_{Q^\alpha(\mathbb{R}^3)}, \\ \|\tilde{\mathbb{S}}_N \mathbf{L}(g_N, g_N)\|_{L^2([0, T]; Q^{\alpha-2}(\mathbb{R}^3))} &\leq C \|g_0\|_{Q^\alpha(\mathbb{R}^3)}. \end{aligned}$$

So that the equation (4.3) implies that the sequence $\{\frac{d}{dt} \tilde{\mathbb{S}}_N g(t)\}$ is uniformly bounded in $Q^{\alpha-3}(\mathbb{R}^3)$ with respect to $N \in \mathbb{N}$ and $t \in [0, T]$. The Arzelà-Ascoli Theorem implies that

$$g_N \rightarrow g \in C^0([0, +\infty[; Q^{\alpha-2}(\mathbb{R}^3)) \subset C^0([0, +\infty[; \mathcal{S}'(\mathbb{R}^3)),$$

and

$$g(0) = g_0.$$

Secondly, for any $\phi(t) \in C^1(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^3))$, the Cauchy problem (4.3) can be rewrite as follows

$$\begin{aligned} \langle g_N(t), \phi(t) \rangle - \langle g_N(0), \phi(0) \rangle &- \int_0^t \langle g_N(\tau), \partial_\tau \phi(\tau) \rangle d\tau \\ &= - \int_0^t \langle \mathcal{L}g_N(\tau), \phi(\tau) \rangle d\tau + \int_0^t \langle \tilde{\mathbb{S}}_N \mathbf{L}(g_N(\tau), g_N(\tau)), \phi(\tau) \rangle d\tau \end{aligned}$$

Let $N \rightarrow +\infty$, we conclude that,

$$\begin{aligned} & \langle g(t), \phi(t) \rangle - \langle g_0, \phi(0) \rangle - \int_0^t \langle g(\tau), \partial_\tau \phi(\tau) \rangle d\tau \\ &= - \int_0^t \langle \mathcal{L}(g(\tau)), \phi(\tau) \rangle d\tau + \int_0^t \langle \mathbf{L}(g(\tau), g(\tau)), \phi(\tau) \rangle d\tau, \end{aligned}$$

which shows $g \in L^\infty([0, +\infty[; Q^\alpha(\mathbb{R}^3))$ is a global weak solution of Cauchy problem (1.3).

Regularity of the solution. For $N \geq 2$, we deduce from the formulas (4.5) and the orthogonality of the basis $(\varphi_{n,l,m})$ that

$$\|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g_N(t)\|_{L^2(\mathbb{R}^3)} \leq \|\mathcal{H}^{\frac{\alpha}{2}} g_0\|_{L^2(\mathbb{R}^3)}, \quad \forall N \geq 2, \quad t \geq 0,$$

by using the monotone convergence theorem, we conclude that, such that

$$\|e^{c_1 t \mathcal{H}} \mathcal{H}^{\frac{\alpha}{2}} g(t)\|_{L^2(\mathbb{R}^3)} \leq \|g_0\|_{Q^\alpha(\mathbb{R}^3)}, \quad \forall t \geq 0.$$

The proof of Theorem 1.1 is completed.

5. THE TECHNICAL COMPUTATIONS

The proof of the main technic part was presented in this section. More precisely, we prepare to prove Proposition 2.1 in Section 2 and Proposition 3.1 in Section 3.

To this ends, we need to state some Lemmas and new notations. Recall firstly

$$v_k v_j \sqrt{\mu} \in \text{span} \{ \varphi_{0,2,0}, \varphi_{0,2,\pm 1}, \varphi_{0,2,\pm 2}, \varphi_{1,0,0}, \varphi_{0,0,0} \}. \quad (5.1)$$

This relation is important in the expansion of the nonlinear operators.

Setting

$$\Psi_{n,l,m}(v) = \sqrt{\mu}(v) \varphi_{n,l,m}(v).$$

Recalled Lemma 7.2 of [3] that the Fourier transformation is ,

$$\widehat{\Psi_{n,l,m}}(\xi) = B_{n,l} |\xi|^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\left(\frac{\xi}{|\xi|}\right), \quad (5.2)$$

where

$$B_{n,l} = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2} n! \Gamma(n+l+\frac{3}{2}) 2^{2n+l}} \right)^{\frac{1}{2}}. \quad (5.3)$$

For the Laplace-Beltrami operator on the unit sphere \mathbb{S}^2 , see also Section 4.2 in [6], we have

$$\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_j^2 \partial_{v_k}^2 - v_k v_j \partial_{v_j} \partial_{v_k}) - 2 \sum_{k=1}^3 v_k \partial_{v_k} = \Delta_{\mathbb{S}^2}. \quad (5.4)$$

And for $n, l \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \leq l$,

$$\left[\sum_{k=1}^3 \partial_{v_k}^2 + v_k \partial_{v_k} \right] \Psi_{n,l,m} = -(2n+l+3) \Psi_{n,l,m}. \quad (5.5)$$

Recall that the family $(Y_l^m(\sigma))_{l \geq 0, |m| \leq l}$ is the orthonormal basis of $L^2(\mathbb{S}^2, d\sigma)$ (see (16) of Chap.1 in [5]). We have the following addition lemma,

Lemma 5.1. For any $\omega \in \mathbb{S}^2$, $l, \tilde{l} \in \mathbb{N}$, $|m| \leq l$, $|\tilde{m}| \leq \tilde{l}$,

$$Y_l^m(\omega)Y_{\tilde{l}}^{\tilde{m}}(\omega) = \sum_{0 \leq p \leq \min(l, \tilde{l})} \left(\int_{\mathbb{S}^2} Y_l^m(\omega)Y_{\tilde{l}}^{\tilde{m}}(\omega)Y_{l+\tilde{l}-2p}^{-m-\tilde{m}}(\omega)d\omega \right) Y_{l+\tilde{l}-2p}^{m+\tilde{m}}(\omega)$$

where we always define $Y_{l+\tilde{l}-2p}^{-m-\tilde{m}}(\omega) \equiv 0$, if $|m + \tilde{m}| > l + \tilde{l} - 2p$.

Proof. For the proof of Lemma 5.1, we refer to (86) in Chap.3 of [5] or Gaunt's formula from (13-12) of Chap.13 in [16]. \square

In particular, for $l = 1$ or $l = 2$ in Lemma 5.1, we have

Corollary 5.2. For all $\omega \in \mathbb{S}^2$, $l \in \mathbb{N}$, $|m| \leq l$, $|m_1| \leq 1$, $|m_2| \leq 2$,

$$\begin{aligned} Y_1^{m_1}(\omega)Y_l^m(\omega) &= \sum_{0 \leq p \leq \min(1, l)} \tilde{C}_{l, l+1-2p}^{m_1, m} Y_{l+1-2p}^{m_1+m}(\omega); \\ Y_2^{m_2}(\omega)Y_l^m(\omega) &= \sum_{0 \leq p \leq \min(2, l)} C_{l, l+2-2p}^{m_2, m} Y_{l+2-2p}^{m_2+m}(\omega) \end{aligned}$$

where

$$\begin{aligned} \tilde{C}_{l, l+1-2p}^{m_1, m} &= \int_{\mathbb{S}^2} Y_1^{m_1}(\omega)Y_l^m(\omega)Y_{l+1-2p}^{-m_1-m}(\omega)d\omega, \\ C_{l, l+2-2p}^{m_2, m} &= \int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l+2-2p}^{-m_2-m}(\omega)d\omega. \end{aligned}$$

More explicitly, for any $l \in \mathbb{N}$, $|m| \leq l$

$$Y_1^{m_1}(\omega)Y_l^m(\omega) = \tilde{C}_{l, l+1}^{m_1, m} Y_{l+1}^{m_1+m}(\omega) + \tilde{C}_{l, l-1}^{m_1, m} Y_{l-1}^{m_1+m}(\omega),$$

and

$$\begin{aligned} Y_2^{m_2}(\omega)Y_l^m(\omega) &= C_{l, l+2}^{m_2, m} Y_{l+2}^{m_2+m}(\omega) + C_{l, l}^{m_2, m} Y_l^{m_2+m}(\omega) + C_{l, l-2}^{m_2, m} Y_{l-2}^{m_2+m}(\omega) \end{aligned}$$

where, for convenience, we note

$$Y_{-1}^{m_1+m}(\omega) = 0, \quad Y_{-2}^{m_2+m}(\omega) = 0.$$

Lemma 5.3. Let $\Psi_{n, l, m} = \sqrt{\mu} \varphi_{n, l, m}$, then for $v \in \mathbb{R}^3$, we have

1) For $|m_1| \leq 1$,

$$\int_{\mathbb{R}_{v_*}^3} (v \cdot v_*) \Psi_{0, 1, m_1}(v_*) dv_* = \sqrt{\frac{4\pi}{3}} |v| Y_1^{m_1}(\sigma). \quad (5.6)$$

2) For $|m_2| \leq 2$,

$$\int_{\mathbb{R}_{v_*}^3} (v \cdot v_*)^2 \Psi_{0, 2, m_2}(v_*) dv_* = \sqrt{\frac{16\pi}{15}} |v|^2 Y_2^{m_2}(\sigma). \quad (5.7)$$

3) and

$$\int_{\mathbb{R}_{v_*}^3} (v \cdot v_*)^2 \Psi_{1, 0, 0}(v_*) dv_* = -\frac{\sqrt{6}}{3} |v|^2. \quad (5.8)$$

Proof. Set $\sigma_* = \frac{v_*}{|v_*|}, \sigma = \frac{v}{|v|} \in \mathbb{S}^2$, then

$$\begin{aligned} \int_{\mathbb{R}_{v_*}^3} (v \cdot v_*) \Psi_{0,1,m_1}(v_*) dv_* &= \int_{\mathbb{R}_{v_*}^3} |v| |v_*| (\sigma \cdot \sigma_*) \Psi_{0,1,m_1}(v_*) dv_* \\ \int_{\mathbb{R}_{v_*}^3} (v \cdot v_*)^2 \Psi_{0,2,m_2}(v_*) dv_* &= \int_{\mathbb{R}_{v_*}^3} |v|^2 |v_*|^2 (\sigma \cdot \sigma_*)^2 \Psi_{0,2,m_2}(v_*) dv_* \\ \int_{\mathbb{R}_{v_*}^3} (v \cdot v_*)^2 \Psi_{1,0,0}(v_*) dv_* &= \int_{\mathbb{R}_{v_*}^3} |v|^2 |v_*|^2 (\sigma \cdot \sigma_*)^2 \Psi_{1,0,0}(v_*) dv_* \end{aligned}$$

By using the formulas (5₃) in Sec.1, Chap. III of [14] that

$$P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

We apply the addition theorem of spherical harmonics (7–34) in Chapter 7 of [16] (see also (VII) in Sec.19, Chapter III of [14]) that,

$$P_k(\sigma_* \cdot \sigma) = \frac{4\pi}{2k+1} \sum_{|m_k| \leq k} Y_k^{-m_k}(\sigma_*) Y_k^{m_k}(\sigma).$$

Then

$$\begin{aligned} \sigma_* \cdot \sigma &= P_1(\sigma_* \cdot \sigma) = \frac{4\pi}{3} \sum_{|\tilde{m}_1| \leq 1} Y_1^{-\tilde{m}_1}(\sigma_*) Y_1^{\tilde{m}_1}(\sigma); \\ (\sigma_* \cdot \sigma)^2 &= \frac{2}{3} P_2(\sigma_* \cdot \sigma) + \frac{1}{3} = \frac{8\pi}{15} \sum_{|\tilde{m}_2| \leq 2} Y_2^{-\tilde{m}_2}(\sigma_*) Y_2^{\tilde{m}_2}(\sigma) + \frac{1}{3}. \end{aligned}$$

Substituting this expansion into the integral and using the orthogonal of the eigenfunctions $\varphi_{n,l,m}$ in $L^2(\mathbb{R}^3)$, one has

$$\begin{aligned} \int_{\mathbb{R}_{v_*}^3} |v| |v_*| (\sigma \cdot \sigma_*) \Psi_{0,1,m_1}(v_*) dv_* &= \sqrt{\frac{4\pi}{3}} |v| Y_1^{m_1}(\sigma) \\ \int_{\mathbb{R}_{v_*}^3} |v|^2 |v_*|^2 (\sigma \cdot \sigma_*)^2 \Psi_{0,2,m_2}(v_*) dv_* &= \sqrt{\frac{16\pi}{15}} |v|^2 Y_2^{m_2}(\sigma) \\ \int_{\mathbb{R}_{v_*}^3} (v \cdot v_*)^2 \Psi_{1,0,0}(v_*) dv_* &= \frac{1}{3} \left(\int_{\mathbb{R}_{v_*}^3} |v_*|^2 \Psi_{1,0,0}(v_*) dv_* \right) |v|^2 = -\frac{\sqrt{6}}{3} |v|^2, \end{aligned}$$

where we used the explicit formula of $\varphi_{0,1,m_1}$, $\varphi_{1,0,0}$ and $\varphi_{0,2,m_2}$ in (2.1) in section 2. This ends the proof of Lemma 5.3. \square

We prove now the 5 parts of the Proposition of 2.1 by the following 5 Lemmas

Lemma 5.4. For $n, l \in \mathbb{N}$, $|m| \leq l$, we have

$$\mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = -(2(2n+l) + l(l+1)) \varphi_{n,l,m}.$$

Proof. Since $\varphi_{0,0,0} = \sqrt{\mu}$, then for all $n, l \in \mathbb{N}$, $|m| \leq l$, one has

$$\begin{aligned} \mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) &= \frac{1}{\sqrt{\mu(v)}} Q(\mu, \Psi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{1 \leq k, j \leq 3} \partial_{v_k} \int_{\mathbb{R}^3} a_{k,j}(v - v_*) \left[\mu(v_*) \partial_{v_j} \Psi_{n,l,m}(v) - \partial_{v_j^*} \mu(v_*) \Psi_{n,l,m}(v) \right] dv_* \end{aligned}$$

where

$$a_{k,k}(v - v_*) = \sum_{\substack{1 \leq j \leq 3 \\ j \neq k}} (v_j - v_j^*)^2;$$

$$a_{k,j}(v - v_*) = -(v_k - v_k^*)(v_j - v_j^*) \text{ when } k \neq j.$$

It follows that

$$\mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = \mathbf{L}_1(\varphi_{0,0,0}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,0,0}, \varphi_{n,l,m})$$

where

$$\begin{aligned} & \mathbf{L}_1(\varphi_{0,0,0}, \varphi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \partial_{v_k} \left(\int_{\mathbb{R}^3} (v_j^2 - 2v_j v_j^* + (v_j^*)^2) \mu(v_*) dv_* \partial_{v_k} \Psi_{n,l,m}(v) \right) \\ & \quad - \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \partial_{v_k} \left(\int_{\mathbb{R}^3} (v_j^2 - 2v_j v_j^* + (v_j^*)^2) \partial_{v_k^*} \mu(v_*) dv_* \Psi_{n,l,m}(v) \right); \\ & \mathbf{L}_2(\varphi_{0,0,0}, \varphi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \partial_{v_k} \left(\int_{\mathbb{R}^3} (v_k v_j - v_k^* v_j - v_k v_j^* + v_k^* v_j^*) \mu(v_*) dv_* \partial_{v_j} \Psi_{n,l,m}(v) \right) \\ & \quad - \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \partial_{v_k} \left(\int_{\mathbb{R}^3} (v_k v_j - v_k^* v_j - v_k v_j^* + v_k^* v_j^*) \partial_{v_j^*} \mu(v_*) dv_* \Psi_{n,l,m}(v) \right). \end{aligned}$$

Then by using (5.2), we have,

$$\begin{aligned} & \int_{\mathbb{R}^3} \partial_{v_k^*} \mu(v_*) dv_* = i \xi_k \widehat{\mu}(\xi) \Big|_{\xi=0} = i \xi_k \widehat{\Psi_{0,0,0}}(\xi) \Big|_{\xi=0} = 0, \quad k = 1, 2, 3; \\ & \int_{\mathbb{R}^3} v_j^* \partial_{v_k^*} \mu(v_*) dv_* = -\xi_k \partial_{\xi_j^*} \widehat{\Psi_{0,0,0}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\ & \int_{\mathbb{R}^3} (v_j^*)^2 \partial_{v_k^*} \mu(v_*) dv_* = -i \xi_k \partial_{\xi_j^*}^2 \widehat{\Psi_{0,0,0}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\ & \int_{\mathbb{R}^3} v_j^* \partial_{v_j^*} \mu(v_*) dv_* = -\widehat{\Psi_{0,0,0}}(\xi) - \xi_j \partial_{\xi_j^*} \widehat{\Psi_{0,0,0}}(\xi) \Big|_{\xi=0} = -1; \\ & \int_{\mathbb{R}^3} v_k^* v_j^* \partial_{v_j^*} \mu(v_*) dv_* = - \int_{\mathbb{R}^3} v_k^* \mu(v_*) dv_* = 0. \end{aligned}$$

It is obviously that

$$\begin{aligned} & \int_{\mathbb{R}^3} \mu(v_*) dv_* = (\varphi_{0,0,0}, \varphi_{0,0,0})_{L^2(\mathbb{R}^3)} = 1, \\ & \int_{\mathbb{R}^3} v_k^* \mu(v_*) dv_* = 0, \\ & \int_{\mathbb{R}^3} v_k^* v_j^* \mu(v_*) dv_* = 0, \quad \text{when } k \neq j. \end{aligned}$$

Then we can deduce that

$$\mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = \mathbf{L}_1(\varphi_{0,0,0}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,0,0}, \varphi_{n,l,m})$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\mu(v)}} \left[\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_j^2 \partial_{v_k}^2 - v_k v_j \partial_{v_j} \partial_{v_k}) + 6 \right] \Psi_{n,l,m}(v) \\
&\quad + \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left(\int_{\mathbb{R}^3} (v_j^*)^2 \mu(v_*) dv_* \right) \partial_{v_k}^2 \Psi_{n,l,m}(v) \\
&= \frac{1}{\sqrt{\mu(v)}} \left[\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_j^2 \partial_{v_k}^2 - v_k v_j \partial_{v_j} \partial_{v_k}) + 6 \right] \Psi_{n,l,m}(v) \\
&\quad + \frac{1}{\sqrt{\mu(v)}} \left(\frac{2}{3} \int_{\mathbb{R}^3} |v_*|^2 \mu(v_*) dv_* \right) \sum_{k=1}^3 \partial_{v_k}^2 \Psi_{n,l,m}(v) \\
&= \frac{1}{\sqrt{\mu(v)}} \left[\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_j^2 \partial_{v_k}^2 - v_k v_j \partial_{v_j} \partial_{v_k}) + 6 + 2\Delta_v \right] \Psi_{n,l,m}(v)
\end{aligned}$$

By using (5.4) and (5.5) that,

$$\begin{aligned}
&\left[\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_j^2 \partial_{v_k}^2 - v_k v_j \partial_{v_j} \partial_{v_k}) + 6 + 2\Delta_v \right] \Psi_{n,l,m}(v) \\
&= \left[2 \sum_{k=1}^3 v_k \partial_{v_k} + 6 + 2\Delta_v \right] \Psi_{n,l,m}(v) + \Delta_{\mathbb{S}^2} \Psi_{n,l,m}(v) \\
&= -2(2n+l) \Psi_{n,l,m}(v) - l(l+1) \Psi_{n,l,m}(v)
\end{aligned}$$

where we used the Laplace-Beltrami operator property

$$\Delta_{\mathbb{S}^2} \Psi_{n,l,m} = -l(l+1) \Psi_{n,l,m}.$$

Therefore, we conclude with the notation $\Psi_{n,l,m} = \sqrt{\mu} \varphi_{n,l,m}$ that

$$\mathbf{L}(\varphi_{0,0,0}, \varphi_{n,l,m}) = -(2(2n+l) + l(l+1)) \varphi_{n,l,m}.$$

We end the proof of Lemma 5.4 . \square

Lemma 5.5. For $n, l \in \mathbb{N}$, $|m| \leq l$, we have

$$\begin{aligned}
&\mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) \\
&= A_{n,l,m,m_1}^- \varphi_{n+1,l-1,m_1+m} + A_{n,l,m,m_1}^+ \varphi_{n,l+1,m_1+m}, \quad \forall |m_1| \leq 1
\end{aligned}$$

with

$$\begin{aligned}
A_{n,l,m,m_1}^- &= 4\sqrt{\frac{\pi}{3}}(l-1)\sqrt{2(n+1)}\tilde{C}_{l,l-1}^{m_1,m}; \\
A_{n,l,m,m_1}^+ &= 4\sqrt{\frac{\pi}{3}}(l+2)\sqrt{2n+2l+3}\tilde{C}_{l,l+1}^{m_1,m}
\end{aligned}$$

where $\tilde{C}_{l,l-1}^{m_1,m}, \tilde{C}_{l,l+1}^{m_1,m}$ were defined in the Corollary 5.2.

Proof. For $|m_1| \leq 1$, for all $n, l \in \mathbb{N}$, $|m| \leq l$, one has

$$\begin{aligned} \mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) &= \frac{1}{\sqrt{\mu(v)}} Q(\Psi_{0,1,m_1}, \Psi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{1 \leq k, j \leq 3} \partial_{v_k} \int_{\mathbb{R}^3} a_{k,j}(v - v_*) \\ &\quad \times \left[\Psi_{0,1,m_1}(v_*) \partial_{v_j} \Psi_{n,l,m}(v) - \partial_{v_j^*} \Psi_{0,1,m_1}(v_*) \Psi_{n,l,m}(v) \right] dv_*. \\ &= \mathbf{L}_1(\varphi_{0,1,m_1}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,1,m_1}, \varphi_{n,l,m}). \end{aligned}$$

Recall,

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_{v_k^*} \Psi_{0,1,m_1}(v_*) dv_* &= i \xi_k \widehat{\Psi_{0,1,m_1}}(\xi) \Big|_{\xi=0} = 0, \quad k = 1, 2, 3; \\ \int_{\mathbb{R}^3} v_j^* \partial_{v_k^*} \Psi_{0,1,m_1}(v_*) dv_* &= -\xi_k \partial_{\xi_j^*} \widehat{\Psi_{0,1,m_1}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\ \int_{\mathbb{R}^3} (v_j^*)^2 \partial_{v_k^*} \Psi_{0,1,m_1}(v_*) dv_* &= -i \xi_k \partial_{\xi_j^*}^2 \widehat{\Psi_{0,1,m_1}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\ \int_{\mathbb{R}^3} v_j^* \partial_{v_j^*} \Psi_{0,1,m_1}(v_*) dv_* &= -\widehat{\Psi_{0,1,m_1}}(\xi) - \xi_j \partial_{\xi_j^*} \widehat{\Psi_{0,1,m_1}}(\xi) \Big|_{\xi=0} = 0; \\ \int_{\mathbb{R}^3} v_k^* v_j^* \partial_{v_j^*} \Psi_{0,1,m_1}(v_*) dv_* &= -\int_{\mathbb{R}^3} v_k^* \Psi_{0,1,m_1}(v_*) dv_*. \end{aligned}$$

By using the relation (5.1), one can verify that

$$\begin{aligned} \int_{\mathbb{R}^3} \Psi_{0,1,m_1}(v_*) dv_* &= (\varphi_{0,1,m_1}, \varphi_{0,0,0})_{L^2(\mathbb{R}^3)} = 0, \\ \int_{\mathbb{R}^3} v_k^* v_j^* \Psi_{0,1,m_1}(v_*) dv_* &= 0, \quad \forall 1 \leq k, j \leq 3. \end{aligned}$$

We can conclude that

$$\begin{aligned} \mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) &= \mathbf{L}_1(\varphi_{0,1,m_1}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,1,m_1}, \varphi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left[-2 \left(\int_{\mathbb{R}^3} v_j^* \Psi_{0,1,m_1}(v_*) dv_* \right) v_j \partial_{v_k}^2 \Psi_{n,l,m}(v) \right. \\ &\quad + \left(\int_{\mathbb{R}^3} v_k^* \Psi_{0,1,m_1}(v_*) dv_* \right) v_j \partial_{v_k} \partial_{v_j} \Psi_{n,l,m}(v) \\ &\quad \left. + \left(\int_{\mathbb{R}^3} v_j^* \Psi_{0,1,m_1}(v_*) dv_* \right) v_k \partial_{v_k} \partial_{v_j} \Psi_{n,l,m}(v) \right]. \end{aligned}$$

By Fourier transformation, we have

$$\begin{aligned} &\mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m})](\xi) \\ &= 2i \sum_{j=1}^3 \left(\int_{\mathbb{R}^3} v_j^* \Psi_{0,1,m_1}(v_*) dv_* \right) \partial_{\xi_j} \left(|\xi|^2 \widehat{\Psi_{n,l,m}}(\xi) \right) \\ &\quad - 2i \left(\int_{\mathbb{R}^3} (v^* \cdot \xi) \Psi_{0,1,m_1}(v_*) dv_* \right) (|\xi| |\partial_{|\xi|} + 4) \widehat{\Psi_{n,l,m}}(\xi) \\ &= \widehat{\mathbf{H}}_1(\xi) - \widehat{\mathbf{H}}_2(\xi) \end{aligned}$$

where $v^* \cdot \xi = \sum_{k=1}^3 v_k^* \xi_k$. We simplify the calculation into the following equality

$$\mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) = \frac{1}{\sqrt{\mu}} [\mathbf{H}_1(v) - \mathbf{H}_2(v)]. \quad (5.9)$$

By using the explicit formula of the Fourier transform of $\Psi_{n,l,m}$ that,

$$\widehat{\Psi_{n,l,m}} = B_{n,l} |\xi|^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\left(\frac{\xi}{|\xi|}\right)$$

with $B_{n,l}$ defined in (5.3), we have

$$|\xi|^2 \widehat{\Psi_{n,l,m}}(\xi) = B_{n,l} |\xi|^{2(n+1)+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\left(\frac{\xi}{|\xi|}\right) = \frac{B_{n,l}}{B_{n+1,l}} \widehat{\Psi_{n+1,l,m}}.$$

It follows from the inverse Fourier transformation of $\widehat{\mathbf{H}}_1(\xi)$ and the equality (5.6) in Lemma 5.3,

$$\begin{aligned} \mathbf{H}_1(v) &= 2 \frac{B_{n,l}}{B_{n+1,l}} \sum_{j=1}^3 \left(\int_{\mathbb{R}^3} v_j^* \Psi_{0,1,m_1}(v_*) dv_* \right) v_j \Psi_{n+1,l,m}(v) \\ &= 2 \frac{B_{n,l}}{B_{n+1,l}} \left(\int_{\mathbb{R}^3} (v_* \cdot v) \Psi_{0,1,m_1}(v_*) dv_* \right) \Psi_{n+1,l,m}(v) \\ &= 2 \sqrt{\frac{4\pi}{3}} \frac{B_{n,l}}{B_{n+1,l}} |v| Y_1^{m_1} \left(\frac{v}{|v|} \right) \Psi_{n+1,l,m}(v) \\ &= 2 \sqrt{\frac{4\pi}{3}} \frac{(n+1)! B_{n,l}}{(-i)^l 2^{(n+1)}} |v|^{l+1} e^{-\frac{|v|^2}{2}} L_{n+1}^{(l+1/2)} \left(\frac{|v|^2}{2} \right) \\ &\quad \times \left(\tilde{C}_{l,l+1}^{m_1,m} Y_{l+1}^{m_1+m}(\sigma) + \tilde{C}_{l,l-1}^{m_1,m} Y_{l-1}^{m_1+m}(\sigma) \right) \end{aligned}$$

Using the formulas (10'),(11),(12) of Sec.1, Chap.IV in [14] for $x = \frac{|v|^2}{2}$, we have

$$L_{n+1}^{(l+\frac{1}{2})}(x) = L_{n+1}^{(l+\frac{3}{2})}(x) - L_n^{(l+\frac{3}{2})}(x), \quad n = 0, 1, \dots$$

and

$$x L_{n+1}^{(l+\frac{1}{2})}(x) = (n+l+\frac{3}{2}) L_{n+1}^{(l-\frac{1}{2})}(x) - (n+2) L_{n+2}^{(l-\frac{1}{2})}(x).$$

Direct calculation shows that

$$\begin{aligned} \mathbf{H}_1(v) &= 4 \sqrt{\frac{\pi}{3}} \tilde{C}_{l,l+1}^{m_1,m} \sqrt{2(n+1)(2n+2l+3)(2n+2l+5)} \Psi_{n+1,l+1,m_1+m}(v) \\ &\quad - 4 \sqrt{\frac{\pi}{3}} \tilde{C}_{l,l+1}^{m_1,m} 2(n+1) \sqrt{2n+2l+3} \Psi_{n,l+1,m_1+m}(v) \\ &\quad + 4 \sqrt{\frac{\pi}{3}} \tilde{C}_{l,l-1}^{m_1,m} (2n+2l+3) \sqrt{2(n+1)} \Psi_{n+1,l-1,m_1+m}(v) \\ &\quad - 4 \sqrt{\frac{\pi}{3}} \tilde{C}_{l,l-1}^{m_1,m} \sqrt{4(n+1)(n+2)(2n+2l+3)} \Psi_{n+2,l-1,m_1+m}(v). \end{aligned}$$

Now we calculate \mathbf{H}_2 . We can deduce from the equality (5.6) in Lemma 5.3 with $v = \xi$ that

$$\begin{aligned} \widehat{\mathbf{H}}_2(\xi) &= 2i \sqrt{\frac{4\pi}{3}} |\xi| Y_1^{m_1} \left(\frac{\xi}{|\xi|} \right) \\ &\quad \times [(2n+l+4) \widehat{\Psi_{n,l,m}} - \sqrt{2(n+1)(2n+2l+3)} \widehat{\Psi_{n+1,l,m}}] \end{aligned}$$

By using the Corollary 5.2 and the explicit formula of $\widehat{\Psi}_{n,l,m}$, one can verify that

$$\begin{aligned}
\widehat{\mathbf{H}}_2(\xi) &= 2i\sqrt{\frac{4\pi}{3}}(2n+l+4)B_{n,l}|\xi|^{2n+l+1}e^{-\frac{|\xi|^2}{2}} \\
&\quad \times \left(\tilde{C}_{l,l+1}^{m_1,m} Y_{l+1}^{m_1+m} + \tilde{C}_{l,l-1}^{m_1,m} Y_{l-1}^{m_1+m} \right) \\
&\quad - 2i\sqrt{\frac{4\pi}{3}}\sqrt{2(n+1)(2n+2l+3)}B_{n+1,l}|\xi|^{2n+l+3}e^{-\frac{|\xi|^2}{2}} \\
&\quad \times \left(\tilde{C}_{l,l+1}^{m_1,m} Y_{l+1}^{m_1+m} + \tilde{C}_{l,l-1}^{m_1,m} Y_{l-1}^{m_1+m} \right) \\
&= -4\sqrt{\frac{\pi}{3}}\tilde{C}_{l,l+1}^{m_1,m}(2n+l+4)\sqrt{2n+2l+3}\mathcal{F}(\Psi_{n,l+1,m_1+m}) \\
&\quad + 4\sqrt{\frac{\pi}{3}}\tilde{C}_{l,l-1}^{m_1,m}(2n+l+4)\sqrt{2(n+1)}\mathcal{F}(\Psi_{n+1,l-1,m_1+m}) \\
&\quad + 4\sqrt{\frac{\pi}{3}}\tilde{C}_{l,l+1}^{m_1,m}\sqrt{2(n+1)(2n+2l+3)}\sqrt{2n+2l+5}\mathcal{F}(\Psi_{n+1,l+1,m_1+m}) \\
&\quad - 4\sqrt{\frac{\pi}{3}}\tilde{C}_{l,l-1}^{m_1,m}\sqrt{4(n+1)(n+2)(2n+2l+3)}\mathcal{F}(\Psi_{n+2,l-1,m_1+m}).
\end{aligned}$$

Then by the inverse Fourier transform of $\widehat{\mathbf{H}}_2(\xi)$ and substituting the equalities of $\mathbf{H}_1, \mathbf{H}_2$ into (5.9), we conclude that

$$\begin{aligned}
\mathbf{L}(\varphi_{0,1,m_1}, \varphi_{n,l,m}) &= \frac{1}{\sqrt{\mu}} [\mathbf{H}_1(v) - \mathbf{H}_2(v)] \\
&= A_{n,l,m,m_1}^- \varphi_{n+1,l-1,m_1+m} + A_{n,l,m,m_1}^+ \varphi_{n,l+1,m_1+m}.
\end{aligned}$$

□

Lemma 5.6. For $n, l \in \mathbb{N}$, $|m| \leq l$, we have

$$\mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m}) = \frac{4\sqrt{3(n+1)(2n+2l+3)}}{3} \varphi_{n+1,l,m}.$$

Proof. Firstly

$$\mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m}) = \mathbf{L}_1(\varphi_{1,0,0}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{1,0,0}, \varphi_{n,l,m}).$$

Using

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_{v_k^*} \Psi_{1,0,0}(v_*) dv_* &= i\xi_k \widehat{\Psi}_{1,0,0}(\xi) \Big|_{\xi=0} = 0, \quad k = 1, 2, 3; \\
\int_{\mathbb{R}^3} v_j^* \partial_{v_k^*} \Psi_{1,0,0}(v_*) dv_* &= -\xi_k \partial_{\xi_j^*} \widehat{\Psi}_{1,0,0}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\
\int_{\mathbb{R}^3} (v_j^*)^2 \partial_{v_k^*} \Psi_{1,0,0}(v_*) dv_* &= -i\xi_k \partial_{\xi_j^*}^2 \widehat{\Psi}_{1,0,0}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\
\int_{\mathbb{R}^3} v_j^* \partial_{v_j^*} \Psi_{1,0,0}(v_*) dv_* &= -\widehat{\Psi}_{1,0,0}(0) - \xi_j \partial_{\xi_j^*} \widehat{\Psi}_{1,0,0}(\xi) \Big|_{\xi=0} = 0; \\
\int_{\mathbb{R}^3} v_k^* v_j^* \partial_{v_j^*} \Psi_{1,0,0}(v_*) dv_* &= -\int_{\mathbb{R}^3} v_k^* \Psi_{1,0,0}(v_*) dv_*,
\end{aligned}$$

and

$$\int_{\mathbb{R}^3} \Psi_{1,0,0}(v_*) dv_* = (\varphi_{1,0,0}, \varphi_{0,0,0})_{L^2(\mathbb{R}^3)} = 0,$$

$$\int_{\mathbb{R}^3} v_k^* \Psi_{1,0,0}(v_*) dv_* = 0, \quad \forall 1 \leq k \leq 3.$$

We can conclude that

$$\begin{aligned} \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m}) &= \mathbf{L}_1(\varphi_{1,0,0}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{1,0,0}, \varphi_{n,l,m}) \\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left(\int_{\mathbb{R}^3} |v_j^*|^2 \Psi_{1,0,0}(v_*) dv_* \right) \partial_{v_k}^2 \Psi_{n,l,m}(v) \\ &\quad + \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left(\int_{\mathbb{R}^3} v_k^* v_j^* \Psi_{1,0,0}(v_*) dv_* \right) \partial_{v_k} \partial_{v_j} \Psi_{n,l,m}(v). \end{aligned}$$

Again by the Fourier transformation of $\sqrt{\mu} \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m})$, we have

$$\begin{aligned} &\mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m})](\xi) \\ &= \int_{\mathbb{R}_{v_*}^3} (\xi \cdot v_*)^2 \Psi_{1,0,0}(v_*) dv_* \widehat{\Psi_{n,l,m}}(\xi) - (|v_*|^2 \sqrt{\mu}, \varphi_{1,0,0})_{L^2(\mathbb{R}^3)} |\xi|^2 \widehat{\Psi_{n,l,m}}(\xi). \end{aligned}$$

We deduce from the equality (5.8) in Lemma 5.3 that

$$\int_{\mathbb{R}_{v_*}^3} (\xi \cdot v_*)^2 \Psi_{1,0,0}(v_*) dv_* = -\frac{\sqrt{6}}{3} |\xi|^2.$$

From the definition of eigenfunctions $\varphi_{1,0,0}, \varphi_{0,0,0}$, one can calculate that

$$|v|^2 \sqrt{\mu} = 3\varphi_{0,0,0} - \sqrt{6}\varphi_{1,0,0}$$

Then

$$-(|v_*|^2 \sqrt{\mu}, \varphi_{1,0,0})_{L^2(\mathbb{R}^3)} = \sqrt{6}.$$

This implies that

$$\begin{aligned} \mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{1,0,0}, \varphi_{n,l,m})](\xi) &= \frac{2\sqrt{6}}{3} |\xi|^2 \widehat{\Psi_{n,l,m}}(\xi) \\ &= \frac{4\sqrt{3(n+1)(2n+2l+3)}}{3} \widehat{\Psi_{n+1,l,m}}(\xi). \end{aligned}$$

We end the proof of the Lemma by inverse Fourier transformation. \square

Lemma 5.7. For $n, l \in \mathbb{N}$, $|m| \leq l$, we have

$$\begin{aligned} \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) &= A_{n,l,m,m_2}^1 \varphi_{n+2,l-2,m+m_2} \\ &\quad + A_{n,l,m,m_2}^2 \varphi_{n+1,l,m+m_2} + A_{n,l,m,m_2}^3 \varphi_{n,l+2,m+m_2}, \quad \forall |m_2| \leq 2 \end{aligned}$$

with

$$\begin{aligned}
A_{n,l,m,m_2}^1 &= -4\sqrt{\frac{\pi}{15}}\sqrt{4(n+2)(n+1)}\int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l-2}^{-m_2-m}(\omega)d\omega; \\
A_{n,l,m,m_2}^2 &= 4\sqrt{\frac{\pi}{15}}\sqrt{2(n+1)(2n+2l+3)} \\
&\quad \times \int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_l^{-m_2-m}(\omega)d\omega; \\
A_{n,l,m,m_2}^3 &= -4\sqrt{\frac{\pi}{15}}\sqrt{(2n+2l+5)(2n+2l+3)} \\
&\quad \times \int_{\mathbb{S}^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l+2}^{-m_2-m}(\omega)d\omega.
\end{aligned} \tag{5.10}$$

Proof. For $|m_2| \leq 2$, for all $n, l \in \mathbb{N}$, $|m| \leq l$, one has

$$\mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) = \mathbf{L}_1(\varphi_{0,2,m_2}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,2,m_2}, \varphi_{n,l,m}).$$

Using now

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_{v_k^*} \Psi_{0,2,m_2}(v_*) dv_* &= i\xi_k \widehat{\Psi_{0,2,m_2}}(\xi) \Big|_{\xi=0} = 0, \quad k = 1, 2, 3; \\
\int_{\mathbb{R}^3} v_j^* \partial_{v_k^*} \Psi_{0,2,m_2}(v_*) dv_* &= -\xi_k \partial_{\xi_j^*} \widehat{\Psi_{0,2,m_2}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\
\int_{\mathbb{R}^3} (v_j^*)^2 \partial_{v_k^*} \Psi_{0,2,m_2}(v_*) dv_* &= -i\xi_k \partial_{\xi_j^*}^2 \widehat{\Psi_{0,2,m_2}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\
\int_{\mathbb{R}^3} v_j^* \partial_{v_j^*} \Psi_{0,2,m_2}(v_*) dv_* &= -\widehat{\Psi_{0,2,m_2}}(0) - \xi_j \partial_{\xi_j^*} \widehat{\Psi_{0,2,m_2}}(\xi) \Big|_{\xi=0} = 0; \\
\int_{\mathbb{R}^3} v_k^* v_j^* \partial_{v_j^*} \Psi_{0,2,m_2}(v_*) dv_* &= -\int_{\mathbb{R}^3} v_k^* \Psi_{0,2,m_2}(v_*) dv_*,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3} \Psi_{0,2,m_2}(v_*) dv_* &= (\varphi_{0,2,m_2}, \varphi_{0,0,0})_{L^2(\mathbb{R}^3)} = 0, \\
\int_{\mathbb{R}^3} v_k^* \Psi_{0,2,m_2}(v_*) dv_* &= 0, \quad \forall 1 \leq k \leq 3.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
\mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) &= \mathbf{L}_1(\varphi_{0,2,m_2}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{0,2,m_2}, \varphi_{n,l,m}) \\
&= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left(\int_{\mathbb{R}^3} |v_j^*|^2 \Psi_{0,2,m_2}(v_*) dv_* \right) \partial_{v_k}^2 \Psi_{n,l,m}(v) \\
&\quad + \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \left(\int_{\mathbb{R}^3} v_k^* v_j^* \Psi_{0,2,m_2}(v_*) dv_* \right) \partial_{v_k} \partial_{v_j} \Psi_{n,l,m}(v).
\end{aligned}$$

By the Fourier transformation, we have

$$\begin{aligned}
\mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m})](\xi) \\
= -(|v_*|^2 \sqrt{\mu}, \varphi_{0,2,m_2})_{L^2(\mathbb{R}^3)} |\xi|^2 \widehat{\Psi_{n,l,m}}(\xi)
\end{aligned}$$

$$+ \int_{\mathbb{R}^3_{v_*}} (\xi \cdot v_*)^2 \Psi_{0,2,m_2}(v_*) dv_* \widehat{\Psi_{n,l,m}}(\xi).$$

By using

$$(|v_*|^2 \sqrt{\mu}, \varphi_{0,2,m_2})_{L^2(\mathbb{R}^3)} = \left(3\varphi_{0,0,0} - \sqrt{6}\varphi_{1,0,0}, \varphi_{0,2,m_2} \right)_{L^2(\mathbb{R}^3)} = 0.$$

and by using the equality (5.7) with $v = \xi$ in Lemma 5.3 that

$$\int_{\mathbb{R}^3_{v_*}} (\xi \cdot v_*)^2 \Psi_{0,2,m_2}(v_*) dv_* = \sqrt{\frac{16\pi}{15}} |\xi|^2 Y_2^{m_2} \left(\frac{\xi}{|\xi|} \right),$$

we obtain

$$\mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m})](\xi) = 4\sqrt{\frac{\pi}{15}} |\xi|^2 Y_2^{m_2} \left(\frac{\xi}{|\xi|} \right) \widehat{\Psi_{n,l,m}}(\xi).$$

We apply Corollary 5.2 with $l = 2$, $\omega = \frac{\xi}{|\xi|}$ that

$$Y_2^{m_2}(\omega) Y_l^m(\omega) = C_{l,l-2}^{m_2,m} Y_{l-2}^{m_2+m}(\omega) + C_{l,l}^{m_2,m} Y_l^{m_2+m}(\omega) + C_{l,l+2}^{m_2,m} Y_{l+2}^{m_2+m}(\omega).$$

Recalled that

$$\widehat{\Psi_{n,l,m}}(\xi) = B_{n,l} |\xi|^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m \left(\frac{\xi}{|\xi|} \right),$$

we have

$$\begin{aligned} & \mathcal{F}[\sqrt{\mu} \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m})](\xi) \\ &= 4\sqrt{\frac{\pi}{15}} B_{n,l} |\xi|^{2n+l+2} e^{-\frac{|\xi|^2}{2}} \\ & \quad \times \left(C_{l,l-2}^{m_2,m} Y_{l-2}^{m_2+m}(\omega) + C_{l,l}^{m_2,m} Y_l^{m_2+m}(\omega) + C_{l,l+2}^{m_2,m} Y_{l+2}^{m_2+m}(\omega) \right) \\ &= 4\sqrt{\frac{\pi}{15}} \frac{B_{n,l}}{B_{n+2,l-2}} C_{l,l-2}^{m_2,m} \widehat{\Psi_{n+2,l-2,m}}(\xi) + 4\sqrt{\frac{\pi}{15}} \frac{B_{n,l}}{B_{n+1,l}} C_{l,l-2}^{m_2,m} \widehat{\Psi_{n+1,l,m}}(\xi) \\ & \quad + 4\sqrt{\frac{\pi}{15}} \frac{B_{n,l}}{B_{n,l+2}} C_{l,l-2}^{m_2,m} \widehat{\Psi_{n,l+2,m}}(\xi). \end{aligned}$$

Direct calculation and the inverse Fourier transform implies that

$$\begin{aligned} \mathbf{L}(\varphi_{0,2,m_2}, \varphi_{n,l,m}) &= A_{n,l,m,m_2}^1 \varphi_{n+2,l-2,m+m_2} \\ & \quad + A_{n,l,m,m_2}^2 \varphi_{n+1,l,m+m_2} + A_{n,l,m,m_2}^3 \varphi_{n,l+2,m+m_2}. \end{aligned}$$

This ends the proof of Lemma. \square

Lemma 5.8. For $n, l \in \mathbb{N}$, $|m| \leq l$, we have

$$\mathbf{L}(\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{n,l,m}) = 0, \quad \forall 2\tilde{n} + \tilde{l} > 2, |\tilde{m}| \leq \tilde{l}.$$

Proof. For any $n, l \in \mathbb{N}$, $\tilde{m} \in \mathbb{Z}$, and $2\tilde{n} + \tilde{l} > 2$, $|\tilde{m}| \leq \tilde{l}$, we have again

$$\mathbf{L}(\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{n,l,m}) = \mathbf{L}_1(\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{n,l,m}) - \mathbf{L}_2(\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{n,l,m}).$$

Using the facts

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_{v_k^*} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= i\xi_k \widehat{\Psi_{\tilde{n}, \tilde{l}, \tilde{m}}}(\xi) \Big|_{\xi=0} = 0, \quad k = 1, 2, 3; \\ \int_{\mathbb{R}^3} v_j^* \partial_{v_k^*} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= -\xi_k \partial_{\xi_j^*} \widehat{\Psi_{\tilde{n}, \tilde{l}, \tilde{m}}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^3} (v_j^*)^2 \partial_{v_k^*} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= -i \xi_k \partial_{\xi_j^*}^2 \widehat{\Psi_{\tilde{n}, \tilde{l}, \tilde{m}}}(\xi) \Big|_{\xi=0} = 0, \quad k \neq j; \\
\int_{\mathbb{R}^3} v_j^* \partial_{v_j^*} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= -\widehat{\Psi_{\tilde{n}, \tilde{l}, \tilde{m}}}(0) - \xi_j \partial_{\xi_j^*} \widehat{\Psi_{\tilde{n}, \tilde{l}, \tilde{m}}}(\xi) \Big|_{\xi=0} = 0; \\
\int_{\mathbb{R}^3} v_k^* v_j^* \partial_{v_j^*} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= - \int_{\mathbb{R}^3} v_k^* \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_*.
\end{aligned}$$

By the relation (5.1), one can verify that, for any $2\tilde{n} + \tilde{l} > 2$, $|\tilde{m}| \leq \tilde{l}$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= (\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{0,0,0})_{L^2(\mathbb{R}^3)} = 0, \\
\int_{\mathbb{R}^3} v_k^* \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= 0, \quad \forall 1 \leq k \leq 3 \\
\int_{\mathbb{R}^3} v_k^* v_j^* \Psi_{\tilde{n}, \tilde{l}, \tilde{m}}(v_*) dv_* &= 0, \quad \forall 1 \leq k, j \leq 3.
\end{aligned}$$

We can conclude that, for all $2\tilde{n} + \tilde{l} > 2$, $|\tilde{m}| \leq \tilde{l}$,

$$\mathbf{L}(\varphi_{\tilde{n}, \tilde{l}, \tilde{m}}, \varphi_{n,l,m}) \equiv 0.$$

□

For the proof of the Proposition 3.1, we recall the elementary result about the Legendre polynomial in the following.

Lemma 5.9. *Let $l \in \mathbb{N}$ be nonnegative integer, $P_l(x)$ is the Legendre polynomial, we have*

$$\left\{ \begin{aligned}
P_2(x)P_0(x) &= P_2(x), \\
P_2(x)P_1(x) &= \frac{3}{5}P_3(x) + \frac{2}{5}P_1(x), \\
P_1(x)P_l(x) &= \frac{l+1}{2l+1}P_{l+1}(x) + \frac{l}{2l+1}P_{l-1}(x), \\
P_2(x)P_l(x) &= \frac{3(l+2)(l+1)}{2(2l+3)(2l+1)}P_{l+2}(x) + \frac{(l+1)l}{(2l+3)(2l-1)}P_l(x) \\
&\quad + \frac{3l(l-1)}{2(2l+1)(2l-1)}P_{l-2}(x) \quad \text{for } l \geq 2.
\end{aligned} \right.$$

For more general case, we can refer to the Example 11 in Chap.XV in [20] or (1.4) in Appendix 1 in [5].

Proof of the Proposition 3.1

Recalled from Lemma 5.7 that

$$\begin{aligned}
A_{n-2, l+2, m, m_2}^1 &= -4 \sqrt{\frac{\pi}{15}} \sqrt{4n(n-1)} \int_{\mathbb{S}^2} Y_2^{m_2}(\omega) Y_{l+2}^m(\omega) Y_l^{-m_2-m}(\omega) d\omega; \\
A_{n-1, l, m, m_2}^2 &= 4 \sqrt{\frac{\pi}{15}} \sqrt{2n(2n+2l+1)} \int_{\mathbb{S}^2} Y_2^{m_2}(\omega) Y_l^m(\omega) Y_l^{-m_2-m}(\omega) d\omega; \\
A_{n, l-2, m, m_2}^3 &= -4 \sqrt{\frac{\pi}{15}} \sqrt{(2n+2l+1)(2n+2l-1)} \int_{\mathbb{S}^2} Y_2^{m_2} Y_{l-2}^m Y_l^{-m_2-m} d\omega.
\end{aligned}$$

We recalled the addition theorem (7-34) of Chapter 7 in [16], (VIII) of Sec.19, Chap. III in [14] or Theorem 1 of Sec.4, Chap. 1 in [13] that, for $\sigma, \kappa \in \mathbb{S}^2$,

$$P_k(\sigma \cdot \kappa) = \frac{4\pi}{2k+1} \sum_{|m| \leq k} Y_k^m(\sigma) Y_k^{-m}(\kappa), \quad \forall k \in \mathbb{N}.$$

Therefore, for any $m^* \in \mathbb{Z}$ and $|m^*| \leq l$,

$$\begin{aligned} & \sum_{\substack{|m| \leq l+2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n-2, l+2, m, m_2}^1|^2 \\ &= \frac{64n(n-1)\pi}{15} \sum_{|m| \leq l+2} \sum_{|m_2| \leq 2} \left| \int_{S^2} Y_{l+2}^m Y_2^{m_2} Y_l^{m^*} d\sigma \right|^2 \\ &= \frac{64n(n-1)\pi}{15} \int_{S_\sigma^2} \int_{S_\kappa^2} \frac{5}{4\pi} \frac{2l+5}{4\pi} P_2(\kappa \cdot \sigma) P_{l+2}(\kappa \cdot \sigma) Y_l^{m^*}(\kappa) Y_l^{-m^*}(\sigma) d\kappa d\sigma. \end{aligned}$$

By using Lemma 5.9 and the orthogonality of $\{Y_l^m\}_{l \in \mathbb{N}, |m| \leq l}$ on \mathbb{S}^2 , we have, for $n \geq 2$,

$$\begin{aligned} & \sum_{\substack{|m| \leq l+2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n-2, l+2, m, m_2}^1|^2 \\ &= \frac{64n(n-1)\pi}{15} \frac{5}{4\pi} \frac{2l+5}{4\pi} \frac{3(l+2)(l+1)}{2(2l+5)(2l+3)} \frac{4\pi}{2l+1} \\ &= \frac{8n(n-1)(l+2)(l+1)}{(2l+3)(2l+1)} \leq \frac{16n(n-1)}{3}. \end{aligned}$$

This is the estimation (3.1). Similar to the proof of (3.1), one can deduce also from Lemma 5.9 and the orthogonality of $\{Y_l^m\}_{l \in \mathbb{N}, |m| \leq l}$ on \mathbb{S}^2 that

$$\begin{aligned} & |A_{n-1, 0, 0, 0}^2|^2 \\ &= \frac{32n(2n+1)\pi}{15} \int_{S_\sigma^2} \int_{S_\kappa^2} \frac{5}{4\pi} \frac{1}{4\pi} P_2(\kappa \cdot \sigma) P_0(\kappa \cdot \sigma) Y_0^0(\kappa) Y_0^0(\sigma) d\kappa d\sigma \\ &= \frac{32n(2n+1)\pi}{15} \frac{5}{4\pi} \frac{1}{4\pi} \int_{S_\sigma^2} \int_{S_\kappa^2} P_2(\kappa \cdot \sigma) Y_0^0(\kappa) Y_0^0(\sigma) d\kappa d\sigma = 0, \quad \forall n \geq 1, \end{aligned}$$

and for any $m^* \in \mathbb{Z}$ and $|m^*| \leq l$

$$\begin{aligned} & \sum_{\substack{|m| \leq l, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n-1, l, m, m_2}^2|^2 \leq \frac{32n(2n+2l+1)\pi}{15} \frac{5}{4\pi} \frac{(l+1)l}{(2l+3)(2l-1)} \\ & \leq \frac{4n(2n+2l+1)}{3}, \quad \forall n \geq 1, l \geq 1. \end{aligned}$$

Finally, one can estimate that

$$\begin{aligned} & \sum_{\substack{|m| \leq l-2, |m_2| \leq 2 \\ m+m_2=m^*}} |A_{n, l-2, m, m_2}^3|^2 \\ &= \frac{16(2n+2l+1)(2n+2l-1)\pi}{15} \frac{5}{4\pi} \frac{3l(l-1)}{2(2l+1)(2l-1)} \\ & \leq \frac{(2n+2l+1)(2n+2l-1)}{2} \quad \forall n \in \mathbb{N}, l \geq 2. \end{aligned}$$

The estimations (3.2) and (3.3) follow. We end the proof of Proposition 3.1.

6. APPENDIX

The proof of the example and the characterization of the Gelfand-Shilov spaces and the Shubin spaces are presented in this section. For the self-content of paper, we will present some proof here.

The proof of the Example 1.1. Now we prove that the function g_0 defined in (1.8) is belongs to $Q^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ and $\|\mathbb{S}_2 g\|_{L^2(\mathbb{R}^3)} = 0$ for $\alpha < -\frac{3}{2}$.

Recalled the spectrum functions $\varphi_{n,l,m}(v)$ with $2n+l \leq 2$, $|m| \leq l$, we have

$$g_0 = \frac{1}{\sqrt{\mu}} \delta_{\mathbf{0}} - \left(\frac{5}{2} - \frac{|v|^2}{2} \right) \sqrt{\mu} = \frac{1}{\sqrt{\mu}} \delta_{\mathbf{0}} - \varphi_{0,0,0} - \sqrt{\frac{3}{2}} \varphi_{1,0,0}.$$

One can calculate directly that

$$\begin{aligned} \langle g_0, \varphi_{0,0,0} \rangle &= \langle \delta_{\mathbf{0}}, 1 \rangle - \langle \varphi_{0,0,0}, \varphi_{0,0,0} \rangle = 0; \\ \langle g_0, \varphi_{1,0,0} \rangle &= \langle \delta_{\mathbf{0}}, \sqrt{\frac{2}{3}} \left(\frac{3}{2} - \frac{|v|^2}{2} \right) \rangle - \sqrt{\frac{3}{2}} \langle \varphi_{1,0,0}, \varphi_{1,0,0} \rangle = 0. \end{aligned}$$

Since g_0 is radial, we can verify that

$$\langle g_0, \varphi_{n,l,m} \rangle \equiv 0, \quad \forall 2n+l \leq 2, |m| \leq l.$$

This shows that $g_0 \in \mathcal{N}^\perp$ and $\|\mathbb{S}_2 g\|_{L^2(\mathbb{R}^3)} = 0$. Now we prove that $g_0 \in Q^\alpha(\mathbb{R}^3)$ for $\alpha < -\frac{3}{2}$. Since $g_0 \in \mathcal{N}^\perp$ and radial, we can write g_0 in the form

$$g_0 = \sum_{k=2}^{+\infty} \langle g_0, \varphi_{k,0,0} \rangle \varphi_{k,0,0},$$

where we can calculate in details that,

$$\langle g_0, \varphi_{k,0,0} \rangle = \langle \mu^{-\frac{1}{2}} \delta_{\mathbf{0}}, \varphi_{k,0,0} \rangle = \sqrt{\frac{2\Gamma(k + \frac{3}{2})}{\sqrt{\pi}k!}}.$$

By using the Stirling equivalent

$$\Gamma(x+1) \sim_{x \rightarrow +\infty} \sqrt{2\pi x} \left(\frac{x}{e} \right)^x,$$

we have that, $\forall k \geq 2$

$$\sqrt{\frac{2\Gamma(k + \frac{3}{2})}{\sqrt{\pi}k!}} \sim k^{\frac{1}{4}}.$$

Therefore, for any $\alpha < -\frac{3}{2}$,

$$\|g_0\|_{Q^\alpha(\mathbb{R}^3)}^2 = \sum_{k=2}^{+\infty} (4k)^\alpha |\langle g_0, \varphi_{k,0,0} \rangle|^2 \lesssim \sum_{k=2}^{+\infty} k^{\alpha + \frac{1}{2}} < +\infty.$$

This implies that $g_0 \in Q^\alpha(\mathbb{R}^3)$, we end the proof of the Example. \square

Gelfand-Shilov spaces. The symmetric Gelfand-Shilov space $S_\nu^\nu(\mathbb{R}^3)$ can be characterized through the decomposition into the Hermite basis $\{H_\alpha\}_{\alpha \in \mathbb{N}^3}$ and the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$. For more details, see Theorem 2.1 in [4]

$$f \in S_\nu^\nu(\mathbb{R}^3) \Leftrightarrow f \in C^\infty(\mathbb{R}^3), \exists \tau > 0, \|e^{\tau \mathcal{H} \frac{1}{2\nu}} f\|_{L^2} < +\infty;$$

$$\begin{aligned} &\Leftrightarrow f \in L^2(\mathbb{R}^3), \exists \epsilon_0 > 0, \left\| \left(e^{\epsilon_0 |\alpha|^{\frac{1}{2\nu}}} (f, H_\alpha)_{L^2} \right)_{\alpha \in \mathbb{N}^3} \right\|_{l^2} < +\infty; \\ &\Leftrightarrow \exists C > 0, A > 0, \|(-\Delta + \frac{|v|^2}{4})^{\frac{k}{2}} f\|_{L^2(\mathbb{R}^3)} \leq AC^k (k!)^\nu, \quad k \in \mathbb{N} \end{aligned}$$

where

$$H_\alpha(v) = H_{\alpha_1}(v_1)H_{\alpha_2}(v_2)H_{\alpha_3}(v_3), \quad \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$,

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \pi}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-x^2}) = \frac{1}{\sqrt{2^n n! \pi}} \left(x - \frac{d}{dx}\right)^n (e^{-\frac{x^2}{2}}).$$

For the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ of 3-dimension and $s > 0$, we have

$$\mathcal{H}^{\frac{k}{2}} H_\alpha = (\lambda_\alpha)^{\frac{k}{2}} H_\alpha, \quad \lambda_\alpha = \sum_{j=1}^3 \left(\alpha_j + \frac{1}{2}\right), \quad k \in \mathbb{N}, \alpha \in \mathbb{N}^3.$$

Shubin spaces. We refer the reader to the works [4, 15] for the Shubin spaces. Let $\tau \in \mathbb{R}$, The Shubin spaces $Q^\tau(\mathbb{R}^3)$ can be also characterized through the decomposition into the Hermite basis :

$$\begin{aligned} f \in Q^\tau(\mathbb{R}^3) &\Leftrightarrow f \in \mathcal{S}'(\mathbb{R}^3), \left\| \mathcal{H}^{\frac{\tau}{2}} f \right\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in \mathcal{S}'(\mathbb{R}^3), \left\| \left((|\alpha| + \frac{3}{2})^{\tau/2} (f, H_\alpha)_{L^2} \right)_{\alpha \in \mathbb{N}^3} \right\|_{l^2} < +\infty, \end{aligned}$$

and for $\tau > 0$,

$$Q^\tau(\mathbb{R}^3) \subsetneq H^\tau(\mathbb{R}^3)$$

where $H^\tau(\mathbb{R}^3)$ is the usual Sobolev space. In fact,

$$\mathcal{H}f \in L^2(\mathbb{R}^3) \Rightarrow \Delta f, |v|^2 f \in L^2(\mathbb{R}^3).$$

So that for the negative index, we have,

$$H^{-\tau}(\mathbb{R}^3) \subsetneq Q^{-\tau}(\mathbb{R}^3).$$

See more details in the Appendix in [10].

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