Recovery of nonlinearly degraded sparse signals through rational optimization
Marc Castella, Jean-Christophe Pesquet

To cite this version:


HAL Id: hal-01575293
https://hal.science/hal-01575293
Submitted on 18 Aug 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Recovery of Nonlinearly Degraded Sparse Signals through Rational Optimization

Marc Castella
SAMOVAR, Télécom SudParis, CNRS, Université Paris-Saclay,
9 rue Charles Fourier, 91011 Evry Cedex, France
Email: marc.castella@telecom-sudparis.eu

Jean-Christophe Pesquet
Center for Visual Computing, CentraleSupelec,
Université Paris-Saclay,
Grande Voie des Vignes, 92295, Château-Malabry, France
Email: jean-christophe@pesquet.eu

Abstract—We show the benefit which can be drawn from recent global rational optimization methods for the minimization of a regularized criterion. The regularization term is a rational Geman-MacClure like potential, approximating the ℓ0 norm and the fit term is a least-squares criterion suited to a wide class of nonlinear degradation models.

I. INTRODUCTION

Over the last decade, much attention has been paid to inverse problems involving sparse signals. A popular approach consists in formulating such problems under a variational form where one minimizes the sum of a data fidelity term and a regularization term incorporating prior information. For sparse signals, the regularization term may involve the ℓ0 norm, or an approximation of it [1]. This generally results in difficult optimization problems with many local minima and weak global convergence guarantees [2]–[5].

In this work, we consider rational optimization algorithms offering global optimality guarantees. In addition, our method allows us to address the challenging case of a nonlinear model [6]–[8].

II. MODEL AND CRITERION

Consider a sparse vector with unknown nonnegative samples \( \mathbf{x} := (x_1, \ldots, x_T) \), only a few of which are nonzero. We aim at recovering it from measurements \( \mathbf{y} := (y_1, \ldots, y_T) \) related to \( \mathbf{x} \) through a linear transformation (typically, a convolution) followed by some nonlinear effects:

\[
\mathbf{y} = \phi(\mathbf{Hx}) + \mathbf{n},
\]

where \( \mathbf{n} := (n_1, \ldots, n_T)^T \) is a realization of a random noise vector, and \( \phi: \mathbb{R}^T \rightarrow \mathbb{R}^T \) is a rational nonlinear function with components \( \phi(u)_k = \phi(u_k) \) depending on the \( k \)-th entry \( u_k \) only. \( \mathbf{H} \in \mathbb{R}^{T \times T} \) is a given convolution matrix, which is assumed Toeplitz banded under suitable vanishing boundary conditions. To estimate \( \mathbf{x} \), we minimize a penalized criterion having the following form:

\[
\mathcal{J} (\mathbf{x}) := \| \mathbf{y} - \phi(\mathbf{Hx}) \|^2 + \sum_{t=1}^{T} \frac{x_t}{\delta + x_t},
\]

where \( \lambda \) and \( \delta \) are positive regularization and smoothing parameters. The last term is a Geman-McClure like potential as in [9]. We assume that an upper-bound \( B \) on the values \( (\mathcal{P}_t)_{t=1}^T \) is available and the minimization is thus performed over a compact set defined and represented by \( \mathcal{K} = \{ \mathbf{x} \in \mathbb{R}^T | x_t (B - x_t) \geq 0, t = 1, \ldots, T \} \). Then, the optimization problem consists in finding \( \mathcal{J}^* := \inf_{\mathbf{x} \in \mathcal{K}} \mathcal{J} (\mathbf{x}) \).

III. RATIONAL MINIMIZATION

Given \( \mathcal{J} \) in (2), the previous minimization is a rational problem. The methodology in [10, 11] builds for different orders \( k \) a hierarchical sequence of semi-definite programming (SDP) relaxations \( \mathcal{P}_k^* \) for which the following optimality result holds: \( \mathcal{P}_k^* \uparrow \mathcal{J}^* \) as \( k \to +\infty \).

By using SDP solvers to solve \( \mathcal{P}_k^* \), one can hence theoretically obtain the global optimum [9]. Due to the maximum tractable size of state-of-the-art SDP solvers, this approach is however limited to small/medium size problems having small degree, even when restricting the hierarchy to a finite and small order \( k \). To overcome this difficulty, we exploit the problem structure in the sum of rational terms in (2). Using the sparse Toeplitz banded shape of \( \mathbf{H} \), it can be noticed that:

\[
\mathcal{J} (\mathbf{x}) = \sum_{t=1}^{T} \left[ y_t - \phi \left( \sum_{i=t}^{L} h_i x_{t-i} \right) \right]^2 + \lambda \sum_{t=1}^{T} \frac{x_t}{\delta + x_t},
\]

where \( J_t = \{ \min \{1, t - L - 1, \ldots, t \} \} \) and \( J_{t+L} = \{ t \} \) for any \( t \in \{1, \ldots, T \} \). These index subsets satisfy the so-called “Running Intersection Property” [12]. As a consequence, it is possible to introduce a much smaller SDP relaxation \( \mathcal{P}_k^* \) instead of \( \mathcal{P}_k^* \).

The fundamental idea is that the SDP relaxations involve variables representing monomials in \( (x_1, \ldots, x_T) \). Using the above split form, many monomials can be discarded, the most striking case being when \( \mathcal{J} \) is fully separable.

IV. EXPERIMENTS

We have generated 100 Monte-Carlo realizations of vector \( \mathbf{x} \) containing \( T = 200 \) sparse samples, exactly 20 of which are nonzero. The nonzero sample values were randomly drawn in \( [\frac{2}{3}; 1] \). We have generated \( \mathbf{y} \) according to (1) with the nonlinearity \( \phi(u_k) = \frac{u_k}{0.4 + u_k} \) and with additive i.i.d. zero-mean Gaussian noise with standard deviation \( \sigma = 0.15 \). The banded Toeplitz matrix \( \mathbf{H} \) has been set in accordance with two choices of FIR filters of length 3 (denoted \( \mathbf{h}^{(o)} \) and \( \mathbf{h}^{(b)} \)). We considered the estimate \( \mathbf{x}_k^* \) given by the optimal point of the SDP relaxation \( \mathcal{P}_k^* \) of order \( k = 3 \).

For comparison, we have implemented a proximal gradient algorithm based on Iterative Hard Thresholding (IHT) [3] extended to the nonlinear model. Also, we tested a convex relaxation based on a linearized reconstruction with \( \ell_1 \) penalization. The local optimization algorithms have been started with different initializations and Table I indicates the existence of local minima.

On Figure 1, we have plotted the value \( \mathcal{P}_k^* \) reached by the SDP relaxation (which is a lower bound on \( \mathcal{J}^* \)), the objective value \( \mathcal{J} (\mathbf{x}_k^*) \) and the objective value reached using IHT using two different initializations. Clearly, our method provides a point close to a global minimizer and is very useful in providing a good initialization point for local optimization algorithms.

Finally, the estimation error has been quantified by \( \| \mathbf{x} - \mathbf{x}_k^* \| \) for a given estimate \( \mathbf{x}_k^* \). The average error and objective values are summarized in Table II.
TABLE I
Final values of the objective \( J(x) \) for the gradient and IHT local optimizations (average over 100 Monte-Carlo realizations)

<table>
<thead>
<tr>
<th>Filter param.</th>
<th>Initialization</th>
<th>Gradient minimization</th>
<th>IHT minimization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( x^* )</td>
<td>( x^* )</td>
</tr>
<tr>
<td>( h^{(a)} )</td>
<td>6.9219</td>
<td>15.136</td>
<td>6.6943</td>
</tr>
<tr>
<td>( h^{(b)} )</td>
<td>6.7078</td>
<td>13.245</td>
<td>6.6292</td>
</tr>
</tbody>
</table>

Fig. 1. Objective values provided by the different algorithms and lower-bound (using filter \( h^{(a)} \)).

TABLE II
Final values of the objective \( J(x) \) and estimation error given by the proposed method and IHT with different initializations (average over 1000 Monte-Carlo realizations).

<table>
<thead>
<tr>
<th>Filter param.</th>
<th>Objective</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method</td>
<td>( h^{(a)} )</td>
<td>( h^{(b)} )</td>
</tr>
<tr>
<td>Proposed method + IHT</td>
<td>( h^{(a)} )</td>
<td>( h^{(b)} )</td>
</tr>
<tr>
<td>linear + ( \ell_1 ) + IHT</td>
<td>( h^{(a)} )</td>
<td>( h^{(b)} )</td>
</tr>
</tbody>
</table>

REFERENCES