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► To cite this version:

Nabil Kazi-Tani, Didier Rulli re. On a construction of multivariate distributions given some multidimensional marginals. 2017. hal-01575169v1

HAL Id: hal-01575169

<https://hal.science/hal-01575169v1>

Preprint submitted on 18 Aug 2017 (v1), last revised 1 Mar 2019 (v3)

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On a construction of multivariate distributions given some multidimensional marginals.*

Nabil Kazi-Tani [†] Didier Rullière [‡]

17/08/2017

Abstract

In this paper, we investigate the link between the joint law of a d -dimensional random vector and the law of some of its multivariate marginals. We introduce and focus on a class of distributions, that we call *projective*, for which we give detailed properties. This allows us to obtain conditions that are easy to verify, to ensure that a given construction is projective. We illustrate our results on elliptical distributions on the first hand, and on a new class of distribution having given bivariate exponential margins on the other hand.

Keywords: Multidimensional marginals, Copulas, Elliptical Distributions.

AMS Subject Classification(2010): 60E05; 60E10; 62H05; 62H20.

1 Introduction

Multivariate analysis is now a cornerstone in probability theory. Constructing multivariate distributions from given marginals is mathematically interesting on its own, but also has huge impact in practical problems.

Let $d \geq 1$ be an integer and let X_1, \dots, X_d be continuous random variables with cumulative distribution functions (CDFs) F_1, \dots, F_d , respectively. Sklar's theorem [25] states that the joint distribution H of (X_1, \dots, X_d) can be written, for all $(x_1, \dots, x_d) \in \mathbb{R}^d$ in the form

$$H(x_1, \dots, x_d) = \mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d) = C\left(F_1(x_1), \dots, F_d(x_d)\right),$$

where C is a copula function, i.e. the cumulative distribution function of a probability measure on \mathbb{R}^d whose marginals are uniform on $[0, 1]$. Copula models have shown their interest in particular because, as it can be seen from the previous equation, they separate the study of the margins and the study of the dependence structure.

In this paper, we want to investigate the link between the joint law of a d -dimensional random vector and the law of its multivariate marginals. For any subset $K = (j_1, \dots, j_k)$ of $\{1, \dots, d\}$ with cardinal k , and any random vector $X = (X_1, \dots, X_d)$, we will write X_K the random vector with values in \mathbb{R}^k given by $(X_{j_1}, \dots, X_{j_k})$ and F_K the cumulative distribution function of X_K . We will abuse notations and call F_K a probability distribution on \mathbb{R}^k . Let $n \leq 2^d$ be a positive integer and let K_1, \dots, K_n be n subsets of $\{1, \dots, d\}$ with cardinals k_1, \dots, k_n .

*This manuscript version: 17/08/2017 15:40:43.

[†]Laboratoire SAF, Université Lyon I, France, nabil.kazi-tani@univ-lyon1.fr

[‡]Laboratoire SAF, Université Lyon I, France, didier.rulliere@univ-lyon1.fr

A question that has been extensively studied in the literature is the following: given n probability measures P_1, \dots, P_n such that P_i is a probability measure on \mathbb{R}^{k_i} , is it possible to construct a probability measure F on \mathbb{R}^d such that

$$F_{K_i} = P_i \text{ for each } i = 1, \dots, n? \quad (1.1)$$

The existence of such a measure F is not guaranteed. In the case where the subsets $\{K_i, i = 1, \dots, n\}$ are disjoint, the product distribution guarantees its existence. One could also try to extend the notion of Copula function to the case of non overlapping multidimensional marginals: Genest & al. [10] showed that this approach is useless, since it only allows to model the product distribution. More precisely, they proved that if

$$H(x_1, \dots, x_m, y_1, \dots, y_n) = C(F(x_1, \dots, x_m), G(y_1, \dots, y_n))$$

defines a proper $(m+n)$ -dimensional distribution function (with $m+n \geq 3$) for every F and G with respective dimensions m and n , then $C(u, v) = u \cdot v$.

When the subsets $\{K_i, i = 1, \dots, n\}$ are not disjoint, an obvious necessary condition is that the prescribed measures $\{P_i, i = 1, \dots, n\}$ have the same marginals on common subspaces. But this condition is not sufficient : Kellerer [18] gave a necessary and sufficient condition only involving the structure of the sets $\{K_i, i = 1, \dots, n\}$. We refer to [15] (in particular, chapters 3 and 4) and [7] and the references therein for further details and related problems, in particular extremal distributions with fixed multidimensional marginals and related optimization problems.

The question of uniqueness is also tricky. In [11], it is proved that if μ is a probability measure on \mathbb{R}^d with a density f with respect to the Lebesgue measure, then there exists a subset U of \mathbb{R}^d such that all the lower dimensional projections of the uniform distribution on U coincide with the lower dimensional projections of μ . This shows, at least in the case where measures have a density, that there is in general no hope for uniqueness of a measure with prescribed projections.

Possible explicit constructions of distribution functions F satisfying (1.1) have been given in [5], [23], [24], [6], [12], [13], [14], [21], [19], [20], [9] for overlapping or non-overlapping marginals.

All these constructions are very useful, in particular in statistics. Indeed, when the dimension d is large, one can first estimate all the bivariate marginals, since fitting a 2-dimensional copula is doable, and then construct a valid d -dimensional vector having the prescribed 2-dimensional margins. One problem of this approach is that it may not provide a unique d -dimensional distribution, but as pointed out in [15], one can then use entropy maximization techniques to choose a distribution among all those that have the prescribed marginals. By comparison, directly fitting the right copula in large dimension is however quite difficult and often makes use of recent research developments (nested, hierarchical or vine copulas, see [22], [4], [16], including pair-copula constructions or copulas with prescribed bivariate projections [1], [2], [8]).

The previously mentioned constructions all make use of the joint CDF, or the joint density. There are however other functions characterizing the joint distribution, for instance the characteristic function. We will call any such function a *characterizing function*. In this paper, we assume that a characterizing function m is given, and that there is a linear explicit decomposition of the characterizing function of the d -dimensional vector with respect to the characterizing functions of certain of its multi-dimensional marginals (Definition 2.1). We will say that a probability distribution satisfying Definition 2.1 is in a *projective class*. The main result of the paper is a complete analysis of the coefficients appearing in a decomposition of a projective class. Indeed, the distributions satisfying our Definition 2.1 are stable by projection, in the sense that they are such that all their multidimensional marginals also satisfy the same Definition 2.1. This allows us to give precise, and simple necessary conditions for a sequence of coefficients to generate a probability distribution on \mathbb{R}^d having fixed multidimensional marginals and belonging to a projective class. In particular, a necessary condition is that a matrix containing these coefficients is idempotent (Proposition 2.2). Notice that the *linear* form of the

decomposition in a projective class is not as restrictive as one would initially imagine, since one could first apply a bijective non linear transformation to a characterizing function, and then obtain a linear relation in the form of (2.1). The case of Elliptical random vectors illustrates well this last point.

In Section 2 we define the projective classes that we are going to work with, and in Section 3 we give and analyse examples of elements of this classes.

2 Projective class of random vectors

Let $D = \{1, \dots, d\}$ and $\mathbf{t} \in \mathbb{R}^d$, consider a random vector $\mathbf{X} = (X_i)_{i \in D}$. For $K \in \mathcal{P}(D)$ a subset of D , we denote by $\mathbf{X}_K = (X_i)_{i \in K}$ the subvector of \mathbf{X} , and by $\mathbf{t}_K = (t_i)_{i \in K}$ the subvector of \mathbf{t} .

We assume the existence of a link between the joint distribution of \mathbf{X} and the joint distributions of its projections \mathbf{X}_K . In general this link could be derived from the characteristic function of \mathbf{X} , from its CDF, or from some other quantity. We thus define a function m for which the link will be investigated.

We will denote by \mathcal{X}_d the space of \mathbb{R}^d -valued random variables that we will work with (with $k \leq d$). In the rest of the paper, the quantities involved (and our constructions) will only depend on the distributions of the random variables, and not on the random variables themselves. Because it has no impact on our results, we will nonetheless use random variables.

Assumption 2.1 (Characterizing function) *We assume that there exists a function $m : \mathbb{R}^d \times \mathcal{X}_d \times \mathcal{P}(D) \mapsto \mathbb{R}$ such that for any non-empty $K \in \mathcal{P}(D)$:*

1. $\{m(\mathbf{t}, \mathbf{X}, K)\}_{\mathbf{t} \in \mathbb{R}^d}$ characterizes the joint distribution of \mathbf{X}_K , i.e. $m(\mathbf{t}, \mathbf{X}, K) = m(\mathbf{t}, \mathbf{Y}, K)$ for all $\mathbf{t} \in \mathbb{R}^d$ if and only if \mathbf{X}_K and \mathbf{Y}_K have the same distribution.
2. there exists $a \in \overline{\mathbb{R}}$ such that for all $\mathbf{t} \in \mathbb{R}^d$, $m(P_K \mathbf{t}, \mathbf{X}, D) = m(\mathbf{t}, \mathbf{X}, K)$, where

$$(P_K \mathbf{t})_i := \begin{cases} t_i & \text{if } i \in K \\ a & \text{if } i \notin K. \end{cases}$$

In the rest of the paper, to simplify notations, we will denote $m(\mathbf{t}, \mathbf{X}, K)$ by $m(\mathbf{t}, \mathbf{X}_K)$.

Remark 2.1 *Assumption 2.1 2. implies that $P_{\emptyset, D} \mathbf{t} = (a)_{i \in D} =: \mathbf{a}$, and that $m(\mathbf{t}, \mathbf{X}, \emptyset) = m(\mathbf{a}, \mathbf{X}, \emptyset)$ for all \mathbf{t} . For simplicity, we will write this quantity $m_0 := m(\mathbf{a}, \mathbf{X}, \emptyset)$, or abuse notation and write $m(\mathbf{t}, \mathbf{X}_\emptyset) = m_0$.*

Remark 2.2 *Such a function m always exists. Typically, m can be given by $m(\mathbf{t}, \mathbf{X}_K) = \mathbb{E} \left[e^{i \mathbf{t}_K^\top \mathbf{X}_K} \right]$, with $a = 0$ and $m_0 = 1$. Another example is $m(\mathbf{t}, \mathbf{X}_K) = F_K(\mathbf{t}_K)$, the CDF of \mathbf{X}_K , with $a = +\infty$ and $m_0 = 1$. A third example is $m(\mathbf{t}, \mathbf{X}_K) = \phi^{-1} \circ F_K(\mathbf{t}_K)$ where $\phi : \mathbb{R}_+ \rightarrow [0, 1]$ is an invertible Archimedean generator, with $a = +\infty$ and $m_0 = 0$. Direct transformations of these functions, as entropy or survival functions, are also suitable characterizing functions.*

Remark 2.3 *Assumption 2.1 2. says that to study the marginal distribution of X on the subset K , it is enough to study the distribution of X , with the characterizing function m restricted to $P_K \mathbf{t}$. However, notice that every function characterizing the distribution of random variables does not satisfy this Assumption. Let us give the example of the potential function ([3]). Define the potential kernel v on \mathbb{R}^d as follows:*

$$v(x) = -|x|, \quad d = 1, \quad v(x) = -\log |x|, \quad d = 2, \quad v(x) = |x|^{-d+2}, \quad d \geq 3.$$

Then the potential U_X of a random vector X on \mathbb{R}^d is defined by

$$U_X(\mathbf{t}) = \mathbb{E}[v(X - \mathbf{t})]$$

when the expectation exists. We have that $U_X = U_Y$ if and only if X and Y have the same distribution, but the potential function does not satisfy Assumption 2.1 2.

We aim at defining the whole distribution of \mathbf{X} , using only some of its projections, i.e. using only the laws of X_K for $K \in \mathcal{S}$, where \mathcal{S} is a given subset of $\mathcal{P}(D)$. For example, \mathcal{S} can gather some subsets of cardinal 3, their subsets, and some singletons, or \mathcal{S} can gather only subsets of cardinal 1, as in copula theory. We assume that \mathcal{S} is *decreasing*, in the sense that for all $K \subset J$, $J \in \mathcal{S}$ implies $K \in \mathcal{S}$: knowing the distribution of a projection easily allows to know the distribution of every sub-vector. In the algebraic topology terminology, \mathcal{S} is a *simplicial complex* [26]. Simplicial complexes can be represented using points, line segments, triangles, and simplices in higher dimensions, which may ease the understanding of the projections and the model (see Figure 1 for an illustration).

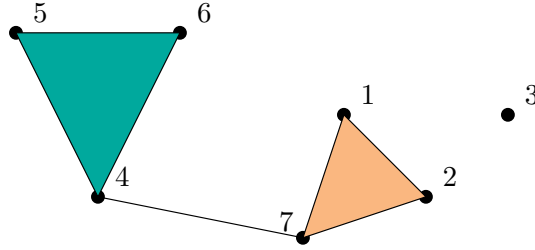


Figure 1: Illustration of a multivariate distribution in the dimension 7, knowing the marginal distribution $\{3\}$, the bivariate projection $\{4, 7\}$, and the trivariate projections $\{1, 2, 7\}$ and $\{4, 5, 6\}$. All subsets of these projections also correspond to known marginal distributions.

Definition 2.1 (Projective class) Let $\mathcal{S} \in \mathcal{P}(D)$ be decreasing. We say that a random vector $\mathbf{X} \in \mathbb{R}^d$ belongs to the projective class $\mathcal{F}_D(\mathcal{S})$ if there exist some real coefficients $\alpha_{K,D}$, $K \subset D$, such that for all $\mathbf{t} \in \mathbb{R}^d$,

$$m(\mathbf{t}, \mathbf{X}) = \sum_{K \in \mathcal{S}} \alpha_{K,D} m(\mathbf{t}, \mathbf{X}_K), \quad (2.1)$$

Remark that if a random vector \mathbf{X} belongs the projective class $\mathcal{F}_D(\mathcal{S})$, then the set of suitable coefficients $\{\alpha_{K,D}, K \subset D\}$ satisfying Equation (2.1) is not necessarily unique. Notice also that if $D \in \mathcal{S}$, any d -dimensional random vector is in $\mathcal{F}_D(\mathcal{S})$, using for example constant coefficients $\alpha_{K,D}$ equal to 1 if $K = D$ and equal to 0 otherwise. This is quite natural, since the class $\mathcal{F}_D(\mathcal{S})$ intends to define multivariate distributions that can be fully determined by some of their projections. This is obviously the case when the initial joint distribution is already in \mathcal{S} .

In the next Proposition, we prove the following projection stability property: if a random vector belongs to the class $\mathcal{F}_D(\mathcal{S})$, then any subvector also belong to $\mathcal{F}_D(\mathcal{S})$, and we compute the corresponding coefficients.

Proposition 2.1 (Projection stability) Let \mathbf{X} be a d -dimensional random vector in $\mathcal{F}_D(\mathcal{S})$, with associated coefficients $\{\alpha_{K,D}, K \in \mathcal{S}\}$. Then for any non-empty $L \subset D$, the subvector \mathbf{X}_L

belongs to $\mathcal{F}_L(\mathcal{S})$, where for any non-empty subset J of L , a suitable set of associated coefficients is given by

$$\alpha_{J,L} = \sum_{K \in \mathcal{S}, K \cap L = J} \alpha_{K,D}, \quad (2.2)$$

which implies in particular that $\alpha_{J,L} = 0$ if $J \not\subset L$.

Let d_0 be an integer such that $1 \leq d_0 \leq d$. When $\mathcal{S} = \{K \subset D, |K| \leq d_0\}$, and when the coefficients $\{\alpha_{J,L}, J \subset L\}$ only depend on the subsets cardinals, i.e. $\alpha_{J,L} = \alpha_{j,\ell}$ with $j = |J| \in \{1, \dots, d\}$ and $\ell = |L| \in \{0, \dots, d\}$, then a suitable set of associated coefficients is given by

$$\alpha_{j,\ell} = \sum_{k=j}^{\min(d_0, d-\ell+j)} \binom{d-\ell}{k-j} \alpha_{k,d} \quad \text{if } j \leq \min(\ell, d_0), \quad (2.3)$$

and $\alpha_{j,\ell} = 0$ otherwise.

Proof. First remark that, due to the second assumption on $m(\mathbf{t}, \mathbf{X})$, we have $m(\mathbf{t}, \mathbf{X}_{K \cap L}) = m(P_{K \cap L} \mathbf{t}, \mathbf{X}) = m(P_K P_L \mathbf{t}, \mathbf{X}) = m(P_L \mathbf{t}, \mathbf{X}_K)$. Thus, using (2.1), we have

$$\begin{aligned} m(\mathbf{t}, \mathbf{X}_L) &= m(P_L \mathbf{t}, \mathbf{X}) = \sum_{K \in \mathcal{S}} \alpha_{K,D} m(P_L \mathbf{t}, \mathbf{X}_K) = \sum_{K \in \mathcal{S}} \alpha_{K,D} m(\mathbf{t}, \mathbf{X}_{K \cap L}) \\ &= \sum_{K \in \mathcal{S}} \alpha_{K,D} \sum_{J=K \cap L} m(\mathbf{t}, \mathbf{X}_J) \\ &= \sum_{J \subset L} \sum_{K \in \mathcal{S}, K \cap L = J} \alpha_{K,D} m(\mathbf{t}, \mathbf{X}_J). \end{aligned}$$

By assumption, $K \in \mathcal{S}$ implies that $J = K \cap L \in \mathcal{S}$. Setting $\alpha_{J,L} = \sum_{K \in \mathcal{S}, K \cap L = J} \alpha_{K,D}$ as in Equation (2.2) we finally get

$$m(\mathbf{t}, \mathbf{X}_L) = \sum_{J \subset L, J \in \mathcal{S}} \alpha_{J,L} m(\mathbf{t}, \mathbf{X}_J), \quad (2.4)$$

hence \mathbf{X}_L belongs to $\mathcal{F}^L(\mathcal{S})$.

Eventually remark that $K \cap L = J$ if and only if $J \subset L$ and $J \subset K$ and $K \cap L \subset J$. As a consequence, if $J \not\subset L$, the sum in (2.2) is empty and $\alpha_{J,L} = 0$.

We prove now the second part of the Proposition. When $\mathcal{S} = \{K \subset D, |K| \leq d_0\}$, we have using (2.2)

$$\alpha_{J,L} = \sum_{K' \subset D \setminus L, |K'| \leq d_0 - |J|} \alpha_{K' \cup J, D}.$$

Now, let $j = |J|$ and $\ell = |L|$. If $j > \ell$, it is clear that $J \not\subset L$ and that $\alpha_{J,L} = 0$. As $K' \subset D \setminus L$ and $|K'| \leq d_0 - |J|$, we get $0 \leq |K'| \leq \min(d_0 - j, d - \ell)$. Thus when $J \subset L$,

$$\alpha_{J,L} = \sum_{k=0}^{\min(d_0-j, d-\ell)} \sum_{K' \subset D \setminus L, |K'|=k} \alpha_{K' \cup J, D} \quad (2.5)$$

and if the coefficients $\alpha_{J,L}$ do only depend on the cardinals,

$$\alpha_{j,\ell} = \sum_{k=0}^{\min(d_0-j, d-\ell)} \sum_{K' \subset D \setminus L, |K'|=k} \alpha_{k+j,d} = \sum_{k=0}^{\min(d_0-j, d-\ell)} \binom{d-\ell}{k} \alpha_{k+j,d}. \quad (2.6)$$

Hence the second result. \square

Remark 2.4 (case of \emptyset) Notice that \emptyset necessarily belongs to \mathcal{S} . In the case where $m(\mathbf{t}, \mathbf{X}_\emptyset) \neq 0$, Equation (2.1) may involve a constant $\alpha_{\emptyset, D}$, and implies that $\sum_{K \subset D, K \in \mathcal{S}} \alpha_{K, D} = 1$. In this case, it becomes useful to determine the coefficients $\alpha_{\emptyset, L}$. For any non-empty L , the Equation (2.2) writes

$$\alpha_{\emptyset, L} = \sum_{K \subset D \setminus L, K \in \mathcal{S}} \alpha_{K, D}. \quad (2.7)$$

When $L = \emptyset$, the Equation (2.1) remains valid for $D = \emptyset$ if one defines $\alpha_{\emptyset, \emptyset} = 1$.

Corollary 2.1 (Given projections up to dimension d_0) Let \mathbf{X} be a d -dimensional random vector in $\mathcal{F}_D(\mathcal{S})$. Assume that all projections of \mathbf{X} are given up to a dimension d_0 , so that $\mathcal{S} = \{K \subset D, |K| \leq d_0\}$. Assume that the associated coefficients $\{\alpha_{J, L}, J \subset L\}$ only depend on the subsets cardinals, i.e. $\alpha_{J, L} = \alpha_{j, \ell}$ with $j = |J| \in \{1, \dots, d\}$ and $\ell = |L| \in \{0, \dots, d\}$. Assume furthermore that $\alpha_{k, k} = 1$ for all $k \leq d_0$. Then the coefficients $\alpha_{d_0-z, d}$ can be obtained recursively, using

$$\alpha_{d_0-z, d} = 1 - \sum_{i=1}^z \binom{d-d_0+z}{i} \alpha_{i+d_0-z, d}, \quad (2.8)$$

for $z = 1, \dots, d_0$, starting with $\alpha_{d_0, d} = 1$. In particular for $2 \leq d_0 \leq d$, we get $\alpha_{d_0, d} = 1$, $\alpha_{d_0-1, d} = -(d-d_0)$, $\alpha_{d_0-2, d} = 1 + \frac{d-d_0+2}{2}(d-d_0-1)$.

When $d_0 \geq 3$, $\alpha_{d_0-3, d} = 1 - (d-d_0+3)\{1 - \frac{d-d_0+2}{2}(1 - \frac{d-d_0+1}{3})\}$. For higher orders, one can check by induction that these coefficients do only depend on $d-d_0$, but their expression is omitted here.

Proof. This follows directly from Proposition 2.1, by writing Equation (2.3) in the case where $j = \ell \leq d_0$, and setting $i = k - j$, $j = d_0 - z$. \square

The case where all bivariate projections are given is a very natural and interesting case: in practical applications, bivariate projections can be graphically visualized, and the estimation of the dependence structure among each pair of random variable is still tractable. The following Remark shows that in this case, under some simple conditions, the coefficients $\alpha_{J, L}$ can be computed explicitly.

Remark 2.5 (Given bivariate projections) Consider the same assumptions as in Corollary 2.1 and assume that all bivariate projections of a multivariate distribution are given, so that $d_0 = 2$ and $\mathcal{S} = \{J \subset D, |J| \leq 2\}$. Then for all non-empty $L \subset D$, we can reformulate equation (2.1) as follows:

$$m(\mathbf{t}, \mathbf{X}_L) = \alpha_{0, |L|} m_0 - (|L| - 2) \sum_{J \subset L, |J|=1} m(\mathbf{t}, \mathbf{X}_J) + \sum_{J \subset L, |J|=2} m(\mathbf{t}, \mathbf{X}_J), \quad (2.9)$$

where $\alpha_{0,0} = 1$, $\alpha_{0, \ell} = 1 + \frac{1}{2}\ell(\ell-3)$, $\ell \geq 1$ and where $m_0 = m(\mathbf{t}, \mathbf{X}_\emptyset)$ is defined in Remark 2.1.

Let $\mathbf{X} \in \mathbb{R}^d$ be a random vector in $\mathcal{F}^D(\mathcal{S})$. Since $D = \{1, \dots, d\}$ is a finite set, the set of subsets of D is also finite, so that we can define the following matrix, indexed by the subsets of D :

$$A = (\alpha_{J, L})_{J \subset D, L \subset D}. \quad (2.10)$$

We will write $A_{\cdot, L}$ for the column vector relative to the subset L .

Proposition 2.2 The coefficients in the matrix A satisfy the following constraints:

1. If the set of associated coefficients $\{\alpha_{J,L}\}$ is unique, then the matrix A defined in (2.10) is idempotent: $A^2 = A$.
2. If furthermore the coefficients do only depend on the subset's cardinal, i.e. $\alpha_{J,L} = \alpha_{j,\ell}$ with $j = |J|$ and $\ell = |L|$, we get for $0 \leq j \leq \ell \leq d$:

$$\alpha_{j,\ell} = \sum_{k=j}^{\min(d_0,\ell)} \binom{\ell-j}{k-j} \alpha_{j,k} \alpha_{k,\ell}. \quad (2.11)$$

Now define $s_{j,\ell} := \binom{\ell-2}{j-2} \alpha_{j,\ell}$ when $j \geq 2$. The previous equation writes, when $j \geq 2$,

$$s_{j,\ell} = \sum_{k=j}^{\min(d_0,\ell)} s_{j,k} s_{k,\ell}. \quad (2.12)$$

Proof. Let $L \subset D$, $|L| \geq 1$. From Proposition 2.1,

$$m(\mathbf{t}, \mathbf{X}_L) = \sum_{K \subset L, K \in \mathcal{S}} \alpha_{K,L} m(\mathbf{t}, \mathbf{X}_K)$$

From Proposition 2.1, we also have $m(\mathbf{t}, \mathbf{X}_K) = \sum_{J \subset K, J \in \mathcal{S}} \alpha_{J,K} m(\mathbf{t}, \mathbf{X}_J)$, so that finally

$$\begin{aligned} m(\mathbf{t}, \mathbf{X}_L) &= \sum_{K \subset L, K \in \mathcal{S}} \sum_{J \subset K, J \in \mathcal{S}} \alpha_{J,K} \alpha_{K,L} m(\mathbf{t}, \mathbf{X}_J) \\ &= \sum_{J \subset L, J \in \mathcal{S}} \sum_{K \subset L, K \in \mathcal{S}} \alpha_{J,K} \alpha_{K,L} m(\mathbf{t}, \mathbf{X}_J), \end{aligned}$$

as $\alpha_{J,K} = 0$ if $J \not\subset K$. Then for all \mathbf{t} ,

$$m(\mathbf{t}, \mathbf{X}_L) = \sum_{J \subset L, J \in \mathcal{S}} \left\{ \sum_{K \subset L, K \in \mathcal{S}} \alpha_{J,K} \alpha_{K,L} \right\} m(\mathbf{t}, \mathbf{X}_J)$$

so that, using the unicity of the set of coefficients $\{\alpha_{J,L}\}$,

$$\alpha_{J,L} = \sum_{K \subset L, K \in \mathcal{S}} \alpha_{J,K} \alpha_{K,L}$$

and thus A is idempotent.

Let us now focus on the second part of the proposition. For a subset $L \subset D$ with cardinal ℓ and $k \leq \ell$, define $[L]^k := \{K \subset L \text{ such that } |K| = k\}$. Assume that when $K \subset L$ the coefficients $\alpha_{K,L}$ do only depend on the cardinals $k = |K|$ and $\ell = |L|$ of the considered sets, and assume $\alpha_{K,L} = 0$ if $K \notin \mathcal{S}$, i.e. $\alpha_{k,\ell} = 0$ if $k > d_0$. Then

$$\begin{aligned} m(t, \mathbf{X}_L) &= \sum_{K \subset L} \alpha_{K,L} m(t, \mathbf{X}_K) \\ &= \sum_{k=0}^{\min(d_0,\ell)} \alpha_{k,\ell} \sum_{K \in [L]^k} m(t, \mathbf{X}_K) \\ &= \sum_{k=0}^{\min(d_0,\ell)} \alpha_{k,\ell} \sum_{K \in [L]^k} \left\{ \sum_{j=0}^{\min(d_0,k)} \alpha_{j,k} \sum_{J \in [K]^j} m(t, \mathbf{X}_J) \right\} \end{aligned}$$

Notice that, by a simple combinatorial argument, for $j \leq k$,

$$\sum_{K \in [L]^k} \sum_{J \in [K]^j} m(t, \mathbf{X}_J) = \binom{\ell - j}{k - j} \sum_{J \in [L]^j} m(t, \mathbf{X}_J),$$

which entails that

$$m(\mathbf{t}, \mathbf{X}_L) = \sum_{k=0}^{\min(d_0, \ell)} \sum_{j=0}^{\min(d_0, k)} \alpha_{k, \ell} \alpha_{j, k} \binom{\ell - j}{k - j} \sum_{J \in [L]^j} m(\mathbf{t}, \mathbf{X}_J).$$

On the other hand we have

$$m(\mathbf{t}, \mathbf{X}_L) = \sum_{j=0}^{\min(d_0, \ell)} \alpha_{j, \ell} \sum_{J \in [L]^j} m(\mathbf{t}, \mathbf{X}_J)$$

so that for all \mathbf{t} , for all $j \leq k$,

$$\alpha_{j, \ell} = \sum_{k=j}^{\min(d_0, \ell)} \binom{\ell - j}{k - j} \alpha_{j, k} \alpha_{k, \ell}$$

and the second result holds. \square

Remark 2.6 Notice that due to the projection stability property of Proposition 2.1, any column of the matrix A can be deduced from the last one by multiplication by a matrix with values in $\{0, 1\}$: $A_{\cdot, L} = P^{(L)} A_{\cdot, D}$, where the $2^d \times 2^d$ matrix $P^{(L)}$ is defined by its components

$$P_{J, K}^{(L)} = \begin{cases} 1 & \text{if } J = K \cap L \text{ and } K \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases},$$

for J, K, L subsets of D . Indeed, we have from Proposition 2.1:

$$\alpha_{J, L} = \sum_{K \in \mathcal{S}, K \cap L = J} \alpha_{K, D} = \sum_{K \in \mathcal{S}} P_{J, K}^{(L)} \alpha_{K, D}. \quad (2.13)$$

3 Examples

3.1 Elliptical random vectors

Recall [15] that d -dimensional elliptical distributions are characterized by the fact that their characteristic function can be written in the following form: for any $\mathbf{t} \in \mathbb{R}^d$,

$$\mathbb{E} \left[e^{i \mathbf{t}^\top (\mathbf{X} - \boldsymbol{\mu})} \right] = \phi(\mathbf{t}^\top \Sigma \mathbf{t}) \quad (3.1)$$

for a given function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ which is called the generator of the distribution, and where $\boldsymbol{\mu}$ is the mean of the vector \mathbf{X} and Σ is a non-negative definite matrix. We assume here that the generator ϕ does not depend on the dimension d of the random vector, i.e. that the elliptical distribution is consistent in the sense of [17].

Let us first remark that when one considers a centered multivariate elliptical distribution, the distribution is fully characterized by all components σ_{ij} of the matrix Σ , that is to say by all bivariate elliptical projections of the distribution (it does not mean that the multivariate elliptical distribution is the only one having those projections).

It is thus quite natural to analyse, in the case of elliptical distributions, the links between the matrix Σ and a given set of submatrices $\Sigma_{K_1}, \dots, \Sigma_{K_n}$, for K_1, \dots, K_n subsets of $D = \{1, \dots, d\}$. This is easier to do using the matrix Σ rather than its inverse Σ^{-1} . It thus seems easier to work with characteristic functions or entropy (which are expressed using Σ) rather than densities of cumulative distribution functions (which are expressed using Σ^{-1}).

We will try to express the quantity $\mathbf{t}^\top \Sigma \mathbf{t}$ as a linear combination of products $\mathbf{t}_K^\top \Sigma_K \mathbf{t}_K$, where K belongs to known projections indexes in \mathcal{S} .

Definition 3.1 (\mathcal{S} -admissible sequence) Let $D \subset \mathbb{N}$, and \mathcal{S} be a decreasing subset of D . A sequence of coefficients $\alpha_{K,D}$, $K \in \mathcal{S}$, is said to be \mathcal{S} -admissible if for all matrix Σ , for all \mathbf{t} ,

$$\mathbf{t}^\top \Sigma \mathbf{t} = \sum_{K \subset D, K \in \mathcal{S}} \alpha_{K,D} \cdot \mathbf{t}_K^\top \Sigma_K \mathbf{t}_K, \quad (3.2)$$

The following Lemma provides a characterization of such coefficients:

Lemma 3.1 (Characterization of \mathcal{S} -admissible sequence) Let $d_0 \in \mathbb{N}$ be such that $2 \leq d_0 \leq d$, and assume that $\mathcal{S} = \{K \subset D, |K| \leq d_0\}$. Assume furthermore that for any sets K, D , $\alpha_{K,D} = \alpha_{|K|,|D|}$. A sequence $\alpha_d = (\alpha_{k,d})_{k \leq d}$ of coefficients is \mathcal{S} -admissible if and only if it can be written

$$\alpha_{1,d} = - \sum_{k=2}^{d_0} s_k \frac{d-k}{k-1} \quad \text{and} \quad \alpha_{k,d} = s_k / \binom{d-2}{k-2}, \quad k = 2, \dots, d_0 \quad (3.3)$$

for some real values s_2, \dots, s_{d_0} such that $s_2 + \dots + s_{d_0} = 1$.

In the particular case where the coefficients are deduced from only two given dimensions, i.e. if there exists $k_0 \geq 2$ such that $s_i = 0$ whenever $i \notin \{1, k_0\}$, we get a particular \mathcal{S} -admissible sequence

$$\alpha_{1,d} = - \frac{d-k_0}{k_0-1} \quad \text{and} \quad (3.4)$$

$$\alpha_{k,d} = \frac{1}{\binom{d-2}{k_0-2}}, \text{ if } k = k_0 \text{ and } \alpha_{k,d} = 0 \text{ otherwise, } k = 2, \dots, d_0. \quad (3.5)$$

Furthermore, when $d_0 = 2$, the only \mathcal{S} -admissible sequence is

$$\alpha_{1,d} = -(d-2) \quad \text{and} \quad \alpha_{2,d} = 1 \quad (3.6)$$

Proof. Assume that α_d is \mathcal{S} -admissible, and that $\alpha_{K,D}$ does only depend on $|K|$ and $|D|$. Let $(i_0, j_0) \in D^2$, $i_0 \neq j_0$. Isolating the coefficient $\mathbf{t}_{i_0}^\top \Sigma_{i_0, j_0} \mathbf{t}_{j_0}$ on both sides of Equation (3.2), we get

$$1 = \sum_{k=2}^{d_0} |\{K : K \subset D, |K| = k, \{i_0, j_0\} \subset K\}| \alpha_{k,d} = \sum_{k=2}^{d_0} \binom{d-2}{k-2} \alpha_{k,d}$$

Denoting $s_k = \alpha_{k,d} \binom{d-2}{k-2}$ for all k , we get $\sum_{k=2}^{d_0} s_k = 1$. Now considering the coefficient $\mathbf{t}_{i_0}^\top \Sigma_{i_0, i_0} \mathbf{t}_{i_0}$ on both sides of Equation (3.2), we get

$$1 = \sum_{k=1}^{d_0} \alpha_{k,d} |\{K : K \subset D, |K| = k, i_0 \in K\}| = \alpha_{1,d} + \sum_{k=2}^{d_0} \binom{d-1}{k-1} \alpha_{k,d}$$

Now as $s_k = \alpha_{k,d} \binom{d-2}{k-2}$ for all k ,

$$1 = \alpha_{1,d} + \sum_{k=2}^{d_0} s_k \frac{\binom{d-1}{k-1}}{\binom{d-2}{k-2}} = \alpha_{1,d} + \sum_{k=2}^{d_0} s_k \frac{d-1}{k-1}$$

Finally, using $1 = \sum_{k=2}^{d_0} s_k$, Equation (3.3) holds. The rest of the proposition are direct applications of this last equation. \square

A direct application of such \mathcal{S} -admissible sequence is the following one:

Proposition 3.1 (Elliptical distributions are projective) *Consider a d -dimensional random vector \mathbf{X} having elliptical distribution with mean $\boldsymbol{\mu}$, matrix Σ and invertible generator ϕ . Let $D = \{1, \dots, d\}$ and $[D]^k = \{K \subset D, |K| = k\}$. Consider that all projections are given up to a dimension d_0 , $2 \leq d_0 \leq d$, so that $\mathcal{S} = \{K \subset D, |K| \leq d_0\}$. Then for any \mathcal{S} -admissible sequence $\boldsymbol{\alpha}_d = (\alpha_{1,d}, \dots, \alpha_{d_0,d})$, the following equality holds*

$$\phi(\mathbf{t}^\top \Sigma \mathbf{t}) = \phi \left(\sum_{k=1}^{d_0} \alpha_{k,d} \sum_{K \in [D]^k} \mathbf{t}_K^\top \Sigma_{KK} \mathbf{t}_K \right). \quad (3.7)$$

In other words, setting $m(\mathbf{t}, \mathbf{X}) = \phi^{-1} \left(\mathbb{E} \left[e^{i\mathbf{t}^\top (\mathbf{X} - \boldsymbol{\mu})} \right] \right)$, and $\mathcal{S} = \{K \subset D, |K| \leq 2\}$, we have

$$\mathbf{X} \in \mathcal{F}_D(\mathcal{S}) \quad (3.8)$$

In particular, when $d_0 = 2$ (i.e. starting from all bivariate projections), the admissible sequence is $\alpha_{1,d} = -(d-2)$ and $\alpha_{2,d} = 1$.

Proof. By definition, for any \mathcal{S} -admissible sequence, $\mathbf{t}^\top \Sigma \mathbf{t} = \sum_{k=1}^{d_0} \alpha_{k,d} \sum_{K \in [D]^k} \mathbf{t}_K^\top \Sigma_{KK} \mathbf{t}_K$. One can also check that the functions m are suitable characterizing functions satisfying Assumption 2.1 with $a = 0$ and $m_0 = \phi^{-1}(1) = 0$. Hence the result. \square

Remark 3.1 (Matrix A in the elliptical case, $d_0 = 2$) *Consider the elliptical case with $d_0 = 2$. Then we get by Lemma 3.1,*

$$\alpha_{J,L} = \begin{cases} 0 & \text{if } J \not\subset L, \\ -(|L| - 2) & \text{if } |J| = 1 \text{ and } J \subset L, \\ 1 & \text{if } |J| = 2 \text{ and } J \subset L. \end{cases}$$

In particular, if $d = 3$ and $D = \{1, 2, 3\}$, the matrix $A = (\alpha_{J,L})_{J \subset D, L \subset D}$ is

$$A = \left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad (3.9)$$

where the seven rows and columns correspond to successive subsets of D :

$$\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

As $m_0 = 0$ in the elliptical case, it is not necessary to compute the coefficients $\alpha_{J,L}$ for $J = \emptyset$ or $L = \emptyset$ (see Remark 2.4). One easily check that we can apply Proposition 2.2 to deduce that A is idempotent, which can also be verified by hand in this example.

3.2 Vectors built from bivariate exponential distributions

Consider a random variable \mathbf{X} distributed in \mathbb{R}_+^d . Let $D = \{1, \dots, d\}$ and define

$$m(\mathbf{t}, \mathbf{X}) = \psi^{-1}(\mathbb{P}(X_i > t_i, i \in D)) . \quad (3.10)$$

where ψ is a decreasing bijection from \mathbb{R}_+ to $(0, 1]$, such that $\psi(0) = 1$ and that derivatives of ψ exist up to the order d . Denote by ψ^{-1} the inverse function of ψ . It is clear that the function m satisfies Assumption 2.1, with constant $a = +\infty$ and $m_0 = m(\mathbf{t}, \emptyset) = 0$. The function m is thus a suitable characterizing function.

Now consider the decreasing set $\mathcal{S} = \{J \subset D, |J| \leq 2\}$, and assume that m belongs to the class $\mathcal{F}_D(\mathcal{S})$ in Definition 2.1: each multivariate distribution is assumed here to depend only on its bivariate projections. Assume furthermore that the associated coefficients $\alpha_{J,K}$ in Definition 2.1, $J \subset K$, do only depend on cardinals $j = |J|$ and $k = |K|$, so that $\alpha_{J,K} = \alpha_{j,k}$.

When all bivariate projections $\mathbb{P}(X_i > t_i, X_j > t_j)$ are given for all $i, j \in K$, $K \subset D$, then a direct application of Remark 2.5 gives the shape of the candidate multivariate survival function. As an example, assume that following bivariate survival functions are given:

$$\mathbb{P}(X_i > t_i, X_j > t_j) = \psi(\theta_i t_i + \theta_j t_j + \theta_{i,j} t_i t_j) ,$$

for some given non-negative parameters θ_i , θ_j and $\theta_{i,j}$, $i, j \in D$. Due to Remark 2.5, if a valid multivariate distribution belongs to the class $\mathcal{F}_D(\mathcal{S})$, then its survival function must be written

$$\bar{F}_K(\mathbf{t}) = \psi \left(-(k-2) \sum_{J \subset K, |J|=1} m(\mathbf{t}, \mathbf{X}_J) + \sum_{J \subset K, |J|=2} m(\mathbf{t}, \mathbf{X}_J) \right) ,$$

Notice that $m(t, \mathbf{X}_{\{i\}}) = \theta_i t_i$ and $m(t, \mathbf{X}_{\{i,j\}}) = \theta_i t_i + \theta_j t_j + \theta_{i,j} t_i t_j$. Now using $\sum_{\{i,j\} \subset K} (\theta_i t_i + \theta_j t_j) = (k-1) \sum_{\{i\} \subset K} \theta_i t_i$ we get

$$\bar{F}_K(\mathbf{t}) = \psi \left(\sum_{\{i\} \subset K} \theta_i t_i + \sum_{\{i,j\} \subset K} \theta_{i,j} t_i t_j \right) . \quad (3.11)$$

The following Proposition shows that under some sufficient conditions this expression is a proper multivariate survival function.

Proposition 3.2 *The following three conditions ensure that, for any fixed subset K , $\bar{F}_K(\mathbf{t})$ is a proper multivariate survival function:*

- (i) ψ and its derivatives goes fast enough to zero, i.e. $\lim_{x \rightarrow +\infty} x \psi^{(n)}(x) = 0$.
- (ii) ψ is k -monotone, where $k := |K|$,
- (iii) for all distinct i, j in K , $\theta_{i,j} \in [0, \theta_i \theta_j \rho_{\psi,k}]$,

where $\rho_{\psi,k} = \inf_{x \in \mathbb{R}^+, r \leq k/2, r \text{ odd}} \gamma_{k,r}^{-1} \left| \frac{\psi^{(k+1-r)}(x)}{\psi^{(k-r)}(x)} \right|$ and $\gamma_{k,r} = \frac{1}{r} \binom{k-2r+2}{2}$.

For example, if $|K| = k = 3$ and $\psi(x) = \exp(-x)$, then ψ is a k -monotone function satisfying condition (i) and (ii). It also satisfies (iii) with coefficient $\rho_{\psi,k} = \frac{1}{3}$, and the function \bar{F}_K defined in Equation (3.11) is a valid multivariate survival function if $\theta_{i,j} \leq \theta_i \theta_j / 3$ for all $i, j \in K$.

Proof. We have

$$\bar{F}_K(\mathbf{t}) = \psi \left(\sum_{\{i\} \subset K} \theta_i t_i + \sum_{\{i,j\} \subset K} \theta_{i,j} t_i t_j \right) = \psi(Q(\mathbf{t})) ,$$

with

$$Q(\mathbf{t}) = \sum_{\{i\} \subset K} \theta_i t_i + \sum_{\{i,j\} \subset K} \theta_{i,j} t_i t_j.$$

Let us consider without loss of generality $K = \{1, \dots, k\}$ and $\mathbf{t} = (t_1, \dots, t_k)$. Let us define

$$f_K(\mathbf{t}) = (-1)^k \frac{\partial^k}{\partial t_1 \dots \partial t_k} \bar{F}_K(\mathbf{t}).$$

If f_K is a non-negative function whose integral is one, then it will be the density of a random vector, and $\bar{F}_K(\mathbf{t})$ will be a proper multivariate cdf.

Positivity Let us first establish conditions under which f_K is a non-negative function. The multivariate Faa Di Bruno's formula gives

$$f_K(\mathbf{t}) = (-1)^k \sum_{\pi \in \Pi_K} \psi^{(|\pi|)}(Q(\mathbf{t})) \cdot \prod_{B \in \pi} \frac{\partial^{|B|} Q(\mathbf{t})}{\prod_{j \in B} \partial t_j}. \quad (3.12)$$

where Π_K is the set of all partitions of K . $B \in \pi$ means that B runs through all non-empty blocks of a considered partition π . In the following, we will write $\frac{\partial^2}{\partial \mathbf{t}_B} Q(\mathbf{t}) = \frac{\partial}{\partial t_i \partial t_j} Q(\mathbf{t})$, and $\theta_B = \theta_{i,j}$, where $B = \{i, j\}$.

Notice that $\partial^{|B|} Q(\mathbf{t})$ is 0 when $|B| \geq 3$. Thus the only partitions π involved in the calculation contain blocks of 1 or 2 elements only. Hereafter, we denote by Π_K^r the partitions in Π_K that contains exactly r distinct blocs of size 2. For a partition $\pi \in \Pi_K^r$, these r blocs will be denoted B_1^π, \dots, B_r^π . Such a partition $\pi \in \Pi_K^r$ contains r blocs of size 2 and $k - 2r$ blocs of size 1, so that $|\pi| = k - r$. Thus we get (with the convention $\prod_{i=1}^0 = 1$)

$$f_K(\mathbf{t}) = (-1)^k \sum_{r=0}^{\lfloor k/2 \rfloor} \sum_{\pi \in \Pi_K^r} \psi^{(k-r)}(Q(\mathbf{t})) \cdot \prod_{i=1}^r \frac{\partial^2 Q(\mathbf{t})}{\partial \mathbf{t}_{B_i^\pi}} \prod_{j \in K \setminus \cup_i B_i^\pi} \frac{\partial Q(\mathbf{t})}{\partial t_j}.$$

If ψ is k -monotone, $\psi^{(k-r)} = (-1)^{k-r} |\psi^{(n-r)}|$, and setting $\mathcal{N}_k = \{0, \dots, \lfloor k/2 \rfloor\}$,

$$f_K(\mathbf{t}) = \sum_{r \in \mathcal{N}_k} \sum_{\pi \in \Pi_K^r} \left| \psi^{(k-r)}(Q(\mathbf{t})) \right| \cdot \prod_{i=1}^r \left(-\frac{\partial^2 Q(\mathbf{t})}{\partial \mathbf{t}_{B_i^\pi}} \right) \prod_{j \in K \setminus \cup_i B_i^\pi} \frac{\partial Q(\mathbf{t})}{\partial t_j}.$$

One can write $f_K(\mathbf{t}) = \sum_{r \in \mathcal{N}_k} \xi(r)$. Assume that all $\theta_i \geq 0$ and $\theta_{i,j} \geq 0$, $i, j \in K$. Under this assumption, one can check that, when r is even, $\xi(r) \geq 0$. As a consequence,

$$\sum_{r \in \mathcal{N}_k} \xi(r) \geq \sum_{r \in \mathcal{N}_k, r \text{ odd}} [\xi(r) + \xi(r-1)].$$

Let us try to simplify $\xi(r) + \xi(r-1)$. First remark that for $r \geq 1$, $|\Pi_K^r| = \binom{k}{2} \dots \binom{k-2(r-1)}{2} / r! = 2^{-r} k! / (k-2r)! r!$, and $|\Pi_K^r| = \gamma_{k,r} |\Pi_K^{r-1}|$, with $\gamma_{k,r} = \frac{1}{r} \binom{k-2r+2}{2}$. Let us write $\xi(r) = \sum_{\pi \in \Pi_K^r} z(B_1^\pi, \dots, B_r^\pi)$. The term $\xi(r-1)$ can be written

$$\sum_{\pi \in \Pi_K^{r-1}} z(B_1^\pi, \dots, B_{r-1}^\pi) = \gamma_{k,r}^{-1} \sum_{\pi \in \Pi_K^r} z(B_1^\pi, \dots, B_{r-1}^\pi).$$

and thus

$$\xi(r) + \xi(r-1) = \sum_{\pi \in \Pi_K^r} \left[z(B_1^\pi, \dots, B_r^\pi) - \gamma_{k,r}^{-1} z(B_1^\pi, \dots, B_{r-1}^\pi) \right].$$

As a consequence, a sufficient condition to ensure that $f_K(\mathbf{t}) \geq 0$ is that: for any $B = \{i, j\}$, for any odd r , for any \mathbf{t} ,

$$-\frac{\partial^2 Q(\mathbf{t})}{\partial \mathbf{t}_{B_i}} \left| \psi^{(k-r)}(Q(\mathbf{t})) \right| + \gamma_{k,r}^{-1} \left| \psi^{(k+1-r)}(Q(\mathbf{t})) \right| \frac{\partial Q(\mathbf{t})}{\partial t_i} \frac{\partial Q(\mathbf{t})}{\partial t_j} \geq 0$$

Thus the sufficient condition to ensure the positivity of $f_K(x)$.

Absolute continuity Let us now check if the integral of f_K is summing to one. If so, \bar{F}_K would be a valid absolutely continuous distribution, without singular component. First assume that for all integers $n \leq k-1$,

$$\lim_{x \rightarrow +\infty} x \psi^{(n)}(x) = 0. \quad (3.13)$$

Recall that $\bar{F}_K(\mathbf{u}) = \psi(Q(u_1, \dots, u_k))$. We now make use of the multivariate Faa Di Bruno's formula as is Equation (3.12). Seen as function of u_n , the derivative of $\bar{F}_K(\mathbf{u})$ with respect to u_{k+1}, \dots, u_d writes as a sum of terms $\psi^{(i)}(au_k + b) \cdot P(u_k)$, where P is a polynomial of degree at most 1 and a, b some real values. Thus under chosen assumption,

$$\lim_{u_n \rightarrow +\infty} \frac{\partial^n}{\partial u_{n+1} \dots \partial u_k} \bar{F}_K(\mathbf{u}) = 0. \quad (3.14)$$

first show that for any $\mathbf{u} \in \mathbb{R}^k$, for all integer $n \leq k-1$,

$$\lim_{u_n \rightarrow +\infty} \frac{\partial^n}{\partial u_{n+1} \dots \partial u_k} \bar{F}_K(\mathbf{u}) = 0. \quad (3.15)$$

As a consequence, one can show by recursion that in this case

$$\bar{F}_K(t_1, \dots, t_k) = \int_{t_1}^{+\infty} \dots \int_{t_k}^{+\infty} \frac{\partial^k}{\partial u_1 \dots \partial u_k} \bar{F}_K(u_1, \dots, u_k) du_k \dots du_1. \quad (3.16)$$

Using the fact that $\bar{F}_K(0, \dots, 0) = 1$, we conclude that the derivative function f_K is non-negative and is integrating to one on the whole domain \mathbb{R}_+^k . Under chosen assumptions, it thus defines a proper probability measure and \bar{F}_K is a valid multivariate survival function. \square

Acknowledgment

This paper was written during an invited stay of the authors at the Vietnam Institute for Advanced Study in Mathematics (VIASM). The authors warmly thank the VIASM institute for his support.

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