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Solving the Tree Containment Problem in Linear Time for Nearly Stable Phylogenetic Networks[☆]

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Abstract

A phylogenetic network is a rooted acyclic digraph whose leaves are uniquely labeled with a set of taxa. The *tree containment problem* asks whether or not a phylogenetic network *displays* a phylogenetic tree over the same set of labeled leaves. It is a fundamental problem arising from validation of phylogenetic network models. The tree containment problem is NP-complete in general. To identify network classes on which the problem is polynomial time solvable, we introduce two classes of networks by generalizations of *tree-child networks* through vertex stability, namely nearly stable networks and genetically stable networks. Here, we study the combinatorial properties of these two classes of phylogenetic networks. We also develop a linear-time algorithm for solving the tree containment problem on binary nearly stable networks.

Keywords: phylogenetic trees, phylogenetic networks, tree containment, reticulation visibility, nearly stable networks, genetically stable networks

1. Introduction

With thousands of genomes being fully sequenced, phylogenetic networks have been adopted to study “horizontal” processes that transfer genetic material from a living organism to another without descendant relation. These processes are believed to be a driving force that shapes the genome of a species in evolution [6, 25].

A (*phylogenetic*) *network* over a set X of taxa is a rooted acyclic digraph with a set of leaves (*i.e.*, vertices of outdegree zero) that are uniquely labeled with the taxa. Such a network represents the evolutionary history of the taxa in X . In a network, the *tree vertices* (*i.e.*, vertices of indegree one) represent speciation events, whereas the vertices of indegree

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10 at least two (called *reticulation vertices*, or simply *reticulations*) represent genetic material
11 flow from several ancestral species into an “unrelated” species. A plethora of methods for
12 reconstructing networks and related algorithmic issues have been extensively studied over
13 the past two decades [16, 17, 23, 24, 27].

14 One approach to assessing the quality of a network is to verify whether or not it is
15 consistent with previous biological knowledge about the species. Biologists therefore demand
16 that the network displays existing gene trees, which corresponds to the *tree containment*
17 (TC) problem [17]. This problem is well-known to be NP-complete [19, 26], and great
18 efforts have therefore been devoted to identifying “tractable” subclasses of binary networks,
19 such as *galled trees* [19], *normal networks*, *tree-child networks* and *level- k networks* [26].
20 Recently, Gunawan et al. [12] and Bordewich and Semple [3] independently proved that the
21 TC problem can be solved in cubic time for binary reticulation-visible networks, thereby
22 settling an open problem [17, 26]. The time complexity was further improved to quadratic
23 time even for arbitrary (i.e. non-binary) reticulation-visible networks in [13].

24 To tackle the TC problem for reticulation-visible networks, we introduced *nearly stable*
25 *networks* and *genetically stable networks*, which both generalize tree-child networks, and
26 gave quadratic time algorithms for solving the TC problem on both classes [10, 11]. These
27 results gave an insight on the topological structure of a reticulation-visible network, and
28 eventually led to a solution to the open problem in [12]. In this paper, we establish the tight
29 upper bounds on the numbers of vertices in a nearly stable network and in a genetically
30 stable network. We further show on simulated data that these two new classes cover a
31 significant proportion of phylogenetic networks, compared with binary reticulation-visible
32 networks.

33 In this paper, we also revisit the TC problem for nearly stable networks. In an earlier
34 version of this work [10], we developed a quadratic-time algorithm for the TC problem on
35 that class. The time complexity of the algorithm was further improved to $O(n \log n)$ in
36 [8] using the same approach but with a more efficient data structure. Here, we develop a
37 linear-time TC algorithm by combining the structure analysis in [10] and a decomposition
38 technique introduced in [12]. This technique plays a vital role in the designs of a quadratic-
39 time TC algorithm for reticulation-visible networks and of a fast exponential-time algorithm
40 for arbitrary networks [14].

41 2. Concepts and notions

42 2.1. Phylogenetic trees and networks

43 A (*phylogenetic*) *network* on a set X of taxa is an acyclic digraph N with a single root
44 ρ_N (a unique vertex with indegree 0) whose leaves (vertices with outdegree 0) are in one-
45 to-one correspondence with the taxa in X . The fact that the root is the unique vertex with
46 indegree 0 implies that there is a (directed) path from the root to every other vertex. For
47 convenience, we attach an incoming edge to ρ_N (so the indegree becomes one) with one
48 open-end. We identify each leaf with the taxon corresponding to it.

49 In a network, *reticulation vertices* (or simply *reticulations*) are vertices with indegree
50 at least two and outdegree one; *tree vertices* are vertices with indegree one, which includes

51 the root and leaves. For a given network N , $\mathcal{L}(N)$ denotes its leaf set, $\mathcal{R}(N)$ its set of
 52 reticulation vertices, $\mathcal{T}(N)$ its set of tree vertices (including leaves), $\mathcal{V}(N)$ the entire set of
 53 vertices, and $\mathcal{E}(N)$ the entire set of edges. An edge is a *reticulation edge* if it is an edge
 54 entering a reticulation vertex.

55 Let N be a network. For an edge set $E \subseteq \mathcal{E}(N)$, $N - E$ denotes the subnetwork with
 56 vertex set $\mathcal{V}(N)$ and edge set $\mathcal{E}(N) \setminus E$. Similarly, for a vertex subset $S \subseteq \mathcal{V}(N)$, $N - S$
 57 denotes the subnetwork with the vertex set $\mathcal{V}(N) \setminus S$ and the edge set $\{(u, v) \in \mathcal{E}(N) \mid u \notin$
 58 $S, v \notin S\}$. When E or S contains only one element x , we simply write $N - x$.

59 Let $v \in \mathcal{V}(N)$. The *subnetwork of N induced by v* consists of v and all its descendants
 60 and the edges between them. It is rooted at v and is denoted by $N[v]$.

61 We say that a network obtained by removing all but one incoming edge from each retic-
 62 ulation in a network N is a *spanning tree* of N .

63 A network is *binary* if its leaves are of degree one, and all other vertices are of degree
 64 three. A *phylogenetic tree* is simply a binary network without reticulation vertices. All
 65 networks we shall consider in this paper are binary unless stated otherwise.

66 2.2. Reticulation stability

67 Consider a network N . Let x and y be two vertices of N . We say that x is a *parent* of
 68 y and y is a *child* of x if (x, y) is an edge of N . More generally, we say that $x \neq y$ is an
 69 *ancestor* of y (or *above* y) and equivalently y is a *descendant* of x (or *below* x) if there is
 70 a directed path from x to y in N . Two vertices are *siblings* if they are the children of the
 71 same vertex.

72 A vertex x is a *stable ancestor* of a vertex v (or x is *stable on* v) if it belongs to all
 73 directed paths from ρ_N to v . We say that x is *stable* (or *visible*) if there exists a leaf ℓ such
 74 that x is a stable ancestor of ℓ . A vertex is *unstable* if it is not a stable ancestor of any leaf.
 75 A network is *reticulation-visible* [17] if every reticulation vertex it contains is visible.

76 **Proposition 2.1.** (Lemma 2.3, [15]) *Let N be a network and $u \in \mathcal{V}(N)$. Let R be a set*
 77 *of reticulations below u such that for each $r \in R$, either (i) r is a descendant of another*
 78 *reticulation $r' \in R$, or (ii) there is a path from ρ_N to r that avoids u . Then, u is not stable*
 79 *on any leaf ℓ below a reticulation in R .*

80 By Proposition 2.1, we have the following simple criteria to determine whether a vertex
 81 is stable or not in a network.

82 **Corollary 2.1.** *Let v be a tree vertex in a network N .*

83 (1) *If v has two reticulation children, then v is unstable.*

84 (2) *Assume v has two children u and w such that u is a tree vertex with two reticulation*
 85 *children and w is a reticulation vertex (Fig. 1). If w is different from the children of*
 86 *u , then v is unstable.*

87 **Proposition 2.2.** *The following facts hold for a network.*

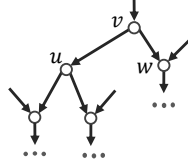


Figure 1: Two types of unstable tree vertices in a network.

- 88 (1) A vertex u is stable if it has a stable tree vertex as a child.
 89 (2) A reticulation u is stable if and only if its unique child is a stable tree vertex.
 90 (3) If u and v are stable ancestors of w , then either u is an ancestor of v or vice versa.

91 *Proof.* (1). Let v be a child of u . Assume it is a tree vertex and stable. Since v is stable,
 92 then it is a stable ancestor of some network leaf ℓ . Since v is a tree vertex, u is the unique
 93 parent of v . Taken together, both facts imply that all paths from ρ_N to ℓ must contain v
 94 and hence u . Therefore, u is also stable on ℓ .

95 (2). The sufficiency follows from (1). To prove the necessity, we assume that u is a
 96 reticulation stable on a leaf ℓ and let x denote its unique child. Since every path from ρ_N
 97 to ℓ goes through u , each such path must also go through x , which is therefore stable on ℓ .
 98 We now show by contradiction that x is a tree vertex: if x is a reticulation, it has another
 99 parent v different from u . For a path from ρ_N to ℓ passing u , it also passes x and so joining a
 100 path from ρ_N to v , the edge (v, x) and a path from x to ℓ gives a path from ρ_N to ℓ avoiding
 101 u , contradicting the fact that u is stable on ℓ . Therefore, x must be a tree vertex.

102 (3) The fact follows from the fact that both u and v must be within every path from ρ_N
 103 to w . □

104 We refer to leaves resulting from the removal of edges or vertices as *dummy leaves* (i.e.
 105 leaves that were not leaves in the original network). A network is *tree-based* if there is an
 106 edge set E that contains an incoming edge for each reticulation vertex such that $N - E$
 107 is a spanning tree of N without dummy leaves [9]. In [10], we used the following result to
 108 establish the first upper bound on the size of reticulation-visible network. Although a better
 109 bound can be obtained using a different approach, this result is interesting in its own right
 110 and therefore presented below.

111 **Theorem 2.1.** *Every reticulation-visible network is tree-based.*

112 *Proof.* Let N be a reticulation-visible network and $E \subset \mathcal{E}(N)$. If E contains two edges
 113 coming out of the same tree vertex, there will be a dummy leaf in $N - E$. Similarly, if E
 114 contains two edges entering the same reticulation vertex, this reticulation vertex will become
 115 a vertex of indegree 0 in $N - E$. Therefore, $N - E$ is a spanning tree of N without dummy
 116 leaves if and only if E is a matching covering every reticulation vertex in N .

117 Since N is reticulation-visible, by the part (2) of Proposition 2.2, the parents of each
 118 reticulation vertex are tree vertices. The existence of such a matching can be found by

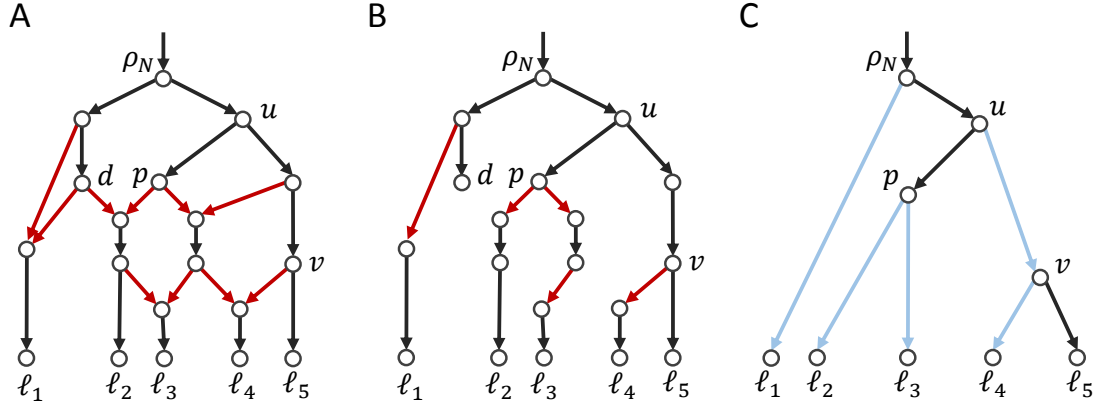


Figure 2: (A) A network N with five leaves labeled with ℓ_i ($1 \leq i \leq 5$). Reticulation edges are colored dark red. (B) A spanning tree T obtained by removing a reticulation edge from each reticulation in the network, in which there is a dummy leaf d . (C) A tree that is displayed in N , which can be obtained from T by contraction. Each shallow blue edge corresponds to a path consisting of two or more edges in T .

119 applying Hall’s Theorem to a bipartite graph with one part consisting of tree vertices and
 120 another consisting of reticulation vertices, and the edges being reticulation edges in N . Since
 121 each reticulation vertex has two incoming reticulation edges and each tree vertex has at most
 122 two outgoing reticulation edges, the existence of such a matching E follows from a theorem
 123 of Alon [2, page 429]. \square

124 2.3. The tree containment problem

Let $(u, v) \in \mathcal{E}(N)$. The *contraction* of (u, v) in N transforms N into the network with the vertex set $\mathcal{V}(N) \setminus \{v\}$ and the following edge set:

$$\{(x, y) \in \mathcal{E}(N) \mid x \neq v \neq y\} \cup \{(x, u) \mid (x, v) \in \mathcal{E}(N) \setminus \{(u, v)\}\} \cup \{(u, y) \mid (v, y) \in \mathcal{E}(N)\}.$$

125 Given a phylogenetic tree G and a network N , we say that N *displays* G if there is a
 126 spanning tree T of N such that G is a *contraction* of T , i.e. T has the same vertex set as N
 127 and G can be obtained from T by repeatedly applying contraction to all edges entering either
 128 the “dummy” leaves or the vertices of both outdegree and indegree one, where a dummy
 129 leaf is a leaf in T but not a leaf in N . Figure 2 shows an example of a network N and a
 130 phylogenetic tree that is displayed by N .

131 The *tree containment* (TC) problem is to determine whether a given phylogenetic network
 132 displays a given phylogenetic tree. We shall discuss the TC problem for network class defined
 133 using vertex stability, namely the nearly stable networks.

134 3. Nearly stable and genetically stable networks

135 3.1. Inclusion Relationship

136 A network is *tree-child* if each of its internal vertices has a child that is a tree vertex [5].
 137 A binary network is *tree-sibling* if every reticulation vertex has a sibling that is a tree

138 vertex [4, 22]. Cordue et al. [7] investigated reticulation-visible networks in which each
139 reticulation has at least a parent p that is connected to some leaf by a path consisting of
140 only tree vertices. Here, such networks are defined to be *nearly tree-child* networks.

141 Huson et al. [17, page 164] noted that if a network is tree-child, then all its vertices are
142 stable. We strengthen their result by proving the other direction.

143 **Proposition 3.1.** *A network is tree-child if and only if every vertex is stable in the network.*

144 *Proof.* The sufficiency follows from Proposition 2.2(2) and Corollary 2.1. \square

145 To extend tree-child networks, we introduced two subclasses of phylogenetic networks. A
146 network is *nearly stable* if for every vertex, either the vertex or its parents are stable [10]. It is
147 *genetically stable* if every reticulation vertex is stable and has at least one stable parent [11].

148 **Proposition 3.2.**

150 (1) *Every genetically stable network is tree-sibling.*

151 (2) *Every tree-child network is nearly stable.*

152 (3) *Every nearly tree-child network is genetically stable.*

153 *Proof.* (2) and (3) follow from the definitions. For (1), let N be a genetically stable network
154 and let $p \in \mathcal{R}(N)$. Since N is genetically stable, p has a stable parent p' . By Proposi-
155 tion 2.2(2), p' is a tree vertex, so it must have another child c . By Corollary 2.1, c is a tree
156 vertex, and N is therefore tree-sibling. \square

157 Based on Proposition 3.2, we summarize the relationships between the classes we study
158 and the other network classes for which the complexity of the TC problem is known in
159 Figure 3.

160 A reticulation-visible and tree-sibling network is not necessarily genetically stable (Fig-
161 ure 4.A). A genetically stable network is not necessarily nearly tree-child (Figure 4.B). A
162 nearly stable and nearly tree-child network is not necessarily tree-child (Figure 4.C). A
163 nearly tree-child network is not necessarily nearly stable (Figure 4.D).

164 3.2. Class sizes

165 Recombination histories of viruses, hybridization histories of plants, and histories of
166 horizontal gene transfers reported in the literature are often found to be nearly stable or
167 reticulation-visible (see e.g. the networks given in [18, 21] which are available at [http:
168 //phylnet.info/recophync/networkDraw.php](http://phylnet.info/recophync/networkDraw.php)).

169 In order to evaluate whether the class of nearly stable networks is relevant in practice,
170 especially combined with the class of reticulation-visible networks for which there also exists
171 a polynomial-time algorithm solving the TC problem, we used a set of phylogenetic networks
172 randomly generated using a simulation program [1] and calculated the proportion of those
173 networks belonging to the classes¹. Figure 5 summarizes the results of our simulation study.

¹A Python script for class recognition as well as the data and the obtained results are available at <http://phylnet.info/recophync/>, and an online demo is provided at <http://phylnet.info/tools/>.

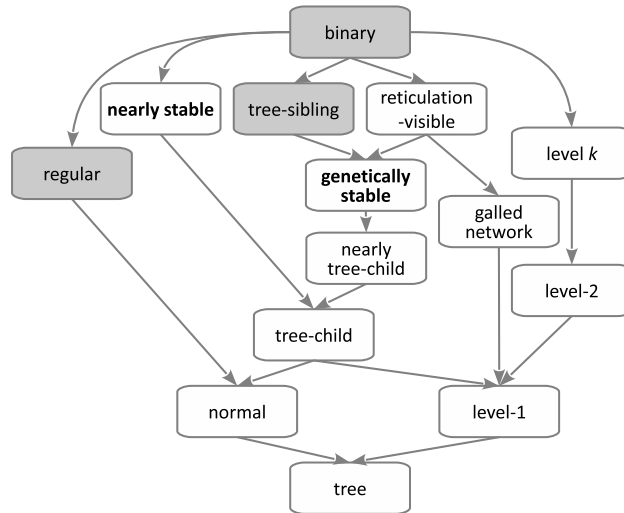


Figure 3: Inclusion relationships between classes of binary phylogenetic networks: an arrow from class A to class B means that A contains B. Class inclusions involving nearly stable and genetically stable are justified in this article (Proposition 3.2); for the other ones, references are available at <http://phylnet.info/isiphync/>. The boxes of classes where the TC problem is NP-complete are colored gray, the ones where the TC problem is solvable in polynomial time are colored white.

174 This experiment shows that among networks generated with the coalescent with recom-
 175 bination model, the proportions of reticulation-visible and especially nearly stable networks
 176 are significantly larger than that of tree-child networks. Furthermore, the proportion of
 177 networks which are reticulation-visible or nearly stable is also significantly larger than the
 178 proportion of reticulation-visible networks.

179 4. How large can nearly stable and genetically stable networks be?

180 A network with n leaves may contain an arbitrary large number of non-leaf vertices and
 181 hence, unlike phylogenetic trees, its size is not bounded from above by a function of the
 182 number of leaves. In [10], we proved that a reticulation-visible network with n leaves has
 183 at most $10n - 9$ vertices (including leaves). Later, the tight size bound $8n - 7$ was proved

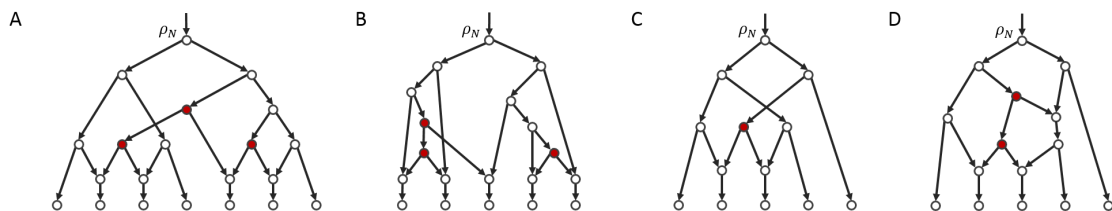


Figure 4: (A) A network which is reticulation-visible and tree-sibling, but not genetically stable. (B) A network which is genetically stable but neither nearly stable nor nearly tree-child. (C) A network which is nearly tree-child and nearly stable but not tree-child. (D) A network which is nearly tree-child but not nearly stable. Here, the filled vertices are unstable.

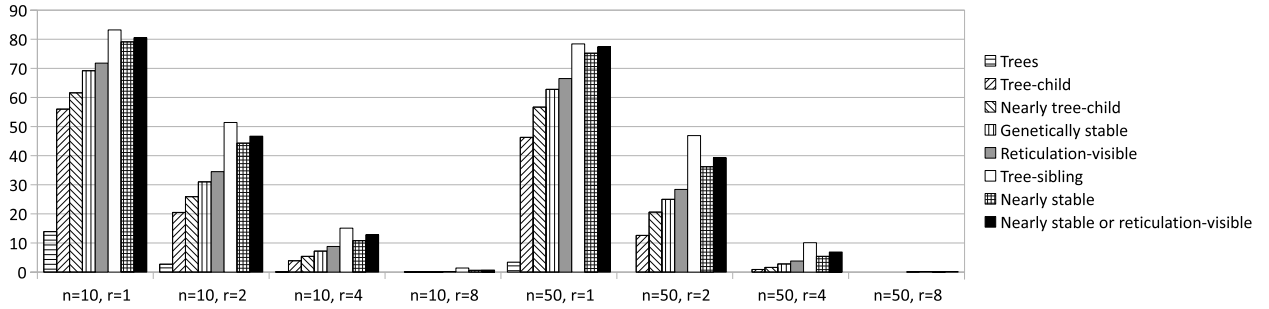


Figure 5: Percentage of binary phylogenetic networks on n leaves generated with the coalescent with recombination model (recombination rate r) in each class of phylogenetic networks. The corresponding data is available at <http://phylnet.info/recophync/>.

184 [3]. In this section, using the decomposition theorem introduced below, we shall first give
 185 a short proof of this tight bound. We also show that nearly stable networks have the same
 186 tight size bound as reticulation-visible networks, whereas genetically stable networks with
 187 n leaves have the tight size bound $6n - 5$.

188 4.1. A decomposition theorem

189 Consider a network N . After the removal of all reticulations, N becomes a forest $N -$
 190 $\mathcal{R}(N)$. One connected component of the forest is rooted at ρ_N , whereas the other components
 191 are each rooted at the child of a reticulation (Figure 6). Since components consist of tree
 192 vertices of N , they are called *tree vertex components* of N . We call them *big* tree vertex
 193 components if they contain more than one vertex.

194 A reticulation is said to be *inner* if its two parents belong to the same tree vertex
 195 components. Otherwise, it is said to be *cross*.

196 We now prove the following decomposition theorem for nearly stable networks, which is
 197 similar to Theorem 1 of [13] for reticulation-visible networks.

198 **Theorem 4.1.** *Let N be a nearly stable or reticulation-visible network with tree vertex*
 199 *components $C_0, C_1, C_2, \dots, C_r$. Then,*

200 (1) *Each component C_j is rooted at a stable tree vertex. Additionally, a vertex is a compo-*
 201 *nent root if and only if it is either the network root or the child of a stable reticulation.*

202 (2) *Each component C_j contains either a network leaf or the two parents of an inner*
 203 *reticulation.*

204 (3) *Each component C_j contains at least two tree vertices if $C_j \neq \{\ell\}$ for any leaf $\ell \in \mathcal{L}(N)$.*

205 *Proof.* (1) By definition of a tree vertex component, its root r must be a tree vertex. If r is
 206 the root of N , then it is obviously stable. Otherwise, as r is a tree vertex, then its parent
 207 p is a reticulation (otherwise r is not a component root). If p is unstable, then r must be
 208 stable as N is a nearly stable network, but this contradicts Proposition 2.2(1). Therefore,

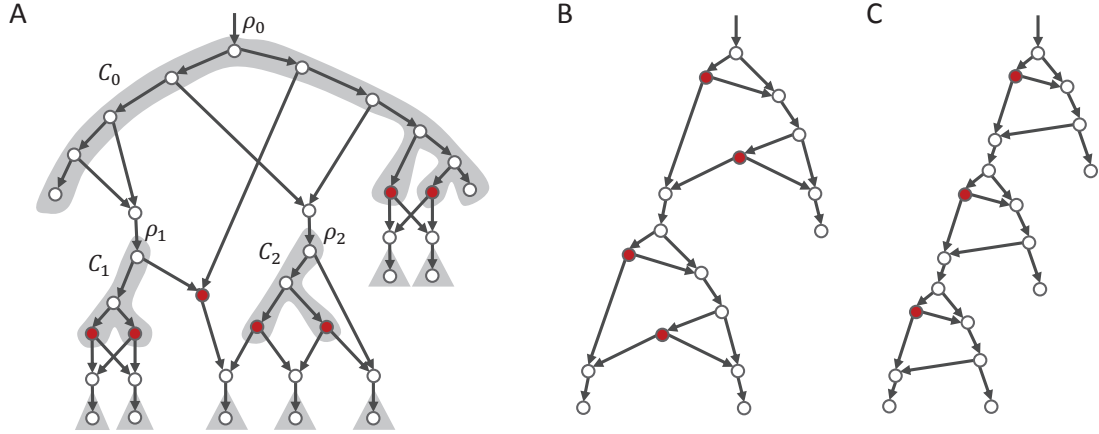


Figure 6: **(A)** The decomposition of a nearly stable network into ten tree vertex components. Seven components consist of a single leaf. The remaining three “big” components are C_0, C_1 , and C_2 , where ρ_i denotes the root of C_i for each $i = 1, 2, 3$. **(B)** A network with 3 leaves and 6 reticulations that is both nearly stable and reticulation-visible. **(C)** A genetically stable network with 4 leaves and 6 reticulations. Here the filled vertices are unstable ones.

209 p is a stable reticulation vertex and r is a stable tree vertex according to the part (2) of
 210 Proposition 2.2. Clearly, every child of a stable reticulation is a tree vertex, and thus is a
 211 component root as well.

212 (2) For a component C_j that does not contain a network leaf, each leaf of C_j is a parent
 213 of some reticulation below C_j . If for any reticulation below C_j not all parents are in C_j ,
 214 then, the set of the reticulations satisfies the conditions in Proposition 2.1 and hence the
 215 root of C_j is not stable, contradicting (1).

216 (3) Suppose for contradiction that C_j contains exactly one tree vertex v and v is not
 217 a leaf. Then both children of v are reticulation vertices, and so v is unstable according to
 218 Corollary 2.1(1), contradicting the condition (1) above (i.e. v is stable).

219 □

220 For a tree vertex component C of N , we denote its root by ρ_C . A tree vertex component
 221 C is *below* another component D , if there is a path from ρ_D to ρ_C in N . A *lowest big tree*
 222 *vertex component* is a component such that every other component below it contains exactly
 223 a single leaf. Such component is always guaranteed to exist, see [12].

224 4.2. Three size bounds

225 Let N be a network. In this subsection, we use r and t to denote the number of reticu-
 226 lations and non-leaf tree vertices in N .

227 **Theorem 4.2.** (i.) ([3]) *If N is reticulation-visible, then $r \leq 3(n - 1)$.*

228 (ii.) *If N is reticulation-visible and tree-sibling, $r \leq 2(n - 1)$. In particular, the bound
 229 holds for genetically stable networks.*

230 (iii.) *If N is nearly stable, then $r \leq 3(n - 1)$.*

231 *Proof.* (i.) Ignoring the open-edge attached to the root, the network root is of indegree 0
 232 and outdegree 2, the other tree vertices are of indegree 1 and outdegree 2 if they are not
 233 leaves, and each reticulation vertex is of indegree 2 and outdegree 1. By the handshaking
 234 lemma, $2t + r = t - 1 + 2r + n$, which is further simplified into:

$$t = r + n - 1. \quad (1)$$

235 Additionally, we let c be the number of tree vertex components of N for the rest of the
 236 proof. By Theorem 4.1(1), $c = r + 1$.

237 Consider a component C . Since N is reticulation-visible, by Theorem 4.1(3), C contains
 238 two distinct parents of an inner reticulation if it does not contain a network leaf. Hence,

$$r + 1 = c \leq n + t/2. \quad (2)$$

239 Replacing t with $r + n - 1$ in Equation (2), we obtain that $r \leq 3(n - 1)$.

240
 241 (ii.) Assume N is reticulation-visible and tree-sibling. We distinguish three types of tree
 242 vertices of N by using T_i to denote the set of tree vertices with exactly i children being also
 243 tree vertices for $i = 0, 1, 2$, respectively.

244 Since N is tree-sibling, each reticulation vertex x has a parent v_x such that v_x is a tree
 245 vertex and has x and another tree vertex as its children. Therefore, mapping x to v_x is an
 246 injective map from $\mathcal{R}(N)$ to T_1 and thus:

$$r \leq |T_1|. \quad (3)$$

247 Consider a tree vertex component C of N that does not contain any network leaf. By
 248 the fact (3) of Theorem 4.1, C contains a leaf v that differs from ρ_C . The children of v must
 249 be reticulation vertices, and hence $v \in T_0$ (Fig 7.A). Since there are at most n tree vertex
 250 components that contain one or more network leaves,

$$r + 1 = c \leq |T_0| + n. \quad (4)$$

Combining Inequalities (3) and (4) with Equation (1), we have:

$$2r + 1 \leq |T_1| + |T_0| + n \leq t + n = r + 2n - 1,$$

251 and thus $r \leq 2(n - 1)$.

252
 253 (iii.) Let N be a nearly stable network. Given that each reticulation vertex may or may
 254 not be stable, we use r_s and r_u to denote the numbers of stable and unstable reticulation
 255 vertices of N , respectively.

256 First, for an unstable reticulation vertex, its parents are both stable tree vertices, as N is
 257 nearly stable. By Corollary 2.1, a tree vertex with two reticulation children is unstable. This
 258 implies that any two different unstable reticulation vertices have distinct parents. Therefore,
 259 there are at least $2r_u$ stable tree vertices that have an unstable reticulation child (Fig 7.B).

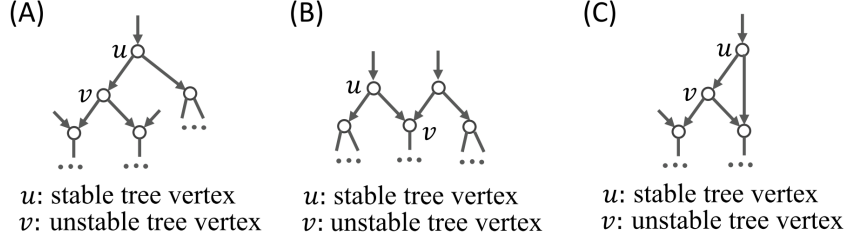


Figure 7: Illustration of three types of tree vertices in a nearly stable network. **(A)** An unstable tree vertex v with two reticulation children and a stable tree vertex u with two tree children. **(B)** A stable tree vertex u for which a child is an unstable reticulation vertex. **(C)** A stable tree vertex u that has a common reticulation child with its tree vertex child.

260 Secondly, for a tree vertex component C not containing any network leaf, its root is a
 261 stable tree vertex. By the fact (3) of Theorem 4.1, C contains a leaf v that differs from
 262 the root. The children of v are both reticulation vertices and thus, by Corollary 2.1, v is
 263 unstable. This implies that:

$$c - n \leq t_u, \quad (5)$$

264 as there are at most n components that contain at least a network leaf. Additionally, since N
 265 is nearly stable, the parent u of v has to be a stable tree vertex. Since v has two reticulation
 266 children, by Corollary 2.1, the stability of u implies that the other child of u has to be
 267 either a tree vertex (Fig. 7.A) or a child of v (Fig. 7.C). Since there are at least $c - n$ tree
 268 vertex components that do not contain any network leaf, there are at least $c - n$ stable tree
 269 vertices such that their children are either two tree vertices or a tree vertex and one (stable)
 270 reticulation vertex.

In summary, we have shown that (i) N contains at least $2r_u$ stable tree vertices that
 have an unstable reticulation child and (ii) there are at least $c - n$ stable tree vertices that
 have no unstable reticulation child. Hence,

$$t_s \geq (c - n) + 2r_u.$$

By Inequality (5),

$$t = t_u + t_s \geq 2(c - n) + 2r_u.$$

Since each tree vertex component is rooted at either the network root or the child of a
 stable reticulation vertex, then, $c = r_s + 1$. Replacing c with $r_s + 1$ and $r_s + r_u$ with r in
 the last inequality, we have:

$$t \geq 2r + 2 - 2n.$$

271 This inequality and Eqn. 1 imply that $3(n - 1) \geq r$. This completes the proof. \square

272 Fig. 6.B gives a nearly stable and reticulation-visible network with 3 leaves and 6 reticu-
 273 lation vertices. Fig. 6.C gives a genetically stable network with 4 leaves and 6 reticulations.
 274 Hence, the second and third bounds in Theorem 4.2 are also tight.

275 Finally, it is not hard to see that in nearly tree-child network, each tree-vertex component
 276 contains at least a network leaf and thus we have $r = c - 1 \leq n - 1$.

277 **5. A linear-time TC algorithm for nearly stable networks**

278 In this section, we present a recursive linear-time algorithm for TC on nearly stable
 279 networks.

280 *5.1. Stable tree vertices*

281 Let N be a nearly stable network. Consider a lowest big tree vertex component C of N ,
 282 and let ρ_C denote the root of C . Let u be a vertex in C . If u is stable, then all vertices in
 283 the path from ρ_C to u are stable by Proposition 2.2(1). Therefore, the stable vertices in C
 284 span a subtree of C with the same root ρ_C (Figure 8.A), called the *stable subtree* of C . The
 285 following proposition characterizes vertices outside the stable subtree.

286 **Proposition 5.1.** *Let N be a nearly stable network, and C be a lowest big tree vertex*
 287 *component of N . Then every unstable tree vertex in C has two reticulation children.*

288 *Proof.* Let x be an unstable tree vertex and y be a child of x in C . Assume y is a tree vertex.
 289 If y was stable, then x would also be stable (Proposition 2.2(1)), a contradiction. Therefore
 290 both y and x should be unstable, which contradicts the fact that N is nearly stable, so y
 291 cannot be a tree vertex. □

292 In the example given in Figure 8, the stable subtree has four leaves ℓ_9, s_1, s_2, s_3 . Clearly,
 293 every network leaf in C is a leaf of the stable subtree. We let $S(C)$ denote the set of leaves
 294 of the stable subtree that are not network leaves, i.e.:

$$S(C) = \{s \in \mathcal{V}(C) \setminus \mathcal{L}(N) : s \text{ is stable but every tree vertex child of } s \text{ is unstable}\}. \quad (6)$$

295 **Proposition 5.2.** *Let x be a vertex in C . Then, $x \in S(C)$ if and only if each child of x in*
 296 *C is a leaf of C but not a network leaf.*

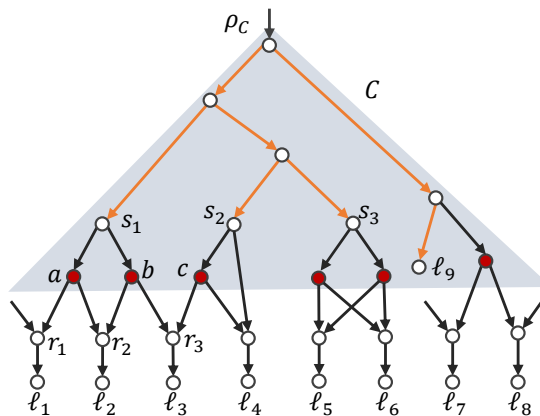


Figure 8: Illustration of the idea of our algorithm. A tree-vertex component C of a nearly stable network, where filled vertices are unstable ones and open edges have their end in other components. The orange subtree is the subtree spanned by the stable vertices with four leaves s_1, s_2, s_3 , and ℓ_9 .

297 *Proof.* Let $x \in S(C)$.

298 Suppose by contradiction that x has no child in C . Then x has two reticulation children
 299 r_1 and r_2 . Thus, by Corollary 2.1, x is unstable: contradiction. Then let y be a child of x in
 300 C . By definition, y is unstable and thus is not a network leaf. If y is not a leaf of C , there
 301 is an unstable tree vertex below y , contradicting that N is nearly stable.

302 Conversely, let y be a leaf of C but not a network leaf. Then, y has two reticulation
 303 children. Thus, by Corollary 2.1, y is unstable. Since N is nearly stable, the parent of y (i.e.
 304 the vertex x) must be stable. If the sibling of y is either a reticulation or a leaf of C but not
 305 a network leaf, its parent is then a lowest stable vertex in C and hence is in $S(C)$. \square

306 5.2. Two key lemmas

307 Let N and G be the given network and phylogenetic tree. We assume that N does not
 308 contain any parallel edge nor a vertex with indegree and outdegree one.

309 The following lemma limits the possible mini-structures below a vertex $s \in S(C)$ for a
 310 lowest tree vertex component C . We use $N[s]$ to denote the subnetwork consisting of all the
 311 vertices below s (including reticulation vertices below C).

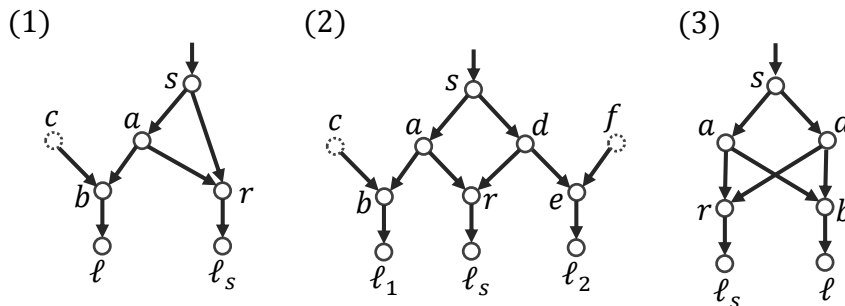


Figure 9: Three possible mini-structures below a vertex $s \in S(C)$ in a lowest component C . Here, the vertices c and f have undetermined vertex type, each of them can be a tree or reticulation vertex.

312 **Lemma 5.1.** *Let C be a lowest tree vertex component of a network N . For $s \in S(C)$, $N[s]$*
 313 *has only three possible mini-structures given in Figure 9.*

314 *Proof.* Assume s is stable on a leaf ℓ_s . The leaf ℓ_s cannot be in C . Otherwise all the vertices
 315 in the path from s to ℓ_s are stable, contradicting the fact that $s \in S(C)$. Thus, ℓ_s is the
 316 unique child of an inner reticulation r below s . Let u be a parent of r . If u is a reticulation,
 317 it is then unstable and its parents are two tree vertices equal to or below s in C . Since
 318 $s \in S(C)$, u and each of its parents below s are unstable, contradicting that N is nearly
 319 stable. Hence the parents of r are two tree vertices below s in C .

320 Since N does not contain parallel edges, at least a parent of r is not s . Let a be such a
 321 parent of r . Note that a is a child of s , otherwise, since $s \in S(C)$, any parent of the tree
 322 vertex a below s in C would not be a leaf of C , therefore contradicting Proposition 5.2. The
 323 fact that $s \in S(C)$ also implies that a is unstable. Since N is nearly stable, there are no

324 two unstable vertices that appear consecutively in a path. This implies that a must also be
 325 a child of s and the other child b of a is a stable reticulation for which the child is a network
 326 leaf ℓ , as C is a lowest component of N .

327 Let d be the other parent of r . If $d = s$, we obtain the mini-structure (1) in Fig. 9.

328 If $d \neq s$, the other child of d is also a stable reticulation with a network leaf as its child, as
 329 N is nearly stable. If a and d have distinct child other than r , we obtain the mini-structure
 330 (2) in Fig. 9. If a and d have the same children, we obtain the mini-structure (3) in Fig. 9.
 331 This completes the proof. \square

332 Now, we consider network leaves in C . For a network leaf ℓ in C , let p be the parent of
 333 ℓ . The following lemma gives all possible mini-structures of $N[p]$ if the sibling of ℓ is not a
 334 stable tree vertex.

335 **Lemma 5.2.** *If the sibling of ℓ is not a stable tree vertex in C , $N[p]$ has only three possible*
 336 *mini-structures given in Figure 10.*

337 *Proof.* Let v be the sibling of ℓ . It is either a tree vertex or a reticulation vertex.

338 If v is a stable reticulation, the child of v must be a network leaf. Then, we obtain the
 339 mini-structure (1) in Figure 10.

340 If v is an unstable reticulation, then its child w is a stable reticulation. Clearly, the child
 341 of w is a leaf. This gives the mini-structure (2) in Figure 10.

342 Finally, if v is an unstable tree vertex, the children of v must be stable, as N is nearly
 343 stable. By the part (1) of Proposition 2.2, the children of v are stable reticulation vertices,
 344 for which the unique child is a network leaf. Otherwise, v is stable. Thus, we obtain the
 345 mini-structure (3) in Figure 10. \square

346 5.3. Dissolving the lowest components

347 Now, we show how to dissolve a lowest big tree vertex component C by working one by
 348 one on the subnetworks below the vertices in $S(C)$ and then the parents of network leaves
 349 in C .

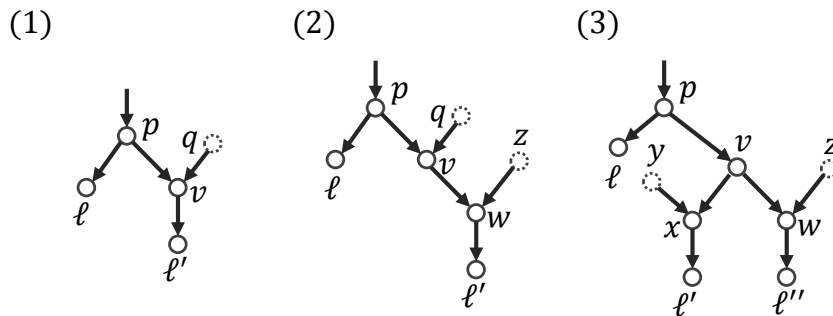


Figure 10: Three possible mini-structures below a tree vertex for which one child is a network leaf and the other is not a stable tree vertex in a lowest component. The vertices q , y and x have undetermined vertex type.

350 In the rest of this discussion, we use $x \wedge_G y$ (resp. $x \text{A}_G y$) to denote that x and y are
 351 (resp. not) siblings in G . We also use $\text{par}_G(v)$ to denote the parent of v in G .

352 First, we consider the mini-structure (1) in Figure 10, which is named the “uncle-nephew
 353 structure” in [10], where two leaves ℓ and ℓ' have the uncle and nephew relationship and q
 354 is either a tree or reticulation vertex.

Lemma 5.3. *If N contains the mini-structure (1) in Figure 10, define*

$$E = \begin{cases} (q, v) & \text{if } \ell \wedge_G \ell'; \\ (p, v) & \text{otherwise.} \end{cases}$$

355 *Then N displays G if and only if $N - E$ displays G .*

356 *Proof.* Let $N' = N - E$. If N' displays G , then N also displays G , as N' is a subnetwork of
 357 N . To prove the necessity, we assume that N displays G with a spanning tree T such that
 358 G is a contraction of T . There are two possible cases.

359 1. $\ell \wedge_G \ell'$: note that T contains either (p, v) or (q, v) exclusively. If T contains (p, v) , N'
 360 displays G , as T is also a spanning tree of N' .

361 If T contains (q, v) , let h be the lowest common ancestor of ℓ and ℓ' in T . Since any
 362 path to ℓ must contain p , p must be in the path from h to p in T and $T[h]$ only has
 363 two labeled leaves, namely ℓ and ℓ' .

364 Define $T' = T - (q, v) + (p, v)$. Then, the vertex p becomes the lowest common ancestor
 365 of ℓ and ℓ' in T' , so $T[h]$ and $T'[h]$ do not contain any network leaves other than ℓ and
 366 ℓ' . Any edge of T that is not below h is also an edge in T' and conversely, so G is a
 367 contraction of both T and T' . The tree T' is a spanning tree of $N - (q, v)$, and thus
 368 is the evidence that $N - (q, v)$ displays G .

369 2. $\ell \text{A}_G \ell'$: if (p, v) is an edge in T , then ℓ and ℓ' has p as their lowest common ancestor
 370 in T , contradicting that ℓ and ℓ' are not sibling in G . Thus, T does not contain (p, v)
 371 and T is also a spanning tree of $N - (p, v)$, implying that N' also displays G .

372 □

373 Using the above lemma, we can prove the following facts. These facts suggest that we
 374 can dissolve the subnetwork below each $s \in S(C)$ by using the structural information on G .

375 **Lemma 5.4.** *Let $s \in S(C)$. Define N' as follows.*

376 (i) *If $N[s]$ has the mini-structure (1) in Figure 9,*

$$N' = N - \begin{cases} \{(a, r), (c, b)\} & \text{if } \ell \wedge_G \ell_s, \\ \{(a, r), (a, b)\} & \text{otherwise.} \end{cases} \quad (7)$$

377 (ii) If $N[s]$ has the mini-structure (2) in Figure 9,

$$N' = N - \begin{cases} \{(a, b), (a, r), (d, e)\} & \text{if neither } \ell_1 \wedge_G \ell_s \text{ nor } \ell_2 \wedge_G \ell_s, \\ \{(c, b), (d, r), (f, e)\} & \text{if } \ell_1 \wedge_G \ell_s \text{ and } \text{par}_G(\ell_s) \wedge_G \ell_2, \\ \{(c, b), (d, r), (d, e)\} & \text{if } \ell_1 \wedge_G \ell_s \text{ but } \text{par}_G(\ell_s) \nwedge_G \ell_2, \\ \{(c, b), (a, r), (f, e)\} & \text{if } \ell_2 \wedge_G \ell_s \text{ and } \text{par}_G(\ell_s) \wedge_G \ell_1, \\ \{(a, b), (a, r), (f, e)\} & \text{if } \ell_2 \wedge_G \ell_s \text{ but } \text{par}_G(\ell_s) \nwedge_G \ell_1. \end{cases} \quad (8)$$

378 (iii) If $N[s]$ has the mini-structure (3) in Figure 9, $N' = N - \{(a, b), (d, r)\}$.

379 Then N displays G if and only if N' displays G .

380 *Proof.* Since N' is a subnetwork of N in each case, N displays G if N' displays G . To prove
381 the other direction, assume that N displays G . There is a spanning tree T of N such that
382 G is a contraction of T .

383 (i) The reticulation edge (a, r) is redundant. $N - (a, r)$ is essentially an uncle-nephew
384 structure. Hence by Lemma 5.3, we have N displays G only if N' defined in Eqn. (7) displays
385 G .

386 (ii) In Eqn. (8), the fourth and fifth cases are symmetric to the second and third cases,
387 respectively. Hence, we only consider the first three cases.

388 CASE 1. Neither $\ell_1 \wedge_G \ell_s$ nor $\ell_2 \wedge_G \ell_s$.

389 Clearly, neither (a, b) nor (d, e) are in T . Otherwise, either ℓ_1 or ℓ_2 is the sibling of ℓ_s ,
390 contradicting the assumption. So T is a spanning tree of $N - \{(a, b), (d, e)\}$. If (d, r) is in
391 T , then, T is also a spanning tree of N' . If (a, r) is in T , then $T - \{(a, r)\} + \{(d, r)\}$ also
392 displays G and is a spanning tree of N' . Thus, N displays G only if N' displays G .

393 CASES 2 and 3. $\ell_1 \wedge_G \ell_s$.

394 We first claim that $N - (d, r)$ displays G . If (d, r) is not in T , then the claim is true. If
395 (d, r) is in T , then (d, e) is not in T . Otherwise, ℓ_s and ℓ_2 are siblings in T . Hence, G is also
396 a contraction of $T - (d, r) + (a, r)$, and $N - (d, r)$ displays G .

397 Next, note that there is essentially an uncle-nephew structure on the leaves ℓ_1 and ℓ_s in
398 $N - (d, r)$. Hence, the assumption that $\ell_1 \wedge_G \ell_s$ implies that $N - \{(d, r), (c, b)\}$ displays G .

399 Finally, in $N - \{(d, r), (c, b)\}$, we have essentially an uncle-nephew structure after con-
400 traction of the subtree below a . Therefore, if ℓ_2 is the sibling of the parent of ℓ_1 and ℓ_s in
401 G , $N - \{(d, r), (c, b), (f, e)\}$ displays G . Otherwise, $N - \{(d, r), (c, b), (d, e)\}$ displays G .

402 (iii) In this case, deleting which edge entering at r and b makes no difference. Therefore,
403 N displays G if and only if N' displays G . \square

404 After we modify $N[s]$ according to the rules suggested in Lemma 5.4 for every $s \in S(C)$,
405 the subtree below each s consists of network leaves, vertices of degree 2 and/or dummy
406 vertices in the resulting network N' (Figure 11). We can further replace $N[s]$ and the
407 corresponding subtree in G by the same leaf ℓ_s , if they are compatible. Otherwise, we
408 conclude that N does not display G and stop the algorithm.

409 In summary, the procedure for simplifying the subnetwork below a lowest stable vertex
410 in C is given in Algorithm 1.

Algorithm 1: Dissolving lowest stable vertices

Procedure *Dissolve_Lowest_Stable_Vertices*(N, C, s)

Input: a network N , a component C , a vertex s

Output: simplified N with reticulations below s being eliminated

```
1 if Case 1 holds (Fig. 9.1) then
2   if  $l \wedge_G l_s$  then
3      $\lfloor$  delete  $(a, r)$  and  $(c, b)$  (Fig. 11.1); contract the edge(s) entering  $c$ ;
4   else
5      $\lfloor$  delete  $(a, r)$  and  $(a, b)$  (Fig. 11.2); contract  $(c, b)$ ;
6 if Case 2 holds (Fig. 9.2) then
7   if neither  $l_1 \wedge_G l_s$  nor  $l_2 \wedge_G l_s$  then
8      $\lfloor$  delete  $(a, b)$ ,  $(a, r)$ , and  $(d, e)$  (Fig. 11.4);
9      $\lfloor$  contract the edges  $(c, b)$  and  $(f, e)$ ;
10  else if  $l_1 \wedge_G l_s$  and  $\text{par}_G(l_s) \wedge_G l_2$  then
11     $\lfloor$  delete  $(c, b)$ ,  $(d, r)$ , and  $(f, e)$  (Fig. 11.5);
12     $\lfloor$  contract the edge(s) entering  $c$  or  $f$ ;
13  else if  $l_1 \wedge_G l_s$  but  $\text{par}_G(l_s) \nwedge_G l_2$  then
14     $\lfloor$  delete  $(c, b)$ ,  $(d, r)$ , and  $(f, e)$ ;
15     $\lfloor$  contract the edge(s) entering  $f$  and the edge  $(c, b)$ ;
16  else if  $l_2 \wedge_G l_s$  and  $\text{par}_G(l_s) \wedge_G l_1$  then
17     $\lfloor$  delete  $(c, b)$ ,  $(a, r)$ , and  $(f, e)$ ;
18     $\lfloor$  contract the edge(s) entering  $c$  or  $f$ ;
19  else if  $l_2 \wedge_G l_s$  but  $\text{par}_G(l_s) \nwedge_G l_1$  then
20     $\lfloor$  delete  $(a, b)$ ,  $(a, r)$ , and  $(f, e)$ ;
21     $\lfloor$  contract the edge(s) entering  $f$  and the edge  $(c, b)$ ;
22 if Case 3 holds (Fig. 9.3) then
23   if  $l \wedge_G l_s$  then
24      $\lfloor$  skip;
25   else
26      $\lfloor$  output “ $G$  is not displayed” and exit;
27 contract  $N[s]$  into  $l_s$ ;
28 contract  $G[\text{par}_G(\text{par}_G(l_s))]$  or  $G[\text{par}_G(l_s)]$  into  $l_s$  if necessary;
```

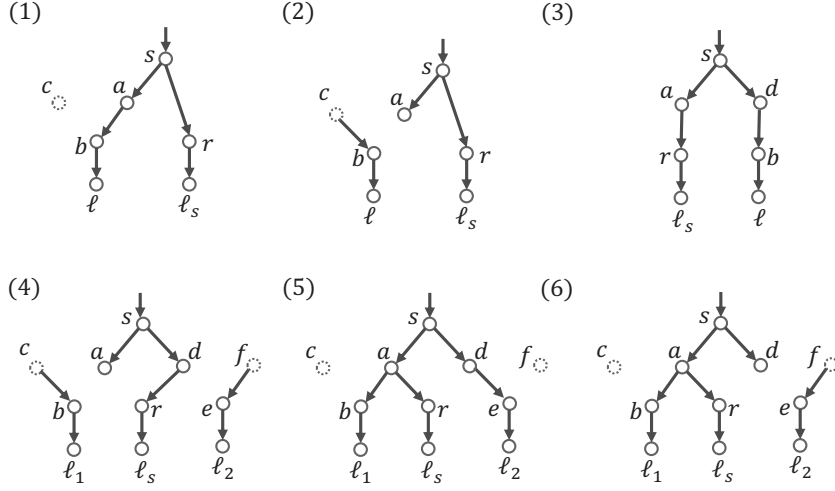


Figure 11: The resulting networks after $N[s]$ is simplified. (1) and (2) for the two cases of the mini-structure (1). (3) for the mini-structure (3). (4)-(6) for the first three cases of the mini-structure (2). The cases 4 and 5 of the mini-structure (2) are symmetric to the cases 2 and 3, respectively.

411 Note that no stable reticulation vertex becomes unstable after removing reticulation
 412 edges in a network if at least one reticulation edge is kept for each reticulation vertex, which
 413 is the case for all the rules of this simplification process. Therefore, after the simplification
 414 process terminates, N' is still nearly stable. Each of the components different from C may
 415 be simplified into a smaller one in N' . Additionally, two big tree vertex components may
 416 even be merged into one component in N' .

417 When working on $N[s']$ for some $s' \in S(C)$, we may transform another $N[s]$ into a
 418 subtree or a subnetwork of different mini-structure through the elimination of the reticulation
 419 vertices that have a parent in both $N[s]$ and $N[s']$. The following result describes the possible
 420 modifications on the subnetwork $N[s]$ below $s \in S(C)$.

421 **Proposition 5.3.** *Let $s, s' \in S(C)$ such that $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s']) \neq \emptyset$. Calling*
 422 *Dissolve_Lowest_Stable_Vertices on s' first will change $N[s]$ into (i) a subnetwork of the mini-*
 423 *structure in Figure 9.1, (ii) a three-vertex tree with two network leaves, (iii) a length-2 path*
 424 *from s to a network leaf on which s is stable, or (iv) a subnetwork in which at least a child*
 425 *of s is a stable tree vertex.*

426 *Proof.* To distinguish $N[s]$ before and after the procedure Dissolve_Lowest_Stable_Vertices is
 427 called on s' , we use $\bar{N}[s]$ to denote the subnetwork obtained when the procedure terminates.
 428 The existence of s' such that $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s']) \neq \emptyset$ implies that $N[s]$ has the mini-structure
 429 either (1) or (2) in Figure 9.

430 If $N[s]$ has the mini-structure (1) in Figure 9, calling the procedure first on s' affects $N[s]$
 431 only if $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s']) = \{b, \ell\}$. Therefore, calling the procedure on s' may alter $N[s]$
 432 in two possible ways. One possibility is that (a, b) is deleted and then (s, a) is contracted,
 433 resulting in $\bar{N}[s]$ equivalent to a path: $s \rightarrow r \rightarrow \ell_s$. The second possibility is that (c, b) is

434 deleted and (a, b) is then contracted, changing the tree vertex a into a tree vertex stable on
 435 ℓ in $\bar{N}[s]$ and hence making s not in $S(C)$.

436 If $N[s]$ has the mini-structure (2) in Figure 9, calling the procedure first on s' affects
 437 $N[s]$ only if $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s'])$ is equal to $\{b, \ell_1\}$, $\{e, \ell_2\}$, or $\{b, e, \ell_1, \ell_2\}$. The first two cases
 438 are symmetric and thus we consider the following two cases.

439 CASE 1. $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s']) = \{b, \ell_1\}$ (resp. $\{e, \ell_2\}$).

440 Running the procedure on s' may alter $N[s]$ in two possible ways. One possibility is that
 441 (a, b) (resp. (d, e)) is deleted and then (s, a) (resp. (s, d)) is contracted, simplifying $\bar{N}[s]$
 442 into a subnetwork of the mini-structure (1) in Figure 9.

443 The second possibility is that (c, b) (resp. (f, e)) is deleted and (a, b) (resp. (d, e)) is
 444 then contracted. This changes the vertex a from an unstable tree vertex into a stable tree
 445 vertex. As such, $\bar{N}[s]$ contains an uncle-nephew structure and s is no longer in $S(C)$.

446 CASE 2. $\mathcal{V}(N[s]) \cap \mathcal{V}(N[s']) = \{b, e, \ell_1, \ell_2\}$.

447 In this case, $N[s']$ has the same mini-structure (2) in Figure 9. Running the procedure
 448 on s' may alter $N[s]$ in three possible ways. One possibility is that both (a, b) and (d, e) are
 449 deleted and thus both (s, a) and (s, d) are contracted. This makes $\bar{N}[s]$ equivalent to the
 450 path $s \rightarrow r \rightarrow \ell_s$.

451 Another possibility is that (a, b) and (f, e) (resp. (c, b) and (d, e)) are deleted and thus
 452 (s, a) and (d, e) (resp. (a, b) and (s, d)) are contracted. This makes $\bar{N}[s]$ essentially equivalent
 453 to a 3-vertex tree $(s, (\ell_1, \ell_s))$ (resp. $(s, (\ell_2, \ell_s))$).

454 The third possibility is that both (c, b) and (f, e) are deleted and thus both (a, b) and
 455 (d, e) are contracted. This changes the vertices a and d from unstable tree vertices into
 456 stable tree vertices, making s not in $S(C)$. \square

457 In N' , each vertex of $S(C)$ becomes a new network leaf. Importantly, the following fact
 458 is true for the simplified component, which is still called C for convenience. It says that the
 459 simplification process will not create new elements for $S(C)$.

460 **Proposition 5.4.** $S(C) = \emptyset$ in N' .

461 *Proof.* Assume the resulting stable subtree of the component contains a leaf u that is not
 462 a network leaf of N' . Then, u must be an unstable tree vertex in N . Otherwise, either u
 463 has a stable tree child in N (and thus in N'), or u is in $S(C)$ and thus becomes a leaf in
 464 N' , a contradiction. By Proposition 5.1, u has two reticulation children x and y . Since C
 465 is a lowest component of N , the unique child for x and y must be a network leaf. If the
 466 reticulation edges entering x and y have never been removed, u could not become stable.
 467 If some edge entering x or y is deleted when $N[s]$ is modified for some $s \in S(C)$, then the
 468 involved child of u is contracted and thus u becomes either a dummy leaf or the parent of
 469 a network leaf. By Proposition 5.2, u is not in $S(C)$, a contradiction. \square

470 Now $S(C) = \emptyset$. However, there may be reticulation vertices below C . We further simplify
 471 C using each of the rules in the following lemma.

472 **Lemma 5.5.** *Let ℓ be a network leaf in C and p be the parent of ℓ such that the sibling of*
 473 *ℓ is not a stable tree vertex in C . Define N' as follows.*

474 (i) If $N[p]$ has the mini-structure (1) in Figure 10,

$$N' = N - \begin{cases} (q, v) & \text{if } \ell' \wedge_G \ell, \\ (p, v) & \text{otherwise.} \end{cases} \quad (9)$$

475 (ii) If $N[p]$ has the mini-structure (2) in Figure 10,

$$N' = N - \begin{cases} \{(q, v), (z, w)\} & \text{if } \ell' \wedge_G \ell, \\ \{(p, v)\} & \text{otherwise.} \end{cases} \quad (10)$$

476 (iii) If $N[p]$ has the mini-structure (3) in Figure 10,

$$N' = N - \begin{cases} \{(y, x), (v, w)\} & \text{if } \ell' \wedge_G \ell, \\ \{(v, x), (z, w)\} & \text{if } \ell'' \wedge_G \ell, \\ \{(y, x), (z, w)\} & \text{if } \ell' \wedge_G \ell'' \text{ and } \text{par}_G(\ell') \wedge_G \ell, \\ \{(v, x), (v, w)\} & \text{otherwise.} \end{cases} \quad (11)$$

477 Then, N displays G if and only if N' displays G .

478 *Proof.* N' is a subnetwork of N , so clearly N displays G if N' displays G .

479 Assume that N displays G . There exists a spanning tree T of N such that G is a
480 contraction of T .

481 By Lemma 5.2, there are three possible mini-structures for $N[p]$ shown in Fig. 10. The
482 mini-structure (1) is an uncle-nephew structure and thus the sufficiency follows from
483 Lemma 5.3. The sufficiency for the mini-structure (2) in Fig. 10 can be proven similarly.

484 Suppose $N[p]$ has the mini-structure (3) in Fig. 10, the first and second case in the
485 definition of N' are symmetric, so we only need to consider the cases 1, 3 and 4.

486 1: $\ell' \wedge_G \ell$: since T must contain p , T does not contain (v, w) . Otherwise, ℓ'' is the sibling
487 of either ℓ' or ℓ depending whether or not (v, x) is in T , contradicting the assumption
488 in this case. This implies that T is a spanning tree of $N - (v, w)$.

489 In $N - (v, w)$, $(N - (v, w))[p]$ is essentially an uncle-nephew structure. By Lemma 5.3,
490 $N' = (N - (v, w)) - (y, x)$ displays G .

491 3: $\ell' \wedge_G \ell''$ and $\text{par}_G(\ell') \wedge_G \ell$.

492 Notice that T contains p . If T contains exactly one of (v, x) and (v, w) , ℓ is a sibling
493 of either ℓ' or ℓ'' in G , contradicting the assumption in this case.

494 Therefore, either both (v, x) and (v, w) are in T or none of them is in T . If the former
495 holds, then T is a spanning tree of N' and N' also displays G . If the latter holds, let
496 t_1 denote the lowest common ancestor of ℓ' and ℓ'' and let t_2 be the lowest common
497 ancestor of t_1 and ℓ in T . Then, $T[t_2]$ contains only three labeled leaves: ℓ , ℓ' , and ℓ'' .
498 Let $T' = T - \{(y, x), (z, w)\} + \{(v, x), (v, w)\}$. The tree in which ℓ' and ℓ'' are siblings
499 and ℓ is their uncle is a contraction of $T'[t_2]$ and $T[t_2]$, and T' is a spanning tree of
500 $N - \{(y, x), (z, w)\}$. Therefore, $N - \{(y, x), (z, w)\}$ displays G .

501 4: All the first three cases are not true.

502 As shown in the third case, either both (v, x) and (v, w) are both in T or none of them
 503 is in T . If (v, x) and (v, w) are both in T , then ℓ' and ℓ'' are sibling and their parent
 504 is a sibling of ℓ in G , contradiction. Therefore, T is a spanning tree of N' . Hence, N'
 505 also display G .

506 □

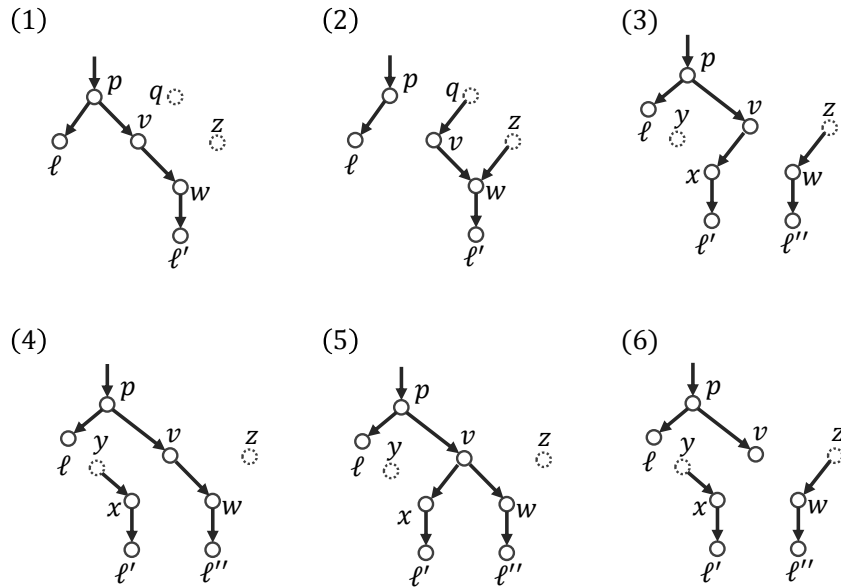


Figure 12: The resulting networks after $N[p]$ is simplified using each of the rule in Lemma 5.5. (1) and (2) for the two cases of the mini-structure (2) in Fig. 10. (3)-(6) for the four cases of the mini-structure (3) in Fig. 10.

507 By Lemma 5.5, we can simplify the subnetwork below the parent of each network leaf in
 508 C using the following procedure called Dissolve_SubntkNearNtkLeaf, which is illustrated in
 509 Fig. 12. When the procedure is called on a network leaf, it may alter the subnetwork below
 510 the parent of another network leaf. Because of this, when a network leaf is examined, the
 511 subnetwork below its parent may have one of the degenerated structures listed in Fig. 13.
 512 For this case, Dissolve_DegeneratedCases is called.

513 By repeatedly simplifying the subnetwork below the parent of each network leaf through
 514 the removal of edges entering reticulations and contraction of edges, we will further transform
 515 the component into a single vertex or else we will discover that the input network does not
 516 display the tree G .

517 5.4. The algorithm

518 We are now ready to describe a linear-time TC algorithm (Algorithm 4) for nearly stable
 519 networks. It is divided into the pre-processing and simplification parts. Roughly speaking,

Algorithm 2: Dissolving subnetwork near network leaf

Procedure *Dissolve_SubntkNearNtkLeaf*(N, C, ℓ)

Input: a network N , a component C , a leaf ℓ

Output: N with reticulations below the parent p of ℓ being eliminated

```
1 if Case 1 holds (Fig. 10.1) then
2   if  $\ell \wedge_G \ell'$  then
3      $\lfloor$  delete  $(q, v)$ ;
4   else
5      $\lfloor$  delete  $(p, v)$ ;
6 if Case 2 holds (Fig. 10.2) then
7   if  $\ell \wedge_G \ell'$  then
8      $\lfloor$  delete  $(q, v)$  and  $(z, w)$  (Fig. 12.1);
9   else
10     $\lfloor$  delete  $(p, v)$  (Fig. 12.2);
11 if Case 3 holds (Fig. 10.3) then
12   if  $\ell \wedge_G \ell'$  then
13      $\lfloor$  delete  $(y, x)$  and  $(v, w)$  (Fig. 12.3);
14   else if  $\ell \wedge_G \ell''$  then
15      $\lfloor$  delete  $(v, x)$  and  $(z, w)$  (Fig. 12.4);
16   else if  $(\ell' \wedge_G \ell'')$  and  $(\text{par}_G(\ell') \wedge_G \ell)$  then
17      $\lfloor$  delete  $(y, x)$  and  $(z, w)$  (Fig. 12.5);
18   else
19      $\lfloor$  delete  $(v, x)$  and  $(v, w)$  (Fig. 12.6);
20  $\lfloor$  contract  $N[p]$  into  $\ell$ ; contract  $G[\text{par}(\ell)]$  into  $\ell$  if necessary (Fig. 12).
```

Algorithm 3: Dissolving degenerated cases

Procedure *Dissolve_DegeneratedCases*(N, C, ℓ)

Input: a network N , a component C , a leaf ℓ

Output: N with reticulations below the parent p of ℓ being eliminated

1 **if** *Case 1 or 2 holds* (Fig. 13.1 or Fig. 13.2) **then**

2 | skip;

3 **if** *Case 3 or 4 holds* (Fig. 13.3 or Fig. 13.4) **then**

4 | **if** $\ell \wedge_G \ell'$ **then**

5 | skip;

6 | **else**

7 | output “No” and exit;

8 **if** *Case 5 holds* (Fig. 13.5) **then**

9 | **if** $(\ell' \wedge_G \ell'')$ and $(\text{par}_G(\ell') \wedge_G \ell)$ **then**

10 | skip;

11 | **else**

12 | output “No” and exit;

13 **if** *Case 6 holds* (Fig. 13.6) **then**

14 | **if** $\ell \wedge_G \ell'$ **then**

15 | delete (z, w) ;

16 | **else**

17 | delete (v, w) ;

18 **if** *Case 7 holds* (Fig. 13.7) **then**

19 | **if** $(\ell' \wedge_G \ell'')$ and $(\text{par}_G(\ell') \wedge_G \ell)$ **then**

20 | delete (z, w) ;

21 | **else if** $\ell' \wedge_G \ell$ **then**

22 | delete (v, w) ;

23 | **else**

24 | output “No” and exit;

25 | contract $N[p]$ into ℓ ; contract $G[\text{par}_G(\ell)]$ into ℓ if necessary.

Algorithm 4: A linear-time tree containment algorithm for nearly stable networks

```
Procedure Tree_containment( $N, G$ )
  Input: a network  $N$ , a tree  $G$ 
  Output: true if  $N$  contains  $G$ , false otherwise
  // 1. Pre-process  $N$ 
1  Compute and topologically sort the big tree-vertex components  $C_i$  ( $0 \leq i \leq k$ ) so
   that  $\rho(C_j)$  is a descendant of  $\rho(C_i)$  only if  $i < j$ ;
  /* Some big components may contain only leaves that are in  $\mathcal{L}(N)$ 
   (i.e., below which there are no reticulation vertices) */
2  for  $i = k$  to 0 do
3    simplify  $C = C_i$  by repeatedly contracting degree-2 vertices and dummy leaves
   so that it becomes binary;
4    compute  $S(C)$  by traversing  $C$  in pre-order (see Proposition 5.2);
5    foreach  $s \in S(C)$  do
6      if  $N[s]$  has a mini-structure in Fig. 9 then
7        Dissolve_Lowest_Stable_Vertices( $N, C, s$ );
8      else
9        simplify  $N[s]$ ;
10   Resimplify  $C$  by repeatedly contracting degree-2 vertices and dummy leaves;
   /* Assume the left child of a vertex is always a stable tree
   vertex, if it has any stable tree vertex child. Exchange the
   left and right children of a vertex if needed. */
11   (Traversing  $C$  in post-order:  $u_1, u_2, \dots, u_t = \rho(C)$ )
12   foreach  $\ell = u_i$  do
13     if  $\ell$  does not have a sibling then
14       replace its parent  $u_{i+1}$  with  $\ell$ ;
15     else if its sibling  $v$  is also a network leaf then // Case C1
16       if  $\ell \wedge_G v$  then
17         return false;
18       else if  $v$  has been visited then
19         contract  $N[\text{par}(\ell)]$  and  $G[\text{par}(\ell)]$  into  $\ell$ ;
20       else skip ;
21     else if its sibling is not a stable tree vertex then // Cases C2 & C3
22       if the subnetwork below the parent of  $\ell$  has a mini-structure in Fig. 10
23       then
24         Dissolve_SubntkNearNtkLeaf( $N, C, \ell$ );
25       else
26         Dissolve_DegenerateCases( $N, C, \ell$ );
27     else skip ; // Case C4: the sibling of  $\ell$  is a stable tree vertex
  return true;
```

520 the algorithm first topologically sorts the big tree vertex components and then dissolves these
 521 components one by one in a bottom-up manner. Each component is dissolved in two stages.
 522 In the first stage, Dissolve_Lowest_Stable_Vertices is called for each $s \in S(C)$. In the second
 523 stage, Dissolve_SubntkNearNtkLeaf and Dissolve_DegenerateCases are repeatedly called. In
 524 the rest of this section, we discuss the correctness and running time of the algorithm step
 525 by step.

526 Let N and G be the input nearly stable network and phylogenetic tree with the same set
 527 of n labeled leaves, respectively. Then, $|\mathcal{V}(N)| = O(n)$, which is proved in Section 4, and
 528 hence $|\mathcal{E}(N)| = O(n)$, as N is degree-bounded. If G is displayed in N , $|\mathcal{V}(G)| \leq |\mathcal{V}(N)|$.
 529 We shall show that the proposed TC algorithm takes $O(n)$ operations. Here, the operations
 530 include (i) edge contraction, (ii) edge deletion, (iii) membership check for $\mathcal{T}(N)$, $\mathcal{R}(N)$ and
 531 $\mathcal{L}(N)$, (iv) verification of sibling or parent-son relationship and (v) traverse to a vertex in
 532 N or G .

533 First, in the pre-processing (line 1), we first identify and topologically sort the roots of
 534 the big tree vertex components in $O(|\mathcal{E}(N)| + |\mathcal{V}(N)|)$ [20, page 103]. Note that the root of
 535 each component is the unique child of a stable reticulation vertex. We then use breadth-first
 536 search to compute each big component by starting from its root and order these tree vertex
 537 components in the same way as their roots are ordered.

538 After the pre-processing, the components are simplified one by one in a bottom-up man-
 539 ner. In line 3, the number of operations taken is linear in the number of contracted edges.
 540 Because each edge is contracted at most once, in a bottom-up manner, this step takes at
 541 most $O(|\mathcal{E}(N)|)$ operations.

542 For a component C , by Proposition 5.2, a vertex x in $S(C)$ if and only if each child of x
 543 in C is a leaf of C but not a network leaf. Hence, when traversing C in pre-order, a vertex
 544 x is in $S(C)$ if and only if (a) the next vertex is in $\mathcal{L}(C) \setminus \mathcal{L}(N)$ and (b) the next vertex
 545 is either not a child of x or also in $\mathcal{L}(C) \setminus \mathcal{L}(N)$, each of which can be checked in constant
 546 time. Hence, the total number of operations taken on line 3 is linear in $\sum_{0 \leq i \leq k} |\mathcal{V}(C_i)|$, for
 547 all $k + 1$ big tree-vertex components of N , which is at most $|\mathcal{V}(N)|$.

548 For each component C , $|S(C)| \leq |\mathcal{L}(C)|$ and thus $|S(C)| \leq |\mathcal{V}(C)|/2$. By the **foreach**
 549 loop on line 5–9, we simplify the networks $N[s]$ below $s \in S(C)$ one-by-one. Since there
 550 are only three possible mini-structures in Figure 9, each containing 4 to 6 vertices, the

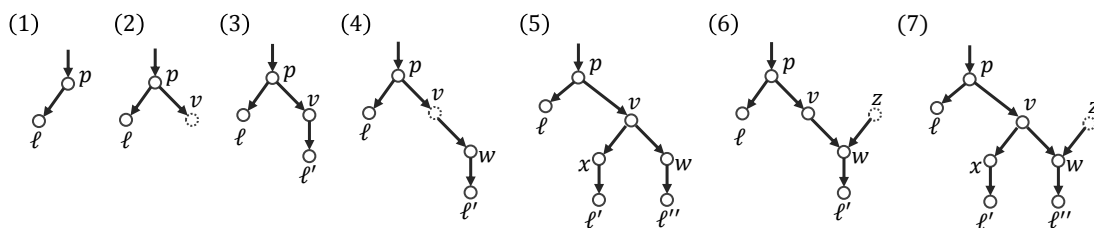


Figure 13: The possible degenerated structures of the subnetwork below the parent of a network leaf in a lowest tree component C . In (2) and (4), v can be either an unstable tree vertex or a reticulation vertex.

551 conditional statement can be verified in constant time. If the conditional statement is
552 true, calling the procedure `Dissolve_Lowest_Stable_Vertices` on s takes constant time. If the
553 conditional statement is false, $N[s]$ has been modified when working on some $s' \in S(C)$
554 earlier. By Proposition 5.3, s is no longer in $S(C)$ or $N[s]$ is equivalent to a length-2 path
555 from s to ℓ_s on which s is stable. Hence, the simplification of $N[s]$ also takes constant
556 time. Overall, the first **foreach** loop takes $O(|S(C)|)$ operations for each C . Hence the
557 total number of the operations taken by the algorithm is $O(|\mathcal{V}(N)|)$ for line 5–9. Note that
558 when the leaves of C are all network leaves, $S(C) = \emptyset$ and the **foreach** loop will stop.

559 $S(C)$ is empty when line 10 starts. Then, either an internal tree vertex or its parent is
560 in the path from $\rho(C)$ to a network leaf. Moreover, if an internal tree vertex is stable, as
561 assumed in the beginning of this step, its left child is also a stable tree vertex.

562 Let u be an internal tree vertex in the current component C . Assume at least one child
563 of $u \in \mathcal{V}(C)$ is a network leaf. Since the vertices in C are traversed in post-order, a network
564 leaf child of u must be visited before u . In particular, line 12 always starts with a network
565 leaf, which is the leftmost leaf of C .

566 Assume the currently visited vertex $\ell = u_i$ is a network leaf and it is the left child of its
567 parent p . Let v be the other child of p in N . There are the following four cases.

568 C1. The vertex v is also a network leaf. Then $v = u_{i+1}$ and $p = u_{i+2}$. If ℓ and v are not
569 siblings in G , we have the evidence that G is not displayed and hence the algorithm
570 terminates. If ℓ and v are siblings in G , then, the algorithm simply moves on to process
571 $v = u_{i+1}$. At v , the algorithm will replace p with v and also replace the parent of ℓ and
572 v with v in G . As such, when the algorithm processes u_{i+2} , it has become a network
573 leaf now and $\{u_{i+2}, \dots, u_t\}$ is still the vertex set of the modified component, listed in
574 the post-order. This explain why the algorithm simply does not act when visiting ℓ in
575 this case.

576 C2. The vertex v is not in C , that is, a reticulation vertex. Then, $u_{i+1} = p$. If $N[p]$ has
577 the mini-structures (1) and (2) in Fig. 10, the procedure `Dissolve_SubntkNearNtkLeaf`
578 is called on ℓ . Otherwise, `Dissolve_DegenerateCases` is called on ℓ . After p is replaced
579 with a network leaf, then $\{p = u_{i+1}, \dots, u_t\}$ is the vertex set of the modified component
580 listed in the post-order.

581 C3. The vertex v is a tree vertex but unstable. Then, $v = u_{i+1}$ and $N[p]$ has the mini-
582 structure (3) in Fig. 10 or a degenerated structure in Fig. 13. Similar to the case C2,
583 after `Dissolve_SubntkNearNtkLeaf` or `Dissolve_DegenerateCases` is called on ℓ , $\{p =$
584 $u_{i+2}, \dots, u_t\}$ is the vertex set of the modified component listed in the post-order.

585 C4. The vertex v is an internal, stable tree vertex in C . This indicates that the subnetwork
586 $N[v]$ contains at least one stable tree vertex and possibly some reticulation vertices.
587 In this case, u_{i+1} is the leftmost network leaf in $C[v]$. The algorithm will move on to
588 simplify $C[v]$ into a network leaf and then contracting the parent of both ℓ and v into
589 a network leaf. In order to simplify $C[v]$ first, the algorithm simply skips at ℓ .

590 Clearly, the second **foreach** loop takes constant operations in each execution.

591 Taken together, the above facts show the following theorem.

592 **Theorem 5.1.** *Algorithm 4 uses $O(|\mathcal{L}(N)|)$ operations to solve the TC problem for nearly*
593 *stable networks N .*

594 6. Conclusion

595 In this paper, we have studied nearly stable and genetically stable networks, two classes
596 that were introduced in the study of the tree containment problem. In particular, we have
597 proved that each nearly (resp. genetically) stable network with n leaves contains at most
598 $3(n - 1)$ (resp. $2(n - 1)$) reticulation vertices. We have also designed a linear-time TC
599 algorithm for binary nearly stable networks.

600 This study arises several problems for future study. The cluster containment (CC) prob-
601 lem asks whether or not a given cluster appears in some phylogenetic tree displayed by a
602 given network. Gunawan et al. [12] designed a linear-time algorithm for the CC problem.
603 Now, we have obtained a linear-time algorithm for the TC problem. Is there a linear-time
604 TC algorithm for reticulation-visible networks? If the answer to this question is negative, is
605 there a linear-time TC problem for genetically stable networks?

606 In each binary nearly stable network, there are at most $3(n - 1)$ reticulation vertices.
607 This is also true for reticulation-visible networks. Is it possible to define a superclass of
608 networks that contains both reticulation-visible networks and nearly stable networks for
609 which the TC and CC problems are still polynomial-time solvable?

610 Binary nearly stable networks have simple local structures compared with reticulation-
611 visible networks. Hence, it is also interesting to investigate how to efficiently reconstruct
612 nearly stable network models from gene trees or sequence data.

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