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ON THE HEDGING OF LIABILITIES WITH AN ENDOGENOUS PROFIT SHARING MECHANISM

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Abstract:

The fair replication method is a method designed to value liabilities with an *endogenous* profit sharing mechanism, i.e. based on the book yield of the backing assets. The basic idea is to construct a hypothetical portfolio, the fair replicating portfolio (FRP), whose cash flows are scenario-invariant. The method is a computationally efficient alternative to traditional stochastic modeling. It may be particularly useful in applications where extensive calculations of best estimate of liabilities are required.

Keywords: Best estimate of liabilities, life insurance, profit sharing mechanism, replicating portfolio, Solvency II.

JEL Classification: G22, G32.

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² The views expressed in this paper are those of the author and do not necessarily reflect those of Delta Lloyd Life.

1. INTRODUCTION

In life insurance, it is quite common that the annual bonuses of participating contracts are based on the performance of the backing assets. However, it is not an easy task to properly model such a profit sharing mechanism.

In this paper, we will consider *closed* ALM models, i.e. models where, at the end of each year, the book value of the backing assets is equal to the mathematical reserve of the corresponding contracts. In accordance with MCEV and Solvency II principles, we will also assume that liabilities run off.

In such a setting, the realization of capital gains/losses can occur quite frequently, with potential impact on the liability side (through the profit sharing mechanism). This is likely to bias the valuation process. Since we believe that this undesirable phenomenon can be avoided by using investment strategies that minimize risk, we propose to investigate the possibility of constructing such investment strategies.

The remainder of the paper is organized as follows. In Section 2, we review the literature. Section 3 presents the rationale for the proposed method. Section 4 describes the construction of the fair replicating portfolio (FRP), the key element of the method. Section 5 is a brief digression on the status of realized capital gains/losses, while Section 6 discusses the existence and uniqueness of the FRP. Finally, Section 7 illustrates with an example how our method works. In Section 8, we conclude.

2. LITERATURE REVIEW

There are many papers in the literature that consider annual bonuses as a function of the performance of an underlying asset portfolio. They can be divided into two categories.

In the first category, the underlying portfolio is treated as a single stock, beyond the control of the insurer. This approach, which emerged in the late 1990s, has been used e.g. in Bacinello (2001), Bauer et. al. (2006) and Printems (2015).

In the second category, the underlying portfolio is supposed to coincide with the assets backing the liabilities. This creates a dynamic in which the two sides of the balance sheet influence each other. These complex interactions, which strongly depend on the investment strategy, have been studied in Kleinow and Willder (2007), Kleinow (2009) and Delong (2010).

This paper definitely belongs to the second category, but differs in two key respects from the aforementioned studies. First, it assumes that the annual bonuses are based on the

book yield of the underlying portfolio, i.e. the accounting rate of return on this portfolio. Such a practice is fairly common in the insurance industry, especially in continental Europe (see e.g. IASB (2009)). Second, it assumes that the behavior of policyholders is *endogenous*, i.e. unaffected by changes in market prices. This point has been discussed e.g. in Eling and Kochanski (2013). Their study, which reviews more than 50 papers on the subject, concludes that there is no clear evidence of financial market dependence.

3. THE IDEA BEHIND THE FAIR REPLICATION METHOD

Let us assume a closed ALM model coupled with a set of stochastic economic scenarios. Let us also assume that profit sharing and policyholder behavior are endogenous. In this setting, our objective is to determine an investment strategy that makes the liability cash flows scenario-invariant. This goal will be achieved if the book yield paths are the same in all scenarios. In turn, this condition will be satisfied if the backing portfolio is static and composed of fixed-income securities.

So the idea is to instantaneously switch – in a self-financing way – from the current backing assets to a portfolio of ad hoc fixed-income securities. This hypothetical portfolio, we call it the fair replicating portfolio (FRP). As we will see later on, the FRP is not entirely different from the current backing portfolio. Rather, it appears to be its closest de-risked version.

The main consequence of the scenario-invariance is that the contracts considered can be valued using only (1) the book yield path of the FRP to project the cash flows and (2) the certainty equivalent scenario to discount them. In other words, by appropriately choosing the investment strategy, the valuation problem becomes deterministic.

The idea of converting a seemingly stochastic problem into a deterministic one (by using the investment strategy that eliminates risk) can be traced back to Black, Merton and Scholes. According to them, pricing does not involve arbitrary but optimal investment strategies. This relies on the law of one price and ultimately on the no-arbitrage principle. Whereas a partial differential equation is derived in the Black-Scholes model, we will show that the present study leads to a fixed point equation.¹

¹ The fair replication method should not be confused with traditional replication methods, which still rely on stochastic projections of cash flows. For the latter, see e.g. Natolski and Werner (2014).

4. FIXED POINT EQUATION

In “Pricing in Incomplete Markets” (2011), Pelsser claims that it is impossible for an insurance company to construct a portfolio that serves simultaneously as a basis for profit sharing and as a replicating portfolio.

In the case of profit-sharing on your own portfolio, it is impossible to hedge: at the moment you start buying instruments to hedge your own profit-sharing options, you start changing the composition of your asset portfolio, which then starts changing the nature of the profit-sharing.

We believe however that the circle can be broken by casting the problem in terms of a fixed point equation. In general, such an equation can be solved iteratively. From an initial guess, successive approximations are constructed, which are expected to converge to the desired solution. If all goes well, a good approximation is obtained after a few iterations. In the following subsections, we will see how this can be used to get a perfect hedge. The existence and uniqueness of the solution – the FRP – will be discussed later in Section 6.

4.1 Iterative procedure

- 1) We start with a tuple of book yields, $BY_0 = (BY_{0_1}, BY_{0_2}, \dots, BY_{0_T})$. Our choice is to take BY_0 equal to the current yield curve.
- 2) Let $CF_1 = (CF_{1_1}, CF_{1_2}, \dots, CF_{1_T})$ be the cash flows to be replicated assuming BY_0 as the book yield path.¹ For each t , CF_{1_t} is defined as $BY_{0_t} \text{Res}_{t-1}(BY_0) - \text{Inc_Res}_t(BY_0)$.²
- 3) To this first tuple of cash flows, CF_1 , we associate a first replicating portfolio, RP_1 . We will show in the next subsection how to obtain it.
- 4) We then repeat the procedure using BY_1 , the book yield path of RP_1 , instead of BY_0 . This gives a new replicating portfolio, RP_2 , and hence a new book yield path, BY_2 , and so on...

¹ The cash flows used to determine the FRP are those affecting the policyholder account, a fictitious deposit account earning the book yield and whose year-end value is equal to the mathematical reserve of the contracts considered.

² The term “ $\text{Res}_t(BY)$ ” (resp. “ $\text{Inc_Res}_t(BY)$ ”) denotes the mathematical reserve (resp. the increase in mathematical reserve) of the contracts considered in t years along the book yield path BY . The assumption that policyholder behavior is endogenous makes the definition unambiguous.

- 5) As a final step, we define the *fair path* as the limit of the sequence (BY_n) and the *fair replicating portfolio* (FRP) as the limit of the sequence (RP_n). It is easy to convince oneself that the former is the book yield of the latter.

4.2 The n^{th} replicating portfolio

Let CF_n = (CF_{n1}, CF_{n2}, ..., CF_{nT}) be the n^{th} tuple of cash flows to be replicated. RP_n, the n^{th} replicating portfolio, is defined as follows.

- 1) Divide the backing assets into two categories: the fixed-income securities, assumed to be already adjusted for credit spread, and the rest.
- 2) Liquidate immediately the second category.
- 3) For each relevant t , eliminate the asset surplus by immediately selling $\text{MAX}(0, \text{Fixed_Income}_t - \text{CF}_{n_t})$.¹
- 4) For each relevant t , eliminate the asset deficit by immediately buying a zero-coupon bond that pays $\text{MAX}(0, \text{CF}_{n_t} - \text{Fixed_Income}_t)$ in t years.²

4.3 Recurrence relation

The procedure described in Subsections 4.1 and 4.2 can be expressed as a recurrence relation between the terms BY_{n+1} and BY_n. Specifically, BY_{n+1} can be written as a vector function, $F = (F_1, F_2, \dots, F_T)$, of BY_n,

$$\text{where } F_t(\text{BY}_n) \text{ is equal to } \frac{\sum_{i=t}^T Y_i^* (1+Y_i^*)^{t-1} \frac{\text{CF}_i(\text{BY}_n)}{(1+Y_i^*)^i}}{\sum_{i=t}^T (1+Y_i^*)^{t-1} \frac{\text{CF}_i(\text{BY}_n)}{(1+Y_i^*)^i}} \text{ and}$$

$$Y_i^* \text{ is such that } \frac{\text{CF}_i(\text{BY}_n)}{(1+Y_i^*)^i} = \frac{K_i(\text{BY}_n)}{(1+Y_i^E)^i} + \frac{\text{CF}_i(\text{BY}_n) - K_i(\text{BY}_n)}{(1+Y_i^C)^i}, \quad 3$$

with (BY₀₁, BY₀₂, ..., BY_{0T}), the seed value, equal to the current yield curve.

In Section 6, we will explore the conditions under which the function F produces the desired results.

¹ The term "Fixed_Income_t" denotes the expected cash flows of the current fixed-income portfolio in t years.

² It is assumed that it is always possible to purchase such bonds, even for very long durations.

³ The term " Y_t^C " denotes the current yield of a t -year zero-coupon bond, while the term " Y_t^E " denotes the effective yield of Fixed_Income_t. The constant $K_t(\text{BY}_n)$ is defined as $\text{MIN}(\text{Fixed_Income}_t, \text{CF}_t(\text{BY}_n))$.

5. CAPITAL GAINS/LOSSES REALIZED AT TIME 0

We are now ready to specify the perfect hedging strategy postulated in Section 3. It consists in instantaneously switching from the current backing assets to the FRP, as defined in Section 4 (see Appendix 1 for proof).

The main feature of this strategy is that all realizations of capital gains/losses occur at the measurement date (time 0). But what about their status? Should they be shared with policyholders?

We think not. In our opinion, it is unfair that policyholders participate in realizations that are from assets (or portions of assets) at risk. A typical example is when huge capital gains are realized due to massive lapses. In that case, it seems unlikely that the insurer would share this extra income with policyholders.

Although this is an important issue, it is not essential for our purpose. Should these realizations be shared with policyholders (e.g. to comply with legal requirements), this would not alter the principle of the FRP, but only its construction. Therefore, to avoid further complications, we will assume from now on that the capital gains/losses realized at time 0 are not shared with policyholders.

6. EXISTENCE AND UNIQUENESS OF THE FAIR PATH

What is the guarantee that the sequence (BY_n), defined in Section 4, *always* converges, i.e. whatever the data may be? We will try to answer this question in the following subsections, first from a theoretical and then from a practical point of view.

6.1 Theoretical results

6.1.1 Existence of the fair path

If, for each t , $\text{Inc_Res}_t^* < 0$, $Y_t^C > 0$ and $Y_t^E > 0$, then F , the vector function that generates the sequence (BY_n), is continuous and maps the hypercube $[0, Z]^T$ into itself (see Appendix 2 for proof).¹ Therefore, the Brouwer fixed point theorem applies. In other words, F admits at least one fixed point in the hypercube. However, this is not enough to ensure that the sequence (BY_n) always converges.

¹ The term “Inc_Res_t^{*}” denotes the increase in mathematical reserve of the contracts considered in t years in the absence of profit sharing, while the constant Z is defined as $\text{MAX}_{t=1}^T(Y_t^C, Y_t^E)$.

6.1.2 Uniqueness of the fair path

A natural idea to prove the uniqueness of the fixed point of F and, by the same token, the convergence of the sequence (BY $_n$) is to try to apply a Banach-style theorem to F (contraction mapping principle). This is the approach we took in “Fair valuation of universal life policies via a replicating portfolio” (2010). In that paper, we showed that, if the set of fixed-income securities is assumed to be empty,¹ F admits, under reasonable conditions, a unique fixed point in $[0, Z]^T$ and the sequence (BY $_n$) always converges to it.

However, a formal proof is still lacking in the *general* case, i.e. the case where the set of fixed-income securities is not empty. In this regard, it might be instructive to see how similar problems have been addressed in other scientific fields. Inspiration can come from anywhere.

6.2 Empirical evidence

It is true that so far we have no proof that the procedure described in Section 4 is always effective. Nevertheless, we extensively tested it and always found that the sequence (BY $_n$) converged. This seems to indicate that the fair path exists under quite general conditions and that, when it exists, the current yield curve belongs to its basin of attraction.

Furthermore, we never needed more than five iterations to obtain a very accurate value of the fair path and hence of the best estimate of liabilities. This should be compared with the thousands of simulations to be run in stochastic ALM models. In applications where extensive calculations of best estimate of liabilities are needed, this could result in significant time (and cost) savings.

7. NUMERICAL EXAMPLE

It is now time to illustrate the method proposed in this paper. We will use it to value a pool of stylized participating contracts.

7.1 Assumptions

7.1.1 Liability side

- 1) The contracts of the pool have a guaranteed rate, g , of 1.00%.
- 2) $\text{Bns}_t(\text{BY})$, the bonus rate credited in year t along the book yield path BY, is equal to $\text{MAX}(0, \text{BY}_t - g - 0.50\%)$ (endogenous profit sharing mechanism).

¹ This assumption makes sense for valuing new business.

- 3) For convenience, the different sources of benefit outflows (mortality, lapse and maturity) are not distinguished. Furthermore, the benefit outflows in year t are assumed to be equal to a fixed percentage, λ_t , of the mathematical reserve at the end of year $t - 1$.¹

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----------|---|-----|-----|-----|-----|-----|-----|-----|-----|------|
| λ | | 20% | 25% | 30% | 35% | 40% | 45% | 50% | 65% | 100% |

- 4) Res_0 , the mathematical reserve at time 0, is equal to 1,000 CU.
 5) At the end of year t , $\text{Res}_t(\text{BY}) = (1 + g + \text{Bns}_t(\text{BY}) - \lambda_t) \text{Res}_{t-1}(\text{BY})$.
 6) The expenses in year t along BY are equal to $0.20\% \text{Res}_{t-1}(\text{BY})$.

7.1.2 Asset side

- 7) At time 0, the book value of the backing portfolio is equal to 1,000 CU (50.0 CU for the equity position and 950.0 CU for the bond portfolio).
 8) For the sake of clarity, the bond portfolio is composed of only nine zero-coupon bonds (one per maturity year). Their respective face values and effective yields are given in the table below.

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Face value | | 175.0 | 275.0 | 225.0 | 175.0 | 100.0 | 50.0 | 25.0 | 5.0 | 5.0 |
| Y^E | | 3.00% | 3.00% | 3.00% | 2.90% | 2.80% | 2.70% | 2.50% | 2.40% | 2.40% |

- 9) At time 0, the market value of the equity position is equal to 48.0 CU.

7.1.3 Yield curve

- 10) The current yield curve at time 0 is given by the following table.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Y^C | | 0.10% | 0.25% | 0.50% | 0.80% | 1.10% | 1.40% | 1.70% | 1.95% | 2.15% |

¹ This implicitly means that the behavior of the policyholders is endogenous.

7.2 Determination of the fair path

The iterative procedure described in Section 4 is the core of the fair replication method. It provides us with the fair path, the book yield path of the FRP.

7.2.1 Initial step

As stated in Subsection 4.1, we take the current yield curve as initial guess for the fair path.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY0 | | 0.10% | 0.25% | 0.50% | 0.80% | 1.10% | 1.40% | 1.70% | 1.95% | 2.15% |

This initial guess leads to a first plausible tuple of bonus rates.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Bns1 | | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.20% | 0.45% | 0.65% |

On this basis, the mathematical reserve and the FRP cash flows are projected over the time horizon.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|-------|-------|-------|-------|-------|-------|------|------|------|------|
| Res1 | 1,000 | 810.0 | 615.6 | 437.1 | 288.5 | 176.0 | 98.5 | 50.5 | 18.4 | 0.0 |
| CF1 | | 191.0 | 196.4 | 181.6 | 152.1 | 115.7 | 79.9 | 49.8 | 33.0 | 18.8 |

We then move on to the construction of the portfolio RP1, which replicates the tuple CF1.

This operation, described in Subsection 4.2, consists of a two-stage transformation of the current backing assets, hereafter represented by their face value (bonds) or their book value (equities).

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|------|-------|-------|-------|-------|-------|------|------|-----|-----|
| Bonds FV | | 175.0 | 275.0 | 225.0 | 175.0 | 100.0 | 50.0 | 25.0 | 5.0 | 5.0 |
| Equities BV | 50.0 | | | | | | | | | |

Stage one: liquidate the equity position and the portion of the bond portfolio in excess of the cash flows to be replicated.

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|-------|---|-------|-------|-------|---|---|---|---|---|
| Sales | -50.0 | | -78.6 | -43.4 | -22.9 | | | | | |

Stage two: fill the gap between the bond portfolio and the cash flows to be replicated by investing in new zero-coupon bonds.

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|---|-------|---|---|---|-------|-------|-------|-------|-------|
| Purchases | | +16.0 | | | | +15.7 | +29.9 | +24.8 | +28.0 | +13.8 |

These transactions, which all occur at time 0, result in a new tuple of book yields, BY1. It is calculated using the formula given in Subsection 4.3.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY1 | | 2.56% | 2.59% | 2.55% | 2.44% | 2.27% | 2.12% | 2.08% | 2.08% | 2.21% |

7.2.2 First iteration

We then repeat the procedure using BY1 instead of BY0. This gives rise to a new tuple of cash flows, CF2.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY1 | | 2.56% | 2.59% | 2.55% | 2.44% | 2.27% | 2.12% | 2.08% | 2.08% | 2.21% |
| Bns2 | | 1.06% | 1.09% | 1.05% | 0.94% | 0.77% | 0.62% | 0.58% | 0.58% | 0.71% |
| Res2 | 1,000 | 820.6 | 632.6 | 455.8 | 305.1 | 188.5 | 106.7 | 55.0 | 20.1 | 0.0 |
| CF2 | | 205.0 | 209.2 | 192.9 | 161.8 | 123.6 | 85.7 | 53.9 | 36.1 | 20.6 |

In order to replicate CF2, the following purchases and sales are made on the current backing portfolio.

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Bonds FV | | 175.0 | 275.0 | 225.0 | 175.0 | 100.0 | 50.0 | 25.0 | 5.0 | 5.0 |
| Equities BV | 50.0 | | | | | | | | | |
| Sales | -50.0 | | -65.8 | -32.1 | -13.2 | | | | | |
| Purchases | | +30.0 | | | | +23.6 | +35.7 | +28.9 | +31.1 | +15.6 |

These transactions, performed at time 0, result in the following book yield path (BY2).

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY2 | | 2.68% | 2.71% | 2.63% | 2.47% | 2.25% | 2.11% | 2.07% | 2.08% | 2.21% |

7.2.3 *Second iteration*

FRP cash flows (CF3)

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY2 | | 2.68% | 2.71% | 2.63% | 2.47% | 2.25% | 2.11% | 2.07% | 2.08% | 2.21% |
| Bns3 | | 1.18% | 1.21% | 1.13% | 0.97% | 0.75% | 0.61% | 0.57% | 0.58% | 0.71% |
| Res3 | 1,000 | 821.8 | 634.5 | 457.7 | 306.5 | 189.3 | 107.1 | 55.3 | 20.2 | 0.0 |
| CF3 | | 205.0 | 209.6 | 193.5 | 162.5 | 124.1 | 86.1 | 54.1 | 36.2 | 20.7 |

Transactions performed on the current backing portfolio at time 0

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Bonds FV | | 175.0 | 275.0 | 225.0 | 175.0 | 100.0 | 50.0 | 25.0 | 5.0 | 5.0 |
| Equities BV | 50.0 | | | | | | | | | |
| Sales | -50.0 | | -65.4 | -31.5 | -12.5 | | | | | |
| Purchases | | +30.0 | | | | +24.1 | +36.1 | +29.1 | +31.2 | +15.7 |

Resulting book yield path (BY3)

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY3 | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |

7.2.4 *Third iteration*

FRP cash flows (CF4)

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY3 | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |
| Bns4 | | 1.19% | 1.22% | 1.13% | 0.97% | 0.74% | 0.60% | 0.57% | 0.58% | 0.71% |
| Res4 | 1,000 | 821.9 | 634.6 | 457.7 | 306.5 | 189.3 | 107.1 | 55.2 | 20.2 | 0.0 |
| CF4 | | 205.0 | 209.6 | 193.6 | 162.5 | 124.1 | 86.1 | 54.1 | 36.2 | 20.7 |

Transactions performed on the current backing portfolio at time 0

| Maturity year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Bonds FV | | 175.0 | 275.0 | 225.0 | 175.0 | 100.0 | 50.0 | 25.0 | 5.0 | 5.0 |
| Equities BV | 50.0 | | | | | | | | | |
| Sales | -50.0 | | -65.4 | -31.4 | -12.5 | | | | | |
| Purchases | | +30.0 | | | | +24.1 | +36.1 | +29.1 | +31.2 | +15.7 |

Resulting book yield path (BY4)

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY4 | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |

7.2.5 Convergence to the fair path

The table below shows how fast the sequence (BY_n) converges. It is remarkable that only three iterations are needed to obtain an accurate approximation of the fair path.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| BY0 | | 0.10% | 0.25% | 0.50% | 0.80% | 1.10% | 1.40% | 1.70% | 1.95% | 2.15% |
| BY1 | | 2.56% | 2.59% | 2.55% | 2.44% | 2.27% | 2.12% | 2.08% | 2.08% | 2.21% |
| BY2 | | 2.68% | 2.71% | 2.63% | 2.47% | 2.25% | 2.11% | 2.07% | 2.08% | 2.21% |
| BY3 | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |
| BY4 | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |

7.3 Valuation of the liability

The fair replication method consists in valuing the liability by instantaneously switching from the current backing assets to the FRP. Practically, it means using the fair path (FP) – closely approximated by BY3 – and the bonus rates based on it.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| FP | | 2.69% | 2.72% | 2.63% | 2.47% | 2.24% | 2.10% | 2.07% | 2.08% | 2.21% |
| Bns(FP) | | 1.19% | 1.22% | 1.13% | 0.97% | 0.74% | 0.60% | 0.57% | 0.58% | 0.71% |

7.3.1 Liability cash flows

The liability cash flows (Liab_CF) are the sum of the benefit outflows (Out) and the expenses (Exp), which are easily derived from the assumptions made at the beginning of this section.¹

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------------|---|-------|-------|-------|-------|-------|------|------|------|------|
| Out(FP) | | 200.0 | 205.5 | 190.4 | 160.2 | 122.6 | 85.2 | 53.6 | 35.9 | 20.6 |
| Exp(FP) | | 2.0 | 1.6 | 1.3 | 0.9 | 0.6 | 0.4 | 0.2 | 0.1 | 0.0 |
| Liab_CF(FP) | | 202.0 | 207.1 | 191.7 | 161.1 | 123.2 | 85.5 | 53.8 | 36.0 | 20.6 |

7.3.2 Best estimate of the liability

By discounting the liability cash flows with the current yield curve, we obtain the best estimate of the liability (BEL). It amounts to 1,043.8 CU. At the same time, the market value of the current backing assets (MVBA) is equal to 1,058.9 CU.

7.3.3 Consistent figures

We conclude this section by showing that the present value of the future gross profits (PVFGP) is equal to the MVBA minus the BEL.

According to our assumptions, $GP_t(\text{FP})$, the gross profit in year t (on a book value basis), is equal to $(0.50\% - 0.20\%) \text{Res}_{t-1}(\text{FP})$.

¹ As we can see, the liability cash flows differ slightly from the FRP cash flows by including the expenses and excluding the interest margins.

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|--------|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| GP(FP) | | 3.0 | 2.5 | 1.9 | 1.4 | 0.9 | 0.6 | 0.3 | 0.2 | 0.1 |

The discounted value of these items is equal to 10.5 CU. By correcting it for the capital gains/losses realized at time 0 (-2.0 CU on equities and $+6.6$ CU on bonds),¹ we obtain the PVFGP. It amounts to 15.1 CU, which corresponds to the difference between the MVBA and the BEL ($1,058.9$ CU $- 1,043.8$ CU).

8. CONCLUSION

In this paper, we proposed a method to value liabilities with an endogenous profit sharing mechanism. The method relies on a fixed point equation, the derivation of which is made possible by assuming that policyholder behavior is also endogenous. Although we have no formal proof that this equation always has a (unique) solution, empirical evidence suggests that this is indeed the case under a wide range of economic and non-economic conditions. Another important feature of the fair replication method is that it is much more efficient than traditional stochastic modeling. In a context where extensive calculations of best estimate of liabilities are becoming the norm, this is nothing to sneeze at.

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¹ We recall that these realizations are integral to the construction of the FRP.

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11. APPENDIX 1

In this appendix, we will show that the FRP perfectly hedges the corresponding insurance contracts.

Notation

The term “ $Res_t(FP)$ ” denotes the mathematical reserve of the contracts considered in t years along the fair path, while the term “ FRP_t ” denotes the book value of the FRP in t years.

Theorem

For each t ($0 \leq t \leq T$), $FRP_t = Res_t(FP)$.

Proof (by backward induction on t)

Base case

Obviously, $FRP_T = Res_T(FP) = 0$.

Inductive step

By the definition of the FRP (see Section 4),

$$(1 + FP_t) FRP_{t-1} = FRP_t + CF_t(FP) \quad \text{and} \quad (1)$$

$$CF_t(FP) = FP_t Res_{t-1}(FP) - Inc_Res_t(FP). \quad (2)$$

Combining (1) and (2),

$$(1 + FP_t) FRP_{t-1} = FRP_t + FP_t Res_{t-1}(FP) - Inc_Res_t(FP).$$

Using the induction hypothesis,

$$\begin{aligned} (1 + FP_t) FRP_{t-1} &= Res_t(FP) + FP_t Res_{t-1}(FP) - Inc_Res_t(FP) \\ &= (1 + FP_t) Res_{t-1}(FP). \end{aligned}$$

Therefore, $FRP_{t-1} = Res_{t-1}(FP)$.

12. APPENDIX 2

The existence of the fair path follows from the Brouwer fixed point theorem. The proof consists in showing that F , the function that generates the sequence (BY_n) , is a continuous self-map on a compact convex set.

Theorem

Let us assume that, for each t , $Inc_Res_t^* < 0$, $Y_t^C > 0$ and $Y_t^E > 0$.

Then, the vector function $F(BY) = (F_1(BY), F_2(BY), \dots, F_T(BY))$,

$$\begin{aligned} \text{where } F_t(BY) \text{ is equal to } & \frac{\sum_{i=t}^T Y_i^* (1+Y_i^*)^{t-1} \frac{CF_i(BY)}{(1+Y_i^*)^i}}{\sum_{i=t}^T (1+Y_i^*)^{t-1} \frac{CF_i(BY)}{(1+Y_i^*)^i}} \text{ and} \\ Y_i^* \text{ is such that } & \frac{CF_i(BY)}{(1+Y_i^*)^i} = \frac{K_i(BY)}{(1+Y_i^E)^i} + \frac{CF_i(BY) - K_i(BY)}{(1+Y_i^C)^i}, \end{aligned}$$

is a continuous self-map on $[0, Z]^T$.

Proof**Compact convex set**

Clearly, $[0, Z]^T$ is a compact convex set.

Continuity on $[0, Z]^T$

The continuity of F on $[0, Z]^T$ follows from its construction and the fact that, for each t ($1 \leq t \leq T$), $CF_t(BY)$ and $1 + Y_t^*$ are positive (Lemmas 1 and 2).

Self-map on $[0, Z]^T$

For each t ($1 \leq t \leq T$), F_t is a weighted average of the yields Y_i^* ($t \leq i \leq T$). This follows from the definition of F_t and the fact that $CF_t(BY)$ and $1 + Y_t^*$ are positive. In other words, $\text{MIN}_{i=t}^T(Y_i^*) < F_t < \text{MAX}_{i=t}^T(Y_i^*)$. By Lemma 2, this means that $0 < F_t < \text{MAX}_{t=1}^T(Y_t^C, Y_t^E) = Z$. Therefore, F is a self-map on $[0, Z]^T$.

For the sake of completeness, we give the proofs of the following two lemmas.

Lemma 1

For each t ($1 \leq t \leq T$), if $\text{Inc_Res}_t^*(BY) < 0$, then $CF_t(BY) > 0$.

Proof

By definition (see Section 4),

$$\begin{aligned} CF_t(BY) &= BY_t \text{Res}_{t-1}(BY) - \text{Inc_Res}_t(BY) \\ &= BY_t \text{Res}_{t-1}(BY) - (\text{Res}_t(BY) - \text{Res}_{t-1}(BY)) \\ &= BY_t \text{Res}_{t-1}(BY) - (\text{Res}_t^* + \text{Bns}_t(BY) - \text{Res}_{t-1}^* - \text{Bns}_{t-1}(BY)) \\ &= [\text{Res}_{t-1}^* - \text{Res}_t^*] + [\text{Bns}_{t-1}(BY) + BY_t \text{Res}_{t-1}(BY) - \text{Bns}_t(BY)]. \end{aligned}$$

On one hand, by assumption,

$$\text{Res}_{t-1}^* - \text{Res}_t^* > 0.$$

On the other hand, since the rate credited to $\text{Bns}_{t-1}(BY)$ should be lower than BY_t ,

$$\text{Bns}_{t-1}(BY) + BY_t \text{Res}_{t-1}(BY) - \text{Bns}_t(BY) > 0.$$

Therefore, $CF_t(BY) > 0$.

Lemma 2

For each t ($1 \leq t \leq T$), Y_t^* lies between Y_t^C and Y_t^E .

Proof

By definition, Y_t^* is such that $\frac{CF_t(BY)}{(1+Y_t^*)^t} = \frac{K_t(BY)}{(1+Y_t^E)^t} + \frac{CF_t(BY) - K_t(BY)}{(1+Y_t^C)^t}$.

Therefore, $\frac{1}{(1+Y_t^*)^t} = \frac{K_t(BY)}{CF_t(BY)} \frac{1}{(1+Y_t^E)^t} + \left(1 - \frac{K_t(BY)}{CF_t(BY)}\right) \frac{1}{(1+Y_t^C)^t}$.

This means that $\frac{1}{(1+Y_t^*)^t}$ lies between $\frac{1}{(1+Y_t^E)^t}$ and $\frac{1}{(1+Y_t^C)^t}$.

In other words, $1 + Y_t^*$ lies between $1 + Y_t^C$ and $1 + Y_t^E$.

Thus Y_t^* lies between Y_t^C and Y_t^E .