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The Memory Effect for Plane Gravitational Waves

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Abstract

We give an account of the gravitational memory effect in the presence of the exact plane wave solution of Einstein's vacuum equations. This allows an elementary but exact description of the soft gravitons and how their presence may be detected by observing the motion of freely falling particles. The theorem of Bondi and Pirani on caustics (for which we present a new proof) implies that the asymptotic relative velocity is constant but not zero, in contradiction with the permanent displacement claimed by Zel'dovich and Polnarev. A non-vanishing asymptotic relative velocity might be used to detect gravitational waves through the "velocity memory effect", considered by Braginsky, Thorne, Grishchuk, and Polnarev.

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I. INTRODUCTION

By "gravitational memory effect" is meant that a system of freely falling particles (viewed as detectors) initially at relative rest are displaced after the passing of a burst of gravitational radiation [1, 2]. The memory effect has risen to prominence recently because of the possibility of its observation using LISA [3, 4] and/or aLIGO [5], and is also relevant to recent work by Hawking, Perry, and Strominger [6] on soft gravitons and black holes.

The original proposal of Zel'dovich and Polnarev [1] claimed that detectors originally at rest will be simply displaced and will have vanishingly small relative velocity, in contrast with later findings of Bondi and Pirani [7]. A "velocity memory effect" highlighted by non-vanishing asymptotic relative velocity was considered by Braginsky, Grishchuk, Thorne, and Polnarev [2, 8, 9] and more recently, by Lasenby [4].

The analysis of Braginsky and Grishchuk [2], who also coined the expression "memory effect", is concerned with the motion of test masses moving in a weak gravitational wave at the linear level. Non-linear generalizations were considered in [10–13]. In the neighborhood of a detector, far from the source, we may approximate the wave by an exact plane wave. Carroll symmetry [14, 15] provides us with a description of the trajectories, leading to a new proof of the Bondi-Pirani theorem [7]. Our results might have some relevance for detecting gravitational waves through the "velocity memory effect" [2, 4, 8, 9, 16].

II. PLANE GRAVITATIONAL WAVES

Plane gravitational waves are often described in *Brinkmann Coordinates* (B) [17, 18] by the metric

$$\mathrm{g} = \delta_{ij}\, dX^i dX^j + 2dUdV + K_{ij}(U)X^iX^j\, dU^2, \tag{II.1}$$

where the symmetric and traceless 2×2 matrix $K(U) = (K_{ij}(U))$ characterizes the profile of the wave. In this Letter we only consider linearly polarized waves with

$$K_{ij}(U)X^iX^j = \frac{1}{2}\mathcal{A}(U)\Big((X^1)^2 - (X^2)^2\Big), \tag{II.2}$$

where $\mathcal{A}(U)$ is an arbitrary function. Our particular interest lies in "sandwich waves" i.e. such that $K(U) \neq 0$ only for $U_i \leq U \leq U_f$ [1].

[1] Our conventions are the opposite of that of Bondi and Pirani [7].

The metric (II.1) is an example of a "Bargmann" space which provides a general framework for describing non-relativistic dynamical systems : any natural non-relativistic system with an n dimensional configuration space may be "Eisenhart-lifted" to a system of null geodesics in an $(n+2)$-dimensional Lorentzian spacetime endowed with a covariantly constant null Killing vector field ∂_V [19, 20]. From the Bargmann point of view, (II.1)-(II.2) describe a non-relativistic particle subjected to an (attractive or repulsive) harmonic anisotropic oscillator potential, (II.2).

Returning to gravitational radiation and following the suggestion of [21], the wave produced by gravitational collapse might be modelled by the (approximate) sandwich wave,

$$\mathcal{A}(U) = \frac{1}{2}\,\frac{d^3(e^{-U^2})}{dU^3} \tag{II.3}$$

shown in fig.1.

In a (naive) model the "detector" could consist of freely falling particles initially at relative rest. This applies to the mirrors of the LIGO interferometers of those attached to the three satellites making up the proposed LISA detector [4].

The profile function in the geodesic equations[2],

$$\frac{d^2\mathbf{X}}{dU^2} - \frac{1}{2}\mathrm{diag}(\mathcal{A}, -\mathcal{A})\,\mathbf{X} = 0 \tag{II.4}$$

acts as an effective potential which is attractive or repulsive depending on the sign of $\mathcal{A}$. Eqn. (II.4) can be solved numerically, yielding Fig.1.

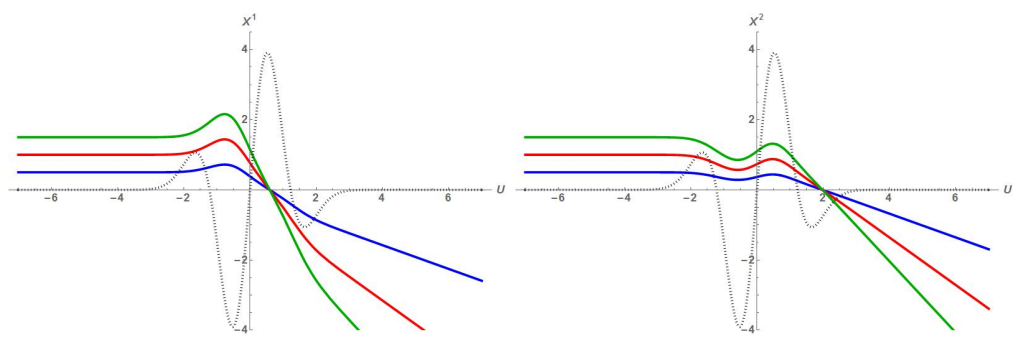

FIG. 1: *Geodesics in Brinkmann coordinates for particles initially at rest for profile $\mathcal{A}(U)$ in (II.3), modelling gravitational collapse. The trajectories are focused where the effective potential is attractive, and are pushed apart where it is repulsive. In the flat afterzone, the acceleration vanishes.*

[2] V satisfies a more complicated equation, (IV.2.c) in [16].

Gravitational waves can also be discussed in *Baldwin-Jeffery-Rosen Coordinates* (BJR) [22, 23], for which

$$\mathrm{g} = a_{ij}(u)\, dx^i dx^j + 2du\, dv \tag{II.5}$$

with $a(u) = (a_{ij}(u))$ a positive matrix. The BJR coordinates are not harmonic and typically not global, exhibiting coordinate singularities [14, 18, 23]. Let us recall the proof [14, 16]. Putting $\chi = \big(\det a\big)^{\frac{1}{4}} > 0$ and $\omega^2(u) = \frac{1}{8}\mathrm{Tr}\left((\gamma^{-1}\dot{\gamma})^2\right)$ where $\gamma = \chi^{-2}a$, the vacuum Einstein equations reduce to the Sturm-Liouville equation

$$\ddot{\chi}(u) + \omega^2(u)\chi(u) = 0, \tag{II.6}$$

see eqn (II.22) of [16]. Then (II.6) implies that $\ddot{\chi} < 0$, which in turn implies the vanishing of χ for some $u_1 > u_i$,

$$\chi(u_1) = 0, \tag{II.7}$$

signalling a singularity of the metric (II.5) and illustrated by fig.2.

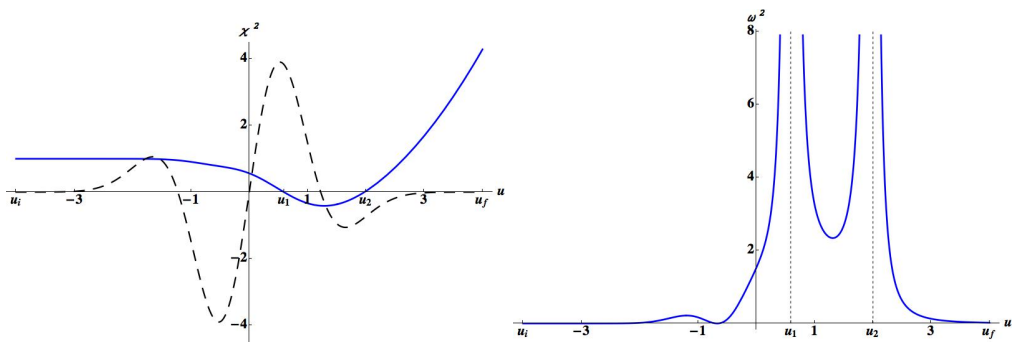

FIG. 2: *The numerical solution of eqn. (II.6) for the "collapse profile" (II.3) indicates that the coordinates become singular when* $u_1 = 0.593342$ *and when* $u_2 = 1.97472$.

Plane gravitational waves have a 5-parameter isometry group [7, 14, 18] identified as the *Carroll group in* $2+1$ *dimensions with broken rotations* [15] and implemented in BJR coordinates as follows : u is fixed, and

$$\mathbf{x} \to \mathbf{x} + H(u)\,\mathbf{b} + \mathbf{c}, \tag{II.8a}$$

$$v \to v - \mathbf{b}\cdot\mathbf{x} - \tfrac{1}{2}\mathbf{b}\cdot H(u)\,\mathbf{b} + f, \tag{II.8b}$$

where H is the symmetric 2×2 matrix calculated using the transverse-space matrix,

$$H(u) = \int_{u_0}^{u} a(t)^{-1} dt. \tag{II.9}$$

Then Noether's theorem provides us with 5 conserved quantities associated with these isometries [14, 15],

$$\boldsymbol{p}=a(u)\,\dot{\mathbf{x}}, \quad \mathbf{k}=\mathbf{x}(u)-H(u)\,\boldsymbol{p}, \tag{II.10}$$

interpreted as conserved *linear and boost-momentum*, supplemented by $m = \dot{u} = 1$. An extra constant of the motion is $e = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$. Geodesics are timelike, lightlike, or spacelike if e is negative, zero, or positive, respectively. The geodesics are thus given by [14, 15]

$$\mathbf{x}(u) = H(u)\,\boldsymbol{p} + \mathbf{k}, \tag{II.11a}$$

$$v(u) = -\frac{1}{2}\boldsymbol{p}\cdot H(u)\,\boldsymbol{p} + e\,u + d, \tag{II.11b}$$

with d a constant. The trajectories can be determined once the matrix-valued function $H(u)$ in (II.9) had been calculated.

The relation between the two coordinates systems is given by [15, 24]

$$\mathbf{X} = P(u)\,\mathbf{x}, \quad V = v - \frac{1}{4}\mathbf{x}\cdot\dot{a}(u)\mathbf{x}\,, \tag{II.12}$$

where $U = u$ [3], $a(u) = P(u)^T P(u)$ and

$$\ddot{P} = K\,P\,, \qquad P^T\dot{P} - \dot{P}^T P = 0\,. \tag{II.13}$$

Now we consider a sandwich wave. In the "beforezone" $U < U_i$ $K = 0$ and therefore eqn. (II.13) can be solved by $P(u) = \mathbf{1}$ so that the Brinkmann and BJR coordinates coincide.

The crucial fact for us is that, by (II.10), the momentum of a particle at rest vanishes, $\boldsymbol{p} = 0$ for $u = U \leq U_i$ — and thus for *all* $u = U$ including the "inside zone" $U_i \leq U \leq U_f$ and the "afterzone" $U_f \leq U$, because $\boldsymbol{p}$ is conserved. By (II.11a) such geodesics are, *for any* H *i.e. for any transverse metric* a and therefore *for any profile* $\mathcal{A}$, simply

$$\mathbf{x}(u) = \mathbf{x}_0, \quad v(u) = e\,u + v_0. \tag{II.14}$$

Thus the *particles initially at rest are, when described in BJR coordinates, undisturbed by the passage of the wave* [4] .

In Brinkmann coordinates both the waves and the geodesics are global with no singularity whatsoever. Solving the equations (II.13) numerically for P yields,

$$\mathbf{X}(U) = P(u)\mathbf{x}^0, \qquad \mathbf{x}^0 = \text{const.} \tag{II.15}$$

[3] The notation u or U is used only to distinguish which coordinates we are working with.

[4] If the initial momentum does not vanish, though, $\boldsymbol{p} \neq 0$, then the trajectories are obviously not simple straight lines; they can be determined by calculating $H(u)$ in (II.9), cf. [16].

cf. (II.12). Those complicated-looking trajectories in B coordinates in Fig.1 are recovered: the plots overlap perfectly *up to the point where the BJR coordinate system becomes singular*; all the curling comes from $P(u)$.

Our plots show that, consistently with the theorem of Bondi and Pirani [7], *after the wave has passed the worldlines will first be focused in a finite and initial-position-independent time, and then diverge with* constant but non-zero asymptotic velocity – contradicting the claim of Zel'dovich and Polnarev [1].

We can actually prove the Bondi-Pirani theorem independently using our framework. First we note that $\mathcal{A} \approx 0$ for large values of U; then it follows from the geodesic equations (II.4) that the acceleration vanishes, implying constant asymptotic velocity [5].

The geodesics we are interested in are of the form (II.15); in the beforezone $U \leq U_i$ we have $P = \mathbf{1}$ and the geodesics are at rest, $\mathbf{X}(U) = \mathbf{x}^0$. Let us now consider two such special geodesics, $\mathbf{X}_a(U)$ and $\mathbf{X}_b(U)$; we show that there exist $\mathbf{x}^0_a \neq \mathbf{x}^0_b$ such that the two geodesics meet, $\mathbf{X}_a(U_1) = \mathbf{X}_b(U_1)$ for some finite and universal value $u_1 > u_i$. $\mathbf{X}_{a,b}(U) = P(u)\mathbf{x}^0_{a,b}$, so we need $\mathbf{x}^0 = \mathbf{x}^0_a - \mathbf{x}^0_b \neq 0$ such that $P(u_1)\mathbf{x}^0 = 0$, which happens when $P(u_1)$ is a singular matrix,

$$\det(P(u_1)) = 0. \tag{II.16}$$

But $a = P^T P$ so that $\det a = (\det P)^2 = \chi^4$ and therefore $\det P = 0$ iff $\chi = 0$. However, as recalled above, (II.6) implies that χ necessarily vanishes at some $u_1 > u_i$ – at the same value where the BJR coordinates become singular. This is not an accident: the singular BJR coordinate system can only be regularized by a singular transformation. Thus $P(u_1)$ has a kernel and all geodesics for which $\mathbf{x}^0_a - \mathbf{x}^0_b$ belongs to the kernel will meet at the same u_1. If $\mathbf{x}^0_a - \mathbf{x}^0_b$ does not belong to the kernel, then we get a caustic. For a linearly polarized wave as in (II.1)-(II.2), $P = \mathrm{diag}(P_{11}, P_{22})$ and the x^1 resp x^2 coordinates meet at critical (caustic) point when either P_{11} or P_{22} vanishes, which is where BJR coordinates become singular, cf. fig. 1.

In the flat afterzone $u \geq u_f$, in particular, we have the exact analytic solution [14, 16]

$$P(u) = \big(\mathbf{1} + (u - u_0)c_0\big)a_0^{1/2} \quad \Rightarrow \quad a(u) = a_0^{\frac{1}{2}}\left[\mathbf{1} + (u - u_0)c_0\right]^2 a_0^{\frac{1}{2}}, \tag{II.17}$$

where we have chosen $u_0 = u_f$, $a_0 = a(u_0)$, $c_0 = \frac{1}{2}a_0^{-1/2}\dot{a}(u_0)a_0^{-1/2} \neq 0$ [14, 16]. Setting

[5] Another proof using BJR coordinates is presented in [16].

$\chi_0 = \chi(u_0)$,

$$\chi(u) = \chi_0 \left[1 + (u - u_0)\mathrm{Tr}(c_0) + (u - u_0)^2 \det(c_0)\right]^{\frac{1}{2}} . \tag{II.18}$$

Then the requirement of continuity implies $\mathrm{Tr}(c_0) = 2\,\dot{\chi}_0/\chi_0$, valid as long as $\chi \neq 0$. Thus $\mathrm{Tr}(c_0) \neq 0$ unless $\dot{\chi}_0 = 0$. Combining (II.15) and (II.17) yields the trajectory in the afterzone, plotted in fig.3. The asymptotic velocity is inferred from (II.17),

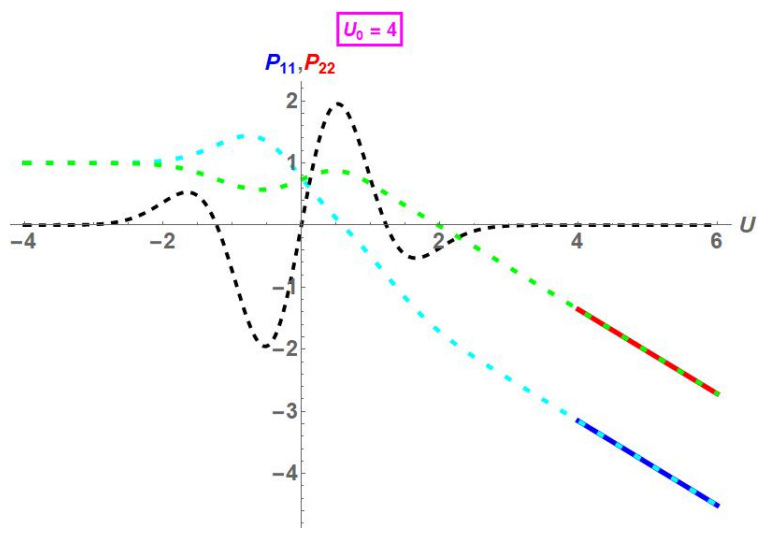

FIG. 3: *The analytic (heavy line) and numerical (dashed line) solutions overlap perfectly in the afterzone* $u \geq u_0 = 4$.

$$\dot{\mathbf{X}}(u) = c_0 a_0^{1/2} \mathbf{x}^0 = \frac{1}{2} a_0^{-1/2} \dot{a}(u_0) \mathbf{x}^0 = \text{const.} \tag{II.19}$$

Thus $\dot{\mathbf{X}}_a \neq \dot{\mathbf{X}}_b$ for $\mathbf{x}_a^0 \neq \mathbf{x}_b^0$ in general, as confirmed by numerical calculations. At last, the linear form of P in (II.17) implies that each component of P can have at most one caustic in the afterzone.

III. ILLUSTRATION BY TISSOT DIAGRAM

The action of gravitational waves on a ring of freely falling particles can be illustrated by a series of time-frames showing how the ring is squashed and stretched as the wave passes over it [25]. An initially (i.e. before the pulse) circular disc of geodesics, $\mathbf{X} \cdot \mathbf{X} \leq 1$ for $U < U_i$, projects to a time-independent circle, $\mathbf{x} \cdot \mathbf{x} \leq 1$, which extends to all u, i.e., during and after the sandwich, $u \geq u_i$. However their inverse images in Brinkmann coordinates are time dependent ellipses,

$$1 = \mathbf{x} \cdot \mathbf{x} = \mathbf{X} \cdot (PP^T)^{-1} \mathbf{X} \tag{III.1}$$

as shown in Fig. 4.

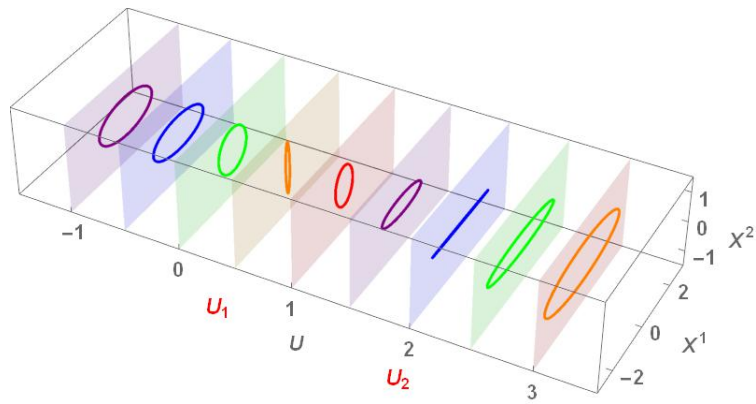

FIG. 4: *Tissot diagram for the "collapse profile" (II.3). The initial circle is deformed to an ellipse which at $u = u_1$ degenerates to a line segment, and so on.*

IV. SOFT GRAVITONS

We end with a description of the flat spacetime after the burst as being dressed by soft gravitons, a concept which plays a central role in the considerations of [6]. Before the burst $a_{ij} = \delta_{ij}$, and the BJR coordinates coincide with standard inertial coordinates in Minkowski spacetime. After the burst a_{ij} will be u-dependent but nevertheless will have vanishing curvature. Thus there exist coordinates $\hat{u}, \hat{v}, \hat{\mathbf{x}}$ such that the 2-metric is $\hat{a}_{ij} = \delta_{ij}$. The transformation relating $u, v, \mathbf{x}$ to the $\hat{u}, \hat{v}, \hat{\mathbf{x}}$ coordinates is given by (II.12) with $\ddot{P} = 0$, as follows from the vanishing of K. In the quantum field theory language adopted by Hawking et al. in [6], the coordinate transformation is a "gauge transformation which does not tend to the identity at infinity". In quantum field theory it is customary to distinguish states which so differ and regard them as containing "soft" [i.e. zero-energy] quanta. In the present case the quanta are gravitons given by pp-waves, whose metric is given by (II.5).

V. CONCLUSION

We found that test particles initially at rest move, after the wave has passed, with *constant but non-zero relative velocity*, contradicting Zel'dovich and Polnarev [1] but consistently with Bondi and Pirani [7], whose theorem we proved independently. The non-vanishing asymptotic relative velocity might allow to observe the *velocity memory effect* by improved

Doppler tracking [2, 8, 9] [6].

The intuitive explanation is that (II.1)-(II.2) is (as said above), the Bargmann metric of an anisotropic oscillator with time-dependent frequencies, one of which is attractive and the other repulsive. The components in the repulsive sector are pushed apart whereas in the attractive sector they are focused and then meet where the matrix P is singular.

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