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1 **Dynamic mass density of resonant metamaterials with homogeneous inclusions**

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## Abstract

7       The occurrence of a negative dynamic mass density is a striking property of meta-  
8 materials. It appears when an inner local resonance is present. Results coming from an  
9 asymptotic theory are recalled briefly, showing the scaling of physical properties leading to  
10 inner resonance in elastic composites containing homogeneous soft inclusions, with negli-  
11 ble scattering of waves travelling through the matrix. This appears for a large contrast of  
12 elastic properties between matrix and inclusion. The frequency-dependent dynamic mass  
13 density depends on the resonance frequencies of the inner inclusions and on their related  
14 participation factors. Having solved the dynamic elasticity problem, these physical quan-  
15 tities are provided in the case of homogeneous cylindrical and spherical inclusions. It is  
16 shown that numerous resonance frequencies do not contribute to the dynamic mass density  
17 or have small participation factors, which simplifies significantly the physics involved in the  
18 concerned inner resonance phenomena. Finally, non-dimensional resonance frequencies and  
19 participation factors are given for both cases of inclusions as functions of the Poisson's ratio,  
20 defining completely the dynamic mass density.

## 21 I. INTRODUCTION

22 Even if the existence of metamaterials has been predicted initially in the case of  
23 electromagnetic properties<sup>41;36</sup>, acoustic metamaterials have been studied extensively  
24 during the last few decades, more specially since the production of experiments showing  
25 band gaps when studying the propagation of elastic waves in phononic crystals<sup>40</sup> followed a  
26 few years after by experiments of wave propagation through acoustic metamaterials<sup>20</sup> .  
27 Acoustic metamaterials can have more or less complex internal structures as prove the  
28 numerous papers in this field either studying properties of metamaterials from a theoretical  
29 approach<sup>17;18;19;7;11</sup> and/or showing experimentally the properties of these  
30 materials<sup>20;38;28;46;33;44</sup>. These materials can have "effective" negative dynamic density  
31 and/or elastic moduli for some frequency ranges, leading to forbidden frequency bands. A  
32 recent review paper on acoustic metamaterials<sup>23</sup> draws a large picture of this domain,  
33 showing that the interesting properties of metamaterials are enhanced by inserting local  
34 resonators. These resonators can be a more or less complex combination of plates, beams,  
35 spring-mass systems, Helmholtz resonators or (homogeneous or heterogeneous) inclusions of  
36 different shapes. This review paper reports also several recent studies on the conception of  
37 materials having simultaneously negative effective mass density and elastic moduli<sup>16;14;18</sup> .

38 In the following, the study will be restricted to elastic composites containing  
39 inclusions of different shapes. Even so, the amount of literature devoted to this field is

40 impressive. It is known since a long time that every periodic composite containing  
41 inclusions is characterized by forbidden frequency bands, its response to wave propagation  
42 being studied usually by Bloch waves. However, these forbidden frequencies appear at high  
43 frequencies, when the wavelength is of the order of the inclusion size,i.e. the material being  
44 considered as a "phononic crystal". A few decades ago, it was recognized that the  
45 frequency range of forbidden frequency bands is lowered by introducing inner resonators  
46 and that the most simple way to induce inner resonance is to use matrix-inclusion  
47 composites containing rigid inclusions surrounded by a soft coating (i.e. "composite  
48 inclusions"), the whole being immersed within an elastic matrix. This led to the concept of  
49 acoustic metamaterials. The composite inclusions act as "spring-mass" resonators, the  
50 mass being provided by the inner inclusion and the spring by the surrounding coating. In  
51 the case of a large contrast between elastic properties of coating and matrix, the low  
52 resonance frequency of the spring-mass resonator can be reached practically without  
53 scattering of the waves by the inclusions. It is the extreme case of "inner-resonance with  
54 negligible scattering". Obviously, when the elastic properties of the coating become of the  
55 same order as the one of the matrix, it is necessary to take into account the coupling of the  
56 inner resonance with the scattering of waves by the inclusions. Several methods were used  
57 to account for this coupling<sup>48;47;43;21;45;24</sup>, the most currently used being the Coherent  
58 Potential Approximation<sup>37</sup>. However, in the case of a soft coating, the case of "inner

59 resonance with negligible scattering” is certainly an interesting comparison model, which  
60 becomes fully valid for large contrasts of elastic properties of constituents.

61 Less work was devoted to homogeneous inclusions, and these works deal mainly with  
62 scattering by inner inclusions. However, early works using homogenization theory based on  
63 asymptotic expansion<sup>1;2</sup> have shown that ”inner resonance with negligible scattering” can  
64 occur also in composites containing homogeneous inclusions. This is the domain of  
65 application of the present paper.

66 The physics of wave propagation through composite materials is complex and a strong  
67 physical insight can be gained by using an asymptotic series expansion<sup>8</sup> of the dynamic  
68 solution for different assumptions of the ratios between physical parameters: elastic  
69 coefficients and mass densities. First homogenization results showing the occurrence of  
70 bandgaps in periodic elastic composites with ”inner resonance with negligible scattering”  
71 were obtained in early works<sup>1;2</sup>, but with the development of the ideas on metamaterials,  
72 new results were obtained with the use of asymptotic expansions in dynamic  
73 elasticity<sup>34;4;5;3;39;6;10;32;31;25;26</sup>. More specifically, Auriault and Boutin<sup>3</sup> have recently  
74 extended the method of asymptotic expansion to materials containing composite inclusions  
75 and characterized by ”inner resonance with negligible scattering”. It is worthwhile noticing  
76 that, compared with results coming from other publications in this field, an important  
77 aspect does appear in the papers written by Auriault and al. that are at the basis of this

78 paper<sup>1;2;3</sup>: it is shown indeed in these papers that *the use of different scalings on physical*  
79 *properties can lead to different macroscopic behaviours*. In the case of a strong contrast, the  
80 method proves the occurrence of "inner resonance with negligible scattering" ; it provides  
81 the structure of the effective constitutive equations and the local elasticity problems  
82 (usually called "localization problems" or "cell problems") to be solved in order to obtain  
83 the macroscopic physical properties, more particularly the dynamic mass density. It is  
84 worth noticing that in absence of strong contrasts, the predicted dynamic behaviour can be  
85 non-local in space<sup>12;42;29;30</sup>, while the behaviour described in the case of "inner resonance  
86 with negligible scattering" is local in space (while non-local in time). This point is of  
87 importance for modelling wave propagation through a structure which would be made of  
88 such a metamaterial.

89 In a recent paper<sup>9</sup> the dynamic mass density has been obtained by solving the  
90 localization problem described by the homogenization theory in the case of materials  
91 displaying "inner resonance with negligible scattering" and containing spherical or  
92 cylindrical composite inclusions that act as "spring-mass" resonators. Under suitable  
93 conditions, such a dynamic mass density can be obtained by studying the static behaviour  
94 (spring effect) of the composite inclusion. By comparison, in the case of homogeneous  
95 inclusions described thereafter, the dynamic mass density involves the full dynamic  
96 behaviour of the inclusions.

97 In section 2, the results of asymptotic expansions in the case of materials containing  
98 homogeneous inclusions are synthesized and discussed, showing that suitable ratios of the  
99 physical parameters can lead to inner resonance with negligible scattering. The method  
100 specifies also the dynamic boundary value problem ("cell problem") to be solved in order  
101 to provide the dynamic mass density. In section 3, the inner motion of inclusions made of  
102 cylindrical fibers is studied: the eigenfrequencies are obtained and then participation  
103 factors and dynamic mass densities of these resonators are given in a closed form. The case  
104 of spherical inclusions is studied in section 4 and the full solution of the local elasticity  
105 problem is given in this case, providing again the eigenfrequencies, participation factors  
106 and dynamic mass densities. Finally, numerical applications are presented in section 5.

## 107 **II. DYNAMIC BEHAVIOUR OF ELASTIC COMPOSITES CONTAINING** 108 **HOMOGENEOUS INCLUSIONS**

109 This section recalls briefly the main results from homogenization using asymptotic  
110 theory<sup>1;2;3</sup> and the underlying assumptions that justify the method used in the following  
111 sections to obtain the dynamic mass density. Let us consider a periodic elastic composite  
112 material made up of a matrix containing homogeneous inclusions. This composite material  
113 is defined by the geometry of the periodic cell containing the inclusions and by the physical  
114 properties of the constituent materials. These constituents are assumed elastic and  
115 isotropic. They are characterized by their mass densities  $\rho^{(s)}$ , their volume concentrations



116  $c^{(s)}$  and their Lamé elastic parameters  $\lambda^{(s)}, \mu^{(s)}$ . ( $s$ ) corresponds to the matrix ( $m$ ) or the  
 117 inclusion ( $i$ ). The effective behaviour of the composite depends strongly on the scaling  
 118 parameters. In a first step, these scaling parameters are defined and next, the scaling  
 119 assumptions leading to the occurrence of a negative mass density will be provided.

## 120 **A. Scaling parameters**

121 A first parameter is the geometric scaling ratio defined by  $\epsilon = \frac{l}{L}$ , where  $l$  is the size of  
 122 the periodic cell and  $L$  is the order of magnitude of the wavelength within the matrix in a  
 123 chosen frequency range. The case of interest is when the size of the periodic cell is small  
 124 compared with  $L$  and therefore  $\epsilon \ll 1$ .

125 Other scaling parameters contain the various physical parameters that characterize  
 126 the constituents of the composite.

127 They comprise:

- 128 - the ratio between the mass densities of the constituents  $\frac{\rho^{(i)}}{\rho^{(m)}}$ .
- 129 - the ratio between the orders of magnitude of the elastic coefficients of the  
 130 different constituents.

131 The order of magnitude of the elasticity coefficients of a given material will be defined  
 132 by the value of  $a^{(s)} = \lambda^{(s)} + 2\mu^{(s)}$  (i.e. a norm of the elasticity tensor) . The scaling ratio  
 133 related to elastic coefficients will be defined by  $\frac{\lambda^{(i)}+2\mu^{(i)}}{\lambda^{(m)}+2\mu^{(m)}} = \frac{a^{(i)}}{a^{(m)}}$ :

## 134 **B. Scaling assumptions leading to the occurrence of a negative dynamic mass**

135 **density**

136 Using asymptotic expansion along the scaling parameter  $\epsilon$  , Auriault and Bonnet<sup>1</sup>  
 137 (see also<sup>2;3</sup>) studied the effect of different values of the previously defined scaling ratios on  
 138 the overall behaviour of the composite. These authors showed that the resonance of the  
 139 inner inclusions can be obtained when the inclusions are very soft compared with the  
 140 matrix, the densities being of the same order of magnitude. More precisely, the inner  
 141 resonance is obtained when the scaling ratios meet the following conditions:  $\frac{a^{(i)}}{a^{(m)}} = O(\epsilon^2)$   
 142 for the elastic parameters and  $\frac{\rho^{(i)}}{\rho^{(m)}} = O(1)$  for the mass densities.

143 **C. The dynamic mass density**

Under the previously defined conditions, the composite has an overall effective  
 behaviour for harmonic time excitation at radial frequency  $\omega$  that is characterized by the  
 dynamic equation

$$\frac{\partial(a_{ijkl}^{(eff)} \varepsilon_{kl})}{\partial x_j} + \omega^2 \rho_{ij}^{(eff)} u_j = 0 \quad (1)$$

144 where  $a_{ijkl}^{(eff)}$  are the "effective" elastic coefficients of the composite ,  $\varepsilon_{kl}$  are the components  
 145 of the overall strain tensor,  $u_j$  are the components of the displacement and  $\rho_{ij}^{(eff)}$  are the  
 146 components of a frequency-dependent dynamic mass density. With a periodic array of  
 147 inclusions, the effective elastic behaviour is not isotropic, contrarily to the behaviour of the  
 148 constituents. Due to the scaling assumptions, the inclusions are very soft compared with

149 the matrix. As a consequence, the effective elastic coefficients  $a_{ijkl}^{(eff)}$  can be computed as for  
 150 a matrix that contains voids in place of inclusions by usual means to obtain effective  
 151 properties of heterogeneous elastic materials.

Concerning the components  $\rho_{ij}^{(eff)}$  of the mass density, they must be considered as the  
 ones of a tensor of second rank. In the following, it will be assumed that the inclusions  
 have three orthogonal symmetry planes. In this case, the dynamic mass density is diagonal  
 for a coordinate system whose axes are parallel to the symmetry planes, with:

$$\rho_{ii}^{(eff)}(\omega) = \langle \rho \rangle + c^{(i)} \rho^{(i)} h_{ii}(\omega) \quad (2)$$

152 where Einstein's convention summation must not be applied to the repeated index  $i$  in  
 153  $\rho_{ii}^{(eff)}$ ,  $h_{ii}$ .  $\langle \rho \rangle$  is the volume average of the density within the periodic cell and  $c^{(i)}$  is the  
 154 volume concentration of inclusions.  $h_{ii}$  are the components of a diagonal second order  
 155 tensor.

The homogenization method provides the localization problems ("cell problems")  
 allowing to compute the effective elastic properties and the tensor of components  $h_{ij}$ , as  
 recalled in Appendix I. The asymptotic process proves that, at the first order, due to the  
 large wavelength within the matrix, the fluctuation of displacement of the matrix  $\vec{u}^{(m)}$   
 within the periodic cell is of the order of  $\epsilon$  (while the fluctuations of strain field are of the  
 same order as the macroscopic strain). It means that the displacement and acceleration of  
 the matrix are nearly constant inside a periodic cell. As a consequence, the localization

problem obtained by homogenization theory recalled in Appendix I corresponds to a dynamic localization problem within the inclusion moved by a uniform acceleration induced by the matrix. It can be expressed by using the relative displacement

$\vec{w} = \vec{u}^{(i)} - \vec{u}^{(m)}$ . This relative displacement is null over the boundary of the inclusion and is the solution of

$$\mu^{(i)} \Delta \vec{w} + (\lambda^{(i)} + \mu^{(i)}) \overrightarrow{\text{grad}} (\text{div}(\vec{w})) + \rho^{(i)}(\omega)^2 \vec{w} = -\rho^{(i)}(\omega)^2 \vec{u}^{(m)} \quad (3)$$

156 It means that the motion within a given inclusion, in a local reference frame moving with  
157 the matrix, is impulsed by the inertial acceleration  $-\rho^{(i)}(\omega)^2 \vec{u}^{(m)}$ .

The solution of this problem can be expressed by using the eigenfrequencies  $\omega^p$  and eigenmodes  $\vec{u}^p$  of the inclusions *for fixed boundaries in the reference frame moving with the matrix*, i.e. the solutions of

$$\mu^{(i)} \Delta \vec{u}^p + (\lambda^{(i)} + \mu^{(i)}) \overrightarrow{\text{grad}} (\text{div}(\vec{u}^p)) + \rho^{(i)}(\omega^p)^2 \vec{u}^p = 0 \quad (4)$$

158 with  $\vec{u}^p = 0$  on the boundary of the inclusion.

Finally, for an acceleration of the matrix given by  $\gamma \vec{e}_j$  where  $\vec{e}_j$  is a unit vector along one of the axes, the related component  $h_{jj}(\omega)$  is given by:

$$h_{jj}(\omega) = \sum_{p=1}^{\infty} K_{jj}^p \cdot \frac{1}{\left(\frac{\omega^p}{\omega}\right)^2 - 1} \quad (5)$$

where the participation factors  $K_{jj}^p$  do not depend on  $\omega$  and are given by

$$K_{jj}^p = \frac{(\langle \vec{u}^p \rangle^{(i)} \cdot \vec{e}_j)^2}{\langle \|\vec{u}^p\|^2 \rangle^{(i)}}, \quad (6)$$

159 the volume average  $\langle f \rangle^{(i)}$  being computed over the volume of the inclusion. Due to the  
 160 dependence on  $\omega$ ,  $h_{jj}$  tends to minus infinity just above each resonance frequency. As a  
 161 consequence, the related component of the overall dynamic mass density  $\rho_{jj}^{(eff)}$  tends also  
 162 to minus infinity and remains negative until its negative frequency dependent part is  
 163 equilibrated by the static overall mass density  $\langle \rho \rangle$  in equation (4).

164 It is worthwhile noticing that, due to the condition at the boundary of the inclusion  
 165 (induced obviously by the assumption of "negligible scattering"), the contributions of  
 166 different separate inclusions within a periodic cell are completely independent and can be  
 167 summed in order to compute the overall dynamic mass density for any distribution of  
 168 inclusions.

169 Finally, the eigenfrequencies and eigenmodes  $(\omega^p, \vec{u}^p)$  must be obtained for each  
 170 direction of  $\vec{e}_j$  by solving a "localized eigenmodes problem" (4). Next, the averages in (6)  
 171 are computed to produce the three components of the participation factors. In the following  
 172 sections, the localization problems will be solved for cylindrical and spherical inclusions.

#### 173 **D. Physical discussion of the results**

174 The results obtained from asymptotic expansion may be surprising by some aspects.  
 175 Indeed, these results show that the inclusion behaves as if the surrounding material moves  
 176 uniformly at its boundary. In addition, the field inside the inclusion does not depend on  
 177 the wavelength within the surrounding material and is therefore not affected by the

178 scattering of these waves by the inclusions.

179        Obviously, the displacement within the matrix is not exactly uniform, due to the  
180 varying strain around the inclusion. However, the ratio of strain fields between matrix and  
181 inclusion is very large, due to the elasticity contrast, and as a consequence, the fluctuation  
182 of displacement around the inclusion is very small compared with the displacement inside  
183 the inclusion and may be considered as negligible. The absence of scattering comes  
184 naturally from the initial assumption of a large ratio between the wavelength related to the  
185 propagation within the matrix and the size of inclusions. Finally, the results coming from  
186 the asymptotic theory appear as physically sound. Indeed, numerous devices entering the  
187 composition of metamaterials use inner resonators whose resonance appears at low  
188 frequencies, below frequencies inducing a scattering by the cells containing the resonators  
189 (Helmholtz resonators, spring-mass resonators,...) .

190        Another point of discussion is the fact that the dynamic density tends to infinity at  
191 the resonance frequencies. Obviously, this result comes from the assumption of perfectly  
192 elastic materials. For real materials, the physical damping must be taken into account.  
193 This can be effected by assuming the inclusion material as viscoelastic.

194        In this case, the viscoelastic behaviour of the inclusion material can be taken into  
195 account by using complex and frequency dependent elastic moduli. As a result of the use of  
196 complex moduli, the orthogonality of modes is no more ensured. Due to this loss of

197 orthogonality, the method presented previously cannot be used as such in the case of  
198 viscoelasticity and will not be studied thereafter. For a slight damping, an asymptotic  
199 procedure can again be used in order to provide the viscoelastic solution<sup>3</sup>. It leads to a  
200 finite value of the dynamic density and to a small shift of the resonance frequency.

### 201 **III. DYNAMIC MASS DENSITY FOR CYLINDRICAL INCLUSIONS**

202 Let us consider a composite containing long parallel cylindrical inclusions, i.e. long  
203 cylindrical fibers of radius  $a$ . As seen in section 2, the dynamic contribution to the  
204 dynamic mass density depends only on the properties of the inclusions. Therefore, in the  
205 following, the notation  $\lambda^{(i)} = \lambda, \mu^{(i)} = \mu, \rho^{(i)} = \rho$  will be used. In a first step, the general  
206 dynamic displacement within the fiber is given by using separate variables. Next, it will be  
207 shown that numerous contributions to this general field do not provide contributions to the  
208 participation factors. Finally, the expression of the dynamic mass density will be given.

#### 209 **A. Dynamic displacement field within a cylindrical domain**

210 Different solutions exist for the dynamic displacement field within a cylindrical  
211 domain<sup>22;15</sup> using separate variables. The expression below is given in Eringen and  
212 Suhubi<sup>15</sup>( eq.8.9.17-8.9.18) where it is shown that the displacement field is the

213 superposition for integer values of  $n$  of the following partial displacements:

$$\begin{aligned}
 u_r &= \frac{1}{r} [A_1 U_1(\alpha r) \cos n\theta e^{\pm i\gamma_p z} + B_1 U_2(\beta r) \sin n\theta e^{\pm i\gamma_q z} + C_1 U_3(\beta r) \cos n\theta e^{\pm i\gamma_q z}] \\
 u_\theta &= \frac{1}{r} [A_1 V_1(\alpha r) \sin n\theta e^{\pm i\gamma_p z} + B_1 V_2(\beta r) \cos n\theta e^{\pm i\gamma_q z} + C_1 V_3(\beta r) \sin n\theta e^{\pm i\gamma_q z}] \\
 u_z &= A_1 W_1(\alpha r) \cos n\theta e^{\pm i\gamma_p z} + C_1 W_3(\beta r) \cos n\theta e^{\pm i\gamma_q z}
 \end{aligned}$$

214 for  $n = 0..∞$ , where  $\alpha = \frac{\omega}{c_p}$  ,  $\beta = \frac{\omega}{c_s}$  are the wave numbers related to the celerities of

215 compressional and shear waves given by  $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ ,  $c_s = \sqrt{\mu/\rho}$ , with an

216 "alternative expression" obtained by replacing the *cos* terms by *sin* ones (and reversely)

217 and the constants  $A_1, B_1, C_1$  by another set of constants  $A_2, B_2, C_2$ .

218  $U_1, U_2, U_3, V_1, V_2, V_3, W_1, W_3$  are functions of  $r$  which are expressed by using Bessel  
 219 functions.

220 As shown previously, the displacement field is sought for an acceleration parallel to  
 221 one of the base vectors  $\vec{e}_j$ . This displacement field has the symmetry with respect to the

222 planes containing two coordinate axes. As a consequence, the symmetry induces that

223  $\gamma_p = \gamma_q = 0$ . From the expression of the radial components, it comes that  $U_3, V_3$  and  $W_1$

224 are null. As another consequence of the symmetry, the constants  $B_1, A_2, C_2$  are null.



225 Finally, replacing  $A_1, B_2, C_1$  by  $A, B, C$ , it comes that the displacement field is given by:

$$\begin{aligned} u_r &= \frac{1}{r}[AU_1(\alpha r) \cos(n\theta) + BU_2(\beta r) \cos(n\theta)] = u_r^0 \cos(n\theta) \\ u_\theta &= \frac{1}{r}[AV_1(\alpha r) \sin(n\theta) + BV_2(\beta r) \sin(n\theta)] = u_\theta^0 \sin(n\theta) \\ u_z &= CW_3(\beta r) \cos(n\theta) \end{aligned}$$

226 The radial dependence is described by the following functions:

$$\begin{aligned} U_1(\alpha r) &= \alpha r J_{n-1}(\alpha r) - n J_n(\alpha r) \\ U_2(\beta r) &= n J_n(\beta r) \\ V_1(\alpha r) &= -n J_n(\alpha r) \\ V_2(\beta r) &= -\beta r J_{n-1}(\beta r) + n J_n(\beta r) \\ W_3(\beta r) &= \beta^2 J_n(\beta r) \end{aligned}$$

227 where  $J_n$  is the  $n^{th}$  order Bessel function of first kind. It can be seen that this displacement  
 228 field contains two independent fields: a first one with components  $(u_r, u_\theta)$  and a second one  
 229 corresponding to  $u_z$ . The first one corresponds to a transversal displacement and the  
 230 second one to a longitudinal displacement.

231 At this stage, the expression of the displacement field contains the components related  
 232 to any value of  $n$ . However, it will be shown now that only a few of them contribute to the  
 233 participation factors.

234 **B. Selection of the modes contributing to the dynamic mass density**

235 As seen in section 2, the participation factors are functions of the average of the  
 236 projection of the displacement field along the direction of the unit vector  $\vec{e}_j$ . In a first  
 237 step, this average projection is studied, because it is important to select only the modes  
 238 which contribute to the dynamic mass density.

239 **1. Transversal motion**

240 It is assumed that the acceleration is transversal, i.e.  $\vec{e}_j$  is perpendicular to the axis of the  
 241 cylinder. Due to the symmetry, the direction of  $\vec{e}_j$  can be chosen as  $\vec{e}_x$  corresponding to  
 242  $\theta = 0$ . The average of the projection of the displacement field over the section is given by  
 243  $I_t/S$ , where  $S = \pi a^2$  and  $I_t$  is the integral of the projection of the displacement field onto  
 244  $\vec{e}_x = \cos \theta \vec{e}_r - \sin \theta \vec{e}_\theta$  given by  $I_t = \int \vec{u} \cdot \vec{e}_x r dr d\theta$ , where  
 245  $\vec{u} \cdot \vec{e}_x = u_r^0 \cos n\theta \cos \theta - u_\theta^0 \sin n\theta \sin \theta$ . The integral over  $\theta$  is null except for  $n = 1$ . It  
 246 implies that only the contributions corresponding to  $n = 1$  must be taken into account in  
 247 the computation of the dynamic mass density. Finally, the transversal displacement field is  
 248 given by:

$$u_r = \frac{1}{r} [AU_1(\alpha r) \cos \theta + BU_2(\beta r) \cos \theta] = u_r^0 \cos \theta$$

$$u_\theta = \frac{1}{r} [AV_1(\alpha r) \sin \theta + BV_2(\beta r) \sin \theta] = u_\theta^0 \sin \theta$$

249 with

$$\begin{aligned} u_r^0 &= \frac{1}{r}[AU_1(\alpha r) + BU_2(\beta r)] \\ u_\theta^0 &= \frac{1}{r}[AV_1(\alpha r) + BV_2(\beta r)] \end{aligned}$$

250 and

$$\begin{aligned} U_1(\alpha r) &= \alpha r J_0(\alpha r) - J_1(\alpha r) = r \frac{dJ_1(\alpha r)}{dr} \\ U_2(\beta r) &= J_1(\beta r) \\ V_1(\alpha r) &= -J_1(\alpha r) \\ V_2(\beta r) &= -\beta r J_0(\beta r) + J_1(\beta r) = -r \frac{dJ_1(\beta r)}{dr} \end{aligned}$$

## 251 **2. Longitudinal motion**

252 The longitudinal motion is obtained for  $\vec{e}_j = \vec{e}_z$ . In this case, the integral of the projection  
 253 of the displacement field is given by  $I_l = \int \vec{u} \cdot \vec{e}_z r dr d\theta = \int u_z r dr d\theta$ . It involves the  
 254 integral over  $\theta$  of  $\cos n\theta$  which is null except for  $n = 0$ . The longitudinal motion is therefore  
 255 given by  $u_z = CW_3(\beta r) = DJ_0(\beta r)$ , where  $D$  is an undetermined constant.

## 256 **C. Computation of the eigenfrequencies**

257 **1. Transversal motion**

258 The eigenfrequencies correspond to the displacement fields which comply with the  
 259 condition of null displacement over the boundary of the cylinder ( $r = a$ ) and therefore to  
 260 the linear system

$$AU_1(\alpha a) + BU_2(\beta a) = 0$$

$$AV_1(\alpha a) + BV_2(\beta a) = 0$$

This homogeneous system has a solution only when its determinant is null and finally the  
 eigenfrequencies equation is given by  $U_1(\alpha a)V_2(\beta a) - U_2(\beta a)V_1(\alpha a) = 0$ . Using the  
 expressions of functions  $U_i, V_i$ , this relation can be written by using only the  
 non-dimensional frequency  $\omega^* = \frac{\omega a}{c_s}$  as:

$$-k_c \omega^* J_0(k_c \omega^*) J_0(\omega^*) + k_c J_0(k_c \omega^*) J_1(\omega^*) + J_0(\omega^*) J_1(k_c \omega^*) = 0 \quad (7)$$

261 where  $k_c = c_s/c_p$  is the ratio between celerities of shear and compressional waves, which  
 262 depends only on the Poisson's ratio  $\nu$  and is given by  $k_c = \sqrt{\frac{1-2\nu}{2(1-\nu)}}$ .

263 The solutions of this equation in  $\omega^*$  are the non-dimensional eigenfrequencies  
 264  $\omega^{p*}$  ( $p = 1.. \infty$ ) related to the transversal modes. Having computed these eigenfrequencies,  
 265 the eigenmodes are obtained by noticing that  $A$  and  $B$  must comply with the homogeneous  
 266 equations and therefore  $B = q^p \cdot A$  with  $q^p = -U_1(k_c \omega^{p*})/U_2(\omega^{p*})$ . These eigenmodes are  
 267 thus defined up to the multiplying constant  $A$ .

268 The expression of the displacement field is no more valid for the case of an  
 269 incompressible material, which corresponds to  $k_c = 0$ . However, the limit of the equation  
 270 for eigenfrequencies when  $k_c$  tends to 0 can be obtained and leads to  $J_2(\omega^*) = 0$ . Therefore,  
 271 the eigenfrequencies tend to the zeros of Bessel function  $J_2$  when  $k_c$  tends to 0 (and also  $\nu$   
 272 tends to 0.5).

## 273 **2. Longitudinal motion**

274 In this case, the non-dimensional eigenfrequencies are the solutions of  $J_0(\omega^{p*}) = 0$ . The  
 275 non-dimensional eigenfrequencies  $\omega^{p*}$  are therefore the zeros of Bessel function  $J_0$ .

### 276 **D. Computation of the participation factors**

The participation factors  $K_{jj}^p$  are given by averages of the projection of the displacement field and of its quadratic norm. These averages involve the integrals of the projection  $I_j = \int \vec{u} \cdot \vec{e}_j dS$  and of the square of norm of the displacement field  $M_j = \int || \vec{u} ||^2 dS$ . Using these quantities,  $K_p$  is given by:

$$K_{jj}^p = \frac{I_j^2}{S.M_j} \quad (8)$$

277 where  $S$  is the cross section area of the fiber. The integrals  $I_j$  and  $M_j$  are given below in a  
 278 closed form.

279 **1. Transversal motion**

The integrals involved in the transversal motion are given by:

$$I_x = aA\pi[J_1(k_c\omega^{p*}) + q^p J_1(\omega^{p*})] \quad (9)$$

and

$$M_x = \pi A^2[G(k_c\omega^{p*}) + q^{p2}G(\omega^{p*}) + 2q^p J_1(k_c\omega^{p*})J_1(\omega^{p*})] \quad (10)$$

where

$$G(x) = \frac{1}{2}[(x^2 - 2)J_1^2(x) + x^2 J_0^2(x)] \quad (11)$$

A further simplification can be effected on the participation factors by taking into account the equation for eigenfrequencies (10). Finally, the participation factors are given by:

$$K_{xx}^p = \frac{2\alpha^*\beta^* J_2(\alpha^*)J_2(\beta^*)}{J_1(\alpha^*)J_1(\beta^*)(\alpha^{*2} + \beta^{*2}) - \alpha^*\beta^* [J_1(\beta^*)\beta^* J_0(\alpha^*) + J_1(\alpha^*)\alpha^* J_0(\beta^*)]} \quad (12)$$

280 where  $\alpha^* = k_c\omega^{p*}$  and  $\beta^* = \omega^{p*}$ .

281 It is noteworthy that, in the case of incompressibility, the participation factors tend to  
 282 0, because their numerators contain  $J_2(\beta^*)$  which tends to zero with  $k_c$  as observed  
 283 previously.

284 **2. Longitudinal motion**

In the same way, the participation factors for the longitudinal motion are given by:

$$I_z = \frac{2\pi Da^2}{\beta^2} J_1(\omega^{p*}) \quad (13)$$

$$M_z = \pi D^2 a^2 [J_0^2(\omega^{p*}) + J_1^2(\omega^{p*})] \quad (14)$$

$$K_{zz}^p = \frac{4J_1^2(\omega^{p*})}{\omega^{p*2} [J_0^2(\omega^{p*}) + J_1^2(\omega^{p*})]} \quad (15)$$

285 Taking into account the properties of eigenfrequencies leads finally to  $K_{zz}^p = 4/\beta^{*2}$ .

## 286 E. Synthesis of the results for cylindrical inclusions

287 The dynamic mass density (4) for cylindrical inclusions of radius  $a$  is characterized by  
288 two sets of eigenfrequencies.

-The set of eigenfrequencies for transversal motion that are solutions of

$$-k_c \omega^* J_0(k_c \omega^*) J_0(\omega^*) + k_c J_0(k_c \omega^*) J_1(\omega^*) + J_0(\omega^*) J_1(k_c \omega^*) = 0 \quad (16)$$

289 where  $\omega^*$  is the non dimensional frequency given by  $\omega^* = \frac{\omega a}{c_s}$ ,  $k_c$  is given by  $k_c = \sqrt{\frac{1-2\nu}{2(1-\nu)}}$

290 and  $J_0, J_1$  are Bessel functions.

291 -The set of eigenfrequencies for longitudinal motion, solutions of  $J_0(\omega^{p*}) = 0$ .

292 The participation factors  $K_{xx}^p = K_{yy}^p, K_{zz}^p$  for the transversal motion which enter the  
293 expression of the dynamic density through (4), (6) and (7) are given

294

- for transversal motion by:

$$K_{xx}^p = \frac{2k_c J_2(k_c \omega^{p*}) J_2(\omega^{p*})}{J_1(k_c \omega^{p*}) J_1(\omega^{p*}) (1 + k_c^2) - k_c \omega^{p*} [J_1(\omega^{p*}) J_0(k_c \omega^{p*}) + k_c J_1(k_c \omega^{p*}) J_0(\omega^{p*})]} \quad (17)$$

295 -for longitudinal motion by  $K_{zz}^p = 4/(\omega^{p*})^2$ .

## 296 IV. DYNAMIC MASS DENSITY FOR SPHERICAL INCLUSIONS

297 We consider now a composite containing spherical inclusions having a radius  $a$ .  
 298 Similarly to the previous section, it is necessary to compute the eigenmodes related to null  
 299 displacement at the boundary of the inclusions. Numerous papers dealt with  
 300 eigenfrequencies in the case of free boundary, but the case of fixed boundary is less  
 301 frequently seen. This problem was studied early by Debye<sup>13</sup> for the estimation of the  
 302 phonon contribution to the specific heat of solids. Some eigenfrequencies for the fixed  
 303 boundary problem were obtained also by Schafbuch et al.<sup>35</sup> However, to our knowledge, the  
 304 computation of the participation factors and of the dynamic mass density has never been  
 305 produced.

306 Taking into account the symmetry of the problem, the components of the dynamic  
 307 mass density are the same for any orientation of the acceleration of the solid. Therefore,  
 308 the participation factors  $K_{ij}^p$  are the components of an isotropic tensor with  $K_{ij}^p = k^p \delta_{ij}$ ,  
 309 where  $\delta_{ij}$  is the Kronecker symbol.

### 310 A. Displacement field within a sphere

The displacement field within a sphere is given in classical books. The solution  
 proposed in Eringen and Suhubi<sup>15</sup>(equations 8.13.14-8.13.15) for an harmonic motion leads  
 to the following expression of the partial components of the displacement field in spherical



coordinates:

$$\begin{aligned}
 u_r &= \frac{1}{r} [AU_1(\alpha r) + CU_3(\beta r)] P_n^m(\cos \theta) \exp i(m\varphi) \\
 u_\theta &= \frac{1}{r} \left[ [AV_1(\alpha r) + CV_3(\beta r)] [n \cot \theta \cdot P_n^m(\cos \theta) - \frac{n+m}{\sin \theta} P_{n-1}^m(\cos \theta)] \right. \\
 &\quad \left. + BV_2(\beta r) \frac{im}{\sin \theta} P_n^m(\cos \theta) \right] \exp i(m\varphi) \\
 u_\varphi &= \frac{1}{r} \left[ [AV_1(\alpha r) + CV_3(\beta r)] \cdot \frac{im}{\sin \theta} P_n^m(\cos \theta) \right. \\
 &\quad \left. - rBV_2(\beta r) [n \cot \theta \cdot P_n^m(\cos \theta) - \frac{n+m}{\sin \theta} P_{n-1}^m(\cos \theta)] \right] \exp i(m\varphi)
 \end{aligned} \tag{18}$$

311 where  $A, B, C$  are constants,  $P_n^m$  are associated Legendre polynomials and the radial  
 312 dependence is given by

$$\begin{aligned}
 U_1(\alpha r) &= nj_n(\alpha r) - \alpha r j_{n+1}(\alpha r) \\
 U_3(\beta r) &= n(n+1)j_n(\beta r) \\
 V_1(\alpha r) &= j_n(\alpha r) \\
 V_2(\beta r) &= j_n(\beta r) \\
 V_3(\beta r) &= (n+1)j_n(\beta r) - \beta r j_{n+1}(\beta r)
 \end{aligned} \tag{19}$$

313 These functions are expressed using spherical Bessel functions<sup>27</sup>  $j_n$  given by  
 314  $j_n(kr) = \sqrt{\pi/2kr} J_{n+1/2}(kr)$ . The computation of these spherical Bessel functions can be  
 315 significantly alleviated, because these functions can be obtained by using trigonometric  
 316 functions as:  $j_1(z) = \sin z/z^2 - \cos z/z$  and  $j_2(z) = (3/z^3 - 1/z)/\sin z - 3 \cos z/z^2$ . The  
 317 dynamic mass density does not depend on the direction of the matrix acceleration. It is

318 convenient to choose this direction along axis  $z$ . As for the case of cylindrical inclusions,  
 319 the integral of  $\vec{u} \cdot \vec{e}_z$  over the volume of the sphere must be different from zero for the  
 320 components which contribute to the dynamic density. The computation of these integrals  
 321 has been performed, leading to the result that the only contributions of the partial  
 322 solutions correspond to  $m = 0, n = 1$ . All other components do not contribute to the  
 323 dynamic mass density.

Finally, the dynamic displacement field is given by:

$$u_r = \frac{1}{r} [AU_1(\alpha r) + CU_3(\beta r)] \cos \theta$$

$$u_\theta = \frac{-1}{r} [AV_1(\alpha r) + CV_3(\beta r)] [\sin \theta]$$

324 with:

$$U_1(\alpha r) = j_1(\alpha r) - \alpha r j_2(\alpha r)$$

$$U_3(\beta r) = 2j_1(\beta r)$$

$$V_1(\alpha r) = j_1(\alpha r)$$

$$V_3(\beta r) = 2j_1(\beta r) - \beta r j_2(\beta r)$$

325 It is noteworthy that the structure of the displacement field presents strong similarities  
 326 with the transversal displacement field within the cylinder, but involves spherical Bessel  
 327 functions instead of usual Bessel functions.

328 **B. Resonance frequencies**

The non-dimensional resonance frequencies are obtained again by the determinant of the homogeneous system corresponding to a null displacement at  $r = a$ . It leads to the equation in  $\omega^* = \frac{\omega a}{c_s}$ :

$$k_c \omega^* j_2(k_c \omega^*) j_2(\omega^*) - j_1(k_c \omega^*) j_2(\omega^*) - 2k_c j_1(\omega^*) j_2(k_c \omega^*) = 0 \quad (20)$$

329 Having solved this equation and obtained the eigenfrequencies  $\omega^{p*}$ , the boundary  
330 condition leads also to the ratio  $q^p = \frac{C}{A}$  for each eigenfrequency, with

$$\begin{aligned} q^p &= \frac{C}{A} = -\frac{U_1(k_c \omega^{p*})}{U_3(\omega^{p*})} = -\frac{V_1(k_c \omega^{p*})}{V_3(\omega^{p*})} \\ &= \frac{k_c \omega^{p*} j_2(k_c \omega^{p*}) - j_1(k_c \omega^{p*})}{2j_1(\omega^{p*})} \end{aligned}$$

331 As for the case of cylindrical inclusions, it is useful to study the case of incompressibility  
332 when  $k_c$  tends to 0. In this case, the eigenfrequencies equation becomes  $j_2(\omega^*) = 0$  and the  
333 eigenfrequencies are the zeros of spherical Bessel function  $j_2$ .

334 **C. Participation factors**

The participation factors are computed from the integrals  $I = \int \vec{u} \cdot \vec{e}_z dV$  and  
 $M = \int || \vec{u} ||^2 dV$ . These quantities are given by:

$$I = \frac{4\pi A a^2}{3} [j_1(k_c \omega^{p*}) + 2q^p j_1(\omega^{p*})]$$

and

$$M = \int \|\mathbf{u}_p\|^2 = \frac{4\pi a A^2}{3} \{Q_{11}(k_c \omega^{p*}) + 4q^p j_1(k_c \omega^{p*}) j_1(\omega^{p*}) + (q^p)^2 Q_{33}(\omega^{p*})\}$$

335 where

$$Q_{11}(x) = j_1^2 - \frac{5}{2} x j_1 j_2 + \frac{x^2}{2} (j_1^2 + j_2^2)$$

$$Q_{33}(x) = 4j_1^2 - 5x j_1 j_2 + x^2 (j_1^2 + j_2^2)$$

336 with  $j_1 = j_1(x)$ ,  $j_2 = j_2(x)$ .

As previously, the participation factors are finally given by introducing the averages

$I/V, M/V$  over the volume  $V = \frac{4}{3}\pi a^3$  of the sphere by  $k^p = I^2/V M$  or:

$$k^p = \frac{[j_1(k_c \omega^{p*}) + 2q^p j_1(\omega^{p*})]^2}{\{Q_{11}(k_c \omega^{p*}) + 4q^p j_1(k_c \omega^{p*}) j_1(\omega^{p*}) + q^{p2} Q_{33}(\omega^{p*})\}} \quad (21)$$

Introducing the values of the factors  $q^p$  and taking into account the equation for eigenfrequencies leads finally to:

$$k^p = \frac{2\alpha^* \beta^* j_2(\alpha^*) j_2(\beta^*)}{\alpha^* \beta^* (\beta^* j_1(\beta^*) j_2(\alpha^*) + \alpha^* j_1(\alpha^*) j_2(\beta^*)) - j_1(\alpha^*) j_1(\beta^*) (2\alpha^{*2} + \beta^{*2})} \quad (22)$$

337 When  $k_c$  tends to zero, the participation factors tend also to 0, because their numerators

338 contain  $j_2(\beta^*) = j_2(\omega^{p*})$  which tends to 0 with  $k_c$ , as observed previously.

339 **D. Synthesis of the results for spherical inclusions**

The dynamic mass density (4) for spherical inclusions of radius  $a$  is characterized by the eigenfrequencies that are solutions of:

$$k_c \omega^* j_2(k_c \omega^*) j_2(\omega^*) - j_1(k_c \omega^*) j_2(\omega^*) - 2k_c j_1(\omega^*) j_2(k_c \omega^*) = 0 \quad (23)$$

340 where  $j_1, j_2$  are spherical Bessel functions.

The participation factors which enter the expression of the dynamic density through (6) and (7) are given for each solution  $\omega^{p*}$  of the previous equation by

$K_{xx}^p = K_{yy}^p = K_{zz}^p = k^p$ , where  $k^p$  is given by:

$$k^p = \frac{2k_c j_2(k_c \omega^{p*}) j_2(\omega^{p*})}{k_c \omega^{p*} (j_1(\omega^{p*}) j_2(k_c \omega^{p*}) + k_c j_1(k_c \omega^{p*}) j_2(\omega^{p*})) - j_1(k_c \omega^{p*}) j_1(\omega^{p*}) (1 + 2k_c^2)} \quad (24)$$

## 341 V. APPLICATIONS

### 342 A. Cylindrical inclusions

343 As seen in sections 3 and 4 , the resonance frequencies and participation factors  
 344 depend only on the non-dimensional frequency  $\omega^* = \omega.a/c_s$ . However, the coefficients  
 345 appearing in the equation giving  $\omega^*$  for the transversal motion depend on the ratio  
 346  $k_c = c_s/c_p$ , which itself depends only on Poisson's ratio  $\nu$ .

347 Figure 1 displays the non-dimensional resonance frequencies for the transversal  
 348 motion related to the first six modes as a function of the Poisson's ratio. The part near  
 349  $\nu = 0.5$  has been enhanced on the right part of the figure. The resonance frequencies

350 increase moderately with  $\nu$  for any mode, except when the Poisson's ratio is nearing  
351  $\nu = 0.5$  where higher modes display a significant increase near this value of  $\nu$ . In all cases,  
352 the value of the eigenfrequency for  $\nu = 0.5$  has been computed by using the zeros of  $J_2$ ,  
353 confirming the limit of the eigenfrequencies when  $\nu$  tends to 0.5. All curves display a  
354 plateau near  $\nu = 0.5$ , which is very short for the sixth resonance frequency, but is clear on  
355 the enhanced figure at the right.

356 The participation factors related to these first six modes have been displayed on  
357 Figure 2. The left part corresponds to all values of Poisson's ratios and the right part to  $\nu$   
358 between 0.45 and 0.5. It shows that the first mode contains the main participation to the  
359 dynamic mass density with a participation factor being near to 0.7 for small values of the  
360 Poisson's ratio. However, when the Poisson's ratio increases, the first participation factor  
361 decreases, the highest participation factor being successively the one related to increasing  
362 ranks of modes, as show the values corresponding to  $\nu$  nearing 0.5.

363 All participation factors tend to 0 for  $\nu$  reaching 0.5, as expected from the expression  
364 of the participation factors. These results show therefore that the use of nearly  
365 incompressible materials leads to a prediction of the dynamic mass density which is very  
366 sensitive to the value of the Poisson's ratio.

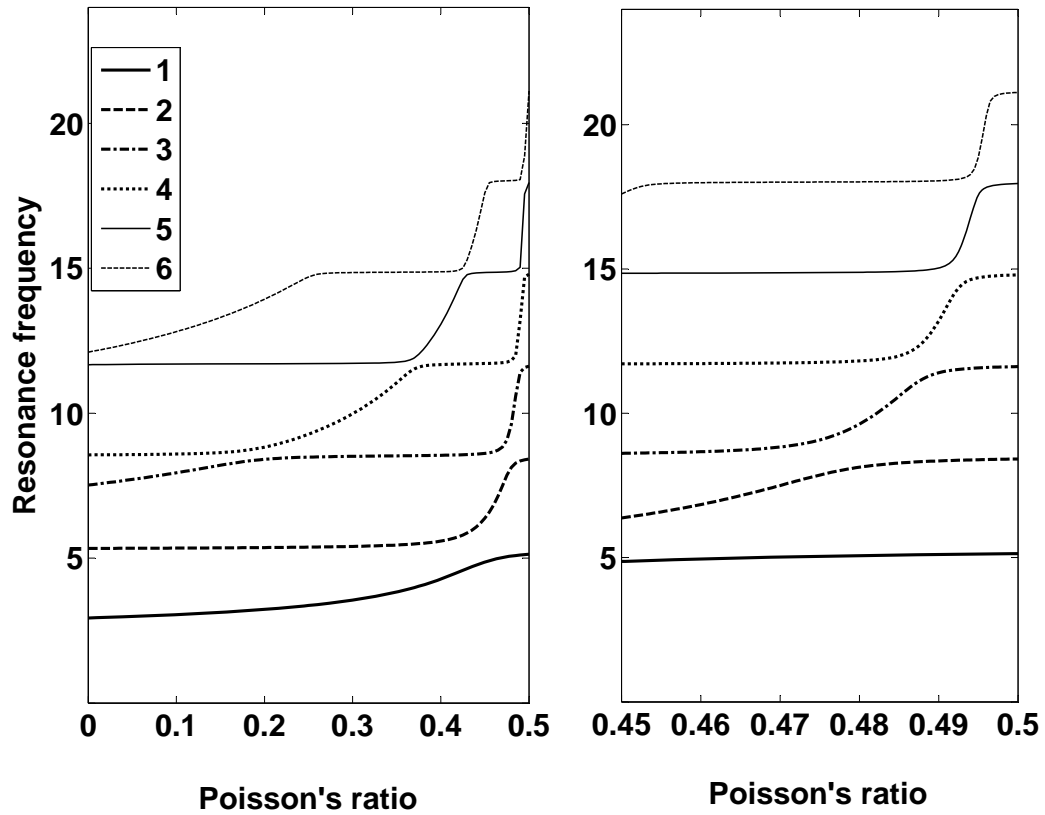


Figure 1: Cylindrical inclusions. First six non-dimensional resonance frequencies  $\omega^{p*} = \frac{\omega^p a}{c_s}$ ,  $p = 1..6$  for the transversal motion as functions of the Poisson's ratio  $\nu$ . Right: enlargement of the range of  $\nu$  between 0.45 and 0.5.

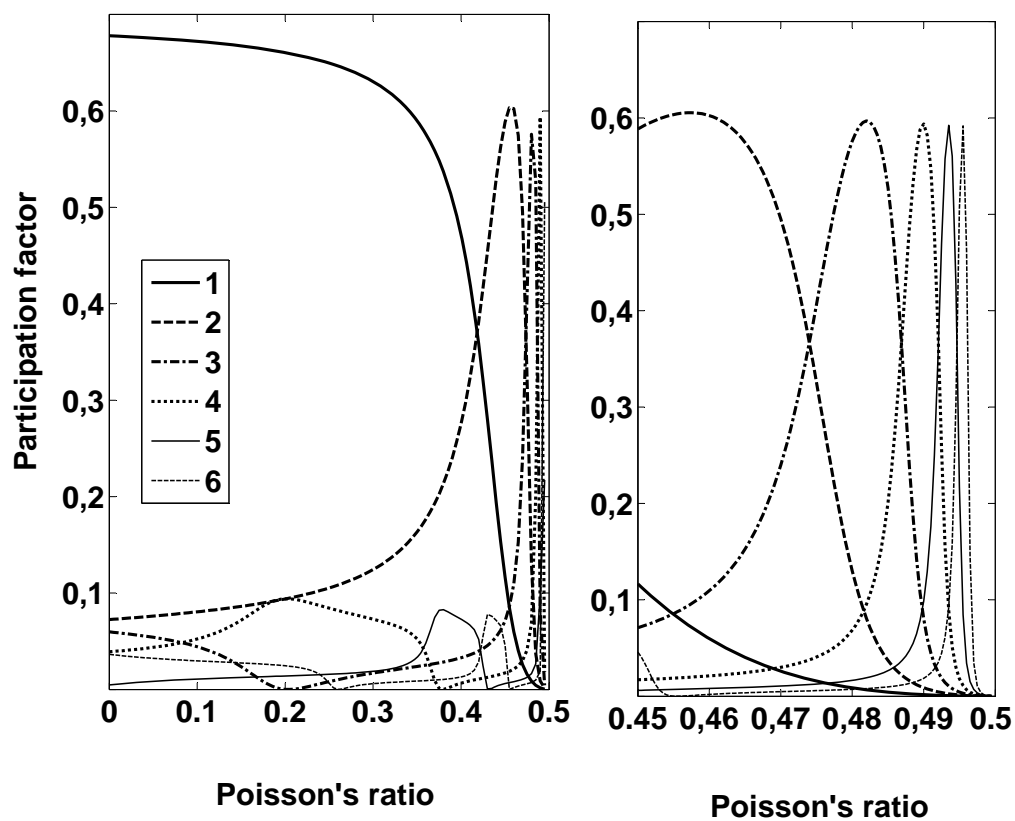


Figure 2: Cylindrical inclusions. First six participation factors for the transversal motion as functions of the Poisson's ratio  $\nu$ . Right : enlargement of the range of  $\nu$  between 0.45 and 0.5.



368 The nondimensional eigenfrequencies for spherical inclusions are reported in Figure 3  
 369 and show values of eigenfrequencies that are higher than the ones for cylindrical inclusions  
 370 with a trend being overall very similar to the one observed in the case of cylindrical  
 371 inclusions.

372 The values of the related participation factors displayed on Figure 4 are also very  
 373 similar to the ones observed in the case of cylindrical inclusions, the value for the first  
 374 mode being however lower than for cylindrical inclusions, at low values of  $\nu$ . The results  
 375 show also a trend similar when the Poisson's ratio is nearing 0.5, the highest participation  
 376 factors being related to higher modes. It is noteworthy that the participation factors are  
 377 closed form expressions of the eigenfrequencies. However, the eigenfrequencies are more  
 378 difficult to obtain. So, in order to make easier a further use of this work, precise values of  
 379 nondimensional eigenfrequencies have been reported in Appendix II.

380 An example of dynamic mass density has been reported in Figure 5 for the case of a  
 381 composite containing a matrix made of alumina (Young modulus  $E_a = 350MPa$ ,  
 382  $\nu_a = 0.25$ ,  $\rho_a = 3950kgm^3$ ) and spherical inclusions made of polystyren  
 383 ( $E_p = 2MPa$ ,  $\nu_p = 0.11$ ,  $\rho_p = 900kg/m^3$ ) with a concentration of 0.5 and a radius of 2mm.  
 384 The frequency range has been chosen to display the dynamic mass density  $\rho^{eff}$  around the  
 385 first two eigenfrequencies, showing the ranges where  $\rho^{eff}$  is negative (highlighted by a bold  
 386 line along the axis  $\rho^{eff} = 0$ ). The frequency range of practical interest where  $\rho^{eff}$  is

387 negative could be enlarged by using reinforced polystyren containing inner inclusions with  
388 a material of higher density, which would increase significantly the density of the inclusion  
389 material and as a consequence the dynamic part of the density.

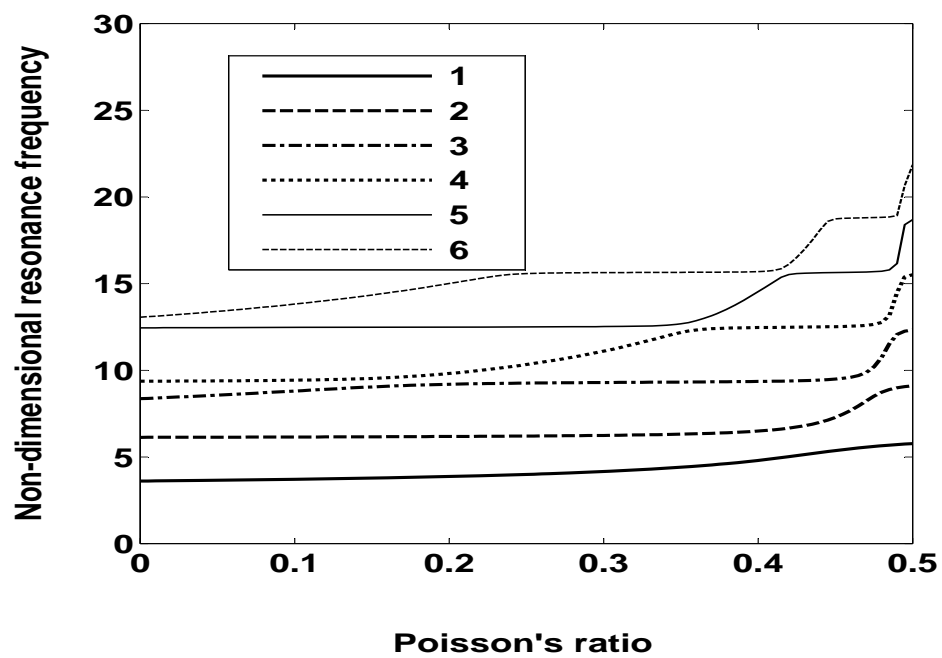


Figure 3: Spherical inclusions. First six non-dimensional resonance frequencies as functions of the Poisson's ratio  $\nu$

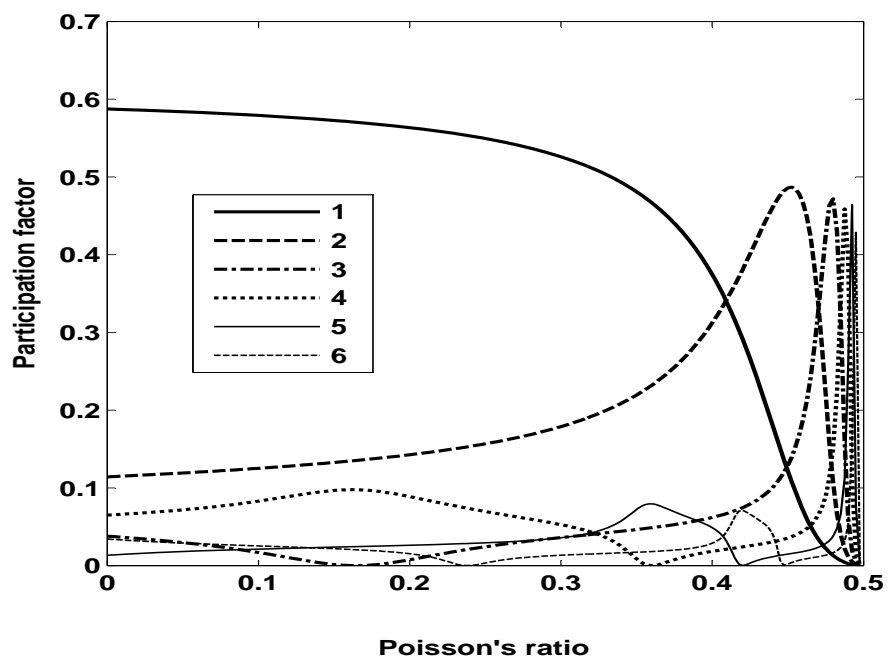


Figure 4: Spherical inclusions. First six participation factors as functions of the Poisson's ratio  $\nu$

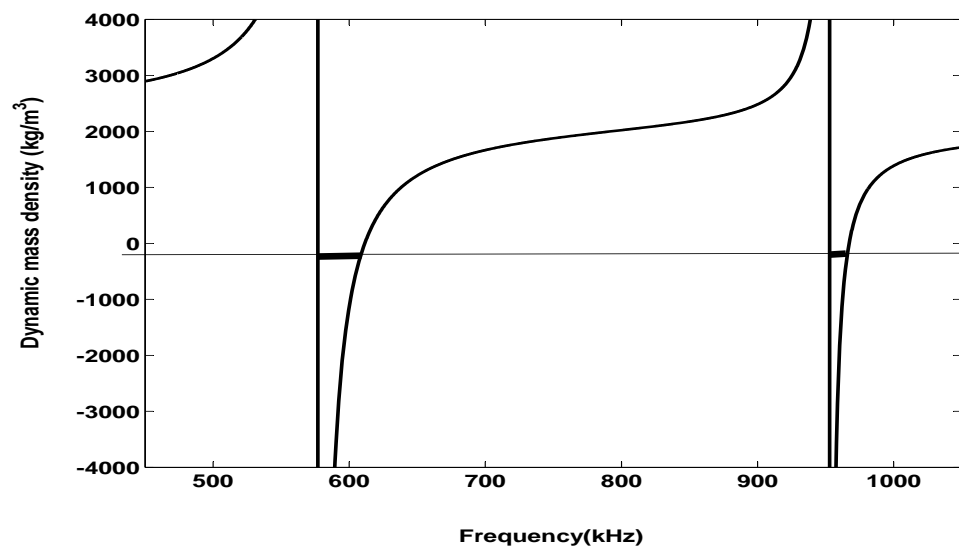


Figure 5: Dynamic mass density as a function of the frequency for a material with an alumina matrix containing spherical inclusions of polystyrene

390 **VI. CONCLUSION**

391 In this paper, the basic results obtained from the homogenization theory using  
392 asymptotic expansions have been summarized for composite elastic metamaterials with  
393 homogeneous inclusions displaying "inner resonance with negligible scattering". These  
394 results show, under specific conditions corresponding to very soft inclusions with a  
395 relatively high density, that the dynamic behaviour of the composite displays a  
396 frequency-dependent and tensorial dynamic mass density. This dynamic mass density can  
397 be computed knowing the resonance frequencies of the inclusions *for the condition of null*  
398 *value at the surface of the inclusion of the relative displacement between matrix and*  
399 *inclusion*. This greatly simplifies the computation of the dynamic mass density. In  
400 addition, it shows that all resonators are independent.

401 The resonance frequencies of these resonators have been computed successfully in the  
402 case of long cylindrical fibers or spherical inclusions. Having obtained these resonance  
403 frequencies, the participation factors related to the different modes are given by simple  
404 closed-form expressions. The results show that the participation factors decrease with  
405 increasing ranks of resonance frequencies at moderate values of the Poisson's ratio. In this  
406 case, the first modes constitute the main contribution to the dynamic mass density. When  
407 the material is nearly incompressible, the contribution of modes related to higher  
408 frequencies become more important and the dynamic mass density is very sensitive to the

409 exact value of the Poisson's ratio of the inclusions.

410 It can be noticed that the dynamic mass density tends to infinity at the resonance

411 frequencies, because the results have been obtained for a perfectly elastic material.

412 However, these results can be extended to viscoelastic materials in order to account for

413 physical damping, leading to a finite dynamic mass density by using an asymptotic solution

414 for a slight damping<sup>3</sup>. Such an extension will be the subject of a further paper.

415 **APPENDICES**

416 **I. THE CELL PROBLEM WITHIN THE INCLUSION**

417 The cell problem over the inclusion can be obtained<sup>1;2</sup> by using a homogenization  
 418 method based on double scale asymptotic expansion<sup>8</sup>, whose main aspects are recalled  
 419 thereafter.

420 The microscopic structure is assumed periodic with a periodic cell  $\Omega$ , splitted into  $\Omega_m$   
 421 and  $\Omega_i$  for matrix and inclusion, separated by the interface  $\Gamma$  between matrix and  
 422 inclusion. One considers that the ordering is such that  $\Omega_m$  is connected over all the  
 423 domain, while  $\Omega_i$  is made of separate inclusion domains.

424 The double scale method can be described as follows: the position of a point at the  
 425 macroscopic scale is determined by using the so-called "slow position vector"  $\vec{x}$ , while the  
 426 position of a point at the microscopic scale (i.e. inside the periodic cell) is determined by  
 427 the "fast position vector"  $\vec{y} = \vec{x}/\epsilon$ , leading to an amplification of the fluctuation of physical  
 428 variables when looking at the microscopic scale. The components of the displacement field  
 429 are assumed to be functions of fast and slow variables by  $\vec{u} = \vec{u}(\vec{x}, \vec{y})$  and their dependence  
 430 in  $\vec{y}$  is periodic. Elasticity parameters and densities are also periodic functions of  $\vec{y}$ , by  
 431 construction of the composite material.

432 The displacement field is searched by using an asymptotic expansion of the form:

433  $\vec{u}(\vec{x}, \vec{y}) = \vec{u}_{(0)}(\vec{x}, \vec{y}) + \epsilon \vec{u}_{(1)}(\vec{x}, \vec{y}) + \dots$ . Finally, the spatial derivatives  $\frac{\partial}{\partial x_j}$  within local



434 dynamic equations are replaced by "double scale spatial derivatives"  $\frac{\partial}{\partial x_j} + \frac{1}{\epsilon} \frac{\partial}{\partial y_j}$ . The factor  
 435 in front of the second derivative accounts for the enhancement of the fluctuations at the  
 436 local scale.

437 The asymptotic expansion of the displacement field can be introduced into the  
 438 dynamic equations (within matrix and inclusion) and continuity equations of displacement  
 439 and traction over  $\Gamma$ . Using the double scale derivatives, it produces a set of cell problems  
 440 that can be solved sequentially to obtain the displacement fields over matrix and inclusion.

441 The displacement field within the matrix is found to be<sup>1;2</sup> of the form  
 442  $\vec{u}^{(m)}(\vec{x}, \vec{y}) = \vec{U}^{(m)}(\vec{x}) + \epsilon \vec{u}_{(1)}^{(m)}(\vec{x}, \vec{y})$ , where  $\vec{u}_{(1)}^{(m)}$  is the solution of the cell problem  
 443 corresponding to the static homogenization within the matrix for a macroscopic strain  
 444 tensor computed from  $\vec{U}^{(m)}(\vec{x})$ , with null traction over  $\Gamma$ . Indeed, at this stage, due to the  
 445 large wavelength within the matrix, the inertia term does not contribute to the localization  
 446 problem within the matrix.

447 The first order components  $u_{j(0)}^{(i)}$  of the displacement field  $\vec{u}_{(0)}^{(i)}$  within the inclusions  
 448 are then obtained by solving the first order cell problem for  $\vec{u}_{(0)}^{(i)}$ :

$$\begin{aligned} \frac{\partial}{\partial y_k} (\lambda^{(i)} \varepsilon_{u(0)(y)}^{(i)} \delta_{jk} + \mu^{(i)} \varepsilon_{ik(0)(y)}^{(i)}) + \rho^{(i)} \omega^2 u_{j(0)}^{(i)} &= 0 \\ u_{j(0)}^{(i)}(\vec{x}, \vec{y}) &= \vec{U}^{(m)}(\vec{x}), \vec{y} \in \Gamma \end{aligned}$$

449 where  $\varepsilon_{ik(0)(y)}^{(i)}$  stands for the components of the strain tensor within the inclusion computed  
 450 from  $\vec{u}_{(0)}^{(i)}$  by using  $y$ -derivatives.

451 Equation (5) is then obtained by making the change of variable  $\vec{w} = \vec{u}_{(0)}^{(i)} - \vec{U}^{(m)}(\vec{x})$

452 **II. DETAILED VALUES OF EIGENFREQUENCIES**

Frequency rank→	1	2	3	4	5	6	7	8
↓Poisson's ratio								
0	2.9345	5.334	7.519	8.561	11.670	12.106	14.861	16.554
0.1	3.050	5.346	7.944	8.590	11.694	12.812	14.865	17.545
0.2	3.230	5.364	8.408	8.828	11.704	13.930	14.876	18.009
0.3	3.553	5.403	8.511	9.981	11.718	14.851	15.977	18.019
0.4	4.277	5.588	8.548	11.678	13.069	14.877	18.013	20.865
0.5	5.136	8.417	11.620	14.796	17.960	21.117	24.270	27.421

Table 1: Values of non-dimensional eigenfrequencies for transversal motion of cylindrical inclusions.

Frequency rank→	1	2	3	4	5	6	7	8
↓Poisson's ratio								
0	3.607	6.129	8.357	9.365	12.444	13.057	15.642	17.538
0.1	3.709	6.148	8.800	9.425	12.472	13.815	15.650	18.553
0.2	3.870	6.178	9.193	9.813	12.488	15.000	15.678	18.789
0.3	4.157	6.241	9.291	11.101	12.518	15.632	17.223	18.807
0.4	4.798	6.494	9.354	12.461	14.521	15.690	18.797	21.908
0.5	5.763	9.095	12.323	15.515	18.689	21.854	25.013	28.168

Table 2: Values of non-dimensional eigenfrequencies for spherical inclusions.

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