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Abstract. In this work, we prove the existence of local convex solution to the following $k-$Hessian equation

$$S_k[u] = K(y)g(y, u, Du)$$

in the neighborhood of a point $(y_0, u_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, where $g \in C^\infty$, $g(y_0, u_0, p_0) > 0$, $K \in C^\infty$ is nonnegative near y_0, $K(y_0) = 0$ and Rank $(D_y^2 K)(y_0) \geq n - k + 1$.

1. Introduction

In this work, we study the following k-Hessian equation :

$$S_k[u] = f(y, u, Du), \tag{1.1}$$

on the open domain $\Omega \subset \mathbb{R}^n$ with $2 \leq k \leq n$, where $f \geq 0$ is defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with $f(y_0, u_0, p_0) = 0$. When $u \in C^2$, the k-Hessian operator $S_k[u]$ is defined by

$$S_k[u] = S_k(D^2u) = \sigma_k[\lambda(D^2u)] = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k},$$

where $S_k(D^2u)$ is the sum of all k-order principal minors of the Hessian matrix (D^2u), and $\lambda(D^2u) = (\lambda_1(D^2u), \ldots, \lambda_n(D^2u))$ are the eigenvalues of the matrix (D^2u). One origin of $k-$Hessian operator is from Christoffel-Minkowski problem, see [5, 6, 7, 8] and references therein, another one is from calibrated geometries in [12]. The background of $k-$Hessian operator in terms of differential geometry can also be found in Section 4, [15].

When $f > 0$, the solutions u of (1.1) is considered with $\lambda(D^2u)$ in the so-called Gårding cone :

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n;\ \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

If $f \geq 0$, the equation (1.1) is called degenerate, in this case, we consider the solutions with

$$\lambda(D^2u) \in \bar{\Gamma}_k(n) = \{\lambda \in \mathbb{R}^n;\ \sigma_j(\lambda) \geq 0, 1 \leq j \leq k\}.$$

A function $u \in C^2$ is called to be k-convex, if $\lambda(D^2u) \in \bar{\Gamma}_k(n)$. The n-convex function is simply called convex.

The convexity of solutions of (1.1) is an important problem in the field of geometries analysis, including usual convexity, power convexity, log-convexity, or quasi-convexity.

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For example, in the study of Christoffel-Minkowski problem (see [5, 6, 7, 8]), an important subject is to prove the existence of a convex body with prescribed area measure of suitable order, this is equivalent to prove the microscopic convexity principle (constant rank theorem) for some k-Hessian type equation on the unit sphere $\mathbb{S}^n$. There is also a strong connection between convexity properties of solutions to elliptic and parabolic partial differential equations and Brunn-Minkowski type inequalities for associated variational functionals, see [16, 17, 18]. When Guan [4] uses subsolution in place of all curvature restrictions on $\partial\Omega$ to construct local barriers for boundary estimates, it is an assumption that there exists a locally strictly convex function in $C^2(\overline{\Omega})$, see Theorem 1.1 and 1.3, [4].

The microscopic convexity principle, with applications in geometric equations on manifolds, has been established in [2] for the very general fully nonlinear elliptic and parabolic operators of second order. Guan, Spruck and Xiao [9] point out that the asymptotic Plateau problem for finding a complete strictly locally convex hypersurface is reduced to the Dirichlet problem for a fully nonlinear equation, a special form of which is the $k-$Hessian equation, see Corollary 1.11, [9] and they have proved the existence of such hypersurface, it is especially interesting that they have proved that, if $\partial\Omega$ is strictly (Euclidean) star-shaped about the origin, so is the unique solution, see Theorem 1.5, [9]. For the k-Hessian equation with $k = 2, n = 3$, the power convexity for Dirichlet problem of equation (1.1) with $f = 1$, and log-convexity for the eigenvalue problem have been studied in [16, 17], see also [18]. The above convexity results are established on two facts, one is that the equations are elliptic, another is that the existence of classical (at least C^2) solution has already known or can be proved. However, in many important geometric problem, the associated k-Hessian equation is degenerate (see [11]), and for the degenerate elliptic k-Hessian equation, one can only prove the existence of $C^{1,1}$ solution for Dirichlet problem ([14]).

In this paper, we study the convexity of solution with following definition: for a convex domain E, the function $v \in C(E)$ is said to be *strictly convex* if

$$v(ty + (1-t)z) < t\,v(y) + (1-t)v(z),\ 0 < t < 1,\ y, z \in E,\ y \neq z.$$

Which, in case $v \in C^2(E)$, is equivalent to

$$\sum_{i,j=1}^{n}(y_i - z_i)(y_j - z_j)\int_0^1\int_0^1 \frac{\partial^2 v}{\partial x_i \partial x_j}(x(s,\mu))\,d\mu\,ds > 0 \tag{1.2}$$

with $x(s,\mu) = (s\mu + (1-s)t)y + (s(1-\mu) + (1-s)(1-t))z$, this shows that the positive definiteness of Hessian matrix $(D^2 v)$ is a sufficient condition for the strict convexity, but not necessary.

However if $u \in C^2$ is a k-convex solution of $S_k[u] = f(y) \geq 0$ with $2 \leq k < n$ and $f(y_0) = 0$, then $S_{k+1}[u](y_0) > 0$ will never occur, so that there are two possibilities: 1) $S_{k+1}[u](y_0) < 0$, in this case, u is not $(k+1)$-convex; 2) $S_{k+1}[u](y_0) = 0$, in this case, it is shown in Theorem 1.1 of [21] that, if $S_k[u](y_0) = S_{k+1}[u](y_0) = 0$, then $S_l[u](y_0) = 0$ for $k \leq l \leq n$. In particular, if $f \equiv 0$, since the case $S_{k+1}[u] > 0$ will never occur, then either $S_{k+1}[u] < 0$ in some open subset of Ω or $S_{k+1}[u] \equiv 0$ in Ω itself, in the former case u is not $(k+1)-$convex and let alone strictly convex; in the latter case, $S_l[u] \equiv 0$ for $k \leq l \leq n$, then the graph $(y; u(y))$ for $k = 2$ has the vanishing sectional curvature and must be a cylinder or

a plane, see [19] and [22], meanwhile the graph $(y; u(y))$ for $k > 2$, by Lemma 3.1 of [10], is a surface of constant nullity (at least) $n - k + 1$ and then is a $(n - k + 1)$-ruled surface. Therefore if $f \equiv 0$, the solution u to (1.1) is not strictly convex (at least) along the rulings.

Motivated by above analysis, in this work, we study the local solutions for the following equation, $2 \leq k \leq n$,

$$S_k[u] = K(y)g(y, u, Du)\,, \tag{1.3}$$

with the following assumptions

$$(H) \qquad \begin{cases} K \in C^\infty \text{ is nonnegative in a neighbourhood of } y_0 \in \mathbb{R}^n, \\ K(y_0) = 0,\ \text{Rank}\,(D^2K)(y_0) \geq n - k + 1, \\ g \in C^\infty \text{ near } Z_0 = (y_0, u_0, p_0) \text{ and } g(Z_0) > 0. \end{cases}$$

This assumption is independent of coordinates. Our main Theorem is :

Theorem 1.1. *If K, g satisfy the assumption (H), then for any $s \geq 2[\frac{n}{2}] + 5$, the equation* (1.3) *admits a strictly convex H^s-local solution in a neighbourhood of $y_0 \in \mathbb{R}^n$.*

Remark that $u = \frac{1}{2}\sum_{i=1}^{n-1} y_i^2 + \frac{1}{12}y_n^4$ is a strictly convex solution of the following Monge-Ampère equation :

$$\det D^2u = y_n^2\,.$$

But the Hessian matrix (D^2u) is not positive definite at origin.

This article is arranged as follow: In Section 2, we will introduce the idea of how to construct convex local solution in terms of $K(y)$. In Section 3, we will construct the first order approximate solution $\psi(y)$ which is strictly convex. The Section 4 will be devoted to proving the degenerate ellipticity of the linearized operator of $S_k[u]$ around ψ. In Section 5, we will use Lax-Milgram theorem to prove the existence of H^0-weak solution and their a priori H^s-estimates. In Section 6, we will prove the existence of k-convex solution by the Nash-Moser-Hörmander iteration procedure. In section 7, we will prove the strict convexity of the local solution obtained in Section 6. Section 8, as an appendix, is devoted to the estimates of eigenvalues and eigenvectors for a matrix after a small perturbation, the conclusion of which is a generalization of Lemma 1.1 of [13].

2. Schema of construction of convex local solutions

The assumption (H) is independent of coordinates. Now we choose special coordinates under which the leader term of the solution can be explicitly expressed. Since the degeneracy is come from the term K, for the simplicity of notations and also computation, we suppose that

$$(B) \qquad g \equiv 1,\ \text{Rank}\,(D^2K)(y_0) = n - k + 1.$$

By a translation $y \to y + y_0$ and a change of unknown function $u \to u - u(y_0) - Du(y_0) \cdot y$, we can assume $Z_0 = (0, 0, 0)$. On the other hand, the solution is searched for locally, that is, we assume without loss of generality that $K(y)$ is defined in some neighborhood of origin and

$$K(y) = \sum_{j=k}^{n} c_j y_j^2 + O(|y|^3) \tag{2.1}$$

where $2c_j > 0$, $k \le j \le n$ are the positive eigenvalues of $(D^2K)(0)$. In order to describe the "localness", small $\varepsilon > 0$ is introduced by the change of variables $y = \varepsilon^2 x$, we fix a domain

$$\Omega = Q_\pi \times Q_{\delta_0} \subset \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1}$$

with $\delta_0 > 0$ being chosen small later and $x' = (x_1, \cdots, x_{k-1})$, $x'' = (x_k, \cdots, x_n)$,

$$Q_\pi = \{x' \in \mathbb{R}^{k-1};\ |x_i| < \pi, 1 \le i \le k-1\}, \quad Q_{\delta_0} = \{x'' \in \mathbb{R}^{n-k+1};\ \sum_{i=k}^{n} |x_i|^2 < \delta_0^2\}.$$

We will determine some $\varepsilon_0 > 0$ and study the equation (1.3) in the following form

$$S_k[u] = \tilde{K} \ \text{ in } \ \Omega_{\varepsilon_0} = \left\{y = \varepsilon_0^2 x; x \in \Omega\right\} \tag{2.2}$$

with

$$\tilde{K}(y) = (1 - \chi(\varepsilon^{-2}y')) \sum_{i=k}^{n} c_i y_i^2 + \chi(\varepsilon^{-2}y')K(y), \tag{2.3}$$

where $\chi(x') \in C_0^\infty(Q_\pi)$ is a cutoff function equal to 1 if $|x'| \le \frac{\pi}{2}$, equal to zero if $|x'| \ge \pi$, and $0 \le \chi \le 1$. The local solution of (2.3) is also the one of (1.3). The aim of introduce of function $\chi(x')$ is to guarantee the periodicity with respect to variable x' for nonhomogeneous terms and the coefficients of all the linearized equations, which is important and convenient for existence of solution because the linearized operator $L_G(w)$ of (4.1) may be degenerate in the direction x'.

We will construct the local solution of equation (2.2) in the following form

$$u(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y). \tag{2.4}$$

So the construction of solution is by three steps:

1) Solutions of the homogeneous equation

Let $\tau = (\tau_1, \cdots, \tau_{k-1}, 0, \cdots, 0)$ with $\tau_1 > \cdots > \tau_{k-1} > 0$, then the convex function

$$\varphi(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 \tag{2.5}$$

satisfies the homogenous equation $S_k[\varphi] = 0$, and the linearized operators

$$\mathcal{L}_\varphi = \sum_{j=1}^{n} \sigma_{k-1,j}(\tau)\partial_j^2 \tag{2.6}$$

is degenerate elliptic with

$$\sigma_{k-1,j}(\tau) = 0, 1 \le j \le k-1;\ \ \sigma_{k-1,j}(\tau) = \sigma_{k-1}(\tau) = \prod_{l=1}^{k-1} \tau_l > 0,\ \ k \le j \le n.$$

We have also $S_{k+1}[\varphi] = \cdots = S_n[\varphi] = 0$.

Remark that, in [20, 21], we choose $\tau \in \partial\Gamma_k(n)$ with $\sigma_{k+1}(\tau) < 0$, so φ is not $(k+1)$-convex; also the solutions constructed in [3] must not be $(k+1)$-convex, because in these cases every linearized operator (2.6) is uniformly elliptic. But in present work, we want to construct the local strictly convex solution, so we can't make that choice. On the other hand, the function φ defined in (2.5) is only *weakly* convex, so it is difficult to guarantee the convexity after a perturbation.

2) Approximate strictly convex solution

Using the assumption (H) on K, we construct a function P such that

$$\psi(y) = \frac{1}{2}\sum_{j=1}^{k-1}\tau_j y_j^2 + P(y) \tag{2.7}$$

satisfies

$$S_k[\psi] = \tilde{K} + O(1)\varepsilon^{\frac{9}{2}},$$

and ψ is strictly convex on Ω_{ε_0}. The construction of the function P is algebraic by using the assumption (H) of K.

3) Nash-Moser-Hörmander iteration

We construct finally the smooth function w such that the function u defined by (2.4) is a local solution of equation (2.2). We use the Nash-Moser-Hörmander iteration procedure:

$$\begin{cases} w_0 = 0, w_{m+1} = w_m + S_m\rho_m \\ L_G(w_m)\rho_m + \theta_m\triangle\rho_m = g_m, \quad \text{in} \quad x \in \Omega. \end{cases} \tag{2.8}$$

where $\{S_m\}$ is a family of smoothing operators,

$$g_m = -G(w_m) = \frac{1}{\varepsilon^{\frac{9}{2}}}\left\{S_k(\psi + \varepsilon^{\frac{17}{2}}w_m(\varepsilon^{-2}y)) - \tilde{K}\right\},\ \theta_m = \sup_{\Omega}|G(w_m)| = \|g_m\|_{L^\infty}, \tag{2.9}$$

and

$$L_G(w) = \sum_{i,j=1}^{n}\frac{\partial S_k(\mathbf{r}(w))}{\partial r_{ij}}\partial_i\partial_j,$$

where

$$\mathbf{r}(w) = \left(\sum_{l=1}^{k-1}\delta_i^j\delta_j^l\tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}}w_{ij}(x)\right)_{1\le i,j\le n}.$$

The procedure is to prove the existence and the convergence of the sequence $w_m \to w$ in some Sobolev space with $\theta_m \to 0$ which imply that the function u defined in (2.4) by w is a local solution of equation (2.2). We also need to prove that the perturbation doesn't destruct the strictly convexity of ψ constructed by (2.7).

Remark that the linearized equation is degenerate elliptic, so that there is a loss of the regularity for the à priori estimate of solution ρ_m, but the coefficients of linearized operators depends on D^2w_m, so that we need to smoothing the solution ρ_m to continue the iteration (2.8) for $m \in \mathbb{N}$. This is quite different from the procedure of iteration used in [21] where the linearized equation is uniformly elliptic.

3. The first order approximate solutions

Since $K(y)$ attains its minimum 0 at origin, the critical-point theorem implies $\nabla K(0) = 0$. Then we have

Proposition 3.1. *Suppose that $K(y)$ satisfies assumption (H) and (2.1), then we have the following decomposition*

$$K(y) = K(y', 0) + \frac{1}{2}\sum_{i=k}^{n}\frac{\partial^2 K}{\partial y_i^2}(y', 0)y_i^2 + R(y)$$

where $K(y',0)$ vanishes at $y'=0$ up to order greater than four and

$$R(y)=\sum_{i=k}^{n}\frac{\partial K}{\partial y_i}(y',0)y_i+\frac{1}{2}\sum_{i,j=k,i\neq j}^{n}\frac{\partial^2 K}{\partial y_i\partial y_j}(y',0)y_iy_j+O(1)|y''|^3.$$

In particulary, for $y=\varepsilon^2 x$, $x\in\Omega$, we have

$$R(y)=O(1)\varepsilon^6.$$

Formally, if $u\in C^{3,1}$ is a solution to (1.3), by Taylor expansion

$$u(y)=\sum_{i,j=1}^{n}u_{ij}(0)y_iy_j+o(|y|^2) \tag{3.1}$$

substituting (3.1) into (1.3), we see that,

$$S_k(D^2u(0))=S_k(u_{ij}(0))=\lim_{y\to 0}S_k(D^2u(y))=K(0)=0.$$

Therefore, a smooth (at least $C^{3,1}$) local solution to (1.3) is a solution of $S_k[u]=0$ plus a small perturbation $o(|y|^2)$. So we wish to construct the first order approximate solution as following form

$$\psi(y)=\frac{1}{2}\sum_{i=1}^{k-1}\tau_iy_i^2+P(y)$$

with $\tau_1>\tau_2>\ldots>\tau_{k-1}>0$, such that

$$S_k[\psi]=\tilde{K}+O(1)\varepsilon^{\frac{9}{2}},$$

which is difficult to arrive at. Our observation is that $\sigma_{k-1}(\tau)\sum_{j=k}^{n}P_{jj}(y)$ is the main part of $S_k[\psi]$, so we only need to find out $P(y)$ to satisfy the weaker version

$$\sigma_{k-1}(\tau)\sum_{j=k}^{n}P_{jj}(y)=\tilde{K}-\chi(\varepsilon^{-2}y')R(y).$$

This is our trick how to construct $P(y)$. Let

$$\begin{aligned}P(y)\equiv&\frac{1}{2(n-k+1)\sigma_{k-1}(\tau)}\chi(\varepsilon^{-2}y')K(y',0)\sum_{j=k}^{n}y_j^2\\&+\frac{1}{24\sigma_{k-1}(\tau)}\chi(\varepsilon^{-2}y')\sum_{i=k}^{n}\left(\frac{\partial^2K}{\partial y_i^2}(y',0)-2c_i\right)y_i^4\\&+\frac{1}{12}\sum_{i=k}^{n}\left[\frac{c_i}{\sigma_{k-1}(\tau)}-4\alpha(n-k)\right]y_i^4+\alpha\sum_{j=k}^{n}\sum_{i=k,i\neq j}^{n}y_i^2y_j^2\end{aligned} \tag{3.2}$$

where

$$0<\alpha<\frac{1}{16(n-k)^2+4(n-k+1)}\min_{k\le j\le n}\left\{\frac{c_j}{2\sigma_{k-1}(\tau)}\right\}. \tag{3.3}$$

Remark 3.2. Giving an example, $K^1(y)=K^1(y'')=\sum_{i=k}^{n}c_iy_i^2$, then

$$P(y)=P^1(y'')=\frac{1}{12}\sum_{i=k}^{n}\left[\frac{c_i}{\sigma_{k-1}(\tau)}-4\alpha(n-k)\right]y_i^4+\alpha\sum_{j=k}^{n}\sum_{i=k,i\neq j}^{n}y_i^2y_j^2.$$

We have

$$\sigma_{k-1}(\tau)\sum_{j=k}^{n} P^1_{jj}(y'') = K^1(y''),$$

and the strictly convex function $\psi^1(y) = \frac{1}{2}\sum_{i=1}^{k-1}\tau_i y_i^2 + P^1(y'')$ satisfies

$$S_k(\psi^1) = K^1(y'') + O(|y''|^4).$$

The direct calculation gives

Proposition 3.3. *Let $P(y)$ be defined in* (3.2)*, then*

$$\begin{cases} \sigma_{k-1}(\tau)\sum_{j=k}^{n} P_{jj}(y) = \tilde{K} - \chi(\varepsilon^{-2}y')R(y); \\ P_{jj}(y', 0) = O(1)|y'|^4, \quad k \le j \le n, \\ P_{ij}(y) = 8\alpha y_i y_j, \quad i \ne j, \quad k \le i, j \le n, \end{cases} \tag{3.4}$$

and

$$P_{jj}(y) \ge 4\alpha|y''|^2, \quad k \le j \le n. \tag{3.5}$$

For small $|y|$, the minor matrix $(P_{ij})_{k\le i,j\le n}$ is strictly diagonally dominant, more explicitly, for fixed $k \le j_0 \le n$,

$$P_{j_0 j_0}(y) \ge 2\alpha|y''|^2 + \sum_{i=k, i\ne j_0}^{n} |P_{ij_0}(y)|. \tag{3.6}$$

Proof. From (3.2), we obtain, for fixed $k \le j \le n$,

$$\begin{aligned} P_{jj}(y) = {} & \frac{1}{(n-k+1)\sigma_{k-1}(\tau)}\chi(\varepsilon^{-2}y')K(y', 0) \\ & + \frac{1}{\sigma_{k-1}(\tau)}\chi(\varepsilon^{-2}y')\left[\frac{1}{2}\frac{\partial^2 K}{\partial y_j^2}(y', 0) - c_j\right]y_j^2 \\ & + \left[\frac{c_j}{\sigma_{k-1}(\tau)} - 4\alpha(n-k)\right]y_j^2 + 4\alpha\sum_{i=k, i\ne j}^{n} y_i^2. \end{aligned} \tag{3.7}$$

By (3.7) we have

$$\sigma_{k-1}(\tau)\sum_{j=k}^{n} P_{jj}(y) = \chi(\varepsilon^{-2}y')\left(K(y', 0) + \frac{1}{2}\sum_{i=k}^{n}\frac{\partial^2 K}{\partial y_i^2}(y', 0)y_i^2\right) + (1 - \chi(\varepsilon^{-2}y'))\sum_{j=k}^{n} c_j y_j^2$$

which proves the first equality in (3.4). Since $K(y', 0)$ vanishes up to order greater than four, we have

$$P_{jj}(y', 0) = \frac{1}{(n-k+1)\sigma_{k-1}(\tau)}\chi(\varepsilon^{-2}y')K(y', 0) = O(1)|y'|^4,$$

then the second equality in (3.4) is true. The third equality in (3.4) is obvious.

Now we return to (3.7) for $P_{jj}(y)$. Since $K(y', 0) \ge 0$, employing (2.1), choosing small $|y'|$ and then taking α to satisfy (3.3), we have

$$P_{jj}(y) \ge \left[\frac{c_j}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k)\right]y_j^2 + 4\alpha\sum_{i=k, i\ne j}^{n} y_i^2$$

which implies (3.5). By virtue of inequality above , for fixed j_0 with $k \le j_0 \le n$, we obtain by Cauchy inequality,

$$\begin{aligned}
P_{j_0 j_0}(y) \ge & [\frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k)]y_{j_0}^2 + 4\alpha \sum_{i=k, i\neq j_0}^{n} y_i^2 \\
\ge & 2\alpha|y''|^2 + \frac{1}{n-k}\sum_{i=k,i\neq j_0}^{n}\Big[\Big[\frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k+1)\Big]y_{j_0}^2 + 2\alpha y_i^2\Big] \\
\ge & 2\alpha|y''|^2 + \sum_{i=k,i\neq j_0}^{n}\frac{2}{n-k}|y_i y_{j_0}|\sqrt{2\alpha\Big[\frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k+1)\Big]} \\
\ge & 2\alpha|y''|^2 + \sum_{i=k,i\neq j_0}^{n} 8\alpha|y_i y_{j_0}| = 2\alpha|y''|^2 + \sum_{i=k,i\neq j_0}^{n}|P_{ij_0}(y)|,
\end{aligned}$$

so the minor matrix $(P_{ij})_{k\le i,j\le n}$ is strictly diagonally dominant and (3.6) is proved. □

We will construct the solution as a perturbation of the strictly convex function $\psi(y)$ in the following form,

$$u(y) = \psi + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y) = \frac{1}{2}\sum_{j=1}^{k-1}\tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y), \tag{3.8}$$

see (2.4), where $w(x)$ will be proved to be a smooth function later.

By a change of variable $x = \varepsilon^{-2}y$, the Hessian matrix of u defined in (3.8) is

$$(D^2u)(\varepsilon^2 x) = \mathbf{r} = \left(\sum_{l=1}^{k-1}\delta_i^j\delta_j^l\tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)\right). \tag{3.9}$$

We study firstly the minor matrix $\mathbf{r}_{k-1} = (r_{ij})_{1\le i,j\le k-1}$ which is real-valued and symmetric, then there is an orthogonal $(k-1)\times(k-1)$ matrix $\widetilde{Q}$ such that

$$\widetilde{Q}(x,\varepsilon)\,\mathbf{r}_{k-1}\,{}^t\widetilde{Q}(x,\varepsilon) = \mathrm{diag}(\lambda_1(x,\varepsilon),\ldots,\lambda_{k-1}(x,\varepsilon)).$$

Let

$$Q = \begin{pmatrix} \widetilde{Q} & 0 \\ 0 & \mathbf{I}_{n-k+1,n-k+1}\end{pmatrix},$$

then

$$Q\,\mathbf{r}\,{}^tQ = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 & r_{1,k} & \ldots & r_{1n} \\
0 & \lambda_2 & \ldots & 0 & r_{2,k} & \ldots & r_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \lambda_{k-1} & r_{k-1,k} & \ldots & r_{k-1,n} \\
r_{k,1} & r_{k,2} & \ldots & r_{k,k-1} & r_{k,k} & \ldots & r_{k,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_{n,1} & r_{n,2} & \ldots & r_{n,k-1} & r_{n,k} & \ldots & r_{n,n}
\end{pmatrix},$$

where all terms r_{ij} in the above matrix are in the form

$$r_{ij}(x) = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x). \tag{3.10}$$

Noticing the existence of $\chi(\varepsilon^{-2}y')$ in $P(y)$, we have

$$P_{ij}(\varepsilon^2 x', \varepsilon^2 x'') = O(1)\varepsilon^8|x''|^2 + O(1)\varepsilon^4|x''|^4, \quad 1 \le i, j \le k-1,$$

applying Lemma 8.1 to the minor matrix $\mathbf{r}_{k-1}$, we obtain that

$$\lambda_i = \tau_i + O(1)\varepsilon^4, \quad \lambda_1 > \lambda_2 > \dots \lambda_{k-1} > 0, \tag{3.11}$$

for $\varepsilon > 0$ small enough.

Lemma 3.4. *Let* $\mathbf{r}$ *be defined in* (3.9)*, then* $S_k(\mathbf{r}) = ① + ② + ③$ *with*

$$\begin{cases} ① = \sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1})\sigma_1(r_{kk}, \dots, r_{nn}) - \sum_{i=k}^n \sum_{j=1}^{k-1} \sigma_{k-1,j}(\lambda_1, \dots, \lambda_{k-1}) r_{ji}^2 \\ ② = \sigma_{k-2}(\lambda_1, \dots, \lambda_{k-1}) \left[\sigma_2(r_{kk}, \dots, r_{nn}) - \sum_{i=k}^n \sum_{s=k, s\neq i}^n r_{si}^2\right] + O(\sum_{m\ge k; i\ge 1} |r_{mi}|^3) \\ ③ = \sum_{j=3}^{k-1} \sigma_{k-j}(\lambda_1, \dots, \lambda_{k-1}) O(\sum_{m\ge k; i\ge 1} |r_{mi}|^3) + O(\sum_{m\ge k; i\ge 1} |r_{mi}|^3), \end{cases}$$

where $r_{mi} = r_{im} = O(1)\varepsilon^4$ *for* $m \ge k; i \ge 1$.

Proof. Since $S_k(\mathbf{r})$ is invariant under orthogonal transform, then $S_k(\mathbf{r}) = S_k(Q\mathbf{r}^t Q)$. By virtue of (3.11), we can separate $S_k(Q\mathbf{r}^t Q)$, which is the sum of all principal minors of order k of the Hessian $(Q\mathbf{r}^t Q)$, into three parts : ① are the minors containing $\sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}) = \prod_{i=1}^{k-1} \lambda_i$; ② are the ones containing all of the elementary symmetric polynomials of $(k-2)-$order in $\lambda_1, \cdots, \lambda_{k-1}$; ③ are the ones containing all of the elementary symmetric polynomials of order $\le k-3$ in $\lambda_1, \dots, \lambda_{k-1}$. □

Proposition 3.5. *We have for any* $w \in C^2(\overline{\Omega})$*, the function* u *defined by* (3.8) *satisfy*

$$S_k[u](y) = \tilde{K} + O(1)\varepsilon^{\frac{9}{2}}, \tag{3.12}$$

where $O(1)$ *depends on* $\|w\|_{C^2}$ *and* $\tilde{K}$ *is defined in* (2.3)*.*

Proof. By (3.10) and (3.11), we have

$$\sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}) = \sigma_{k-1}(\tau) + O(1)\varepsilon^4.$$

Noticing that $r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)$ for $i \ge k$ or $j \ge k$, we obtain $S_k(\mathbf{r}) = ① + ② + ③$ with

$$\begin{cases} ① & = [\sigma_{k-1}(\tau) + \varepsilon^4 O(1)] \sum_{l=k}^n r_{ll}(\varepsilon^2 x) - \sum_{l=k}^n \sum_{j=1}^{k-1} \sigma_{k-1,j}(\lambda_1, \dots, \lambda_{k-1}) r_{jl}^2(\varepsilon^2 x) \\ & = \sigma_{k-1}(\tau) \sum_{l=k}^n r_{ll}(\varepsilon^2 x) + \varepsilon^8 O(1) = \sigma_{k-1}(\tau) \sum_{l=k}^n P_{ll}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} O(1) \\ ② & = \sigma_{k-2}(\lambda_1, \dots, \lambda_{k-1}) \sum_{k\le l_1, l_2\le n} (r_{l_1 l_1}(\varepsilon^2 x) r_{l_2 l_2}(\varepsilon^2 x) - r_{l_1 l_2}^2(\varepsilon^2 x)) \\ & \quad + O(r_{il}^3(\varepsilon^2 x)) = \varepsilon^8 O(1) \\ ③ & = O(|r_{il}^3(\varepsilon^2 x)|) = \varepsilon^{12} O(1). \end{cases}$$

Therefore,

$$S_k(\mathbf{r}) = \sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(\varepsilon^2 x) + O(1)\varepsilon^{\frac{9}{2}},$$

from which, using (3.4) and recalling $y = \varepsilon^2 x$, we obtain (3.12). □

4. Linearized degenerate elliptic operators

By the construction of Section 3 and Proposition 3.5, we have

$$G(w) = \frac{1}{\varepsilon^{\frac{9}{2}}}\left\{S_k(u) - \tilde{K}\right\} = \frac{1}{\varepsilon^{\frac{9}{2}}}\left\{O(1)\varepsilon^{\frac{9}{2}}\right\} = O(1), \tag{4.1}$$

So that for any $w \in C^2(\bar{\Omega})$,

$$\theta(w) = \sup_{x\in\Omega} |G(w)(x)| < +\infty \tag{4.2}$$

uniformly with respect to ε, and then (4.1) is well-defined for $0 < \varepsilon \le \varepsilon_0 << 1$.

The linearized operator of G at w is

$$L_G(w) = \sum_{i,j=1}^{n} S_k^{ij}(w)\partial_i\partial_j,$$

where

$$S_k^{ij}(w) = \frac{\partial S_k}{\partial r_{ij}}(w) = \frac{\partial S_k(\mathbf{r})}{\partial r_{ij}}(w). \tag{4.3}$$

Since the matrix $(S_k^{ij})(w)$ and $\mathbf{r}$ is simultaneously diagonalizable, see [23], that is, for any smooth function w, we can find out an orthogonal matrix $T(x,\varepsilon)$ satisfying

$$\begin{cases} T(x,\varepsilon)(S_k^{ij})\ {}^tT(x,\varepsilon) = \operatorname{diag}\left[\frac{\partial\sigma_k(\lambda(x,\varepsilon))}{\partial\lambda_1}, \frac{\partial\sigma_k(\lambda(x,\varepsilon))}{\partial\lambda_2}, \ldots, \frac{\partial\sigma_k(\lambda(x,\varepsilon))}{\partial\lambda_n}\right] \\ T(x,\varepsilon)\mathbf{r}\ {}^tT(x,\varepsilon) = \operatorname{diag}\left[\lambda_1(x,\varepsilon), \lambda_2(x,\varepsilon), \ldots, \lambda_n(x,\varepsilon)\right], \end{cases} \tag{4.4}$$

where $T_i(x,\varepsilon)$ is the corresponding unit eigenvectors of λ_i, $i = 1, 2, \ldots, n$.

The linearized operator $L_G(w)$ is not guaranteed to be degenerately elliptic, because $(\lambda_1(x,\varepsilon), \lambda_2(x,\varepsilon), \ldots, \lambda_n(x,\varepsilon))$, as a result of the perturbation by $\varepsilon^{\frac{9}{2}}w_{ij}(x)$, may be not in $\bar{\Gamma}_k$. But we have

Proposition 4.1. *Assume that $\|w\|_{C^2(\Omega)} \le 1$, then the second order differential operators*

$$L_G(w) + \theta\Delta$$

is a degenerate elliptic operator if $\varepsilon > 0$ is sufficiently small, where θ is defined in (4.2)*.*

Proof. By the definition of degenerate elliptic operator, we have to prove

$$A = \theta|\xi|^2 + \sum_{i,j=1}^{n} S_k^{ij}\xi_i\xi_j \ge 0, \quad \text{for any } \xi \in \mathbb{R}^n$$

which is equivalent to prove,

$$A = \theta|\tilde{\xi}|^2 + ((S_k^{ij})\ {}^tT\tilde{\xi}, {}^tT\tilde{\xi}) = \theta|\tilde{\xi}|^2 + (T(S_k^{ij})\ {}^tT\tilde{\xi}, \tilde{\xi}) \ge 0, \quad \text{for any } \tilde{\xi} \in \mathbb{R}^n,$$

where T is the orthogonal matrix in (4.4), then

$$A = \theta|\tilde{\xi}|^2 + \sum_{i=1}^{k-1}\sigma_{k-1,i}(\lambda(x,\varepsilon))\tilde{\xi}_i^2 + \sum_{i=k}^{n}\sigma_{k-1,i}(\lambda(x,\varepsilon))\tilde{\xi}_i^2 \tag{4.5}$$

Since for small ε, we have $\sigma_{k-1,i}(\lambda(x,\varepsilon)) = \prod_{j=1}^{k-1}\tau_j + O(\varepsilon) > 0$, $k \le i \le n$, we only need to prove

$$\theta + \sigma_{k-1,i}(\lambda(x,\varepsilon)) \ge 0, \qquad 1 \le i \le k-1. \tag{4.6}$$

If $\sigma_k(\lambda(x,\varepsilon)) \geq 0$, together with the fact $\sigma_j(\lambda(x,\varepsilon)) = \sigma_j(\tau_1,\tau_2,\ldots,\tau_{k-1}) + O(\varepsilon) > 0$ for $1 \leq j \leq k-1$, then $\lambda(x,\varepsilon) \in \bar{\Gamma}_k$ which yields

$$\sigma_{k-1,i}(\lambda(x,\varepsilon)) \geq 0, \quad 1 \leq i \leq n.$$

It is left to consider the case $\sigma_k(\lambda(x,\varepsilon)) < 0$, in which case, since $\tilde{K} \geq 0$,

$$\begin{aligned}\theta = \theta(w) &= \max_{x\in\Omega} |G(w)| \\ &= \max_{x\in\Omega} \frac{1}{\varepsilon}|\sigma_k(\lambda(x,\varepsilon)) - \tilde{K}| \geq -\frac{1}{\varepsilon}\sigma_k(\lambda(x,\varepsilon)).\end{aligned}$$

Now we prove (4.6) for $i = 1$ with $\sigma_{k-1,1}(\lambda) < 0$, the other cases can be proved similarly. By the definition of $G(w)$ and $\sigma_k(\lambda) = \lambda_1\sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda)$,

$$\begin{aligned}&\theta + 2\sigma_{k-1,1}(\lambda) = \theta + 2\frac{\sigma_k(\lambda) - \sigma_{k,1}(\lambda)}{\lambda_1} \\ &\geq -\frac{1}{\varepsilon}\sigma_k(\lambda) + 2\frac{\sigma_k(\lambda) - \sigma_{k,1}(\lambda)}{\lambda_1} = (-\frac{1}{\varepsilon} + \frac{2}{\lambda_1})\sigma_k(\lambda) - \frac{2}{\lambda_1}\sigma_{k,1}(\lambda).\end{aligned} \tag{4.7}$$

Under the assumption $\sigma_{k-1,1}(\lambda) < 0$ and $\sigma_k(\lambda) < 0$, we will distinguish two cases.

Case 1. If $\sigma_{k,1}(\lambda) \leq 0$, we have by (4.7)

$$\theta + \sigma_{k-1,1} > \theta + 2\sigma_{k-1,1} \geq (-\frac{1}{\varepsilon} + \frac{2}{\lambda_1})\sigma_k(\lambda) - \frac{2}{\lambda_1}\sigma_{k,1}(\lambda) \geq (-\frac{1}{\varepsilon} + \frac{2}{\lambda_1})\sigma_k(\lambda) > 0$$

provided ε is small enough.

Case 2. Next it is left to consider the case in which

$$\sigma_k(\lambda) < 0, \;\; \sigma_{k,1}(\lambda) > 0, \;\; \sigma_{k-1,1}(\lambda) < 0$$

hold simultaneously. Using Newton's inequalities for $(n-1)$-tuple vectors

$$\sigma_{k,1}(\lambda)\sigma_{k-2,1}(\lambda) \leq \frac{(k-1)(n-k)}{k(n-k+1)}[\sigma_{k-1,1}(\lambda)]^2, \quad \lambda \in \mathbb{R}^n$$

and the fact

$$\sigma_{k-2,1}(\lambda) = \prod_{i=2}^{k-1}\tau_i + O(\varepsilon) > 0, \quad \sigma_{k-1,1}(\lambda) = O(\varepsilon),$$

we obtain

$$0 < \sigma_{k,1} \leq \frac{(k-1)(n-k)}{k(n-k+1)}\frac{[\sigma_{k-1,1}(\lambda)]^2}{\sigma_{k-2,1}(\lambda)} \leq |O(\varepsilon)\sigma_{k-1,1}(\lambda)|.$$

Back to (4.7), using $\sigma_k(\lambda) < 0$, then for $\varepsilon > 0$ small, we have

$$\theta + 2\sigma_{k-1,1} \geq \left(-\frac{1}{\varepsilon} + \frac{2}{\lambda_1}\right)\sigma_k(\lambda) - \frac{2}{\lambda_1}\sigma_{k,1}(\lambda) \geq -\frac{2}{\lambda_1}\sigma_{k,1}(\lambda) = -|O(\varepsilon)\sigma_{k-1,1}(\lambda)|,$$

which yields

$$\theta + \sigma_{k-1,1} \geq \theta + 2\sigma_{k-1,1} + |O(\varepsilon)\sigma_{k-1,1}(\lambda)| \geq 0$$

provided $\varepsilon > 0$ small enough. Proof is done. □

Equality (4.5) shows that the operator $L_G(w) + \theta\Delta$ may be degenerate elliptic in the directions of $(\tilde{\xi}_1,\cdots,\tilde{\xi}_{k-1})$ and is uniformly elliptic in the directions of $(\tilde{\xi}_k,\cdots,\tilde{\xi}_n)$ after the orthogonal transform $T(x,\varepsilon)$ which is a perturbation of unit matrix, so we can impose the Dirichlet boundary condition on ∂Q_{δ_0} (the x'' direction), but we can't do that on ∂Q_π (the x' direction). Instead of treating a Dirichlet boundary value problem, we shall prove

the existence, uniqueness and à priori estimates of the solution, which is periodic with respect to x', to the degenerately linearized elliptic equation

$$\begin{cases} L_G(w)\rho + \theta\Delta\rho = g, \text{in } \Omega; \\ \rho(x', x'') \text{ is periodic for } x' \in Q_\pi \text{ and } \rho(x', x'') = 0 \text{ for } x'' \in \partial Q_{\delta_0}, \end{cases} \tag{4.8}$$

in some suitable Hilbert space defined below, this idea is inspired by Hong and Zuily [13] where they consider the case $k = n$. We introduce the space $\mathbf{H}^s(\Omega)$ (s is an integer), which is the completion of the space of trigonometrical polynomials

$$\rho(x) = \sum_{\ell=(l_1,l_2,\ldots,l_{k-1})\in\mathbb{Z}^{k-1}} \alpha_\ell(x'')e^{\sqrt{-1}\sum_{j=1}^{k-1} l_j x_j}$$

with the (complex-valued) coefficients α_ℓ subject to the condition $\overline{\alpha_\ell}(x'') = \alpha_{-\ell}(x'') \in C^\infty(Q_{\delta_0})$, with respect to the norm

$$\|\rho\|_s^2 = \sum_{t+j\le s}\sum_{\ell\in\mathbb{Z}^{k-1}} (1 + \sum_{i=1}^{k-1} l_i^2)^t \|\alpha_\ell\|^2_{H^j(Q_{\delta_0})},$$

where $H^j(Q_{\delta_0})$ is the usual Sobolev space. We define $\mathbf{H}_0^s(\Omega)$ in the same way by taking $\alpha_\ell \in C_0^\infty(Q_{\delta_0})$. We will prove, in the next section, the following Theorem.

Theorem 4.2. *Let w be smooth and $\|w\|_{C^{[\frac{n}{2}]+3+l_0}(\Omega)} \le 1$ with nonnegative integer l_0. Then for any $s_0 \in \mathbb{N}\cup\{0\}$, one can find a constant $\varepsilon(s_0)$ such that the equation* (4.8) *possesses a solution $\rho \in \mathbf{H}_0^s(\Omega)$ provided that $g \in H^s(\Omega)$, $0 \le s \le s_0$ and $0 < \varepsilon \le \varepsilon(s_0)$. If $s \ge 1$, the solution is unique . Moreover,*

$$\begin{cases} \|\rho\|_s \le C_s(\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2}\|\rho\|_{L^\infty}), \quad \text{if } s > \left[\frac{n}{2}\right] + 1 + l_0; \\ \|\rho\|_s \le C_s\|g\|_s, \quad \text{if } s \le \left[\frac{n}{2}\right] + 1 + l_0 \end{cases} \tag{4.9}$$

holds for some constants C_s independent of w and ε. Here

$$W_{ij}(x) = P_{y_iy_j}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{y_iy_j}(x).$$

Remark 4.3. Since the equation (4.8) is degenerate elliptic, we can only get the a priori estimate (4.9) with a loss of order 2. This loss of regularity of linearized equation ask us to use the Nash-Moser-Hörmander iteration to deal with the solution of (4.8). By definition of r_{ij} by (3.9), if $P(y) = 0$, then $\sum_{i,j=1}^n \|(W_{ij})\|_{s+2}$ is reduced to $\|(w)\|_{s+4}$ which is introduced in Theorem 1.3 of [13]. When $l_0 = 0$, Theorem 4.2 ($2 \le k \le n$) is a generalization of Theorem 1.3 ($k = n$) in [13]. The assumption $\|w\|_{C^{[\frac{n}{2}]+4}(\Omega)} \le 1 (l_0 = 1)$ is necessary in the estimates of the quadratic error in Lemma 6.2 for $f = f(y, u, Du)$ in Lemma 6.2, although we will not give its estimates of the quadratic error ; while the assumption $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \le 1 (l_0 = 0)$ is enough in case $f = f(y, u)$. The uniqueness for $s \ge 1$ follows from (5.4) by taking $\nu = 0$.

5. À priori estimates of solutions for linearized equations

First of all, using the change of unknown function $\bar\rho = \rho e^{\mu\sum_{j=k}^n x_j^2}$, we reduce (4.8) to

$$\begin{cases} L(w)\bar\rho = \sum_{i,j=1}^n (S^{ij}_k(w) + \delta_i^j\theta)\partial_i\partial_j\bar\rho + \sum_{i=1}^n b_i\partial_i\bar\rho + c\bar\rho = e^{\mu\sum_{j=k}^n x_j^2} g, \quad \text{in } \Omega \\ \bar\rho = 0 \text{ on } \partial Q_{\delta_0} \text{ and periodic on } Q_\pi. \end{cases} \tag{5.1}$$

The coefficients b_i and c are expressed as follows.

$$b_i = \begin{cases} -4\sum_{j=k}^n (\mu x_j) S_k^{ij}(w), & 1 \le i \le k-1 \\ -4(\mu x_i)\theta - 4\sum_{j=k}^n (\mu x_j) S_k^{ij}(w), & k \le i \le n. \end{cases}$$

$$c = -2\mu \sum_{i=k}^n (S_k^{ii}(w) + \theta) + 4\sum_{i=k}^n \sum_{j=k}^n (\mu x_i)(\mu x_j) S_k^{ij}(w) + 4\theta \sum_{i=k}^n (\mu x_i)^2.$$

We would like to prove Theorem 4.2 for (5.1) rather than (4.8), and write ρ instead of $\bar{\rho}$, but also not do it directly, we will consider the solution ρ_ν to the regularized version of (5.1), i.e., the following uniformly elliptic equation, for $0 < \nu < 1$,

$$\begin{cases} L_\nu \rho \equiv \sum_{i,j=1}^n (S_k^{ij} + \delta_i^j \theta)\partial_i \partial_j \rho + \nu \triangle \rho + \sum_{i=1}^n b_i \partial_i \rho + c\rho = g, \text{in } \Omega, \\ \rho = 0 \text{ on } \partial Q_{\delta_0} \text{ and periodic on } Q_\pi. \end{cases} \tag{5.2}$$

We first need the following Lemmas which is standard for the degenerate elliptic operators. So we only point out some important steps for the proof.

Lemma 5.1. *Suppose that w is smooth and $\|w\|_{C^4(\Omega)} \le 1$. Then there exists two positive constants μ_0 large and ε_0 small such that, for $0 < \varepsilon \le \delta_0$, $\mu_0 \delta_0 \le 1$, $g \in H^0(\Omega)$ and $\nu > 0$, problem* (5.2) *admits an unique solution $\rho_\nu \in \mathbf{H}_0^1(\Omega)$, which satisfies*

$$\|\rho_\nu\|_0 \le C_0 \|g\|_0, \tag{5.3}$$

where C_0 is uniform for $\nu \in]0, 1[$, $\varepsilon \in]0, \delta_0]$ and independent of w.

Proof. We prove the existence and uniqueness of the solution ρ_ν to (5.2) by applying Lax-Milgram Theorem to the bilinear form

$$< -L_\nu \rho, \varrho >$$

where $< \cdot, \cdot >$ is the dual pair on $H^{-1} \times H_0^1$. The condition $\|w\|_{C^4} \le 1$ yields

$$| < -L_\nu \rho, \varrho > | \le C \|\rho\|_1 \|\varrho\|_1, \quad \forall \rho, \varrho \in H_0^1,$$

where C is uniform on $0 < \varepsilon < 1, 0 < \nu < 1$. For the coercivity, the proof of which is almost the same as that of Lemma 1.4, [13], there exist $\varepsilon_0 > 0$ small and large $\mu > 0$ such that

$$- < L_\nu \rho, \rho > \ge \nu \|D\rho\|_0^2 + 2\sigma_{k-1}(\tau)\|\rho\|_0^2, \tag{5.4}$$

then by using Lax-Milgram Theorem, for $g \in H^0(\Omega)$ and $\nu > 0$, problem (5.2) admits an unique solution $\rho_\nu \in \mathbf{H}_0^1(\Omega)$. Since $|- < L_\nu \rho_\nu, \rho_\nu > | = | < g, \rho_\nu > | \le \|g\|_0 \|\rho_\nu\|_0$, then (5.3) follows from (5.4). □

Similarly to the proof of Theorem 1.3, [13], we have the higher order à priori estimates

Lemma 5.2. *Suppose that w is smooth and $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \le 1$. For $s > 0$, then there exists $\varepsilon_0(s) > 0$ small such that, for $0 < \varepsilon \le \varepsilon_0(s)$, and $g \in H^s(\Omega)$, the problem* (5.2) *admits an unique solution $\rho_\nu \in \mathbf{H}_0^{s+1}(\Omega)$, which satisfies*

$$\begin{cases} \|\rho_\nu\|_s \le C_s(\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2} \|\rho_\nu\|_{L^\infty}), & \text{if } s > \left[\frac{n}{2}\right] + 1; \\ \|\rho_\nu\|_s \le C_s \|g\|_s, & \text{if } s \le \left[\frac{n}{2}\right] + 1 \end{cases} \tag{5.5}$$

where C_s is uniform for $\nu \in]0, 1]$, $\varepsilon \in]0, \varepsilon_0(s)]$ and independent of w.

Proof. For $s = 0$, Since $-(L_\nu\rho, \rho) = -(g, \rho)$, then (5.4) and Cauchy inequality yields

$$\nu\|D\rho\|_0^2 + \|\rho\|_0^2 \le C_0(\tau)\|g\|_0^2,$$

where $C_0(\tau)$ is independent of ν, w and ε. On the other hand, for $\alpha \in \mathbb{N}^n, |\alpha| \le s$

$$(L_\nu(\partial^\alpha\rho), (\partial^\alpha\rho)) = (\partial^\alpha g, (\partial^\alpha\rho)) + ([L_\nu, \partial^\alpha]\rho, (\partial^\alpha\rho)),$$

where the commutators is

$$[L_\nu, \partial^\alpha] = -\sum_{\beta\le\alpha, |\beta|\ge 1}\sum_{i,j=1}^{n} C_{\alpha\beta}\Big(\partial^\beta(S_k^{ij})\partial_i\partial_j + \sum_{i=1}^{n}\partial^\beta b_i\partial_i + \partial^\beta c\Big)\partial^{\alpha-\beta},$$

here the coefficients depends on D^2w, by using the interpolation inequalities, we can get:
1) If $s \le \left[\frac{n}{2}\right] + 1$, then the condition $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \le 1$ imply

$$|([L_\nu, \partial^\alpha]\rho, (\partial^\alpha\rho))| \le \varepsilon C\|\rho\|_s^2.$$

2) If $s > [\frac{n}{2}] + 1$, a little involved computation also give

$$|([L_\nu, \partial^\alpha]\rho, (\partial^\alpha\rho))| \le C_s\Big(\varepsilon\|\rho\|_s^2 + \|\rho\|_s(\|g\|_s + \|\rho\|_{L^\infty}\sum_{i,j=1}^{n}\|W_{ij}\|_{s+2})\Big),$$

where C_s depends only on s, so we finish the proof. □

With Lemmas 5.1 and 5.2, we can now prove Theorem 4.2.

Proof. **The proof of Theorem 4.2.** To simplify the computation, we consider only the case $l_0 = 0$, and prove (4.9) under the assumption $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \le 1$. Now for $0 \le s \le [\frac{n}{2}] + 1$, we can apply Banach-Saks Theorem to find a subsequence ρ_{ν_j} with $\nu_j = j^{-2}$ and an element $\rho_0 \in \mathbf{H}_0^s$ such that

$$\|\frac{\rho_{\nu_1} + \cdots + \rho_{\nu_m}}{m} - \rho_0\|_s \to 0 \ (m \to \infty).$$

Since $\frac{\rho_{\nu_1}+\cdots+\rho_{\nu_m}}{m}$ is periodic in x' for each m, so is ρ_0. Back to (5.2), we have

$$\left[\sum_{i,j=1}^{n}(S_k^{ij} + \delta_i^j\theta)\partial_i\partial_j + \sum_{i}^{n} b_i\partial_i + c\right]\frac{\rho_{\nu_1} + \cdots + \rho_{\nu_m}}{m} + \frac{1}{m}\sum_{j=1}^{m}\nu_j\triangle\rho_{\nu_j} = g.$$

For any test function $\phi \in C_0^\infty$, Lemma 5.1 yields

$$|(\frac{1}{m}\sum_{j=1}^{m}\nu_j\triangle\rho_{\nu_j}, \phi)_{L^2}| \le \|\triangle\phi\|_0\,\|\frac{1}{m}\sum_{j=1}^{m}\nu_j\rho_{\nu_j}\|_0 \le C_0\|\triangle\phi\|_0\,\|g\|_0\,\frac{1}{m}\sum_{j=1}^{m}\nu_j \to 0,$$

taking $m \to \infty$, we have that ρ_0 is a solution of (4.8) in the sense of distribution :

$$\sum_{i,j=1}^{n}(S_k^{ij} + \delta_i^j\theta)\partial_i\partial_j\rho_0 + \sum_{i}^{n} b_i\partial_i\rho_0 + c\rho_0 = g.$$

Moreover, by Lemma 5.2

$$\|\rho_0\|_s = \lim_{m\to\infty}\|\frac{\rho_{\nu_1} + \cdots + \rho_{\nu_m}}{m}\|_s \le C_s\|g\|_s, \quad \text{for } 0 \le s \le [\frac{n}{2}] + 1.$$

Now we prove (4.9) for $s > [\frac{n}{2}] + 1$. Since Lemma 5.2 yields

$$\|\rho_\nu\|_{[\frac{n}{2}]+1} \le C_{[\frac{n}{2}]+1}\|g\|_{[\frac{n}{2}]+1},$$

by Sobolev imbedding theorem , $\|\rho_v\|_{L^\infty} < \infty$, moreover if $\sum_{i,j=1}^n \|W_{ij}\|_{s+2} < \infty$, then (5.5) shows that

$$\|\rho_v\|_s \leq C < \infty \ \text{ for } \ s > [\frac{n}{2}] + 1,$$

then, by weak compactness theorem, there is a subsequence ρ_{v_j} of ρ_v and ρ_0 such that $\rho_{v_j} \to \rho_0$ in the weak topology of $W^{s,2}(\Omega)$ and

$$\|\rho_{v_j} - \rho_0\|_{L^\infty(\Omega)} \to 0, \|\rho_{v_j} - \rho_0\|_{W^{1,2}(\Omega)} \to 0,$$

therefore ρ_0 is periodic in x' , $\rho_0 \in H_0^s(\Omega)$, and by Lemma 5.2 again

$$\|\rho_0\|_s \leq \liminf_{j\to\infty} \|\rho_{v_j}\|_s \leq C_s(\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2}\|\rho_0\|_{L^\infty}),$$

so we complete the proof of (4.9). □

6. Nash-Moser-Hörmander iteration

We prove firstly the existence of k-convex local solution of (2.2) as a perturbation of ψ constructed in Section 3 by employing the Nash-Moser-Hömander iteration, which is based on the à priori estimates established in last section. Since the linearized operators is degenerate elliptic, a loss of regularity of order 2 has occurred for the solution of linearized equation, so we need to **mollify** the solution by taking a family of smoothing operators $S(t), t \geq 1$ such that

$$S(t) : \cup_{s\geq 0}\mathbf{H}_0^s(\Omega) \longrightarrow \cap_{s\geq 0}\mathbf{H}^s(\Omega)$$

with the following properties:

$$\|S(t)u\|_{s_1} \leq C_{s_1 s_2}\|u\|_{s_2}, \quad \text{if} \quad s_1 \leq s_2, \tag{6.1}$$

$$\|S(t)u\|_{s_1} \leq C_{s_1 s_2} t^{s_1 - s_2}\|u\|_{s_2}, \quad \text{if} \quad s_1 \geq s_2, \tag{6.2}$$

$$\|S(t)u - u\|_{s_1} \leq C_{s_1 s_2} t^{s_1 - s_2}\|u\|_{s_2}, \quad \text{if} \quad s_1 \leq s_2, \tag{6.3}$$

where $C_{s_1 s_2}$ is independent of t and depends only on s_1, s_2 , see [1] for more detailed properties of smoothing operators.

Now we define $S_m = S(\mu_m)$ with $\mu_m = \sigma^{\gamma^m}$ where $\sigma > 1, \gamma > 1$ to be determined later, we use the following iteration procedure:

$$\begin{cases} w_0 = 0, w_{m+1} = w_m + S_m\rho_m, \\ L(w_m)\rho_m = L_G(w_m)\rho_m + \theta_m\triangle\rho_m = g_m, \quad x \in \Omega \\ \rho_m \in \mathbf{H}_0^s(\Omega), g_m \in H^s(\Omega). \end{cases} \tag{6.4}$$

where g_m and θ_m are defined in (2.9), $G(w)$ is defined in (4.1).

Remember we are studying the equation (4.3) in the variables x rather than $y = \varepsilon^2 x$, and

$$u_m(y) = \frac{1}{2}\sum_{j=1}^{k-1}\tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w_m(\varepsilon^{-2}y).$$

Denoting

$$\mathcal{M}(s) = \|g_0\|_s + \varepsilon^4 \sum_{i,j=1}^{n} \|P_{ij}(\varepsilon^2\,\cdot)\|_{s-1},$$
$$\mathcal{N}(s) = \mathcal{M}(s) + \sum_{i,j=1}^{n} \|P_{ij}(\varepsilon^2\,\cdot)\|_{s+2}\Big[1 + \mathcal{M}\Big(\Big[\frac{n}{2}\Big] + 1\Big)\Big]$$

which are small if $\varepsilon > 0$ small. We prove the à priori estimates by induction.

Lemma 6.1. *Suppose that* $\|w_l\|_{C^{[\frac{n}{2}]+3}} \le 1$ *for* $0 \le l \le m$. *Then*

$$\|g_m\|_s \le C_s(\mathcal{M}(s) + \|w_m\|_{s+2}) \tag{6.5}$$

and

$$\|w_{m+1}\|_{s+4} \le C_s^{m+1}\mu_{m+1}^{\beta}\mathcal{N}(s), \quad \text{for} \quad \beta = \frac{4}{\gamma - 1}, \tag{6.6}$$

where C_s *is independent of* m *and* γ.

Proof. We prove first (6.5). Remembering

$$\mathbf{r}_m = \left(\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}}(w_m)_{ij}(x)\right)$$

using Taylor expansion and $w_0 = 0$, we have

$$-g_m = G(w_m) = G(w_0) + \int_0^1 \frac{\partial}{\partial t}[G(tw_m)]dt = -g_0 + \int_0^1 \sum_{i,j=1}^{n} S_k^{ij}(\mathbf{r}_m(t))\frac{\partial^2 w_m}{\partial x_i \partial x_j}dt.$$

Hence

$$\|g_m\|_s \le \|g_0\|_s + \sum_{i,j=1}^{n} \sum_{|\alpha|+|\beta|\le s} C_\alpha^\beta \|\partial^\alpha S_k^{ij}(\mathbf{r}_m)\partial^\beta\Big(\frac{\partial^2 w_m}{\partial x_i \partial x_j}\Big)\|_0.$$

By using $\|w_m\|_{C^{[\frac{n}{2}]+3}} \le 1$, we have

$$\sum_{|\alpha|+|\beta|\le s, |\alpha|\le[\frac{n}{2}]+1} \|\partial^\alpha S_k^{ij}(\mathbf{r}_m)\partial^\beta\Big(\frac{\partial^2 w_m}{\partial x_i \partial x_j}\Big)\|_0 \le C \sum_{|\beta|\le s} \|\partial^\beta\Big(\frac{\partial^2 w_m}{\partial x_i \partial x_j}\Big)\|_0 \le C\|w_m\|_{s+2}.$$

On the other hand, by interpolation and $\|w_m\|_{C^{[\frac{n}{2}]+3}} \leq 1$,

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|\leq s,|\alpha|>[\frac{n}{2}]+1} \|\partial^\alpha S_k^{ij}(\mathbf{r}_m)\partial^\beta(\frac{\partial^2 w_m}{\partial x_i \partial x_j})\|_0 \\
=&\sum_{|\alpha|+|\beta|\leq s,|\alpha|>[\frac{n}{2}]+1} \|\partial^{\alpha-[\frac{n}{2}]-1}\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\partial^\beta(\frac{\partial^2 w_m}{\partial x_i \partial x_j})\|_0 \\
\leq& C \sum_{|\alpha|+|\beta|\leq s,|\alpha|>[\frac{n}{2}]+1} \Big[\|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{L^\infty}\|\partial_i\partial_j w_m\|_{|\alpha|+|\beta|-[\frac{n}{2}]-1} \\
&+\|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{|\alpha|+|\beta|-[\frac{n}{2}]-1}\|\partial_i\partial_j w_m\|_{L^\infty}\Big] \\
\leq& C \sum_{|\alpha|+|\beta|\leq s,|\alpha|>[\frac{n}{2}]+1} \Big[\|w_m\|_{|\alpha|+|\beta|-[\frac{n}{2}]+1} + \|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{|\alpha|+|\beta|-[\frac{n}{2}]-1}\Big] \\
\leq& C\|w_m\|_{s-[\frac{n}{2}]+1} + C\sum_{i,j=1}^n \|(W_m)_{ij}\|_s,
\end{aligned}$$

where

$$(W_m)_{ij}(x) = P_{y_iy_j}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}}(w_m)_{y_iy_j}(x).$$

Therefore

$$\begin{aligned}
&\|S_k^{ij}(\mathbf{r}_m)\frac{\partial^2 w_m}{\partial x_i \partial x_j}\|_s \\
&\leq C\|w_m\|_{s+2} + C\|w_m\|_{s-[\frac{n}{2}]+1} + C\sum_{i,j=1}^n \|(W_m)_{ij}\|_s \\
&\leq C\|w_m\|_{s+2} + C\sum_{i,j=1}^n \|(W_m)_{ij}\|_s.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|g_m\|_s = \|G(w_m)\|_s \\
&\leq \|G(w_0)\|_s + \int_0^1 \|\frac{\partial}{\partial t}[G(tw_m)]\|_s dt \leq \|g_0\|_s + C\|w_m\|_{s+2} + C\|\partial^3 u_m\|_{s-2}.
\end{aligned}$$

Remembering

$$\partial_x\partial_{x_i}\partial_{x_j}[u_m(y)\,|_{y=\varepsilon^2 x}] = \varepsilon^4\partial_x(W_m)_{ij}(x) = \varepsilon^4\partial_x[P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}}(w_m)_{ij}(x)],$$

we have

$$\|g_m\|_s \leq \|g_0\|_s + C\varepsilon^4\sum_{i,j=1}^n \|P_{ij}(\varepsilon^2\,\cdot)\|_{s-1} + C\|w_m\|_{s+2} \leq C_s(\mathcal{M}(s) + \|w_m\|_{s+2}),$$

this completes the proof of (6.5).

We prove now (6.6). By using (6.2) and (6.4), we have

$$\|w_{m+1}\|_{s+4} \leq \|w_m\|_{s+4} + \|S_m\rho_m\|_{s+4} \leq \|w_m\|_{s+4} + C_s\mu_m^4\|\rho_m\|_s.$$

The à priori estimate (4.9) and Sobolev imbedding theorem yield

$$
\begin{aligned}
\|\rho_m\|_s &\le C(\|g_m\|_s + \sum_{i,j=1}^{n} \|(W_{ij})\|_{s+2}\|\rho_m\|_{L^\infty}) \\
&\le C(\|g_m\|_s + \sum_{i,j=1}^{n} \|(W_{ij})\|_{s+2}\|\rho_m\|_{[\frac{n}{2}]+1}) \\
&\le C(\|g_m\|_s + \sum_{i,j=1}^{n} \|(W_{ij})\|_{s+2}\|g_m\|_{[\frac{n}{2}]+1}) \qquad (6.7) \\
&\le C\Big(\mathcal{M}(s) + \|w_m\|_{s+2} + C(\sum_{i,j=1}^{n} \|P_{ij}(\varepsilon^2\,\cdot)\|_{s+2} + \|w_m\|_{s+4}) \\
&\qquad \times (\mathcal{M}([\frac{n}{2}]+1) + \|w_m\|_{[\frac{n}{2}]+3})\Big) \\
&\le C\,(\mathcal{N}(s) + \|w_m\|_{s+4})\,.
\end{aligned}
$$

Thus we get

$$
\|w_{m+1}\|_{s+4} \le \|w_m\|_{s+4} + C_s\mu_m^4\|\rho_m\|_s \le C\mu_m^4(\mathcal{N}(s) + \|w_m\|_{s+4}), \qquad (6.8)
$$

then by using $w_0 = 0$ and $\mu_m = \sigma^{\gamma^m}$, the iteration of (6.8) gives

$$
\|w_{m+1}\|_{s+4} \le C_s^m(m+1)\sigma^{4\frac{\gamma^{m+1}-1}{\gamma-1}}\mathcal{N}(s),
$$

if we choose $C_s > 1$, $C_s' = 2C_s$, and $\beta = \frac{4}{\gamma-1}$, then

$$
\|w_{m+1}\|_{s+4} \le (C_s')^{m+1}\mu_{m+1}^{\beta}\mathcal{N}(s).
$$

This completes the proof of (6.6). □

Lemma 6.2. *Suppose* $\|w_l\|_{C^{[\frac{n}{2}]+4}} \le 1$ *for* $l = 0, 1, \ldots, m$. *Then for any given* $s^* > s > 1$, *there exists* $\sigma > 1, \gamma > 1$ *and* $a > 0$ *such that*

$$
\|g_{m+1}\|_0 + \|g_{m+1}\|_{L^\infty} \le \mu_{m+1}^{-a}\mathcal{N}(s^*). \qquad (6.9)
$$

Proof. By Taylor formula with remainder and (6.4), we consider the quadratic error of $G(w_{m+1})$

$$
\begin{aligned}
&-g_{m+1} = G(w_{m+1}) = G(w_m + S_m\rho_m) \\
&= G(w_m) + L_G(w_m)S_m\rho_m + \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}[G(w_m + tS_m\rho_m)]dt \\
&= [G(w_m) + L(w_m)\rho_m] + L(w_m)(S_m - I)\rho_m - \theta_m\triangle S_m\rho_m \\
&\quad + \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}[G(w_m + tS_m\rho_m)]dt \\
&= L(w_m)(S_m - I)\rho_m - \theta_m\triangle S_m\rho_m + \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}[G(w_m + tS_m\rho_m)]dt \\
&= (A) + (B) + \int_0^1 (1-t)(C)dt\,.
\end{aligned}
$$

For the last terms, we have

$$\begin{aligned}(C) &= \frac{\partial^2}{\partial t^2}[G(w_m + tS_m\rho_m)]\\ &= \sum_{i,j,p,l=1}^{n} \frac{\partial^2 S_k}{\partial r_{ij}\partial r_{pl}}(w_m + tS_m\rho_m)(S_m\rho_m)_{ij}(S_m\rho_m)_{pl}\end{aligned}$$

Noticing $\frac{\partial^2 S_k}{\partial r_{ij}\partial r_{pl}}(\mathbf{r})$ is a polynomial of order $k-2$, we have, by Sobolev imbedding theorem, à priori estimate (4.9), (6.5) and (6.2),

$$\|S_m\rho_m\|_{[\frac{n}{2}]+3} \le C\mu_m^2\|\rho_m\|_{[\frac{n}{2}]+1} \le C\mu_m^2\|g_m\|_{[\frac{n}{2}]+1} \le C\mu_m^2(\mathcal{M}([\frac{n}{2}]+1) + \|w_m\|_{[\frac{n}{2}]+3}) \le C\mu_m^2,$$

and

$$\begin{aligned}&\|\frac{\partial^2 S_k}{\partial r_{ij}\partial r_{pl}}(w_m + tS_m\rho_m)\|_{L^\infty} \le \|D^2 w_m + tD^2 S_m\rho_m\|_{L^\infty}^{k-2}\\ &\le [\|D^2 w_m\|_{L^\infty} + \|D^2 S_m\rho_m\|_{L^\infty}]^{k-2} \le [1 + \|S_m\rho_m\|_{[\frac{n}{2}]+3}]^{k-2}\\ &\le (C')^{k-2}\mu_m^{2(k-2)}.\end{aligned}$$

Then (6.2) yields

$$\begin{aligned}\|(C)\|_0 \le& \sum_{i,j,p,l=1}^{n} \|\frac{\partial^2 S_k}{\partial r_{ij}\partial r_{pl}}(w_m + tS_m\rho_m)\|_{L^\infty}\|(S_m\rho_m)_{ij}\|_{L^\infty}\|(S_m\rho_m)_{pl}\|_0\\ \le& C^{k-2}\mu_m^{2(k-2)}\|S_m\rho_m\|_{[\frac{n}{2}]+3}\|S_m\rho_m\|_2 \le C^{k-2}\mu_m^{2(k-2)+[\frac{n}{2}]+5}\|\rho_m\|_0^2\\ \le& C^{k-2}\mu_m^{2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2.\end{aligned}$$

By (6.3), (6.7) and (6.6), we have

$$\begin{aligned}\|(A)\|_0 &= \|L(w_m)(S_m - I)\rho_m\|_0 \le C\|(S_m - I)\rho_m\|_2\\ &\le CC_{s*}\mu_m^{-(s^*-2)}\|\rho_m\|_{s^*}\\ &\le C'_{s*}\mu_m^{-(s^*-2)}C_{s*}^{m+1}\mu_m^{\beta}N(s^*).\end{aligned}$$

By (6.2) and the à priori estimate (4.9),

$$\|(B)\|_0 = \|\theta_m\triangle S_m\rho_m\|_0 \le C\theta_m\|S_m\rho_m\|_2 \le C\theta_m\mu_m^2\|\rho_m\|_0 \le C\theta_m\mu_m^2\|g_m\|_0.$$

Now by combining the estimates of (A), (B) and (C) , we obtain

$$\|g_{m+1}\|_0 \le C\Big(\mu_m^{-(s^*-2-\beta)}C_{s*}^{m+1}N(s^*) + \theta_m\mu_m^2\|g_m\|_0 + \mu_m^{2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2\Big). \tag{6.10}$$

Since $\theta_m = \|g_m\|_{L^\infty}$ by (2.9), we need to prove two estimates in (6.9) together. By same computation, using Sobolev imbedding, we have

$$\|g_{m+1}\|_{L^\infty} \le C\mu_m^{[\frac{n}{2}]+1}[\mu_m^{-(s^*-2-\beta)}C_{s*}^{m+1}\mathcal{N}(s^*) + \|g_m\|_{L^\infty}\mu_m^2\|g_m\|_0 + \mu_m^{2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2]. \tag{6.11}$$

By comparing the powers of μ_m on both sides of (6.10) and (6.11), we can choose $a > 0$ and large $s^* > s$ such that

$$\begin{cases} 2(k-2) + 2[\frac{n}{2}] + 6 + a\gamma \le 2a - 1\\ s^* - [\frac{n}{2}] - 3 - \beta \ge a\gamma + 1\,. \end{cases} \tag{6.12}$$

Noticing that $\mu_{m+1}=\mu_m^\gamma$, we can change (6.10) and (6.11) as

$$\begin{aligned}C\mu_{m+1}^a\|g_{m+1}\|_0 \le& C^2\Big[\mu_m^{-(s^*-2-\beta-a\gamma)}C_{s*}^{m+1}\mathcal{N}(s^*)+\|g_m\|_{L^\infty}\mu_m^{a\gamma+2}\|g_m\|_0\\ &+\mu_m^{a\gamma+2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2\Big],\\ C\mu_{m+1}^a\|g_{m+1}\|_{L^\infty} \le& C^2\mu_m^{[\frac{n}{2}]+1}\Big[\mu_m^{-(s^*-2-\beta-a\gamma)}C_{s*}^{m+1}\mathcal{N}(s^*)+\|g_m\|_{L^\infty}\mu_m^{a\gamma+2}\|g_m\|_0\\ &+\mu_m^{a\gamma+2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2\Big],\end{aligned}$$

from which we obtain, by using (6.12) and $\mu_m>1$,

$$\left\{\begin{array}{l}C\mu_{m+1}^a\|g_{m+1}\|_0 \le C^2\left[\mu_m^{-1}C_{s*}^{m+1}\mathcal{N}(s^*)+\|g_m\|_{L^\infty}\mu_m^{2a-1}\|g_m\|_0+\mu_m^{2a-1}\|g_m\|_0^2\right]\\ C\mu_{m+1}^a\|g_{m+1}\|_{L^\infty} \le C^2\left[\mu_m^{-1}C_{s*}^{m+1}\mathcal{N}(s^*)+\|g_m\|_{L^\infty}\mu_m^{2a-1}\|g_m\|_0+\mu_m^{2a-1}\|g_m\|_0^2\right].\end{array}\right. \tag{6.13}$$

Noticing $\gamma>1$, we can choose $\sigma=\sigma(s^*)>1$ so large that $\mu_m^{-1}=\sigma^{-\gamma^m}<\frac{1}{2}$ and

$$C^2\mu_m^{-1}C_{s*}^{m+1}=C^2\sigma^{-\gamma^m}C_{s^*}^{m+1}=C^2\left(\frac{C_{s^*}}{\sigma^{\frac{1}{m+1}\gamma^m}}\right)^{m+1}\le\frac{1}{4}. \tag{6.14}$$

Inserting such $\sigma(s^*)$ into (6.13), we have

$$\left\{\begin{array}{l}C\mu_{m+1}^a\|g_{m+1}\|_0 \le \frac{1}{4}\mathcal{N}(s^*)+\frac{1}{2}C^2\|g_m\|_{L^\infty}\mu_m^{2a}\|g_m\|_0+\frac{1}{2}C^2\mu_m^{2a}\|g_m\|_0^2\\ C\mu_{m+1}^a\|g_{m+1}\|_{L^\infty} \le \frac{1}{4}\mathcal{N}(s^*)+\frac{1}{2}C^2\|g_m\|_{L^\infty}\mu_m^{2a}\|g_m\|_0+\frac{1}{2}C^2\mu_m^{2a}\|g_m\|_0^2.\end{array}\right.$$

Set

$$d_{m+1}=\max\{C\mu_{m+1}^a\|g_{m+1}\|_0,\ C\mu_{m+1}^a\|g_{m+1}\|_{L^\infty}\},$$

we get

$$\left\{\begin{array}{l}C\mu_{m+1}^a\|g_{m+1}\|_0 \le \frac{1}{4}\mathcal{N}(s^*)+d_m^2\\ C\mu_{m+1}^a\|g_{m+1}\|_{L^\infty} \le \frac{1}{4}\mathcal{N}(s^*)+d_m^2.\end{array}\right.$$

So we obtain,

$$d_{m+1}\le\frac{1}{4}\mathcal{N}(s^*)+d_m^2. \tag{6.15}$$

Since

$$\|g_0\|_0=\|G(w_0)\|_0=\|G(0)\|_0=O(\varepsilon),\ \ \|g_0\|_{L^\infty}=O(\varepsilon),$$

we choose $\varepsilon(\sigma)>0$ small such that $\mathcal{N}(s^*)$, $\|g_0\|_{L^\infty}$ and $\|g_0\|_0$ small. By (6.13), (6.14) and $\|g_0\|_0\le\|g_0\|_{s^*}\le\mathcal{N}(s^*)$, we have

$$d_1\le\frac{1}{4}\mathcal{N}(s^*)+\frac{1}{4}\|g_0\|_0\le\frac{1}{2}\mathcal{N}(s^*).$$

By induction and (6.15), we see that

$$d_{m+1}\le\frac{1}{2}\mathcal{N}(s^*),$$

this completes the proof of (6.9). □

Remark 6.3. The estimates in Lemma 6.1 and 6.2, obtained only in the special case $f=f(y)$, are also true for the general case $f=f(y,u)$ or $f=f(y,u,Du)$. Here we only give it

as an example the estimate of $\|(C)\|_0$ in the proof of Lemma 6.2. In fact, from (6.5), and à priori estimate (4.9), it follows that if $\|w_m\|_{[\frac{n}{2}]+3} \leq 1$,

$$\begin{aligned}&\|w_m + tS_m\rho_m\|_{L^\infty} \leq \|w_m\|_{L^\infty} + \|S_m\rho_m\|_{L^\infty}\\ \leq&1 + \|S_m\rho_m\|_{[\frac{n}{2}]+1} \leq 1 + C\|\rho_m\|_{[\frac{n}{2}]+1}\\ \leq&1 + C\|g_m\|_{[\frac{n}{2}]+1} \leq 1 + C(\mathcal{M}([\frac{n}{2}] + 1) + \|w_m\|_{[\frac{n}{2}]+3}) \leq C'.\end{aligned}$$

and similarly, if $\|w_m\|_{[\frac{n}{2}]+4} \leq 1$,

$$\|\nabla w_m + t\nabla S_m\rho_m\|_{L^\infty} \leq \|\nabla w_m\|_{L^\infty} + \|\nabla S_m\rho_m\|_{L^\infty} \leq C',$$

so we have similar estimates

$$\|(C)\|_0 \leq C\mu_m^{2(k-2)+[\frac{n}{2}]+5}\|g_m\|_0^2.$$

for the quadratic error of $f = f(y, u)$ or $f = f(y, u, Du)$.

Existence of k-convex solution of Theorem 1.1

Now we employ the Nash-Moser-Hörmander iteration to prove the existence of solution of main theorem with $s \geq 2[\frac{n}{2}] + 5$. We shall prove by induction that, there exists $\varepsilon_0 > 0$ small such that, for any $\varepsilon \in]0, \varepsilon_0]$

$$\|w_m\|_s \leq 1, \quad \forall m \in \mathbb{N}. \tag{6.16}$$

Since $w_0 = 0$, we may assume that (6.16) holds for $0 \leq l \leq m$, which, by Sobolev imbedding theorem, guarantees the assumption of Lemma 6.1 and 6.2. Interpolation inequality and (6.1) yield, for any $0 \leq s \leq s^*$,

$$\|w_{m+1}\|_s \leq \sum_{l=0}^{m}\|S_l\rho_l\|_s \leq C_s\sum_{l=0}^{m}\|\rho_l\|_s \leq C_s\sum_{l=0}^{m}\|\rho_l\|_{s^*}^{\frac{s}{s^*}}\|\rho_l\|_0^{1-\frac{s}{s^*}}.$$

By (6.6), it follows that, for $0 \leq l \leq m$,

$$\|\rho_l\|_{s^*} \leq (C'_{s^*})^l\mu_l^\beta\mathcal{N}(s^*)$$

with $\mu_l = \sigma^{\gamma^l}, \sigma > 1, \gamma > 1, \beta = \frac{4}{\gamma-1}$, and by (6.9),

$$\|\rho_l\|_0 \leq C\|g_l\|_0 \leq C'\mu_l^{-a}\mathcal{N}(s^*),$$

thus

$$\|w_{m+1}\|_s \leq C_s\sum_{l=0}^{m}(C'_{s^*})^{l\frac{s}{s^*}}\mu_l^{\beta\frac{s}{s^*}-a(1-\frac{s}{s^*})}\mathcal{N}(s^*).$$

So that we can choose s^* large enough such that

$$\beta\frac{s}{s^*} - a(1 - \frac{s}{s^*}) = -\tilde{a} < 0.$$

We choose $\varepsilon_0 = \varepsilon_0(\sigma) > 0$ smaller to make $\mathcal{N}(s^*)$ small enough such that for $s \geq 2[\frac{n}{2}] + 5$,

$$\|w_{m+1}\|_s \leq C_s\sum_{l=0}^{\infty}(C'_{s^*})^{l\frac{s}{s^*}}\mu_l^{-\tilde{a}}\mathcal{N}(s^*) \leq 1$$

This completes the proof of (6.16).

On one hand, by (6.16) there is a subsequence of w_m, still denoted by itself, such that $w_m \to w$ in weak topology of $\mathbf{H}^s(\Omega)$, $s \geq 2[\frac{n}{2}] + 5$ and $w_m \to w$ in $C^{[\frac{n}{2}]+4}$. Hence

$$g_m = -G(w_m) \to -G(w) \quad \text{in} \quad C^{[\frac{n}{2}]+2}(\Omega).$$

On the other hand, by using (6.16), Lemma 6.2 can be applied for all $m \in \mathbb{N}$, letting $m \to \infty$ in (6.9) and recalling (2.9), we see that $G(w) = 0$, thus

$$u(y) = \frac{1}{2}\sum_{i=1}^{k-1} \tau_i y_i^2 + P(y) + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2} y) \in \mathbf{H}^s(\Omega)$$

is a local solution of the Hessian equation (1.1).

7. Strict convexity of local solution

In this section, we will prove that the smooth k-convex local-solution obtained in Section 6 is locally strict convex under the hypothesis of Theorem 1.1, that is, by (1.2) we need to prove that, for $0 < t < 1,\ y,\ z \in \Omega,\ y \neq z$,

$$\sum_{i,j=1}^{n} \int_0^1 \int_0^1 u_{ij}(x(s,\mu)) d\mu ds (y_i - z_i)(y_j - z_j) > 0 \tag{7.1}$$

with $x(s,\mu) = (s\mu + (1-s)t)y + (s(1-\mu) + (1-s)(1-t))z$. Recalling from (3.9),

$$(u_{ij})_{1\le i,j\le n} = \mathbf{r} = \left(\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x) \right),$$

we separate this matrix into two parts: one is

$$r_{ij} = \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x), \qquad 1 \le i, j \le k-1,$$

the principal term of which is $\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l$ and obviously can control the perturbation term $P_{ij}(\varepsilon^2 x) + \varepsilon^5 w_{ij}(x)$ for small $\varepsilon > 0$. The other is

$$r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x), \qquad i \ge k \ \text{ or } \ j \ge k,$$

for which, in order to control the perturbation term, our idea is to prove

$$w_{ij}(x', 0) = w_{ijp}(x', 0) = 0, \qquad k \le p \le n, \tag{7.2}$$

if $i \ge k$ or $j \ge k$. Then the Taylor expansion with respect to $x'' = (x_k, \cdots, x_n)$

$$w_{ij}(x) = w_{ij}(x', x'') = w_{ij}(x', 0) + \sum_{p=k}^{n} w_{ijp}(x', 0) x_p + O(|x''|^2),$$

yields

$$r_{ij}(x) = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} O(|x''|^2), \qquad i \ge k \ \text{ or } \ j \ge k,$$

which, together with (3.6), implies the minor matrix $(r_{ij})_{k\le i,j\le n}$ is strictly diagonally dominant with

$$\begin{cases} |r_{ij}| \le O(1)\varepsilon^4 |x''| + O(1)\varepsilon^{\frac{9}{2}} |x''|^2, \quad 1 \le i \le k-1, k \le j \le n, \\ r_{j_0 j_0}(x) = P_{j_0 j_0}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{j_0 j_0}(x) \ge \alpha \varepsilon^4 |x''|^2 + \sum_{i=k, i\neq j_0}^{n} |r_{i j_0}|, \quad k \le j_0 \le n. \end{cases}$$

Choose $\varepsilon > 0$ small ($\varepsilon \ll \alpha$) enough such that, for $k \le j \le n$,

$$r_{jj}(x) - \sum_{i=k, i\neq j}^{n} |r_{ij}(x)| \ge \alpha\, \varepsilon^4 |x''|^2,$$

Then Cauchy inequality yields

$$\begin{aligned}
&\sum_{i,j=1}^{n} r_{ij}\xi_i\xi_j \\
&\geq \frac{1}{2}\sum_{i=1}^{k-1}\tau_i|\xi_i|^2 + 2\sum_{1\leq i\leq k-1, k\leq j\leq n} O(1)[\varepsilon^4|x''| + \varepsilon^{\frac{9}{2}}|x''|^2]\xi_i\xi_j + \sum_{i=k}^{n}\alpha\varepsilon^4|x''|^2|\xi_i|^2 \\
&\geq \frac{1}{4}\sum_{i=1}^{k-1}\tau_i|\xi_i|^2 + \frac{1}{2}\sum_{i=k}^{n}\alpha\varepsilon^4|x''|^2|\xi_i|^2, \quad \forall \xi\in\mathbb{R}^n
\end{aligned}$$

from which, it follows that (7.1) holds. In fact , setting $\xi = y - z$ and recalling $x'' = x''(s,\mu) = (s\mu + (1-s)t)y'' + (s(1-\mu) + (1-s)(1-t))z''$, we have

$$\begin{aligned}
&\sum_{i,j=1}^{n}\int_0^1\int_0^1 u_{ij}(x)d\mu ds(y_i - z_i)(y_j - z_j) \\
&\geq \left(\frac{1}{4}\sum_{i=1}^{k-1}\tau_i|y_i - z_i|^2 + \frac{1}{2}\alpha\varepsilon^4 b(t)\sum_{i=k}^{n}|y_i - z_i|^2\right) > 0
\end{aligned}$$

with

$$b(t) = \int_0^1\int_0^1 |[s\mu + (1-s)t]y'' + [s(1-\mu) + (1-s)(1-t)]z''|^2 d\mu ds > 0,$$

for any $0 < t < 1$ and $y'' \neq z''$, this inequality is true because

$$\Big\{(s,\mu)\in[0,1]\times[0,1];\ [s\mu + (1-s)t]y'' + [s(1-\mu) + (1-s)(1-t)]z'' = 0\Big\}$$

lies on a hyperbolic curve in the $(s,\mu)-$ plane and then its Lebesgue measure for $d\mu ds$ is zero. So that u is strictly convex on Ω.

We prove now (7.2) by the following two lemmas. From the explicit expression of $P(y)$ in (3.2) together with $u(0) = 0$ and $\nabla u(0) = 0$, we see that $w(0) = 0$ and $\nabla w(0) = 0$. Moreover, we have

Lemma 7.1. *Let u be a solution of equation* (2.2) *in the form of* (3.8) *with $\|w\|_{C^3} \leq 1$, then*

$$w_{ij}(x',0) = 0, \quad i\geq k \ \text{ or } \ j\geq k. \tag{7.3}$$

Proof. By (3.2),

$$P_{ij}(\varepsilon^2 x)\,|_{x''=0} = 0 \quad \text{for} \quad i\neq j,\ i\geq k \text{ or } j\geq k,$$

then

$$r_{ij}\,|_{x''=0} = \varepsilon^{\frac{9}{2}} w_{ij}(x',0),\ \ i\neq j,\ i\geq k \quad \text{or} \quad j\geq k.$$

and

$$\sum_{i=k}^{n}\sum_{j=1}^{k-1} r_{ji}^2(x',0)\sigma_{k-2,j}(\lambda_1,\ldots,\lambda_{k-1}) = O(1)\varepsilon^9.$$

$$P_{ii}(\varepsilon^2 x',0) = O(1)\varepsilon^8 \quad \text{for} \quad k\leq i\leq n,$$

thus, for $k\leq j\leq n$,

$$r_{jj}(x',0) = P_{jj}(\varepsilon^2 x',0) + \varepsilon^{\frac{9}{2}} w_{jj}(x',0) = \varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} w_{jj}(x',0).$$

Inserting the above estimates into $S_k(\mathbf{r}) = ① + ② + ③$ in Lemma 3.4 and then multiplying both sides by $\varepsilon^{-\frac{9}{2}}$, we obtain

$$\sigma_{k-1}(\lambda_1, \ldots, \lambda_{k-1}) \sum_{j=k}^{n} w_{jj}(x', 0) = \varepsilon^{\frac{9}{2}} O(1).$$

Letting $\varepsilon \to 0^+$, we get

$$\sum_{j=k}^{n} w_{jj}(x', 0) = \sigma_1(w_{kk}(x', 0), w_{k+1,k+1}(x', 0), \ldots, w_{nn}(x', 0)) = 0.$$

Using now the identity

$$2\sigma_2(\lambda) = [\sigma_1(\lambda)]^2 - \sum_{i=1}^{m} \lambda_i^2, \quad \forall \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m,$$

it follows that

$$\begin{aligned}
&2\sigma_2(r_{kk}(x', 0), r_{k+1,k+1}(x', 0), \ldots, r_{nn}(x', 0)) \\
=&[\sum_{j=k}^{n} r_{jj}(x', 0)]^2 - \sum_{j=k}^{n} [r_{jj}(x', 0)]^2 \\
=&[\varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} \sum_{j=k}^{n} w_{jj}(x', 0)]^2 - \sum_{j=k}^{n} [\varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0)]^2 \\
=&O(1)\varepsilon^{16} - \sum_{j=k}^{n} [O(1)\varepsilon^8 + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0)]^2.
\end{aligned}$$

Multiplying by ε^{-9} on both sides of $S_k(\mathbf{r}) = ① + ② + ③$ in Lemma 3.4, taking $x'' = 0$ and letting $\varepsilon \to 0$, we obtain

$$\sum_{i=k}^{n} \sum_{j=1}^{k-1} w_{ji}^2(x', 0) = \sum_{j=k}^{n} [w_{jj}(x', 0)]^2 = \sum_{i=k}^{n} \sum_{s=k, s\neq i}^{n} w_{si}^2(x', 0) = 0$$

and then (7.3) is true. □

Using now (7.3), the Taylor expansion of $r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)$ for $i \geq k$ or $j \geq k$ is of the following version

$$r_{ij}(x) = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} \sum_{p=k}^{n} w_{ijp}(x', 0) x_p + \frac{1}{2} \varepsilon^{\frac{9}{2}} \sum_{p,q=k}^{n} w_{ijpq}(x', 0) x_p x_q + O(1)\varepsilon^{\frac{9}{2}} |x''|^3,$$

where $\|w\|_{C^{[\frac{n}{2}]+3}} \leq 1$ is required. Similar to the proof of Lemma 7.1, we can obtain

$$\sum_{i=k}^{n} w_{iip}(x', 0) = \sum_{i=k}^{n} w_{iipp}(x', 0) = 0, \quad k \leq p \leq n.$$

And also :

Lemma 7.2. *Let u be a solution of equation* (2.2) *in the form of* (3.8)*, and* $\|w\|_{C^{[\frac{n}{2}]+4}} \leq 1$*, then*

$$w_{ijp}(x', 0) = 0, \quad i \geq k \ \text{ or } \ j \geq k, \ k \leq p \leq n.$$

8. Appendix

In this section, we will estimate the eigenvalues and eigenvectors of the following Hessian matrix

$$\mathbf{r} = (r_{ij})_{1\le i,j\le n} = \left(\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + \varepsilon \tilde{w}_{ij}(x)\right)_{1\le i,j\le n}$$

which is a small perturbation of diagonal matrix $\left(\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l\right)_{1\le i,j\le n}$. For any function $\tilde{w} \in C^2$, we can find out an orthogonal matrix $T(x,\varepsilon)$ satisfying

$$T(x,\varepsilon)\,\mathbf{r}\,{}^tT(x,\varepsilon) = \mathrm{diag}\,[\lambda_1(x,\varepsilon), \lambda_2(x,\varepsilon), \ldots, \lambda_n(x,\varepsilon)]\,,$$

Let

$$T(x,\varepsilon) = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix}, \quad T(x,\varepsilon)\mid_{\varepsilon=0} = Id$$

with $T_i = (T_{i1}, T_{i2}, \ldots, T_{in})$ the corresponding unit eigenvectors of $\lambda_i, i = 1, 2, \ldots, n$. In [13], there also exists such an orthogonal matrix $T(x,\varepsilon)$, the estimates of good regularity for all of its entries can be easily obtained because all of its eigenvalues are different from each other. Now we can only give the estimates of T_i for $1 \le i \le k-1$, because in our case $\lambda_j(x,\varepsilon)$ $(k \le j \le n)$ are around zero and there is no distinct gap among them, they are not necessarily smooth in x and ε, so are the corresponding eigenvectors $T_i(x,\varepsilon)$ $(k \le i \le n)$. But the following estimates are enough for us.

Proposition 8.1. *Suppose that $\tilde{w}$ is smooth and $\|\tilde{w}\|_{C^3} \le 1$. Then, $T_i(x,\varepsilon)$, $1 \le i \le k-1$, is smooth in $(x,\varepsilon) \in \overline{\Omega} \times [0,\varepsilon_0]$ for some positive $0 < \varepsilon_0 \ll 1$, and*

$$\sum_{i=1}^{k-1} |\lambda_i - \tau_i| \le C\varepsilon \sum_{i,j=1}^{n} |\tilde{w}_{ij}(x)|, \qquad \sum_{i=k}^{n} |\lambda_i| \le C\varepsilon^{\frac{1}{2}} \Big(\sum_{i,j=1}^{n} |\tilde{w}_{ij}(x)|\Big)^{\frac{1}{2}}, \tag{8.1}$$

$$\sum_{i=1}^{k-1} |T_{ii}(x,\varepsilon) - 1| + \sum_{i=1}^{k-1} \sum_{j=1, j\ne i}^{n} |T_{ij}(x,\varepsilon)| \le C\varepsilon \sum_{i,j=1}^{n} |\tilde{w}_{ij}(x)|, \tag{8.2}$$

$$\sum_{i=1}^{k-1} \sum_{j=1}^{n} |DT_{ij}(x,\varepsilon)| \le C\varepsilon \sum_{i,j,l=1}^{n} |\tilde{w}_{ijl}(x)| \tag{8.3}$$

with C independent of ε and $\tilde{w}$.

Moreover, if $\tilde{w}(x)$ is periodic in x_i $(1 \le i \le k-1)$, then so is each of T_{ij}, while T_{ij} has the same regularity as $D^2\tilde{w}$ and is C^∞ in ε for $1 \le i \le k-1, 1 \le j \le n$.

Proof. Noting

$$F(t) = \det(\mathbf{r} - t\,\mathbf{I}).$$

Now we denote $R(x,\varepsilon)$, $R_i(x,\varepsilon)$ and $R_{ij}(x,\varepsilon)$ as the different functions, which are smooth in x , ε and $\tilde{w}$, with the properties

$$|R(x,\varepsilon)| + |R_i(x,\varepsilon)| + |R_{ij}(x,\varepsilon)| \le C\varepsilon \sum_{i,j=1}^{n} |\tilde{w}_{ij}(x)| \tag{8.4}$$

and C being independent of x and ε. Firstly, using the condition $\|\tilde{w}\|_{C^2} \leq 1$ and $0 < \varepsilon_0 \ll 1$, we have

$$F(t) = \prod_{i=1}^{k-1}(\tau_i - t)t^{n-k+1} + R(x, \varepsilon).$$

Noticing $\tau_1 > \tau_2 > \ldots > \tau_{k-1} > 0$, we take $\delta = \frac{1}{4}\min_{1\leq i\leq k-2}(1 - \frac{\tau_{i+1}}{\tau_i}, \frac{\tau_i}{\tau_{i+1}} - 1) > 0$ if $k > 2$ and $\delta = \frac{1}{4}$ if $k = 2$. For fixed i_0 with $1 \leq i_0 \leq k-1$, we have

$$F((1-\delta)\tau_{i_0}) = \prod_{i=1}^{i_0-1}(\tau_i - (1-\delta)\tau_{i_0})\delta\tau_{i_0} \prod_{i=i_0+1}^{k-1}(\tau_i - (1-\delta)\tau_{i_0})((1-\delta)\tau_{i_0})^{n-k+1} + R_1(x, \varepsilon)$$

and

$$F((1+\delta)\tau_{i_0}) = -\prod_{i=1}^{i_0-1}(\tau_i - (1+\delta)\tau_{i_0})\delta\tau_{i_0} \prod_{i=i_0+1}^{k-1}(\tau_i - (1+\delta)\tau_{i_0})((1+\delta)\tau_{i_0})^{n-k+1} + R_2(x, \varepsilon).$$

By the choice of δ, we have

$$\tau_i - (1+\delta)\tau_{i_0} > \delta\tau_{i_0-1}, \quad 1 \leq i \leq i_0 - 1$$

and

$$\tau_i - (1-\delta)\tau_{i_0} < -\delta\tau_{i_0+1}, \quad i_0 + 1 \leq i \leq k-1,$$

therefore, when $0 < \varepsilon \ll \delta\tau_{k-1}$, we obtain

$$F((1-\delta)\tau_{i_0})F((1+\delta)\tau_{i_0}) < 0$$

and, by virtue of intermediate value theorem, there exists an eigenvalue, denoted by λ_{i_0}, such that

$$(1-\delta)\tau_{i_0} < \lambda_{i_0} < (1+\delta)\tau_{i_0}, \quad F(\lambda_{i_0}) = 0,$$

which yields

$$\tau_i - \lambda_{i_0} \geq \tau_i - (1+\delta)\tau_{i_0} > \delta\tau_{k-1}, \quad 1 \leq i \leq i_0 - 1$$

and

$$\tau_i - \lambda_{i_0} \leq \tau_i - (1-\delta)\tau_{i_0} < -\delta\tau_{k-1}, \quad i_0 + 1 \leq i \leq k-1.$$

From $0 = F(\lambda_{i_0}) = \prod_{i=1}^{i_0-1}(\tau_i - \lambda_{i_0})(\tau_{i_0} - \lambda_{i_0}) \prod_{i=i_0+1}^{k-1}(\tau_i - \lambda_{i_0})(\lambda_{i_0})^{n-k+1} + R_3(x, \varepsilon)$, it follows that

$$\tau_{i_0} - \lambda_{i_0} = -\frac{R_3(x, \varepsilon)}{\prod_{i=1}^{i_0-1}(\tau_i - \lambda_{i_0}) \prod_{i=i_0+1}^{k-1}(\tau_i - \lambda_{i_0})(\lambda_{i_0})^{n-k+1}}$$

and then

$$\lambda_{i_0} = \tau_{i_0} + R_4(x, \varepsilon).$$

Now we have proved that there are eigenvalues $(\lambda_i)_{i=1}^{k-1}$ such that

$$\lambda_i = \tau_i + R(x, \varepsilon), \quad 1 \leq i \leq k-1.$$

Let us express all the eigenvalues as $(\lambda_i)_{i=1}^{n}$. Then, since $\|\tilde{w}\|_{C^3} \leq 1$,

$$S_1(r) = \sum_{i=1}^{k-1}\tau_i + R(x, \varepsilon), \quad S_2(r) = \sigma_2(\tau_1, \ldots, \tau_{k-1}) + R(x, \varepsilon).$$

Since $S_k(r)$ is invariant under orthogonal transformation, then

$$\sum_{i=1}^{k-1}\tau_i + R(x,\varepsilon) = S_1(\mathbf{r}) = \sigma_1(\lambda) = \sum_{i=1}^{k-1}\lambda_i + \sum_{i=k}^{n}\lambda_i$$

and

$$\sigma_2(\tau_1,\dots,\tau_{k-1}) + R(x,\varepsilon) = S_2(\mathbf{r}) = \sigma_2(\lambda).$$

from which, using

$$2\sigma_2(\lambda) = \sigma_1(\lambda)^2 - \sum_{i=1}^{n}\lambda_i^2.$$

we see that

$$\sigma_2(\lambda) = \sigma_2(\tau_1,\dots,\tau_{k-1}) - \frac{1}{2}\sum_{i=k}^{n}\lambda_i^2 + R(x,\varepsilon)$$

and then

$$\sum_{i=k}^{n}\lambda_i^2 = R(x,\varepsilon),$$

which implies

$$|\lambda_i| \le C\varepsilon^{\frac{1}{2}}\Big(\sum_{i,j=1}^{n}|\tilde{w}_{ij}(x)|\Big)^{\frac{1}{2}}, \qquad k \le i \le n. \tag{8.5}$$

This completes the proof of (8.1).

Now we pass to prove (8.2). Let T_1 be the eigenvector corresponding to λ_1, then T_1 satisfies the linear equation $\mathbf{r}T_1 - \lambda_1 T_1 = 0$ and $\operatorname{rank}(\mathbf{r} - \lambda_1\mathbf{I}) \le n-1$. To solve T_1, we will use the Gaussian elimination procedure. Noticing $\lambda_i = \tau_i + R(x,\varepsilon)$ for $1 \le i \le k-1$, we can write the matrix $\mathbf{r} - \lambda_1\mathbf{I}$ as

$$\begin{pmatrix}
R_{11} & R_{12} & \dots & R_{1,k-1} & R_{1k} & \dots & R_{1n} \\
R_{21} & R_{22}+\tau_2-\tau_1 & \dots & R_{2,k-1} & R_{2,k} & \dots & R_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
R_{k-1,1} & R_{k-1,2} & \dots & R_{k-1,k-1}+\tau_{k-1}-\tau_1 & R_{k-1,k} & \dots & R_{k-1,n} \\
R_{k,1} & R_{k,2} & \dots & R_{k,k-1} & R_{k,k}-\tau_1 & \dots & R_{k,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
R_{n,1} & R_{n,2} & \dots & R_{n,k-1} & R_{n,k} & \dots & R_{n,n}-\tau_1.
\end{pmatrix}$$

Since each R_{jj}, $2 \le j \le k-1$, is dominated by $\tau_j - \tau_1$ and each R_{ii}, $k \le i \le n$, is dominated by τ_1, then $\operatorname{rank}(\mathbf{r} - \lambda_1\mathbf{I}) = n-1$. Taking the (row) elementary operations repeatedly, we can transform $\mathbf{r} - \lambda_1\mathbf{I}$ and still denote it as

$$\begin{pmatrix}
R_{11} & 0 & \dots & 0 & 0 & \dots & 0 \\
R_{21} & R_{22}+\tau_2-\tau_1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
R_{k-1,1} & 0 & \dots & R_{k-1,k-1}+\tau_{k-1}-\tau_1 & 0 & \dots & 0 \\
R_{k,1} & 0 & \dots & 0 & R_{k,k}-\tau_1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
R_{n,1} & 0 & \dots & 0 & 0 & \dots & R_{n,n}-\tau_1.
\end{pmatrix}$$

Since $\mathrm{rank}(\mathbf{r}-\lambda_1\mathbf{I}) = n-1$, we see that $R_{11} \equiv 0$ and the solutions of the equation $\mathbf{r}T_1 - \lambda_1 T_1 = 0$ is in the form

$$\begin{pmatrix} T_{11} \\ T_{12} \\ \vdots \\ T_{1,k-1} \\ T_{1,k} \\ \vdots \\ T_{1n} \end{pmatrix} = T_{11} \begin{pmatrix} 1 \\ \frac{R_{21}(x,\varepsilon)}{\tau_1-\tau_2-R_{22}(x,\varepsilon)} \\ \vdots \\ \frac{R_{k-1,1}(x,\varepsilon)}{\tau_1-\tau_{k-1}-R_{k-1,k-1}(x,\varepsilon)} \\ \frac{R_{k,1}(x,\varepsilon)}{\tau_1-R_{k-1,k-1}(x,\varepsilon)} \\ \vdots \\ \frac{R_{n,1}(x,\varepsilon)}{\tau_1-R_{nn}(x,\varepsilon)} \end{pmatrix}, \quad \text{with } T_{11} \neq 0, \tag{8.6}$$

and each $R_{ij}(x,\varepsilon)$ has the property (8.4). Because we need $T_1(x,\varepsilon)\mid_{\varepsilon=0} = (1,0,\ldots,0)$, we can choose suitable $T_{11} > 0$ such that $\|T_1\| = 1$, then $T_1 = (T_{11}, T_{12}, \ldots, T_{1n})$ satisfies (8.2). The vectors T_i $(2 \le i \le k-1)$ can be obtained by the same way as T_1. Also it follows from (8.6) that (8.2) and (8.3) hold.

Because each entry of the matrix $\mathbf{r} = (r_{ij})$ is periodic in $x_1, \ldots, x_{k-1}$ and the elementary operations above do not change the periodicity, then each $T_i (1 \le i \le k-1)$ is also periodic. Also by (8.6), when $1 \le i \le k-1$, T_{ij} has the same regularity as $D^2\tilde{w}$ and is C^∞ in ε. □

Remark 8.2. Since λ_i, $k \le i \le n$, are all around 0, then (8.5) shows that they are not necessarily smooth in x and ε, so are the corresponding eigenvectors T_i.

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