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# EXISTENCE AND CONVEXITY OF LOCAL SOLUTIONS TO DEGENERATE HESSIAN EQUATIONS

GUJI TIAN AND CHAO-JIANG XU

ABSTRACT. In this work, we prove the existence of local convex solution to the following  $k$ -Hessian equation

$$S_k[u] = K(y)g(y, u, Du)$$

in the neighborhood of a point  $(y_0, u_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ , where  $g \in C^\infty$ ,  $g(y_0, u_0, p_0) > 0$ ,  $K \in C^\infty$  is nonnegative near  $y_0$ ,  $K(y_0) = 0$  and  $\text{Rank}(D_y^2 K)(y_0) \geq n - k + 1$ .

## 1. INTRODUCTION

In this work, we study the following  $k$ -Hessian equation :

$$(1.1) \quad S_k[u] = f(y, u, Du),$$

on the open domain  $\Omega \subset \mathbb{R}^n$  with  $2 \leq k \leq n$ , where  $f \geq 0$  is defined on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$  with  $f(y_0, u_0, p_0) = 0$ . When  $u \in C^2$ , the  $k$ -Hessian operator  $S_k[u]$  is defined by

$$S_k[u] = S_k(D^2u) = \sigma_k[\lambda(D^2u)] = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k},$$

where  $S_k(D^2u)$  is the sum of all  $k$ -order principal minors of the Hessian matrix  $(D^2u)$ , and  $\lambda(D^2u) = (\lambda_1(D^2u), \dots, \lambda_n(D^2u))$  are the eigenvalues of the matrix  $(D^2u)$ . One origin of  $k$ -Hessian operator is from Christoffel-Minkowski problem, see [4, 5, 6] and references therein, another one is from calibrated geometries in [9]. The background of  $k$ -Hessian operator in terms of differential geometry can also be found in Section 4, [12].

When  $f > 0$ , the solutions  $u$  of (1.1) is considered with  $\lambda(D^2u)$  in the so-called Gårding cone :

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n; \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

If  $f \geq 0$ , the equation (1.1) is called degenerate, in this case, we consider the solutions with

$$\lambda(D^2u) \in \bar{\Gamma}_k(n) = \{\lambda \in \mathbb{R}^n; \sigma_j(\lambda) \geq 0, 1 \leq j \leq k\}.$$

A function  $u \in C^2$  is called to be  $k$ -convex, if  $\lambda(D^2u) \in \bar{\Gamma}_k(n)$ . The  $n$ -convex function is simply called convex.

The convexity of solutions of (1.1) is an important problem in the field of geometries analysis, including usual convexity, power convexity, log-convexity, or quasi-convexity. For example, in the study of Christoffel-Minkowski problem (see [4, 5, 6]), an important

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subject is to prove the existence of a convex body with prescribed area measure of suitable order, this is equivalent to prove the microscopic convexity principle (constant rank theorem) for some  $k$ -Hessian type equation on the unit sphere  $\mathbb{S}^n$ . There is also a strong connection between convexity properties of solutions to elliptic and parabolic partial differential equations and Brunn-Minkowski type inequalities for associated variational functionals, see [3, 13, 14, 15].

The microscopic convexity principle, with applications in geometric equations on manifolds, has been established in [2] for the very general fully nonlinear elliptic and parabolic operators of second order. For the  $k$ -Hessian equation with  $k = 2, n = 3$ , the power convexity for Dirichlet problem of equation (1.1) with  $f = 1$ , and log-convexity for the eigenvalue problem have been studied in [13, 14], see also [15]. The above convexity results are established on two facts, one is that the equations are elliptic, another is that the existence of classical (at least  $C^2$ ) solution has already known. However, in many important geometric problem, the associated  $k$ -Hessian equation is degenerate (see [8]), and for the degenerate elliptic  $k$ -Hessian equation, one can only prove the existence of  $C^{1,1}$  solution for Dirichlet problem ([11]).

In this paper, we study the convexity of solution with following definition: for a convex domain  $E$ , the function  $v \in C(E)$  is said to be *strictly convex* if

$$v(ty + (1-t)z) < tv(y) + (1-t)v(z), \quad 0 < t < 1, \quad y, z \in E, \quad y \neq z.$$

Which, in case  $v \in C^2(E)$ , is equivalent to

$$(1.2) \quad \sum_{i,j=1}^n (y_i - z_i)(y_j - z_j) \int_0^1 \int_0^1 \frac{\partial^2 v}{\partial x_i \partial x_j}(x(s, \mu)) d\mu ds > 0$$

with  $x(s, \mu) = (s\mu + (1-s)t)y + (s(1-\mu) + (1-s)(1-t))z$ , this shows that the positive definiteness of Hessian matrix  $(D^2v)$  is a sufficient condition for the strict convexity, but not necessary.

However if  $u \in C^2$  is a  $k$ -convex solution of  $S_k[u] = f(y) \geq 0$  with  $2 \leq k < n$  and  $f(y_0) = 0$ , then  $S_{k+1}[u](y_0) > 0$  will never occur, so that there are two possibilities: 1)  $S_{k+1}[u](y_0) < 0$ , in this case,  $u$  is not  $(k+1)$ -convex; 2)  $S_{k+1}[u](y_0) = 0$ , in this case, it is shown in Theorem 1.1 of [18] that, if  $S_k[u](y_0) = S_{k+1}[u](y_0) = 0$ , then  $S_l[u](y_0) = 0$  for  $k \leq l \leq n$ . In particular, if  $f \equiv 0$ , since the case  $S_{k+1}[u] > 0$  will never occur, then either  $S_{k+1}[u] < 0$  in some open subset of  $\Omega$  or  $S_{k+1}[u] \equiv 0$  in  $\Omega$  itself, in the former case  $u$  is not  $(k+1)$ -convex and let alone strictly convex; in the latter case,  $S_l[u] \equiv 0$  for  $k \leq l \leq n$ , then the graph  $(y; u(y))$  for  $k = 2$  has the vanishing sectional curvature and must be a cylinder or a plane, see [16] and [19], meanwhile the graph  $(y; u(y))$  for  $k > 2$ , by Lemma 3.1 of [7], is a surface of constant nullity (at least)  $n - k + 1$  and then is a  $(n - k + 1)$ -ruled surface. Therefore if  $f \equiv 0$ , the solution  $u$  to (1.1) is not strictly convex (at least) along the rulings.

Motivated by above analysis, in this work, we study the local solutions for the following equation,  $2 \leq k \leq n$ ,

$$(1.3) \quad S_k[u] = K(y)g(y, u, Du),$$

with the following assumptions

$$(H) \quad \begin{cases} K \in C^\infty \text{ is nonnegative in a neighbourhood of } y_0 \in \mathbb{R}^n, \\ K(y_0) = 0, \text{ Rank}(D^2K)(y_0) \geq n - k + 1, \\ g \in C^\infty \text{ near } Z_0 = (y_0, u_0, p_0) \text{ and } g(Z_0) > 0. \end{cases}$$

This assumption is independent of coordinates. Our main Theorem is :

**Theorem 1.1.** *If  $K, g$  satisfy the assumption (H), then for any  $s \geq 2[\frac{n}{2}] + 5$ , the equation (1.3) admits a strictly convex  $H^s$ -local solution in a neighbourhood of  $y_0 \in \mathbb{R}^n$ .*

Remark that  $u = \frac{1}{2} \sum_{i=1}^{n-1} y_i^2 + \frac{1}{12} y_n^4$  is a strictly convex solution of the following Monge-Ampère equation :

$$\det D^2u = y_n^2.$$

But the Hessian matrix  $(D^2u)$  is not positive definite at origin.

This article is arranged as follow: In Section 2, we will introduce the idea of how to construct convex local solution in terms of  $K(y)$ . In Section 3, we will construct the first order approximate solution  $\psi(y)$  which is strictly convex. The Section 4 will be devoted to proving the degenerate ellipticity of the linearized operator of  $S_k[u]$  around  $\psi$ . In Section 5, we will use Lax-Milgram theorem to prove the existence of  $H^0$ -weak solution and their a priori  $H^s$ -estimates. In Section 6, we will prove the existence of  $k$ -convex solution by the Nash-Moser-Hörmander iteration procedure. In section 7, we will prove the strict convexity of the local solution obtained in Section 6. Section 8, as an appendix, is devoted to the estimates of eigenvalues and eigenvectors for a matrix after a small perturbation, the conclusion of which is a generalization of Lemma 1.1 of [10].

## 2. SCHEMA OF CONSTRUCTION OF CONVEX LOCAL SOLUTIONS

The assumption (H) is independent of coordinates. Now we choose special coordinates under which the leader term of the solution can be explicitly expressed. Since the degeneracy is come from the term  $K$ , for the simplicity of notations and also computation, we suppose that

$$(B) \quad g \equiv 1, \text{ Rank}(D^2K)(y_0) = n - k + 1.$$

By a translation  $y \rightarrow y + y_0$  and a change of unknown function  $u \rightarrow u - u(y_0) - Du(y_0) \cdot y$ , we can assume  $Z_0 = (0, 0, 0)$ . On the other hand, the solution is searched for locally, that is, we assume without loss of generality that  $K(y)$  is defined in some neighborhood of origin and

$$(2.1) \quad K(y) = \sum_{j=k}^n c_j y_j^2 + O(|y|^3)$$

where  $2c_j > 0$ ,  $k \leq j \leq n$  are the positive eigenvalues of  $(D^2K)(0)$ . In order to describe the “localness”, small  $\varepsilon > 0$  is introduced by the change of variables  $y = \varepsilon^2 x$ , we fix a domain

$$\Omega = Q_\pi \times Q_{\delta_0} \subset \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1}$$

with  $\delta_0 > 0$  being chosen small later and  $x' = (x_1, \dots, x_{k-1})$ ,  $x'' = (x_k, \dots, x_n)$ ,

$$Q_\pi = \{x' \in \mathbb{R}^{k-1}; |x_i| < \pi, 1 \leq i \leq k-1\}, \quad Q_{\delta_0} = \{x'' \in \mathbb{R}^{n-k+1}; \sum_{i=k}^n |x_i|^2 < \delta_0^2\}.$$

We will determine some  $\varepsilon_0 > 0$  and study the equation (1.3) in the following form

$$(2.2) \quad S_k[u] = \tilde{K} \text{ in } \Omega_{\varepsilon_0} = \{y = \varepsilon_0^2 x; x \in \Omega\}$$

with

$$(2.3) \quad \tilde{K}(y) = (1 - \chi(\varepsilon^{-2}y')) \sum_{i=k}^n c_i y_i^2 + \chi(\varepsilon^{-2}y') K(y),$$

where  $\chi(x') \in C_0^\infty(Q_\pi)$  is a cutoff function equal to 1 if  $|x'| \leq \frac{\pi}{2}$ , equal to zero if  $|x'| \geq \pi$ , and  $0 \leq \chi \leq 1$ . The local solution of (2.3) is also the one of (1.3). The aim of introduce of function  $\chi(x')$  is to guarantee the periodicity with respect to variable  $x'$  for nonhomogeneous terms and the coefficients of all the linearized equations, which is important and convenient for existence of solution because the linearized operator  $L_G(w)$  of (4.1) may be degenerate in the direction  $x'$ .

We will construct the local solution of equation (2.2) in the following form

$$(2.4) \quad u(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y).$$

So the construction of solution is by three steps:

### 1) Solutions of the homogeneous equation

Let  $\tau = (\tau_1, \dots, \tau_{k-1}, 0, \dots, 0)$  with  $\tau_1 > \dots > \tau_{k-1} > 0$ , then the convex function

$$(2.5) \quad \varphi(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2$$

satisfies the homogenous equation  $S_k[\varphi] = 0$ , and the linearized operators

$$(2.6) \quad \mathcal{L}_\varphi = \sum_{j=1}^n \sigma_{k-1,j}(\tau) \partial_j^2$$

is degenerate elliptic with

$$\sigma_{k-1,j}(\tau) = 0, 1 \leq j \leq k-1; \quad \sigma_{k-1,j}(\tau) = \sigma_{k-1}(\tau) = \prod_{l=1}^{k-1} \tau_l > 0, \quad k \leq j \leq n.$$

We have also  $S_{k+1}[\varphi] = \dots = S_n[\varphi] = 0$ .

Remark that, in [17, 18], we choose  $\tau \in \partial\Gamma_k(n)$  with  $\sigma_{k+1}(\tau) < 0$ , so  $\varphi$  is not  $(k+1)$ -convex. In this case, the linearized operator (2.6) is uniformly elliptic. But in this work, we want to construct the local strictly convex solution, so we can't do that choice. On the other hand, the function  $\varphi$  defined in (2.5) is only *weakly* convex, so it is difficult to guarantee the convexity after a perturbation.

### 2) Approximate strictly convex solution

Using the assumption (H) on  $K$ , we construct a function  $P$  such that

$$(2.7) \quad \psi(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 + P(y)$$

satisfies

$$S_k[\psi] = \tilde{K} + O(1)\varepsilon^{\frac{9}{2}},$$

and  $\psi$  is strictly convex on  $\Omega_{\varepsilon_0}$ . The construction of the function  $P$  is algebraic by using the assumption (H) of  $K$ .

### 3) Nash-Moser-Hörmander iteration

We construct finally the smooth function  $w$  such that the function  $u$  defined by (2.4) is a local solution of equation (2.2). We use the Nash-Moser-Hörmander iteration procedure:

$$(2.8) \quad \begin{cases} w_0 = 0, w_{m+1} = w_m + S_m \rho_m \\ L_G(w_m) \rho_m + \theta_m \Delta \rho_m = g_m, \quad \text{in } x \in \Omega. \end{cases}$$

where  $\{S_m\}$  is a family of smoothing operators,

$$(2.9) \quad g_m = -G(w_m) = \frac{1}{\varepsilon^{\frac{9}{2}}} \left\{ S_k(\psi + \varepsilon^{\frac{17}{2}} w_m(\varepsilon^{-2}y)) - \tilde{K} \right\}, \quad \theta_m = \sup_{\Omega} |G(w_m)| = \|g_m\|_{L^\infty},$$

and

$$L_G(w) = \sum_{i,j=1}^n \frac{\partial S_k(\mathbf{r}(w))}{\partial r_{ij}} \partial_i \partial_j,$$

where

$$\mathbf{r}(w) = \left( \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x) \right)_{1 \leq i, j \leq n}.$$

The procedure is to prove the existence and the convergence of the sequence  $w_m \rightarrow w$  in some Sobolev space with  $\theta_m \rightarrow 0$  which imply that the function  $u$  defined in (2.4) by  $w$  is a local solution of equation (2.2). We also need to prove that the perturbation doesn't destruct the strictly convexity of  $\psi$  constructed by (2.7).

Remark that the linearized equation is degenerate elliptic, so that there is a loss of the regularity for the a priori estimate of solution  $\rho_m$ , but the coefficients of linearized operators depends on  $D^2 w_m$ , so that we need to smoothing the solution  $\rho_m$  to continue the iteration (2.8) for  $m \in \mathbb{N}$ . This is quite different from the procedure of iteration used in [18] where the linearized equation is uniformly elliptic.

### 3. THE FIRST ORDER APPROXIMATE SOLUTIONS

Since  $K(y)$  attains its minimum 0 at origin, the critical-point theorem implies  $\nabla K(0) = 0$ . Then we have

**Proposition 3.1.** *Suppose that  $K(y)$  satisfies assumption (H) and (2.1), then we have the following decomposition*

$$K(y) = K(y', 0) + \frac{1}{2} \sum_{i=k}^n \frac{\partial^2 K}{\partial y_i^2}(y', 0) y_i^2 + R(y)$$

where  $K(y', 0)$  vanishes at  $y' = 0$  up to order greater than four and

$$R(y) = \sum_{i=k}^n \frac{\partial K}{\partial y_i}(y', 0) y_i + \frac{1}{2} \sum_{i,j=k, i \neq j}^n \frac{\partial^2 K}{\partial y_i \partial y_j}(y', 0) y_i y_j + O(1) |y''|^3.$$

In particular, for  $y = \varepsilon^2 x$ ,  $x \in \Omega$ , we have

$$R(y) = O(1) \varepsilon^6.$$

Formally, if  $u \in C^{3,1}$  is a solution to (1.3), by Taylor expansion

$$(3.1) \quad u(y) = \sum_{i,j=1}^n u_{ij}(0)y_i y_j + o(|y|^2)$$

substituting (3.1) into (1.3), we see that,

$$S_k(D^2 u(0)) = S_k(u_{ij}(0)) = \lim_{y \rightarrow 0} S_k(D^2 u(y)) = K(0) = 0.$$

Therefore, a smooth (at least  $C^{3,1}$ ) local solution to (1.3) is a solution of  $S_k[u] = 0$  plus a small perturbation  $o(|y|^2)$ . So we wish to construct the first order approximate solution as following form

$$\psi(y) = \frac{1}{2} \sum_{i=1}^{k-1} \tau_i y_i^2 + P(y)$$

with  $\tau_1 > \tau_2 > \dots > \tau_{k-1} > 0$ , such that

$$S_k[\psi] = \tilde{K} + O(1)\varepsilon^{\frac{9}{2}},$$

which is difficult to arrive at. Our observation is that  $\sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(y)$  is the main part of  $S_k[\psi]$ , so we only need to find out  $P(y)$  to satisfy the weaker version

$$\sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(y) = \tilde{K} - \chi(\varepsilon^{-2}y')R(y).$$

This is our trick how to construct  $P(y)$ , let

$$(3.2) \quad \begin{aligned} P(y) \equiv & \frac{1}{2(n-k+1)\sigma_{k-1}(\tau)} \chi(\varepsilon^{-2}y') K(y', 0) \sum_{j=k}^n y_j^2 \\ & + \frac{1}{24\sigma_{k-1}(\tau)} \chi(\varepsilon^{-2}y') \sum_{i=k}^n \left( \frac{\partial^2 K}{\partial y_i^2}(y', 0) - 2c_i \right) y_i^4 \\ & + \frac{1}{12} \sum_{i=k}^n \left[ \frac{c_i}{\sigma_{k-1}(\tau)} - 4\alpha(n-k) \right] y_i^4 + \alpha \sum_{j=k}^n \sum_{i=k, i \neq j}^n y_i^2 y_j^2 \end{aligned}$$

where

$$(3.3) \quad 0 < \alpha < \frac{1}{16(n-k)^2 + 4(n-k+1)} \min_{k \leq j \leq n} \left\{ \frac{c_j}{2\sigma_{k-1}(\tau)} \right\}.$$

**Remark 3.2.** Taking a example,  $K^1(y) = K^1(y'') = \sum_{i=k}^n c_i y_i^2$ , then

$$P(y) = P^1(y'') = \frac{1}{12} \sum_{i=k}^n \left[ \frac{c_i}{\sigma_{k-1}(\tau)} - 4\alpha(n-k) \right] y_i^4 + \alpha \sum_{j=k}^n \sum_{i=k, i \neq j}^n y_i^2 y_j^2.$$

We have

$$\sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}^1(y'') = K^1(y''),$$

and the strictly convex function  $\psi^1(y) = \frac{1}{2} \sum_{i=1}^{k-1} \tau_i y_i^2 + P^1(y'')$  satisfies

$$S_k(\psi^1) = K^1(y'') + O(|y''|^4).$$

The direct calculation give

**Proposition 3.3.** *Let  $P(y)$  be defined in (3.2), then*

$$(3.4) \quad \begin{cases} \sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(y) = \tilde{K} - \chi(\varepsilon^{-2}y')R(y); \\ P_{jj}(y', 0) = O(1)|y'|^4, \quad k \leq j \leq n, \\ P_{ij}(y) = 8\alpha y_i y_j, \quad i \neq j, \quad k \leq i, j \leq n, \end{cases}$$

and

$$(3.5) \quad P_{jj}(y) \geq 4\alpha|y''|^2, \quad k \leq j \leq n.$$

For small  $|y|$ , the minor matrix  $(P_{ij})_{k \leq i, j \leq n}$  is strictly diagonally dominant, more explicitly, for fixed  $k \leq j_0 \leq n$ ,

$$(3.6) \quad P_{j_0 j_0}(y) \geq 2\alpha|y''|^2 + \sum_{i=k, i \neq j_0}^n |P_{ij_0}(y)|.$$

*Proof.* From (3.2), we obtain, for fixed  $k \leq j \leq n$ ,

$$(3.7) \quad \begin{aligned} P_{jj}(y) &= \frac{1}{(n-k+1)\sigma_{k-1}(\tau)} \chi(\varepsilon^{-2}y') K(y', 0) \\ &+ \frac{1}{\sigma_{k-1}(\tau)} \chi(\varepsilon^{-2}y') \left[ \frac{1}{2} \frac{\partial^2 K}{\partial y_j^2}(y', 0) - c_j \right] y_j^2 \\ &+ \left[ \frac{c_j}{\sigma_{k-1}(\tau)} - 4\alpha(n-k) \right] y_j^2 + 4\alpha \sum_{i=k, i \neq j}^n y_i^2. \end{aligned}$$

By (3.7) we have

$$\sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(y) = \chi(\varepsilon^{-2}y') \left( K(y', 0) + \frac{1}{2} \sum_{i=k}^n \frac{\partial^2 K}{\partial y_i^2}(y', 0) y_i^2 \right) + \sum_{j=k}^n c_j y_j^2$$

which proves the first equality in (3.4). Since  $K(y', 0)$  vanishes up to order greater than four, we have

$$P_{jj}(y', 0) = \frac{1}{(n-k+1)\sigma_{k-1}(\tau)} \chi(\varepsilon^{-2}y') K(y', 0) = O(1)|y'|^4,$$

then the second equality in (3.4) is true. The third equality in (3.4) is obvious.

Now we return to (3.7) for  $P_{jj}(y)$ . Since  $K(y', 0) \geq 0$ , employing (2.1), choosing small  $|y'|$  and then taking  $\alpha$  to satisfy (3.3), we have

$$P_{jj}(y) \geq \left[ \frac{c_j}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k) \right] y_j^2 + 4\alpha \sum_{i=k, i \neq j}^n y_i^2$$



which implies (3.5). By virtue of inequality above, for fixed  $j_0$  with  $k \leq j_0 \leq n$ , we obtain by Cauchy inequality,

$$\begin{aligned}
P_{j_0 j_0}(y) &\geq \left[ \frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k) \right] y_{j_0}^2 + 4\alpha \sum_{i=k, i \neq j_0}^n y_i^2 \\
&\geq 2\alpha |y''|^2 + \frac{1}{n-k} \sum_{i=k, i \neq j_0}^n \left[ \left[ \frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k+1) \right] y_{j_0}^2 + 2\alpha y_i^2 \right] \\
&\geq 2\alpha |y''|^2 + \sum_{i=k, i \neq j_0}^n \frac{2}{n-k} |y_i y_{j_0}| \sqrt{2\alpha \left[ \frac{c_{j_0}}{2\sigma_{k-1}(\tau)} - 4\alpha(n-k+1) \right]} \\
&\geq 2\alpha |y''|^2 + \sum_{i=k, i \neq j_0}^n 8\alpha |y_i y_{j_0}| = 2\alpha |y''|^2 + \sum_{i=k, i \neq j_0}^n |P_{i j_0}(y)|,
\end{aligned}$$

so the minor matrix  $(P_{ij})_{k \leq i, j \leq n}$  is strictly diagonally dominant and (3.6) is proved.  $\square$

We will construct the solution as a perturbation of the strictly convex function  $\psi(y)$  in the following form,

$$(3.8) \quad u(y) = \psi + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w(\varepsilon^{-2}y),$$

see (2.4), where  $w(x)$  is a smooth function to construct.

By a change of variable  $x = \varepsilon^{-2}y$ , the Hessian matrix of  $u$  defined in (3.8) is

$$(3.9) \quad (D^2u)(\varepsilon^2x) = \mathbf{r} = \left( \sum_{l=1}^{k-1} \delta_i^l \delta_j^l \tau_l + P_{ij}(\varepsilon^2x) + \varepsilon^{\frac{9}{2}} w_{ij}(x) \right).$$

We study firstly the minor matrix  $\mathbf{r}_{k-1} = (r_{ij})_{1 \leq i, j \leq k-1}$  which is real-valued and symmetric, then there is an orthogonal  $(k-1) \times (k-1)$  matrix  $\tilde{Q}$  such that

$$\tilde{Q}(x, \varepsilon) \mathbf{r}_{k-1} \tilde{Q}^t(x, \varepsilon) = \text{diag}(\lambda_1(x, \varepsilon), \dots, \lambda_{k-1}(x, \varepsilon)).$$

Let

$$Q = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & \mathbf{I}_{n-k+1, n-k+1} \end{pmatrix},$$

then

$$Q \mathbf{r}^t Q = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & r_{1,k} & \dots & r_{1,n} \\ 0 & \lambda_2 & \dots & 0 & r_{2,k} & \dots & r_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{k-1} & r_{k-1,k} & \dots & r_{k-1,n} \\ r_{k,1} & r_{k,2} & \dots & r_{k,k-1} & r_{k,k} & \dots & r_{k,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n,1} & r_{n,2} & \dots & r_{n,k-1} & r_{n,k} & \dots & r_{n,n} \end{pmatrix},$$

where all terms  $r_{ij}$  in the above matrix is in the form

$$(3.10) \quad r_{ij}(x) = P_{ij}(\varepsilon^2x) + \varepsilon^{\frac{9}{2}} w_{ij}(x).$$

Noticing the existence of  $\chi(\varepsilon^{-2}y')$  in  $P(y)$ , we have

$$P_{ij}(\varepsilon^2 x', \varepsilon^2 x'') = O(1)\varepsilon^8 |x''|^2 + O(1)\varepsilon^4 |x''|^4, \quad 1 \leq i, j \leq k-1,$$

applying Lemma 8.1 to the minor matrix  $\mathbf{r}_{k-1}$ , we obtain that

$$(3.11) \quad \lambda_i = \tau_i + O(1)\varepsilon^4, \quad \lambda_1 > \lambda_2 > \dots > \lambda_{k-1} > 0,$$

for  $\varepsilon > 0$  small enough.

**Lemma 3.4.** *Let  $\mathbf{r}$  be defined in (3.9), then  $S_k(\mathbf{r}) = \textcircled{1} + \textcircled{2} + \textcircled{3}$  with*

$$\begin{cases} \textcircled{1} = \sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1})\sigma_1(r_{kk}, \dots, r_{nn}) - \sum_{i=k}^n \sum_{j=1}^{k-1} \sigma_{k-1,j}(\lambda_1, \dots, \lambda_{k-1})r_{ji}^2 \\ \textcircled{2} = \sigma_{k-2}(\lambda_1, \dots, \lambda_{k-1}) \left[ \sigma_2(r_{kk}, \dots, r_{nn}) - \sum_{i=k}^n \sum_{s=k, s \neq i}^n r_{si}^2 \right] + O(\sum_{m \geq k; i \geq 1} |r_{mi}|^3) \\ \textcircled{3} = \sum_{j=3}^{k-1} \sigma_{k-j}(\lambda_1, \dots, \lambda_{k-1}) O(\sum_{m \geq k; i \geq 1} |r_{mi}|^3) + O(\sum_{m \geq k; i \geq 1} |r_{mi}|^3), \end{cases}$$

where  $r_{mi} = r_{im} = O(1)\varepsilon^4$  for  $m \geq k; i \geq 1$ .

*Proof.* Since  $S_k(\mathbf{r})$  is invariant under orthogonal transform, then  $S_k(\mathbf{r}) = S_k(Q\mathbf{r}'Q)$ . By virtue of (3.11), we can separate  $S_k(Q\mathbf{r}'Q)$ , which is the sum of all principal minors of order  $k$  of the Hessian  $(Q\mathbf{r}'Q)$ , into three parts:  $\textcircled{1}$  are the minors containing  $\sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}) = \prod_{i=1}^{k-1} \lambda_i$ ;  $\textcircled{2}$  are the ones containing all of the elementary symmetric polynomials of  $(k-2)$ -order in  $\lambda_1, \dots, \lambda_{k-1}$ ;  $\textcircled{3}$  are the ones containing all of the elementary symmetric polynomials of order  $\leq k-3$  in  $\lambda_1, \dots, \lambda_{k-1}$ .  $\square$

**Proposition 3.5.** *We have for any  $w \in C^2(\Omega)$ , the function  $u$  defined by (3.8) satisfy*

$$(3.12) \quad S_k[u](y) = \tilde{K} + O(1)\varepsilon^{\frac{9}{2}},$$

with  $O(1)$  depends to  $\|w\|_{C^2}$ , where  $\tilde{K}$  is defined in (2.3).

*Proof.* By (3.10) and (3.11), we have

$$\sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}) = \sigma_{k-1}(\tau) + O(1)\varepsilon^4.$$

Noticing that  $r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)$  for  $i \geq k$  or  $j \geq k$ , we obtain  $S_k(\mathbf{r}) = \textcircled{1} + \textcircled{2} + \textcircled{3}$  with

$$\begin{cases} \textcircled{1} = [\sigma_{k-1}(\tau) + \varepsilon^4 O(1)] \sum_{l=k}^n r_{ll}(\varepsilon^2 x) - \sum_{l=k}^n \sum_{j=1}^{k-1} \sigma_{k-1,j}(\lambda_1, \dots, \lambda_{k-1}) r_{jl}^2(\varepsilon^2 x) \\ \quad = \sigma_{k-1}(\tau) \sum_{l=k}^n r_{ll}(\varepsilon^2 x) + \varepsilon^8 O(1) = \sigma_{k-1}(\tau) \sum_{l=k}^n P_{ll}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} O(1) \\ \textcircled{2} = \sigma_{k-2}(\lambda_1, \dots, \lambda_{k-1}) \sum_{k \leq l_1, l_2 \leq n} (r_{l_1 l_1}(\varepsilon^2 x) r_{l_2 l_2}(\varepsilon^2 x) - r_{l_1 l_2}^2(\varepsilon^2 x)) \\ \quad + O(r_{il}^3(\varepsilon^2 x)) = \varepsilon^8 O(1) \\ \textcircled{3} = O(|r_{ij}^3(\varepsilon^2 x)|) = \varepsilon^{12} O(1). \end{cases}$$

Therefore,

$$S_k(\mathbf{r}) = \sigma_{k-1}(\tau) \sum_{j=k}^n P_{jj}(\varepsilon^2 x) + O(1)\varepsilon^{\frac{9}{2}},$$

from which, using (3.4) and recalling  $y = \varepsilon^2 x$ , we obtain (3.12).  $\square$

## 4. LINEARIZED DEGENERATE ELLIPTIC OPERATORS

By the construction of Section 3 and Proposition 3.5, we have

$$(4.1) \quad G(w) = \frac{1}{\varepsilon^{\frac{9}{2}}} \{S_k(u) - \tilde{K}\} = \frac{1}{\varepsilon^{\frac{9}{2}}} \{O(1)\varepsilon^{\frac{9}{2}}\} = O(1),$$

So that for any  $w \in C^2(\bar{\Omega})$ ,

$$(4.2) \quad \theta(w) = \sup_{x \in \Omega} |G(w)(x)| < +\infty$$

uniformly with respect to  $\varepsilon$ , and then (4.1) is well-defined for  $0 < \varepsilon \leq \varepsilon_0 \ll 1$ .

The linearized operator of  $G$  at  $w$  is

$$L_G(w) = \sum_{i,j=1}^n S_k^{ij}(w) \partial_i \partial_j,$$

where

$$(4.3) \quad S_k^{ij}(w) = \frac{\partial S_k}{\partial r_{ij}}(w) = \frac{\partial S_k(\mathbf{r})}{\partial r_{ij}}(w).$$

Since the matrix  $(S_k^{ij}(w))$  and  $\mathbf{r}$  is simultaneously diagonalizable, see [20], that is, for any smooth function  $w$ , we can find out an orthogonal matrix  $T(x, \varepsilon)$  satisfying

$$(4.4) \quad \begin{cases} T(x, \varepsilon)(S_k^{ij}(w))^t T(x, \varepsilon) = \text{diag} \left[ \frac{\partial \sigma_k(\lambda(x, \varepsilon))}{\partial \lambda_1}, \frac{\partial \sigma_k(\lambda(x, \varepsilon))}{\partial \lambda_2}, \dots, \frac{\partial \sigma_k(\lambda(x, \varepsilon))}{\partial \lambda_n} \right] \\ T(x, \varepsilon) \mathbf{r}^t T(x, \varepsilon) = \text{diag} [\lambda_1(x, \varepsilon), \lambda_2(x, \varepsilon), \dots, \lambda_n(x, \varepsilon)], \end{cases}$$

where  $T_i(x, \varepsilon)$  is the corresponding unit eigenvectors of  $\lambda_i, i = 1, 2, \dots, n$ .

The linearized operator  $L_G(w)$  is not guaranteed to be degenerately elliptic, because  $(\lambda_1(x, \varepsilon), \lambda_2(x, \varepsilon), \dots, \lambda_n(x, \varepsilon))$ , as a result of the perturbation by  $\varepsilon^{\frac{9}{2}} w_{ij}(x)$ , may be not in  $\bar{\Gamma}_k$ . We have

**Proposition 4.1.** *Assume that  $\|w\|_{C^2(\Omega)} \leq 1$ , then the second order differential operators*

$$L_G(w) + \theta \Delta$$

*is a degenerate elliptic operator if  $\varepsilon > 0$  is sufficiently small, where  $\theta$  is defined in (4.2).*

*Proof.* By the definition of degenerate elliptic operator, we have to prove

$$A = \theta |\xi|^2 + \sum_{i,j=1}^n S_k^{ij} \xi_i \xi_j \geq 0, \quad \text{for any } \xi \in \mathbb{R}^n$$

which is equivalent to prove,

$$A = \theta |\tilde{\xi}|^2 + ((S_k^{ij})^t T \tilde{\xi}, T \tilde{\xi}) = \theta |\tilde{\xi}|^2 + (T(S_k^{ij})^t T \tilde{\xi}, \tilde{\xi}) \geq 0, \quad \text{for any } \tilde{\xi} \in \mathbb{R}^n,$$

where  $T$  is the orthogonal matrix in (4.4), then

$$(4.5) \quad A = \theta |\tilde{\xi}|^2 + \sum_{i=1}^{k-1} \sigma_{k-1,i}(\lambda(x, \varepsilon)) \tilde{\xi}_i^2 + \sum_{i=k}^n \sigma_{k-1,i}(\lambda(x, \varepsilon)) \tilde{\xi}_i^2$$

Since for small  $\varepsilon$ , we have  $\sigma_{k-1,i}(\lambda(x, \varepsilon)) = \prod_{j=1}^{k-1} \tau_j + O(\varepsilon) > 0, k \leq i \leq n$ , we only need to prove

$$(4.6) \quad \theta + \sigma_{k-1,i}(\lambda(x, \varepsilon)) \geq 0, \quad 1 \leq i \leq k-1.$$

If  $\sigma_k(\lambda(x, \varepsilon)) \geq 0$ , together with the fact  $\sigma_j(\lambda(x, \varepsilon)) = \sigma_j(\tau_1, \tau_2, \dots, \tau_{k-1}) + O(\varepsilon) > 0$  for  $1 \leq j \leq k-1$ , then  $\lambda(x, \varepsilon) \in \bar{\Gamma}_k$  which yields

$$\sigma_{k-1,i}(\lambda(x, \varepsilon)) \geq 0, \quad 1 \leq i \leq n.$$

It is left to consider the case  $\sigma_k(\lambda(x, \varepsilon)) < 0$ , in which case, since  $\tilde{K} \geq 0$ ,

$$\begin{aligned} \theta &= \theta(w) = \max_{x \in \Omega} |G(w)| \\ &= \max_{x \in \Omega} \frac{1}{\varepsilon} |\sigma_k(\lambda(x, \varepsilon)) - \tilde{K}| \geq -\frac{1}{\varepsilon} \sigma_k(\lambda(x, \varepsilon)). \end{aligned}$$

Now we prove (4.6) for  $i = 1$  with  $\sigma_{k-1,1}(\lambda) < 0$ , the other cases can be proved similarly. By the definition of  $G(w)$  and  $\sigma_k(\lambda) = \lambda_1 \sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda)$ ,

$$(4.7) \quad \begin{aligned} \theta + 2\sigma_{k-1,1}(\lambda) &= \theta + 2 \frac{\sigma_k(\lambda) - \sigma_{k,1}(\lambda)}{\lambda_1} \\ &\geq -\frac{1}{\varepsilon} \sigma_k(\lambda) + 2 \frac{\sigma_k(\lambda) - \sigma_{k,1}(\lambda)}{\lambda_1} = \left(-\frac{1}{\varepsilon} + \frac{2}{\lambda_1}\right) \sigma_k(\lambda) - \frac{2}{\lambda_1} \sigma_{k,1}(\lambda). \end{aligned}$$

Under the assumption  $\sigma_{k-1,1}(\lambda) < 0$  and  $\sigma_k(\lambda) < 0$ , we will distinguish two cases.

**Case 1.** If  $\sigma_{k,1}(\lambda) \leq 0$ , we have by (4.7)

$$\theta + \sigma_{k-1,1} > \theta + 2\sigma_{k-1,1} \geq \left(-\frac{1}{\varepsilon} + \frac{2}{\lambda_1}\right) \sigma_k(\lambda) - \frac{2}{\lambda_1} \sigma_{k,1}(\lambda) \geq \left(-\frac{1}{\varepsilon} + \frac{2}{\lambda_1}\right) \sigma_k(\lambda) > 0$$

provided  $\varepsilon$  is small enough.

**Case 2.** Next it is left to consider the case in which

$$\sigma_k(\lambda) < 0, \quad \sigma_{k,1}(\lambda) > 0, \quad \sigma_{k-1,1}(\lambda) < 0$$

hold simultaneously. Using Newton's inequalities for  $(n-1)$ -tuple vectors

$$\sigma_{k,1}(\lambda) \sigma_{k-2,1}(\lambda) \leq \frac{(k-1)(n-k)}{k(n-k+1)} [\sigma_{k-1,1}(\lambda)]^2, \quad \lambda \in \mathbb{R}^n$$

and the fact

$$\sigma_{k-2,1}(\lambda) = \prod_{i=2}^{k-1} \tau_i + O(\varepsilon) > 0, \quad \sigma_{k-1,1}(\lambda) = O(\varepsilon),$$

we obtain

$$0 < \sigma_{k,1} \leq \frac{(k-1)(n-k)}{k(n-k+1)} \frac{[\sigma_{k-1,1}(\lambda)]^2}{\sigma_{k-2,1}(\lambda)} \leq |O(\varepsilon) \sigma_{k-1,1}(\lambda)|.$$

Back to (4.7), using  $\sigma_k(\lambda) < 0$ , then for  $\varepsilon > 0$  small, we have

$$\theta + 2\sigma_{k-1,1} \geq \left(-\frac{1}{\varepsilon} + \frac{2}{\lambda_1}\right) \sigma_k(\lambda) - \frac{2}{\lambda_1} \sigma_{k,1}(\lambda) \geq -\frac{2}{\lambda_1} \sigma_{k,1}(\lambda) = -|O(\varepsilon) \sigma_{k-1,1}(\lambda)|,$$

which yields

$$\theta + \sigma_{k-1,1} \geq \theta + 2\sigma_{k-1,1} + |O(\varepsilon) \sigma_{k-1,1}(\lambda)| \geq 0$$

provided  $\varepsilon > 0$  small enough. Proof is done.  $\square$

Equality (4.5) shows that the operator  $L_G(w) + \theta \Delta$  may be degenerate elliptic in the directions of  $(\tilde{\xi}_1, \dots, \tilde{\xi}_{k-1})$  and is uniformly elliptic in the directions of  $(\tilde{\xi}_k, \dots, \tilde{\xi}_n)$  after the orthogonal transform  $T(x, \varepsilon)$  which is a perturbation of unit matrix, so we can impose the Dirichlet boundary condition on  $\partial Q_{\delta_0}$  (the  $x''$  direction), but we can't do that on  $\partial Q_\pi$  (the  $x'$  direction). Instead of treating a Dirichlet boundary value problem, we shall prove

the existence, uniqueness and à priori estimates of the solution, which is periodic with respect to  $x'$ , to the degenerately linearized elliptic equation

$$(4.8) \quad \begin{cases} L_G(w)\rho + \theta\Delta\rho = g, & \text{in } \Omega; \\ \rho(x', x'') \text{ is periodic for } x' \in Q_\pi \text{ and } \rho(x', x'') = 0 & \text{for } x'' \in \partial Q_{\delta_0}, \end{cases}$$

in some suitable Hilbert space defined below, this idea is inspired by Hong and Zuily [10] where they consider the case case  $k = n$ . We introduce the space  $\mathbf{H}^s(\Omega)$  ( $s$  is an integer), which is the completion of the space of trigonometrical polynomials

$$\rho(x) = \sum_{\ell=(l_1, l_2, \dots, l_{k-1}) \in \mathbb{Z}^{k-1}} \alpha_\ell(x'') e^{\sqrt{-1} \sum_{j=1}^{k-1} l_j x_j}$$

with the (complex-valued) coefficients  $\alpha_\ell$  subject to the condition  $\overline{\alpha_\ell(x'')} = \alpha_{-\ell}(x'') \in C^\infty(Q_{\delta_0})$ , with respect to the norm

$$\|\rho\|_s^2 = \sum_{t+j \leq s} \sum_{\ell \in \mathbb{Z}^{k-1}} (1 + \sum_{i=1}^{k-1} l_i^2)^t \|\alpha_\ell\|_{H^j(Q_{\delta_0})}^2,$$

where  $H^j(Q_{\delta_0})$  is the usual Sobolev space. We define  $\mathbf{H}_0^s(\Omega)$  in the same way by taking  $\alpha_\ell \in C_0^\infty(Q_{\delta_0})$ . We will prove, in the next section, the following Theorem.

**Theorem 4.2.** *Let  $w$  be smooth and  $\|w\|_{C^{l\frac{3}{2}+3+l_0}(\Omega)} \leq 1$  with nonnegative integer  $l_0$ . Then for any  $s_0 \in \mathbb{N} \cup \{0\}$ , one can find a constant  $\varepsilon(s_0)$  such that the equation (4.8) possesses a solution  $\rho \in \mathbf{H}_0^s(\Omega)$  provided that  $g \in H^s(\Omega)$ ,  $0 \leq s \leq s_0$  and  $0 < \varepsilon \leq \varepsilon(s_0)$ . If  $s \geq 1$ , the solution is unique. Moreover,*

$$(4.9) \quad \begin{cases} \|\rho\|_s \leq C_s (\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2} \|\rho\|_{L^\infty}), & \text{if } s > \left[\frac{n}{2}\right] + 1 + l_0; \\ \|\rho\|_s \leq C_s \|g\|_s, & \text{if } s \leq \left[\frac{n}{2}\right] + 1 + l_0 \end{cases}$$

holds for some constants  $C_s$  independent of  $w$  and  $\varepsilon$ . Here

$$W_{ij}(x) = P_{y_i y_j}(\varepsilon^2 x) + \varepsilon^{\frac{3}{2}} w_{y_i y_j}(x).$$

**Remark 4.3.** Since the equation (4.8) is degenerate elliptic, we can only get the a priori estimate (4.9) with a loss of order 2. This loss of regularity of linearized equation ask us to use the Nash-Moser-Hörmander iteration to deal with the solution of (4.8). By definition of  $r_{ij}$  by (3.9), if  $P(y) = 0$ , then  $\sum_{i,j=1}^n \|(W_{ij})\|_{s+2}$  is reduced to  $\|(w)\|_{s+4}$  which is introduced in Theorem 1.3 of [10]. When  $l_0 = 0$ , Theorem 4.2 ( $2 \leq k \leq n$ ) is a generalization of Theorem 1.3 ( $k = n$ ) in [10]. The assumption  $\|w\|_{C^{l\frac{3}{2}+4}(\Omega)} \leq 1$  ( $l_0 = 1$ ) is necessary in the estimates of the quadratic error in Lemma 6.2 for  $f = f(y, u, Du)$  in Lemma 6.2, although we will not give its estimates of the quadratic error ; while the assumption  $\|w\|_{C^{l\frac{3}{2}+3}(\Omega)} \leq 1$  ( $l_0 = 0$ ) is enough in case  $f = f(y, u)$ . The uniqueness for  $s \geq 1$  follows from (5.4) by taking  $\nu = 0$ .

## 5. À PRIORI ESTIMATES OF SOLUTIONS FOR LINEARIZED EQUATIONS

First of all, using the change of unknown function  $\bar{\rho} = \rho e^{\mu \sum_{j=k}^n x_j^2}$ , we reduce (4.8) to

$$(5.1) \quad \begin{cases} L(w)\bar{\rho} = \sum_{i,j=1}^n (S_k^{ij}(w) + \delta_i^j \theta) \partial_i \partial_j \bar{\rho} + \sum_i^n b_i \partial_i \bar{\rho} + c\bar{\rho} = e^{\mu \sum_{j=k}^n x_j^2} g, & \text{in } \Omega \\ \bar{\rho} = 0 & \text{on } \partial Q_{\delta_0} \text{ and periodic on } Q_\pi. \end{cases}$$

The coefficients  $b_i$  and  $c$  are expressed as follows.

$$b_i = \begin{cases} -4 \sum_{j=k}^n (\mu x_j) S_k^{ij}(w), & 1 \leq i \leq k-1 \\ -4(\mu x_i)\theta - 4 \sum_{j=k}^n (\mu x_j) S_k^{ij}(w), & k \leq i \leq n. \end{cases}$$

$$c = -2\mu \sum_{i=k}^n (S_k^{ii}(w) + \theta) + 4 \sum_{i=k}^n \sum_{j=k}^n (\mu x_i)(\mu x_j) S_k^{ij}(w) + 4\theta \sum_{i=k}^n (\mu x_i)^2.$$

We would like to prove Theorem 4.2 for (5.1) rather than (4.8), and write  $\rho$  instead of  $\bar{\rho}$ , but also not do it directly, we will consider the solution  $\rho_\nu$  to the regularized version of (5.1), i.e., the following uniformly elliptic equation, for  $0 < \nu < 1$ ,

$$(5.2) \quad \begin{cases} L_\nu \rho \equiv \sum_{i,j=1}^n (S_k^{ij} + \delta_i^j \theta) \partial_i \partial_j \rho + \nu \Delta \rho + \sum_i b_i \partial_i \rho + c \rho = g, & \text{in } \Omega, \\ \rho = 0 & \text{on } \partial Q_{\delta_0} \text{ and periodic on } Q_\pi. \end{cases}$$

We first need the following Lemmas which is standard for the degenerate elliptic operators. So we only point out some important steps for the proof.

**Lemma 5.1.** *Suppose that  $w$  is smooth and  $\|w\|_{C^4(\Omega)} \leq 1$ . Then there exists two positive constants  $\mu_0$  large and  $\varepsilon_0$  small such that, for  $0 < \varepsilon \leq \delta_0$ ,  $\mu_0 \delta_0 \leq 1$ ,  $g \in H^0(\Omega)$  and  $\nu > 0$ , problem (5.2) admits a unique solution  $\rho_\nu \in \mathbf{H}_0^1(\Omega)$ , which satisfies*

$$(5.3) \quad \|\rho_\nu\|_0 \leq C_0 \|g\|_0,$$

where  $C_0$  is uniform for  $\nu \in ]0, 1[$ ,  $\varepsilon \in ]0, \delta_0]$  and independent of  $w$ .

*Proof.* We prove the existence and uniqueness of the solution  $\rho_\nu$  to (5.2) by applying Lax-Milgram Theorem to the bilinear form

$$\langle -L_\nu \rho, \varrho \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the dual pair on  $H^{-1} \times H_0^1$ . The condition  $\|w\|_{C^4} \leq 1$  yields

$$|\langle -L_\nu \rho, \varrho \rangle| \leq C \|\rho\|_1 \|\varrho\|_1, \quad \forall \rho, \varrho \in H_0^1,$$

where  $C$  is uniform on  $0 < \varepsilon < 1$ ,  $0 < \nu < 1$ . For the coercivity, the proof of which is almost the same as that of Lemma 1.4, [10], there exist  $\varepsilon_0 > 0$  small and large  $\mu > 0$  such that

$$(5.4) \quad -\langle L_\nu \rho, \rho \rangle \geq \nu \|D\rho\|_0^2 + 2\sigma_{k-1}(\tau) \|\rho\|_0^2,$$

then by using Lax-Milgram Theorem, for  $g \in H^0(\Omega)$  and  $\nu > 0$ , problem (5.2) admits a unique solution  $\rho_\nu \in \mathbf{H}_0^1(\Omega)$ . Since  $|\langle -L_\nu \rho_\nu, \rho_\nu \rangle| = |\langle g, \rho_\nu \rangle| \leq \|g\|_0 \|\rho_\nu\|_0$ , then (5.3) follows from (5.4).  $\square$

Similarly o the proof of Theorem 1.3, [10], we have the higher order à priori estimate

**Lemma 5.2.** *Suppose that  $w$  is smooth and  $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \leq 1$ . For  $s > 0$ , then there exists  $\varepsilon_0(s) > 0$  small such that, for  $0 < \varepsilon \leq \varepsilon_0(s)$ , and  $g \in H^s(\Omega)$ , the problem (5.2) admits a unique solution  $\rho_\nu \in \mathbf{H}_0^{s+1}(\Omega)$ , which satisfies*

$$(5.5) \quad \begin{cases} \|\rho_\nu\|_s \leq C_s (\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2} \|\rho_\nu\|_{L^\infty}), & \text{if } s > \left[\frac{n}{2}\right] + 1; \\ \|\rho_\nu\|_s \leq C_s \|g\|_s, & \text{if } s \leq \left[\frac{n}{2}\right] + 1 \end{cases}$$

where  $C_s$  is uniform for  $\nu \in ]0, 1[$ ,  $\varepsilon \in ]0, \varepsilon_0(s)]$  and independent of  $w$ .

*Proof.* For  $s = 0$ , Since  $-(L_\nu \rho, \rho) = -(g, \rho)$ , then (5.4) and Cauchy inequality yield

$$\nu \|D\rho\|_0^2 + \|\rho\|_0^2 \leq C_0(\tau) \|g\|_0^2,$$

where  $C_0(\tau)$  is independent of  $\nu, w$  and  $\varepsilon$ . On the other hand, for  $\alpha \in \mathbb{N}^n, |\alpha| \leq s$

$$(L_\nu(\partial^\alpha \rho), (\partial^\alpha \rho)) = (\partial^\alpha g, (\partial^\alpha \rho)) + ([L_\nu, \partial^\alpha] \rho, (\partial^\alpha \rho)),$$

where the commutators is

$$[L_\nu, \partial^\alpha] = - \sum_{\beta \leq \alpha, |\beta| \geq 1} \sum_{i, j=1}^n C_{\alpha\beta} (\partial^\beta (S_k^{ij} \partial_i \partial_j + \sum_{i=1}^n \partial^\beta b_i \partial_i + \partial^\beta c)) \partial^{\alpha-\beta},$$

here the coefficients depends on  $D^2 w$ , by using the interpolation inequalities, we can get:

1) If  $s \leq [\frac{n}{2}] + 1$ , then the condition  $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \leq 1$  imply

$$|([L_\nu, \partial^\alpha] \rho, (\partial^\alpha \rho))| \leq \varepsilon C \|\rho\|_s^2.$$

2) If  $s > [\frac{n}{2}] + 1$ , a little involved computation also give

$$|([L_\nu, \partial^\alpha] \rho, (\partial^\alpha \rho))| \leq C_s (\varepsilon \|\rho\|_s^2 + \|\rho\|_s (\|g\|_s + \|\rho\|_{L^\infty} \sum_{i, j=1}^n \|W_{ij}\|_{s+2})),$$

where  $C_s$  depends only on  $s$ , so we finish the proof.  $\square$

With Lemmas 5.1 and 5.2, we can now prove Theorem 4.2.

*Proof. The proof of Theorem 4.2.* To simplify the computation, we consider only the case  $l_0 = 0$ , and prove (4.9) under the assumption  $\|w\|_{C^{[\frac{n}{2}]+3}(\Omega)} \leq 1$ . Now for  $0 \leq s \leq [\frac{n}{2}] + 1$ , we can apply Banach-Saks Theorem to find a subsequence  $\rho_{\nu_j}$  with  $\nu_j = j^{-2}$  and an element  $\rho_0 \in \mathbf{H}_0^s$  such that

$$\|\frac{\rho_{\nu_1} + \dots + \rho_{\nu_m}}{m} - \rho_0\|_s \rightarrow 0 \quad (m \rightarrow \infty).$$

Since  $\frac{\rho_{\nu_1} + \dots + \rho_{\nu_m}}{m}$  is periodic in  $x'$  for each  $m$ , so is  $\rho_0$ . Back to (5.2), we have

$$\left[ \sum_{i, j=1}^n (S_k^{ij} + \delta_i^j \theta) \partial_i \partial_j + \sum_i b_i \partial_i + c \right] \frac{\rho_{\nu_1} + \dots + \rho_{\nu_m}}{m} + \frac{1}{m} \sum_{j=1}^m \nu_j \Delta \rho_{\nu_j} = g.$$

For any test function  $\phi \in C_0^\infty$ , Lemma 5.1 yields

$$|(\frac{1}{m} \sum_{j=1}^m \nu_j \Delta \rho_{\nu_j}, \phi)_{L^2}| \leq \|\Delta \phi\|_0 \|\frac{1}{m} \sum_{j=1}^m \nu_j \rho_{\nu_j}\|_0 \leq C_0 \|\Delta \phi\|_0 \|g\|_0 \frac{1}{m} \sum_{j=1}^m \nu_j \rightarrow 0,$$

taking  $m \rightarrow \infty$ , we have that  $\rho_0$  is a solution of (4.8) in the sense of distribution :

$$\sum_{i, j=1}^n (S_k^{ij} + \delta_i^j \theta) \partial_i \partial_j \rho_0 + \sum_i b_i \partial_i \rho_0 + c \rho_0 = g.$$

Moreover, by Lemma 5.2

$$\|\rho_0\|_s = \lim_{m \rightarrow \infty} \|\frac{\rho_{\nu_1} + \dots + \rho_{\nu_m}}{m}\|_s \leq C_s \|g\|_s, \quad \text{for } 0 \leq s \leq [\frac{n}{2}] + 1.$$

Now we prove (4.9) for  $s > [\frac{n}{2}] + 1$ . Since Lemma 5.2 yields

$$\|\rho_\nu\|_{[\frac{n}{2}]+1} \leq C_{[\frac{n}{2}]+1} \|g\|_{[\frac{n}{2}]+1},$$

by Sobolev imbedding theorem,  $\|\rho_v\|_{L^\infty} < \infty$ , moreover if  $\sum_{i,j=1}^n \|W_{ij}\|_{s+2} < \infty$ , then (5.5) shows that

$$\|\rho_v\|_s \leq C < \infty \text{ for } s > \left[\frac{n}{2}\right] + 1,$$

then, by weak compactness theorem, there is a subsequence  $\rho_{v_j}$  of  $\rho_v$  and  $\rho_0$  such that  $\rho_{v_j} \rightarrow \rho_0$  in the weak topology of  $W^{s,2}(\Omega)$  and

$$\|\rho_{v_j} - \rho_0\|_{L^\infty(\Omega)} \rightarrow 0, \|\rho_{v_j} - \rho_0\|_{W^{1,2}(\Omega)} \rightarrow 0,$$

therefore  $\rho_0$  is periodic in  $x'$ ,  $\rho_0 \in H_0^s(\Omega)$ , and by Lemma 5.2 again

$$\|\rho_0\|_s \leq \liminf_{j \rightarrow \infty} \|\rho_{v_j}\|_s \leq C_s(\|g\|_s + \sum_{i,j=1}^n \|W_{ij}\|_{s+2} \|\rho_0\|_{L^\infty}),$$

so we complete the proof of (4.9).  $\square$

## 6. NASH-MOSER-HÖRMANDER ITERATION

We prove firstly the existence of  $k$ -convex local solution of (2.2) as a perturbation of  $\psi$  constructed in Section 3 by employing the Nash-Moser-Hörmander iteration, which is based on the à priori estimates established in last section. Since the linearized operators is degenerate elliptic, a loss of regularity of order 2 has occurred for the solution of linearized equation, so we need to **mollify** the solution by taking a family of smoothing operators  $S(t)$ ,  $t \geq 1$  such that

$$S(t) : \cup_{s \geq 0} \mathbf{H}_0^s(\Omega) \longrightarrow \cap_{s \geq 0} \mathbf{H}^s(\Omega)$$

with the following properties:

$$(6.1) \quad \|S(t)u\|_{s_1} \leq C_{s_1, s_2} \|u\|_{s_2}, \quad \text{if } s_1 \leq s_2,$$

$$(6.2) \quad \|S(t)u\|_{s_1} \leq C_{s_1, s_2} t^{s_1 - s_2} \|u\|_{s_2}, \quad \text{if } s_1 \geq s_2,$$

$$(6.3) \quad \|S(t)u - u\|_{s_1} \leq C_{s_1, s_2} t^{s_1 - s_2} \|u\|_{s_2}, \quad \text{if } s_1 \leq s_2,$$

where  $C_{s_1, s_2}$  is independent of  $t$  and depends only on  $s_1, s_2$ , see [1] for more detailed properties of smoothing operators.

Now we define  $S_m = S(\mu_m)$  with  $\mu_m = \sigma^{\gamma m}$  where  $\sigma > 1, \gamma > 1$  to be determined later, we use the following iteration procedure:

$$(6.4) \quad \begin{cases} w_0 = 0, w_{m+1} = w_m + S_m \rho_m, \\ L(w_m) \rho_m = L_G(w_m) \rho_m + \theta_m \Delta \rho_m = g_m, & x \in \Omega \\ \rho_m \in \mathbf{H}_0^s(\Omega), g_m \in H^s(\Omega). \end{cases}$$

where  $g_m$  and  $\theta_m$  are defined in (2.9),  $G(w)$  is defined in (4.1).

Remember we are studying the equation (4.3) in the variables  $x$  rather than  $y = \varepsilon^2 x$ , and

$$u_m(y) = \frac{1}{2} \sum_{j=1}^{k-1} \tau_j y_j^2 + P(y) + \varepsilon^{\frac{17}{2}} w_m(\varepsilon^{-2} y).$$



Denoting

$$\begin{aligned}\mathcal{M}(s) &= \|g_0\|_s + \varepsilon^4 \sum_{i,j=1}^n \|P_{ij}(\varepsilon^2 \cdot)\|_{s-1}, \\ \mathcal{N}(s) &= \mathcal{M}(s) + \sum_{i,j=1}^n \|P_{ij}(\varepsilon^2 \cdot)\|_{s+2} \left[1 + \mathcal{M}\left(\left[\frac{n}{2}\right] + 1\right)\right]\end{aligned}$$

which is small if  $\varepsilon > 0$  small. We prove the à priori estimate by induction.

**Lemma 6.1.** *Suppose that  $\|w_l\|_{C^{l\frac{n}{2}+3}} \leq 1$  for  $0 \leq l \leq m$ . Then*

$$(6.5) \quad \|g_m\|_s \leq C_s(\mathcal{M}(s) + \|w_m\|_{s+2})$$

and

$$(6.6) \quad \|w_{m+1}\|_{s+4} \leq C_s^{m+1} \mu_{m+1}^\beta \mathcal{N}(s), \quad \text{for } \beta = \frac{4}{\gamma-1},$$

where  $C_s$  is independents on  $m$  and  $\gamma$ .

*Proof.* We prove first (6.5), noting

$$\mathbf{r}_m = \left( \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} (w_m)_{ij}(x) \right).$$

Using Taylor expansion and  $w_0 = 0$ , we have

$$-g_m = G(w_m) = G(w_0) + \int_0^1 \frac{\partial}{\partial t} [G(tw_m)] dt = -g_0 + \int_0^1 \sum_{i,j=1}^n S_k^{ij}(\mathbf{r}_m(t)) \frac{\partial^2 w_m}{\partial x_i \partial x_j} dt.$$

Hence

$$\|g_m\|_s \leq \|g_0\|_s + \sum_{i,j=1}^n \sum_{|\alpha|+|\beta|\leq s} C_\alpha^\beta \|\partial^\alpha S_k^{ij}(\mathbf{r}_m) \partial^\beta \left( \frac{\partial^2 w_m}{\partial x_i \partial x_j} \right)\|_0.$$

By using  $\|w_m\|_{C^{l\frac{n}{2}+3}} \leq 1$ , we have

$$\sum_{|\alpha|+|\beta|\leq s, |\alpha|\leq [\frac{n}{2}]+1} \|\partial^\alpha S_k^{ij}(\mathbf{r}_m) \partial^\beta \left( \frac{\partial^2 w_m}{\partial x_i \partial x_j} \right)\|_0 \leq C \sum_{|\beta|\leq s} \|\partial^\beta \left( \frac{\partial^2 w_m}{\partial x_i \partial x_j} \right)\|_0 \leq C \|w_m\|_{s+2}.$$

On the other hand, by interpolation and  $\|w_m\|_{C^{[\frac{n}{2}]+3}} \leq 1$ ,

$$\begin{aligned}
& \sum_{|\alpha|+|\beta|\leq s, |\alpha|>[\frac{n}{2}]+1} \|\partial^\alpha S_k^{ij}(\mathbf{r}_m) \partial^\beta (\frac{\partial^2 w_m}{\partial x_i \partial x_j})\|_0 \\
&= \sum_{|\alpha|+|\beta|\leq s, |\alpha|>[\frac{n}{2}]+1} \|\partial^{\alpha-[\frac{n}{2}]-1} \partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m) \partial^\beta (\frac{\partial^2 w_m}{\partial x_i \partial x_j})\|_0 \\
&\leq C \sum_{|\alpha|+|\beta|\leq s, |\alpha|>[\frac{n}{2}]+1} \left[ \|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{L^\infty} \|\partial_i \partial_j w_m\|_{|\alpha|+|\beta|-\lceil\frac{n}{2}\rceil-1} \right. \\
&\quad \left. + \|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{|\alpha|+|\beta|-\lceil\frac{n}{2}\rceil-1} \|\partial_i \partial_j w_m\|_{L^\infty} \right] \\
&\leq C \sum_{|\alpha|+|\beta|\leq s, |\alpha|>[\frac{n}{2}]+1} \left[ \|w_m\|_{|\alpha|+|\beta|-\lceil\frac{n}{2}\rceil+1} + \|\partial^{[\frac{n}{2}]+1} S_k^{ij}(\mathbf{r}_m)\|_{|\alpha|+|\beta|-\lceil\frac{n}{2}\rceil-1} \right] \\
&\leq C \|w_m\|_{s-\lceil\frac{n}{2}\rceil+1} + C \sum_{i,j=1}^n \|(W_m)_{ij}\|_s,
\end{aligned}$$

where

$$(W_m)_{ij}(x) = P_{y_i y_j}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} (w_m)_{y_i y_j}(x).$$

Therefore

$$\begin{aligned}
& \|S_k^{ij}(\mathbf{r}_m) \frac{\partial^2 w_m}{\partial x_i \partial x_j}\|_s \\
&\leq C \|w_m\|_{s+2} + C \|w_m\|_{s-\lceil\frac{n}{2}\rceil+1} + C \sum_{i,j=1}^n \|(W_m)_{ij}\|_s \\
&\leq C \|w_m\|_{s+2} + C \sum_{i,j=1}^n \|(W_m)_{ij}\|_s.
\end{aligned}$$

Thus

$$\begin{aligned}
& \|g_m\|_s = \|G(w_m)\|_s \\
&\leq \|G(w_0)\|_s + \int_0^1 \left\| \frac{\partial}{\partial t} [G(tw_m)] \right\|_s dt \leq \|g_0\|_s + C \|w_m\|_{s+2} + C \|\partial^3 u_m\|_{s-2}.
\end{aligned}$$

Remembering

$$\partial_x \partial_{x_i} \partial_{x_j} [u_m(y) |_{y=\varepsilon^2 x}] = \varepsilon^4 \partial_x (W_m)_{ij}(x) = \varepsilon^4 \partial_x [P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} (w_m)_{ij}(x)],$$

we have

$$\|g_m\|_s \leq \|g_0\|_s + C \varepsilon^4 \sum_{i,j=1}^n \|P_{ij}(\varepsilon^2 \cdot)\|_{s-1} + C \|w_m\|_{s+2} \leq C_s (\mathcal{M}(s) + \|w_m\|_{s+2}),$$

this complete the proof of (6.5).

We prove now (6.6). By using (6.2) and (6.4), we have

$$\|w_{m+1}\|_{s+4} \leq \|w_m\|_{s+4} + \|S_m \rho_m\|_{s+4} \leq \|w_m\|_{s+4} + C_s \mu_m^4 \|\rho_m\|_s.$$

The à priori estimate (4.9) and Sobolev imbedding yields

$$\begin{aligned}
\|\rho_m\|_s &\leq C(\|g_m\|_s + \sum_{i,j=1}^n \|(W_{ij})\|_{s+2} \|\rho_m\|_{L^\infty}) \\
&\leq C(\|g_m\|_s + \sum_{i,j=1}^n \|(W_{ij})\|_{s+2} \|\rho_m\|_{[\frac{n}{2}]+1}) \\
(6.7) \quad &\leq C(\|g_m\|_s + \sum_{i,j=1}^n \|(W_{ij})\|_{s+2} \|g_m\|_{[\frac{n}{2}]+1}) \\
&\leq C(\mathcal{M}(s) + \|w_m\|_{s+2} + C(\sum_{i,j=1}^n \|P_{ij}(\varepsilon^2 \cdot)\|_{s+2} + \|w_m\|_{s+4}) \\
&\quad \times (\mathcal{M}([\frac{n}{2}] + 1) + \|w_m\|_{[\frac{n}{2}]+3})) \\
&\leq C(\mathcal{N}(s) + \|w_m\|_{s+4}).
\end{aligned}$$

Thus we get

$$(6.8) \quad \|w_{m+1}\|_{s+4} \leq \|w_m\|_{s+4} + C_s \mu_m^4 \|\rho_m\|_s \leq C \mu_m^4 (\mathcal{N}(s) + \|w_m\|_{s+4}),$$

then by using  $w_0 = 0$  and  $\mu_m = \sigma^{\gamma^m}$ , the iteration of (6.8) gives

$$\|w_{m+1}\|_{s+4} \leq C_s^m (m+1) \sigma^{4 \frac{\gamma^{m+1}-1}{\gamma-1}} \mathcal{N}(s),$$

if we choose  $C_s > 1$ ,  $C'_s = 2C_s$ , and  $\beta = \frac{4}{\gamma-1}$ , then

$$\|w_{m+1}\|_{s+4} \leq (C'_s)^{m+1} \mu_{m+1}^\beta \mathcal{N}(s).$$

This completes the proof of (6.6).  $\square$

**Lemma 6.2.** *Suppose  $\|w_l\|_{C^l[\frac{n}{2}]+4} \leq 1$  for  $l = 0, 1, \dots, m$ . Then for any given  $s^* > s > 1$ , there exists  $\sigma > 1, \gamma > 1$  and  $a > 0$  such that*

$$(6.9) \quad \|g_{m+1}\|_0 + \|g_{m+1}\|_{L^\infty} \leq \mu_{m+1}^{-a} \mathcal{N}(s^*).$$

*Proof.* By Taylor formula with remainder and (6.4),

$$\begin{aligned}
-g_{m+1} &= G(w_{m+1}) = G(w_m + S_m \rho_m) \\
&= G(w_m) + L_G(w_m) S_m \rho_m + \int_0^1 (1-t) \frac{\partial^2}{\partial t^2} [G(w_m + t S_m \rho_m)] dt \\
&= [G(w_m) + L(w_m) \rho_m] + L(w_m) (S_m - I) \rho_m - \theta_m \Delta S_m \rho_m \\
&\quad + \int_0^1 (1-t) \frac{\partial^2}{\partial t^2} [G(w_m + t S_m \rho_m)] dt \\
&= L(w_m) (S_m - I) \rho_m - \theta_m \Delta S_m \rho_m + \int_0^1 (1-t) \frac{\partial^2}{\partial t^2} [G(w_m + t S_m \rho_m)] dt \\
&= (A) + (B) + \int_0^1 (1-t) (C) dt.
\end{aligned}$$

For the last terms, we have

$$\begin{aligned} (C) &= \frac{\partial^2}{\partial t^2} [G(w_m + tS_m \rho_m)] \\ &= \sum_{i,j,p,l=1}^n \frac{\partial^2 S_k}{\partial r_{ij} \partial r_{pl}} (w_m + tS_m \rho_m) (S_m \rho_m)_{ij} (S_m \rho_m)_{pl} \end{aligned}$$

Noticing  $\frac{\partial^2 S_k}{\partial r_{ij} \partial r_{pl}}(\mathbf{r})$  is a polynomial of order  $k - 2$ , we have, by Sobolev imbedding, à priori estimate (4.9), (6.5) and (6.2),

$$\|S_m \rho_m\|_{[\frac{n}{2}]+3} \leq C\mu_m^2 \|\rho_m\|_{[\frac{n}{2}]+1} \leq C\mu_m^2 \|g_m\|_{[\frac{n}{2}]+1} \leq C\mu_m^2 (\mathcal{M}([\frac{n}{2}] + 1) + \|w_m\|_{[\frac{n}{2}]+3}) \leq C\mu_m^2,$$

and

$$\begin{aligned} &\left\| \frac{\partial^2 S_k}{\partial r_{ij} \partial r_{pl}} (w_m + tS_m \rho_m) \right\|_{L^\infty} \leq \|D^2 w_m + tD^2 S_m \rho_m\|_{L^\infty}^{k-2} \\ &\leq [\|D^2 w_m\|_{L^\infty} + \|D^2 S_m \rho_m\|_{L^\infty}]^{k-2} \leq [1 + \|S_m \rho_m\|_{[\frac{n}{2}]+3}]^{k-2} \\ &\leq (C')^{k-2} \mu_m^{2(k-2)}. \end{aligned}$$

Then (6.2) yields

$$\begin{aligned} \|(C)\|_0 &\leq \sum_{i,j,p,l=1}^n \left\| \frac{\partial^2 S_k}{\partial r_{ij} \partial r_{pl}} (w_m + tS_m \rho_m) \right\|_{L^\infty} \| (S_m \rho_m)_{ij} \|_{L^\infty} \| (S_m \rho_m)_{pl} \|_0 \\ &\leq C^{k-2} \mu_m^{2(k-2)} \|S_m \rho_m\|_{[\frac{n}{2}]+3} \|S_m \rho_m\|_2 \leq C^{k-2} \mu_m^{2(k-2)+[\frac{n}{2}]+5} \|\rho_m\|_0^2 \\ &\leq C^{k-2} \mu_m^{2(k-2)+[\frac{n}{2}]+5} \|g_m\|_0^2. \end{aligned}$$

From (6.5), and à priori estimate (4.9), it follows that if  $\|w_m\|_{[\frac{n}{2}]+3} \leq 1$ ,

$$\begin{aligned} \|w_m + tS_m \rho_m\|_{L^\infty} &\leq \|w_m\|_{L^\infty} + \|S_m \rho_m\|_{L^\infty} \\ &\leq 1 + \|S_m \rho_m\|_{[\frac{n}{2}]+1} \leq 1 + C\|\rho_m\|_{[\frac{n}{2}]+1} \\ &\leq 1 + C\|g_m\|_{[\frac{n}{2}]+1} \leq 1 + C(\mathcal{M}([\frac{n}{2}] + 1) + \|w_m\|_{[\frac{n}{2}]+3}) \leq C'. \end{aligned}$$

and similarly, if  $\|w_m\|_{[\frac{n}{2}]+4} \leq 1$ ,

$$\|\nabla w_m + t\nabla S_m \rho_m\|_{L^\infty} \leq \|\nabla w_m\|_{L^\infty} + \|\nabla S_m \rho_m\|_{L^\infty} \leq C',$$

we have

$$\|(C)\|_0 \leq \mu_m^{2(k-2)+[\frac{n}{2}]+5} \|g_m\|_0^2.$$

By (6.3), (6.7) and (6.6), we have

$$\begin{aligned} \|(A)\|_0 &= \|L(w_m)(S_m - I)\rho_m\|_0 \leq C\|(S_m - I)\rho_m\|_2 \\ &\leq CC_{s^*} \mu_m^{-(s^*-2)} \|\rho_m\|_{s^*} \\ &\leq C'_{s^*} \mu_m^{-(s^*-2)} C_{s^*}^{m+1} \mu_m^\beta N(s^*). \end{aligned}$$

By (6.2) and the à priori estimate (4.9),

$$\|(B)\|_0 = \|\theta_m \Delta S_m \rho_m\|_0 \leq C\theta_m \|S_m \rho_m\|_2 \leq C\theta_m \mu_m^2 \|\rho_m\|_0 \leq C\theta_m \mu_m^2 \|g_m\|_0.$$

Now by combining the estimates of (A), (B) and (C), we obtain

$$(6.10) \quad \|g_{m+1}\|_0 \leq C\left(\mu_m^{-(s^*-2-\beta)} C_{s^*}^{m+1} N(s^*) + \theta_m \mu_m^2 \|g_m\|_0 + \mu_m^{2(k-2)+[\frac{n}{2}]+5} \|g_m\|_0^2\right).$$

Since  $\theta_m = \|g_m\|_{L^\infty}$  by (2.9), we need to prove two estimates in (6.9) together. By same computation, using Sobolev imbedding, we have

$$(6.11) \quad \|g_{m+1}\|_{L^\infty} \leq C\mu_m^{\lfloor \frac{n}{2} \rfloor + 1} [\mu_m^{-(s^*-2-\beta)} C_{s^*}^{m+1} \mathcal{N}(s^*) + \|g_m\|_{L^\infty} \mu_m^2 \|g_m\|_0 + \mu_m^{2(k-2) + \lfloor \frac{n}{2} \rfloor + 5} \|g_m\|_0^2].$$

By comparing the powers of  $\mu_m$  on both sides of (6.10) and (6.11), we can choose  $a > 0$  and large  $s^* > s$  such that

$$(6.12) \quad \begin{cases} 2(k-2) + 2\lfloor \frac{n}{2} \rfloor + 6 + a\gamma \leq 2a - 1 \\ s^* - \lfloor \frac{n}{2} \rfloor - 3 - \beta \geq a\gamma + 1. \end{cases}$$

Noticing that  $\mu_{m+1} = \mu_m^\gamma$ , we can change (6.10) and (6.11) as

$$\begin{aligned} C\mu_{m+1}^a \|g_{m+1}\|_0 &\leq C^2 [\mu_m^{-(s^*-2-\beta-a\gamma)} C_{s^*}^{m+1} \mathcal{N}(s^*) + \|g_m\|_{L^\infty} \mu_m^{a\gamma+2} \|g_m\|_0 \\ &\quad + \mu_m^{a\gamma+2(k-2) + \lfloor \frac{n}{2} \rfloor + 5} \|g_m\|_0^2], \\ C\mu_{m+1}^a \|g_{m+1}\|_{L^\infty} &\leq C^2 \mu_m^{\lfloor \frac{n}{2} \rfloor + 1} [\mu_m^{-(s^*-2-\beta-a\gamma)} C_{s^*}^{m+1} \mathcal{N}(s^*) + \|g_m\|_{L^\infty} \mu_m^{a\gamma+2} \|g_m\|_0 \\ &\quad + \mu_m^{a\gamma+2(k-2) + \lfloor \frac{n}{2} \rfloor + 5} \|g_m\|_0^2], \end{aligned}$$

from which we obtain, by using (6.12) and  $\mu_m > 1$ ,

$$(6.13) \quad \begin{cases} C\mu_{m+1}^a \|g_{m+1}\|_0 \leq C^2 [\mu_m^{-1} C_{s^*}^{m+1} \mathcal{N}(s^*) + \|g_m\|_{L^\infty} \mu_m^{2a-1} \|g_m\|_0 + \mu_m^{2a-1} \|g_m\|_0^2] \\ C\mu_{m+1}^a \|g_{m+1}\|_{L^\infty} \leq C^2 [\mu_m^{-1} C_{s^*}^{m+1} \mathcal{N}(s^*) + \|g_m\|_{L^\infty} \mu_m^{2a-1} \|g_m\|_0 + \mu_m^{2a-1} \|g_m\|_0^2]. \end{cases}$$

Noticing  $\gamma > 1$ , we can choose  $\sigma = \sigma(s^*) > 1$  so large that  $\mu_m^{-1} = \sigma^{-\gamma^m} < \frac{1}{2}$  and

$$(6.14) \quad C^2 \mu_m^{-1} C_{s^*}^{m+1} = C^2 \sigma^{-\gamma^m} C_{s^*}^{m+1} = C^2 \left( \frac{C_{s^*}}{\sigma^{\frac{1}{m+1} \gamma^m}} \right)^{m+1} \leq \frac{1}{4}.$$

Inserting such  $\sigma(s^*)$  into (6.13), we have

$$\begin{cases} C\mu_{m+1}^a \|g_{m+1}\|_0 \leq \frac{1}{4} \mathcal{N}(s^*) + \frac{1}{2} C^2 \|g_m\|_{L^\infty} \mu_m^{2a} \|g_m\|_0 + \frac{1}{2} C^2 \mu_m^{2a} \|g_m\|_0^2 \\ C\mu_{m+1}^a \|g_{m+1}\|_{L^\infty} \leq \frac{1}{4} \mathcal{N}(s^*) + \frac{1}{2} C^2 \|g_m\|_{L^\infty} \mu_m^{2a} \|g_m\|_0 + \frac{1}{2} C^2 \mu_m^{2a} \|g_m\|_0^2. \end{cases}$$

Set

$$d_{m+1} = \max\{C\mu_{m+1}^a \|g_{m+1}\|_0, C\mu_{m+1}^a \|g_{m+1}\|_{L^\infty}\},$$

we get

$$\begin{cases} C\mu_{m+1}^a \|g_{m+1}\|_0 \leq \frac{1}{4} \mathcal{N}(s^*) + d_m^2 \\ C\mu_{m+1}^a \|g_{m+1}\|_{L^\infty} \leq \frac{1}{4} \mathcal{N}(s^*) + d_m^2. \end{cases}$$

So we obtain,

$$(6.15) \quad d_{m+1} \leq \frac{1}{4} \mathcal{N}(s^*) + d_m^2.$$

Since

$$\|g_0\|_0 = \|G(w_0)\|_0 = \|G(0)\|_0 = O(\varepsilon), \quad \|g_0\|_{L^\infty} = O(\varepsilon),$$

we choose  $\varepsilon(\sigma) > 0$  small such that  $\mathcal{N}(s^*)$ ,  $\|g_0\|_{L^\infty}$  and  $\|g_0\|_0$  small. By (6.13), (6.14) and  $\|g_0\|_0 \leq \|g_0\|_{s^*} \leq \mathcal{N}(s^*)$ , we have

$$d_1 \leq \frac{1}{4} \mathcal{N}(s^*) + \frac{1}{4} \|g_0\|_0 \leq \frac{1}{2} \mathcal{N}(s^*).$$

By induction and (6.15), we see that

$$d_{m+1} \leq \frac{1}{2} \mathcal{N}(s^*),$$

this completes the proof of (6.9).  $\square$

### Existence of $k$ -convex solution of Theorem 1.1

We employ now the Nash-Moser-Hörmander iteration to prove the existence of solution of main theorem with  $s \geq 2[\frac{n}{2}] + 5$ . We shall prove by induction that, there exists  $\varepsilon_0 > 0$  small such that, for any  $\varepsilon \in ]0, \varepsilon_0]$

$$(6.16) \quad \|w_m\|_s \leq 1, \quad \forall m \in \mathbb{N}.$$

Since  $w_0 = 0$ , we may assume that (6.16) holds for  $0 \leq l \leq m$ , which, by Sobolev imbedding theorem, guarantees the assumption of Lemma 6.1 and 6.2. Interpolation inequality and (6.1) yield, for any  $0 \leq s \leq s^*$ ,

$$\|w_{m+1}\|_s \leq \sum_{l=0}^m \|S_l \rho_l\|_s \leq C_s \sum_{l=0}^m \|\rho_l\|_s \leq C_s \sum_{l=0}^m \|\rho_l\|_{s^*}^{\frac{s}{s^*}} \|\rho_l\|_0^{1-\frac{s}{s^*}}.$$

By (6.6), it follows that, for  $0 \leq l \leq m$ ,

$$\|\rho_l\|_{s^*} \leq (C'_{s^*})^l \mu_l^\beta \mathcal{N}(s^*)$$

with  $\mu_l = \sigma^{\gamma^l}$ ,  $\sigma > 1$ ,  $\gamma > 1$ ,  $\beta = \frac{4}{\gamma-1}$ , and by (6.9),

$$\|\rho_l\|_0 \leq C \|g_l\|_0 \leq C' \mu_l^{-a} \mathcal{N}(s^*),$$

thus

$$\|w_{m+1}\|_s \leq C_s \sum_{l=0}^m (C'_{s^*})^{l \frac{s}{s^*}} \mu_l^{\beta \frac{s}{s^*} - a(1-\frac{s}{s^*})} \mathcal{N}(s^*).$$

So that we can choose  $s^*$  large enough such that

$$\beta \frac{s}{s^*} - a(1 - \frac{s}{s^*}) = -\tilde{a} < 0.$$

We choose  $\varepsilon_0 = \varepsilon_0(\sigma) > 0$  smaller to make  $\mathcal{N}(s^*)$  small enough such that for  $s \geq 2[\frac{n}{2}] + 5$ ,

$$\|w_{m+1}\|_s \leq C_s \sum_{l=0}^{\infty} (C'_{s^*})^{l \frac{s}{s^*}} \mu_l^{-\tilde{a}} \mathcal{N}(s^*) \leq 1$$

This completes the proof of (6.16).

On one hand, by (6.16) there is a subsequence of  $w_m$ , still denoted by itself, such that  $w_m \rightarrow w$  in weak topology of  $\mathbf{H}^s(\Omega)$ ,  $s \geq 2[\frac{n}{2}] + 5$  and  $w_m \rightarrow w$  in  $C^{[\frac{n}{2}]+4}$ . Hence

$$g_m = -G(w_m) \rightarrow -G(w) \quad \text{in } C^{[\frac{n}{2}]+2}(\Omega).$$

On the other hand, by using (6.16), Lemma 6.2 can be applied for all  $m \in \mathbb{N}$ , letting  $m \rightarrow \infty$  in (6.9) and recalling (2.9), we see that  $G(w) = 0$ , thus

$$u(y) = \frac{1}{2} \sum_{i=1}^{k-1} \tau_i y_i^2 + P(y) + \varepsilon^{\frac{1}{2}} w(\varepsilon^{-2} y) \in \mathbf{H}^s(\Omega)$$

is a local solution of the Hessian equation (1.1).

## 7. STRICT CONVEXITY OF LOCAL SOLUTION

In this section, we will prove that the smooth  $k$ -convex local-solution obtained in Section 6 is locally strict convex under the hypothesis of Theorem 1.1, that is, by (1.2) we need to prove that, for  $0 < t < 1$ ,  $y, z \in \Omega$ ,  $y \neq z$ ,

$$(7.1) \quad \sum_{i,j=1}^n \int_0^1 \int_0^1 u_{ij}(x(s, \mu)) d\mu ds (y_i - z_i)(y_j - z_j) > 0$$

with  $x(s, \mu) = (s\mu + (1-s)t)y + (s(1-\mu) + (1-s)(1-t))z$ . Recalling from (3.9),

$$(u_{ij})_{1 \leq i, j \leq n} = \mathbf{r} = \left( \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x) \right),$$

we separate this matrix into two parts: one is

$$r_{ij} = \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x), \quad 1 \leq i, j \leq k-1,$$

the principal term of which is  $\sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l$  and obviously can control the perturbation term  $P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)$  for small  $\varepsilon > 0$ . The other parts is

$$r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x), \quad i \geq k \text{ or } j \geq k,$$

for which, in order to control the perturbation term, our idea is to prove

$$(7.2) \quad w_{ij}(x', 0) = w_{ijp}(x', 0) = 0, \quad k \leq p \leq n,$$

if  $i \geq k$  or  $j \geq k$ . Then the Taylor expansion with respect to  $x'' = (x_k, \dots, x_n)$

$$w_{ij}(x) = w_{ij}(x', x'') = w_{ij}(x', 0) + \sum_{p=k}^n w_{ijp}(x', 0) x_p + O(|x''|^2),$$

which give,

$$r_{ij}(x) = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} O(|x''|^2), \quad i \geq k \text{ or } j \geq k,$$

which, together with (3.6), implies the minor matrix  $(r_{ij})_{k \leq i, j \leq n}$  is strictly diagonally dominant with

$$\begin{cases} |r_{ij}| \leq O(1)\varepsilon^4 |x''| + O(1)\varepsilon^{\frac{9}{2}} |x''|^2, & 1 \leq i \leq k-1, k \leq j \leq n, \\ r_{j_0 j_0}(x) = P_{j_0 j_0}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{j_0 j_0}(x) \geq \alpha \varepsilon^4 |x''|^2 + \sum_{i=k, i \neq j_0}^n |r_{ij_0}| & k \leq j_0 \leq n. \end{cases}$$

Choose  $\varepsilon > 0$  small ( $\varepsilon \ll \alpha$ ) enough such that, for  $k \leq j \leq n$ ,

$$r_{jj}(x) - \sum_{i=k, i \neq j}^n |r_{ij}(x)| \geq \alpha \varepsilon^4 |x''|^2,$$

Then Cauchy inequality yields

$$\begin{aligned} & \sum_{i,j=1}^n r_{ij}\xi_i\xi_j \\ & \geq \frac{1}{2} \sum_{i=1}^{k-1} \tau_i|\xi_i|^2 + 2 \sum_{1 \leq i \leq k-1, k \leq j \leq n} O(1)[\varepsilon^4|x''| + \varepsilon^{\frac{9}{2}}|x''|^2]\xi_i\xi_j + \sum_{i=k}^n \alpha\varepsilon^4|x''|^2|\xi_i|^2 \\ & \geq \frac{1}{4} \sum_{i=1}^{k-1} \tau_i|\xi_i|^2 + \frac{1}{2} \sum_{i=k}^n \alpha\varepsilon^4|x''|^2|\xi_i|^2, \quad \forall \xi \in \mathbb{R}^n \end{aligned}$$

from which, it follows that (7.1) holds. In fact, setting  $\xi = y - z$  and recalling  $x'' = x''(s, \mu) = (s\mu + (1-s)t)y'' + (s(1-\mu) + (1-s)(1-t))z''$ , we have

$$\begin{aligned} & \sum_{i,j=1}^n \int_0^1 \int_0^1 u_{ij}(x) d\mu ds (y_i - z_i)(y_j - z_j) \\ & \geq \left( \frac{1}{4} \sum_{i=1}^{k-1} \tau_i |y_i - z_i|^2 + \frac{1}{2} \alpha \varepsilon^4 b(t) \sum_{i=k}^n |y_i - z_i|^2 \right) > 0 \end{aligned}$$

with

$$b(t) = \int_0^1 \int_0^1 \|[s\mu + (1-s)t]y'' + [s(1-\mu) + (1-s)(1-t)]z''\|^2 d\mu ds > 0,$$

for any  $0 < t < 1$  and  $y'' \neq z''$ , this inequality is true because

$$\{(s, \mu) \in [0, 1] \times [0, 1]; [s\mu + (1-s)t]y'' + [s(1-\mu) + (1-s)(1-t)]z'' = 0\}$$

lies on a hyperbolic curve in the  $(s, \mu)$ - plane and then its Lebesgue measure for  $d\mu ds$  is zero. So that  $u$  is strictly convex on  $\Omega$ .

We prove now (7.2) by the following 2 lemmas. From the explicit expression of  $P(y)$  in (3.2) together with  $u(0) = 0$  and  $\nabla u(0) = 0$ , we see that  $w(0) = 0$  and  $\nabla w(0) = 0$ . Moreover, we have

**Lemma 7.1.** *Let  $u$  be a solution of equation (2.2) in the form of (3.8) with  $\|w\|_{C^3} \leq 1$ , then*

$$(7.3) \quad w_{ij}(x', 0) = 0, \quad i \geq k \text{ or } j \geq k.$$

*Proof.* By (3.2),

$$P_{ij}(\varepsilon^2 x) |_{x'=0} = 0 \quad \text{for } i \neq j, i \geq k \text{ or } j \geq k,$$

then

$$r_{ij} |_{x'=0} = \varepsilon^{\frac{9}{2}} w_{ij}(x', 0), \quad i \neq j, i \geq k \text{ or } j \geq k.$$

and

$$\sum_{i=k}^n \sum_{j=1}^{k-1} r_{ji}^2(x', 0) \sigma_{k-2,j}(\lambda_1, \dots, \lambda_{k-1}) = O(1)\varepsilon^9.$$

$$P_{ii}(\varepsilon^2 x', 0) = O(1)\varepsilon^8 \quad \text{for } k \leq i \leq n,$$

thus, for  $k \leq j \leq n$ ,

$$r_{jj}(x', 0) = P_{jj}(\varepsilon^2 x', 0) + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0) = \varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0).$$



Then the direct computation give

$$\sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}) \sum_{j=k}^n w_{jj}(x', 0) = \varepsilon^{\frac{9}{2}} O(1).$$

Letting  $\varepsilon \rightarrow 0^+$ , we get

$$\sum_{j=k}^n w_{jj}(x', 0) = \sigma_1(w_{kk}(x', 0), w_{k+1,k+1}(x', 0), \dots, w_{nn}(x', 0)) = 0.$$

Using now the identity

$$2\sigma_2(\lambda) = [\sigma_1(\lambda)]^2 - \sum_{i=1}^m \lambda_i^2, \quad \forall \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m,$$

it follows that

$$\begin{aligned} & 2\sigma_2(r_{kk}(x', 0), r_{k+1,k+1}(x', 0), \dots, r_{nn}(x', 0)) \\ &= \left[ \sum_{j=k}^n r_{jj}(x', 0) \right]^2 - \sum_{j=k}^n [r_{jj}(x', 0)]^2 \\ &= [\varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} \sum_{j=k}^n w_{jj}(x', 0)]^2 - \sum_{j=k}^n [\varepsilon^8 O(1) + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0)]^2 \\ &= O(1)\varepsilon^{16} - \sum_{j=k}^n [O(1)\varepsilon^8 + \varepsilon^{\frac{9}{2}} w_{jj}(x', 0)]^2. \end{aligned}$$

Multiplying by  $\varepsilon^{-9}$  on both side of (2.2), take  $x'' = 0$  and letting  $\varepsilon \rightarrow 0$ , we obtain

$$\sum_{i=k}^n \sum_{j=1}^{k-1} w_{ji}^2(x', 0) = \sum_{j=k}^n [w_{jj}(x', 0)]^2 = \sum_{i=k}^n \sum_{s=k, s \neq i}^n w_{si}^2(x', 0) = 0$$

and then (7.3) is true.  $\square$

Using now (7.3), the Taylor expansion of  $r_{ij} = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} w_{ij}(x)$  for  $i \geq k$  or  $j \geq k$  is of the following version

$$r_{ij}(x) = P_{ij}(\varepsilon^2 x) + \varepsilon^{\frac{9}{2}} \sum_{p=k}^n w_{ijp}(x', 0) x_p + \frac{1}{2} \varepsilon^{\frac{9}{2}} \sum_{p,q=k}^n w_{ijpq}(x', 0) x_p x_q + O(1)\varepsilon^{\frac{9}{2}} |x''|^3,$$

where  $\|w\|_{C^{\frac{9}{2}+3}} \leq 1$  is required. Similar to the proof of above Lemma, we can obtain

$$\sum_{i=k}^n w_{iip}(x', 0) = \sum_{i=k}^n w_{iipp}(x', 0) = 0, \quad k \leq p \leq n.$$

And also :

**Lemma 7.2.** *Let  $u$  be a solution of equation (2.2) in the form of (3.8), and  $\|w\|_{C^{1\frac{9}{2}+4}} \leq 1$ , then*

$$w_{ijp}(x', 0) = 0, \quad i \geq k \text{ or } j \geq k, \quad k \leq p \leq n.$$

## 8. APPENDIX

In this section, we will estimate the eigenvalues and eigenvectors of the following Hessian matrix

$$\mathbf{r} = (r_{ij})_{1 \leq i, j \leq n} = \left( \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l + \varepsilon \tilde{w}_{ij}(x) \right)_{1 \leq i, j \leq n}$$

which is a small perturbation of diagonal matrix  $\left( \sum_{l=1}^{k-1} \delta_i^j \delta_j^l \tau_l \right)_{1 \leq i, j \leq n}$ . For any function  $\tilde{w} \in C^2$ , we can find out an orthogonal matrix  $T(x, \varepsilon)$  satisfying

$$T(x, \varepsilon) \mathbf{r} {}^t T(x, \varepsilon) = \text{diag} [\lambda_1(x, \varepsilon), \lambda_2(x, \varepsilon), \dots, \lambda_n(x, \varepsilon)],$$

Let

$$T(x, \varepsilon) = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix}, \quad T(x, \varepsilon)|_{\varepsilon=0} = Id$$

with  $T_i = (T_{i1}, T_{i2}, \dots, T_{in})$  the corresponding unit eigenvectors of  $\lambda_i, i = 1, 2, \dots, n$ . In [10], there also exists such an orthogonal matrix  $T(x, \varepsilon)$ . The estimates for all of its entries can be obtained because all of its eigenvalues are different from each other. Now we can only give the estimates of  $T_i$  for  $1 \leq i \leq k-1$ , because in our case  $\lambda_j(x, \varepsilon), k \leq j \leq n$  are around zero and there is no distinct gap among them, they are not necessarily smooth in  $x$  and  $\varepsilon$ , so are the corresponding eigenvectors  $T_i(x, \varepsilon) (k \leq i \leq n)$ . But the following estimates are enough for us.

**Proposition 8.1.** *Suppose that  $\tilde{w}$  is smooth and  $\|\tilde{w}\|_{C^3} \leq 1$ . Then,  $T_i(x, \varepsilon), 1 \leq i \leq k-1$ , is smooth in  $(x, \varepsilon) \in \bar{\Omega} \times [0, \varepsilon_0]$  for some positive  $0 < \varepsilon_0 \ll 1$ , and*

$$(8.1) \quad \sum_{i=1}^{k-1} |\lambda_i - \tau_i| \leq C\varepsilon \sum_{i,j=1}^n |\tilde{w}_{ij}(x)|, \quad \sum_{i=k}^n |\lambda_i| \leq C\varepsilon^{\frac{1}{2}} \left( \sum_{i,j=1}^n |\tilde{w}_{ij}(x)| \right)^{\frac{1}{2}},$$

$$(8.2) \quad \sum_{i=1}^{k-1} |T_{ii}(x, \varepsilon) - 1| + \sum_{i=1}^{k-1} \sum_{j=1, j \neq i}^n |T_{ij}(x, \varepsilon)| \leq C\varepsilon \sum_{i,j=1}^n |\tilde{w}_{ij}(x)|,$$

$$(8.3) \quad \sum_{i=1}^{k-1} \sum_{j=1}^n |DT_{ij}(x, \varepsilon)| \leq C\varepsilon \sum_{i,j,l=1}^n |\tilde{w}_{ijl}(x)|$$

with  $C$  independent of  $\varepsilon$  and  $\tilde{w}$ .

Moreover, if  $\tilde{w}(x)$  is periodic in  $x_i (1 \leq i \leq k-1)$ , then so is each of  $T_{ij}$ , while  $T_{ij}$  has the same regularity as  $D^2 \tilde{w}$  and is  $C^\infty$  in  $\varepsilon$  for  $1 \leq i \leq k-1, 1 \leq j \leq n$ .

*Proof.* Noting

$$F(t) = \det(\mathbf{r} - t\mathbf{I}).$$

Now we denote  $R(x, \varepsilon), R_i(x, \varepsilon)$  and  $R_{ij}(x, \varepsilon)$  as the different functions, which are smooth in  $x, \varepsilon$  and  $\tilde{w}$ , with the properties

$$(8.4) \quad |R(x, \varepsilon)| + |R_i(x, \varepsilon)| + |R_{ij}(x, \varepsilon)| \leq C\varepsilon \sum_{i,j=1}^n |\tilde{w}_{ij}(x)|$$

and  $C$  being independent of  $x$  and  $\varepsilon$ . Firstly, using the condition  $\|\tilde{w}\|_{C^2} \leq 1$  and  $0 < \varepsilon_0 \ll 1$ , we have

$$F(t) = \prod_{i=1}^{k-1} (\tau_i - t)t^{n-k+1} + R(x, \varepsilon).$$

Noticing  $\tau_1 > \tau_2 > \dots > \tau_{k-1} > 0$ , we take  $\delta = \frac{1}{4} \min_{1 \leq i \leq k-2} (1 - \frac{\tau_{i+1}}{\tau_i}, \frac{\tau_i}{\tau_{i+1}} - 1) > 0$  if  $k > 2$  and  $\delta = \frac{1}{4}$  if  $k = 2$ . For fixed  $i_0$  with  $1 \leq i_0 \leq k-1$ , we have

$$F((1-\delta)\tau_{i_0}) = \prod_{i=1}^{i_0-1} (\tau_i - (1-\delta)\tau_{i_0})\delta\tau_{i_0} \prod_{i=i_0+1}^{k-1} (\tau_i - (1-\delta)\tau_{i_0})((1-\delta)\tau_{i_0})^{n-k+1} + R_1(x, \varepsilon)$$

and

$$F((1+\delta)\tau_{i_0}) = -\prod_{i=1}^{i_0-1} (\tau_i - (1+\delta)\tau_{i_0})\delta\tau_{i_0} \prod_{i=i_0+1}^{k-1} (\tau_i - (1+\delta)\tau_{i_0})((1+\delta)\tau_{i_0})^{n-k+1} + R_2(x, \varepsilon).$$

By the choice of  $\delta$ , we have

$$\tau_i - (1+\delta)\tau_{i_0} > \delta\tau_{i_0-1}, \quad 1 \leq i \leq i_0 - 1$$

and

$$\tau_i - (1-\delta)\tau_{i_0} < -\delta\tau_{i_0+1}, \quad i_0 + 1 \leq i \leq k-1,$$

therefore, when  $0 < \varepsilon \ll \delta\tau_{k-1}$ , we obtain

$$F((1-\delta)\tau_{i_0})F((1+\delta)\tau_{i_0}) < 0$$

and, by virtue of intermediate value theorem, there exists an eigenvalue, denoted by  $\lambda_{i_0}$ , such that

$$(1-\delta)\tau_{i_0} < \lambda_{i_0} < (1+\delta)\tau_{i_0}, \quad F(\lambda_{i_0}) = 0,$$

which yields

$$\tau_i - \lambda_{i_0} \geq \tau_i - (1+\delta)\tau_{i_0} > \delta\tau_{k-1}, \quad 1 \leq i \leq i_0 - 1$$

and

$$\tau_i - \lambda_{i_0} \leq \tau_i - (1-\delta)\tau_{i_0} < -\delta\tau_{k-1}, \quad i_0 + 1 \leq i \leq k-1.$$

From  $0 = F(\lambda_{i_0}) = \prod_{i=1}^{i_0-1} (\tau_i - \lambda_{i_0})(\tau_{i_0} - \lambda_{i_0}) \prod_{i=i_0+1}^{k-1} (\tau_i - \lambda_{i_0})(\lambda_{i_0})^{n-k+1} + R_3(x, \varepsilon)$ , it follows that

$$\tau_{i_0} - \lambda_{i_0} = -\frac{R_3(x, \varepsilon)}{\prod_{i=1}^{i_0-1} (\tau_i - \lambda_{i_0}) \prod_{i=i_0+1}^{k-1} (\tau_i - \lambda_{i_0})(\lambda_{i_0})^{n-k+1}}$$

and then

$$\lambda_{i_0} = \tau_{i_0} + R_4(x, \varepsilon).$$

Now we have proved that there are eigenvalues  $(\lambda_i)_{i=1}^{k-1}$  such that

$$\lambda_i = \tau_i + R(x, \varepsilon), \quad 1 \leq i \leq k-1.$$

Let us express all the eigenvalues as  $(\lambda_i)_{i=1}^n$ . Then, since  $\|\tilde{w}\|_{C^3} \leq 1$ ,

$$S_1(r) = \sum_{i=1}^{k-1} \tau_i + R(x, \varepsilon), \quad S_2(r) = \sigma_2(\tau_1, \dots, \tau_{k-1}) + R(x, \varepsilon).$$

Since  $S_k(r)$  is invariant under orthogonal transformation, then

$$\sum_{i=1}^{k-1} \tau_i + R(x, \varepsilon) = S_1(\mathbf{r}) = \sigma_1(\lambda) = \sum_{i=1}^{k-1} \lambda_i + \sum_{i=k}^n \lambda_i$$

and

$$\sigma_2(\tau_1, \dots, \tau_{k-1}) + R(x, \varepsilon) = S_2(\mathbf{r}) = \sigma_2(\lambda).$$

from which, using

$$2\sigma_2(\lambda) = \sigma_1(\lambda)^2 - \sum_{i=1}^n \lambda_i^2.$$

we see that

$$\sigma_2(\lambda) = \sigma_2(\tau_1, \dots, \tau_{k-1}) - \frac{1}{2} \sum_{i=k}^n \lambda_i^2 + R(x, \varepsilon)$$

and then

$$\sum_{i=k}^n \lambda_i^2 = R(x, \varepsilon),$$

which implies

$$(8.5) \quad |\lambda_i| \leq C\varepsilon^{\frac{1}{2}} \left( \sum_{i,j=1}^n |\tilde{w}_{ij}(x)| \right)^{\frac{1}{2}}, \quad k \leq i \leq n.$$

This completes the proof of (8.1).

Now we pass to prove (8.2). Let  $T_1$  be the eigenvector corresponding to  $\lambda_1$ , then  $T_1$  satisfies the linear equation  $\mathbf{r}T_1 - \lambda_1 T_1 = 0$  and  $\text{rank}(\mathbf{r} - \lambda_1 \mathbf{I}) \leq n - 1$ . To solve  $T_1$ , we will use the Gaussian elimination procedure. Noticing  $\lambda_i = \tau_i + R(x, \varepsilon)$  for  $1 \leq i \leq k - 1$ , we can write the matrix  $\mathbf{r} - \lambda_1 \mathbf{I}$  as

$$\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1,k-1} & R_{1k} & \dots & R_{1n} \\ R_{21} & R_{22} + \tau_2 - \tau_1 & \dots & R_{2,k-1} & R_{2,k} & \dots & R_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{k-1,1} & R_{k-1,2} & \dots & R_{k-1,k-1} + \tau_{k-1} - \tau_1 & R_{k-1,k} & \dots & R_{k-1,n} \\ R_{k,1} & R_{k,2} & \dots & R_{k,k-1} & R_{k,k} - \tau_1 & \dots & R_{k,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{n,1} & R_{n,2} & \dots & R_{n,k-1} & R_{n,k} & \dots & R_{n,n} - \tau_1 \end{pmatrix}$$

Since each  $R_{jj}$ ,  $2 \leq j \leq k - 1$ , is dominated by  $\tau_j - \tau_1$  and each  $R_{ii}$ ,  $k \leq i \leq n$ , is dominated by  $\tau_1$ , then  $\text{rank}(A) = n - 1$ . The solutions of the equation  $\mathbf{r}T_1 - \lambda_1 T_1 = 0$  is in form

$$(8.6) \quad \begin{pmatrix} T_{11} \\ T_{12} \\ \vdots \\ T_{1,k-1} \\ T_{1,k} \\ \vdots \\ T_{1n} \end{pmatrix} = T_{11} \begin{pmatrix} 1 \\ \frac{R_{21}(x,\varepsilon)}{\tau_1 - \tau_2 - R_{22}(x,\varepsilon)} \\ \vdots \\ \frac{R_{k-1,1}(x,\varepsilon)}{\tau_1 - \tau_{k-1} - R_{k-1,k-1}(x,\varepsilon)} \\ \frac{R_{k,1}(x,\varepsilon)}{\tau_1 - R_{k,k-1}(x,\varepsilon)} \\ \vdots \\ \frac{R_{n,1}(x,\varepsilon)}{\tau_1 - R_{n,n}(x,\varepsilon)} \end{pmatrix}, \quad \text{with } T_{11} \neq 0,$$

and each  $R_{ij}(x, \varepsilon)$  has the property (8.4). Because we need  $T_1(x, \varepsilon)|_{\varepsilon=0} = (1, 0, \dots, 0)$ , we can choose suitable  $T_{11} > 0$  such that  $\|T_1\| = 1$ , then  $T_1 = (T_{11}, T_{12}, \dots, T_{1n})$  satisfies (8.2). The vectors  $T_i (2 \leq i \leq k-1)$  can be obtained by the same way as  $T_1$ . Also it follows from (8.6) that (8.2) and (8.3) hold.

Because each entry of the matrix  $\mathbf{r} = (r_{ij})$  is periodic in  $x_1, \dots, x_{k-1}$  and the elementary operations above do not change the periodicity, then each  $T_i (1 \leq i \leq k-1)$  is also periodic. Also by (8.6), when  $1 \leq i \leq k-1$ ,  $T_{ij}$  has the same regularity as  $D^2\tilde{w}$  and is  $C^\infty$  in  $\varepsilon$ .  $\square$

**Remark 8.2.** Since  $\lambda_i, k \leq i \leq n$ , are all around 0, then (8.5) shows that they are not necessarily smooth in  $x$  and  $\varepsilon$ , so are the corresponding eigenvectors  $T_i$ .

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#### REFERENCES

- [1] S. Alinhac and P. Gérard, Opérateurs pseudo-différentiels et théorème de Nash-Moser, 1991 Editions EDP Sciences CNRS ÉDITIONS, Paris, France.
- [2] B. Bian and P. Guan: A microscopic convexity principle for nonlinear partial differential equations. *Invent math.* **177** (2009), 307-335.
- [3] T. Chen and Q. Han : Smooth local solutions to Weingarten equations and  $k$ -equations, *Discrete Contin. Dyn. Syst.* **36** (2016), 653 - 660.
- [4] P. Guan and X. N. Ma: The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation. *Invent. Math.* **151** (2003), 553-577.
- [5] P. Guan, C. S. Lin and X. N. Ma: The Christoffel-Minkowski problem II: Weingarten curvature equations. *Chin. Ann. Math., Ser. B* **27** (2006), 595-614.
- [6] P. Guan, X. N. Ma and F. Zhou: The Christoffel-Minkowski problem III: existence and convexity of admissible solutions. *Comm. Pure Appl. Math.* **59** (2006), 1352-1376.
- [7] P. Hartman: On isometric immersions in Euclidean space of manifolds with non-negative sectional curvatures. *Trans. Amer. Math. Soc.*, **115** (1965):94-94.
- [8] Q. Han and J.-X., Hong: Isometric embeddings of Riemannian manifolds in Euclidean spaces, *Mathematical Surveys and Monographs*, Volume 130, American Mathematical Society, Providence, RI, 2006.
- [9] R. Harvey and JR, H. B. Lawson: Calibrated geometries. *Acta Math.* **148** (1982), 47-157.
- [10] J. Hong and C. Zuily: Existence of  $C^\infty$  local solutions for the Monge-Ampère equation, *Invent. Math.* **89** (1987), 645-661.
- [11] N. M. Ivochkina, N. S. Trudinger and X.-J. Wang : The Dirichlet problem for degenerate Hessian equations, *Comm. Part. Diff. Equat.* **29** (2004), 219-235.
- [12] N. M. Ivochkina, S. I. Prokofeva and G. V. Yakunina: The Gårding cones in the modern theory of fully nonlinear second order differential equations, *J. Math. Sci.* **184** (2012), 295-315.
- [13] P. Liu, X.-N. Ma and L. Xu : A Brunn-Minkowski inequality for the Hessian eigenvalue in three-dimensional convex domain, *Adv. Math.* **225** (2010), 1616-1633.
- [14] X.-N. Ma and L. Xu : The convexity of solution of a class Hessian equation in bounded convex domain in  $\mathbf{R}^3$ , *J. Funct. Anal.* **255** (2008), 1713-1723.
- [15] S. Paolo: Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in  $\mathbf{R}^3$ , *Adv. Math.* **229** (2012), 1924–1948.
- [16] K. Shiohama: Cylinders in Euclidean space  $E^{2+n}$ . *Kodai Mathematical Seminar Reports* 1967.
- [17] G. Tian, Q. Wang and C.-J. Xu:  $C^\infty$  local solutions of elliptical 2-Hessian equation in  $\mathbf{R}^3$ . *Discrete Contin. Dyn. Syst.* **36** (2016), 1023 - 1039.
- [18] G. Tian, Q. Wang, and C.-J. Xu: Local solvability of the  $k$ -Hessian equations, *Sci. China Math.* **59** (2016), 1753-1768.

- [19] V. Ushakov: Developable surfaces in Euclidean space. *J. Austr. Math. Soc.* **66** (1999):388-402.  
[20] X.-J., Wang: The  $k$ -Hessian equation, Lecture Notes in Math., Springer, Dordrecht (2009).

GUJI TIAN, WUHAN INSTITUTE OF PHYSICS AND MATHEMATICS  
CHINESE ACADEMY OF SCIENCES, WUHAN, P.R. CHINA, 430071  
*E-mail address:* tianguji@wipm.ac.cn

CHAO-JIANG XU, UNIVERSITÉ DE ROUEN, CNRS UMR 6085, LABORATOIRE DE MATHÉMATIQUES  
76801 SAINT-ETIENNE DU ROUVRAY, FRANCE

AND

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA  
*E-mail address:* chao-jiang.xu@univ-rouen.fr