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Antonin Chambolle, Vito Crismale. A DENSITY RESULT IN  $GSBD^p$  WITH APPLICATIONS TO THE APPROXIMATION OF BRITTLE FRACTURE ENERGIES. *Archive for Rational Mechanics and Analysis*, 2019, 232 (3), pp.1329–1378. hal-01573936

**HAL Id: hal-01573936**

**<https://hal.science/hal-01573936v1>**

Submitted on 11 Aug 2017

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# A DENSITY RESULT IN $GSBD^p$ WITH APPLICATIONS TO THE APPROXIMATION OF BRITTLE FRACTURE ENERGIES

ANTONIN CHAMBOLLE AND VITO CRISMALE

**ABSTRACT.** We prove that any function in  $GSBD^p(\Omega)$ , with  $\Omega$  a  $n$ -dimensional open bounded set with finite perimeter, is approximated by functions  $u_k \in SBV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  whose jump is a finite union of  $C^1$  hypersurfaces. The approximation takes place in the sense of Griffith-type energies  $\int_\Omega W(e(u)) \, dx + \mathcal{H}^{n-1}(J_u)$ ,  $e(u)$  and  $J_u$  being the approximate symmetric gradient and the jump set of  $u$ , and  $W$  a nonnegative function with  $p$ -growth,  $p > 1$ . The difference between  $u_k$  and  $u$  is small in  $L^p$  outside a sequence of sets  $E_k \subset \Omega$  whose measure tends to 0 and if  $|u|^r \in L^1(\Omega)$  with  $r \in (0, p]$ , then  $|u_k - u|^r \rightarrow 0$  in  $L^1(\Omega)$ . Moreover, an approximation property for the (truncation of the) amplitude of the jump holds. We apply the density result to deduce  $\Gamma$ -convergence approximation *à la* Ambrosio-Tortorelli for Griffith-type energies with either Dirichlet boundary condition or a mild fidelity term, such that minimisers are *a priori* not even in  $L^1(\Omega; \mathbb{R}^n)$ .

**Keywords:** generalised special functions of bounded deformation, strong approximation, brittle fracture,  $\Gamma$ -convergence, free discontinuity problems

**MSC 2010:** 49Q20, 74R10, 26A45, 49J45, 74G65.

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## INTRODUCTION

A fundamental idea in the variational approach to fracture mechanics is that the formation of fracture is the result of the competition between the surface energy spent to produce the crack and the energy stored in the uncracked region. This idea dates back to the pioneering work of Griffith [41] and is the core of the model for quasistatic crack evolution proposed by Francfort and Marigo [37], which, in turn, is the starting point for a large number of variational models (see e.g. [30, 36, 14, 7, 40] and [28, 29, 45] for brittle fracture in the small and finite strain framework, respectively, and e.g. [13, 31, 25] for cohesive fracture). For brittle fracture models, in small strain assumptions, the sum of the bulk energy and of the surface energy (that in brittle fracture is nothing but the measure of the crack) has usually the form

$$\int_\Omega W(e(u)) \, dx + \mathcal{H}^{n-1}(J_u) \tag{1}$$

in a *reference configuration*  $\Omega \subset \mathbb{R}^n$ . This depends on the *displacement*  $u: \Omega \rightarrow \mathbb{R}^n$  through  $e(u)$ , the *symmetric approximate gradient* of  $u$ , and  $J_u$ , the *jump set* of  $u$ , that represents the crack set. In order to give sense to (1), one assumes that  $u$  admits a measurable (with respect to the Lebesgue measure  $\mathcal{L}^n$ ) *symmetric approximate gradient*  $e(u)(x) \in \mathbb{M}_{sym}^{n \times n}$  for  $\mathcal{L}^n$ -a.e.  $x \in \Omega$ , characterised by

$$\operatorname{ap\,lim}_{y \rightarrow x} \frac{(u(y) - u(x) - e(u)(x)(y - x)) \cdot (y - x)}{|y - x|^2} = 0,$$

(see (1.1) for definition of approximate limit) and that  $J_u$  is countably  $(\mathcal{H}^{n-1}, n-1)$  rectifiable, where  $J_u$  is defined as the set of discontinuity points  $x$  where  $u$  has one-sided approximate limits  $u^+(x) \neq u^-(x)$  with respect to a suitable direction  $\nu_u(x)$  normal to  $J_u$ . The function  $W$  is required to be convex with  $p$ -growth, with  $p > 1$  (cf. e.g. [37, Section 2] in the framework of elastic bulk energies, and [42, Sections 10 and 11] and references therein for a connection with elasto-plastic materials).

The space  $BD(\Omega)$  of *functions of bounded deformation* is an important example of function space in which (1) is well defined. Employed in the mathematical modelling of small strain elasto-plasticity (see e.g. [52, 50, 44, 51]) it consists of the functions  $u \in L^1(\Omega; \mathbb{R}^n)$  whose symmetric distributional derivative  $(Eu)_{ij} := \frac{1}{2}(D_i u_j + D_j u_i)$  is a (matrix-valued) measure with finite total variation in  $\Omega$ . In particular (see for instance [2]),  $J_u$  is countably  $(\mathcal{H}^{n-1}, n-1)$  rectifiable and  $Eu = E^a u + E^c u + E^j u$ , where  $E^a u = e(u)\mathcal{L}^n$ , the *Cantor part*  $E^c$  is singular with respect to  $\mathcal{L}^n$  and vanishes on Borel sets of finite  $\mathcal{H}^{n-1}$  measure, and  $E^j u$  is concentrated on  $J_u$ .

In view of the assumptions on (1), and since in particular it gives no control on the Cantor part of  $Eu$ , in the present context it is useful to focus on the space  $SBD(\Omega)$  of  $BD$  functions with null Cantor part, introduced in [2], and on its subspace

$$SBD^p(\Omega) := \{u \in SBD(\Omega) : e(u) \in L^p(\Omega; \mathbb{M}_{sym}^{n \times n}), \mathcal{H}^{n-1}(J_u) < \infty\}.$$

Indeed, the existence of minimisers for (1) is guaranteed in  $SBD^p(\Omega)$  by the compactness result [8, Theorem 1.1], provided one has an *a priori* bound for  $u$  in  $L^\infty(\Omega; \mathbb{R}^n)$ . Unfortunately, it is hard to obtain such a bound, even if the total energy includes additional lower order terms.

To overcome this drawback, Dal Maso introduced in [27] the spaces  $GBD$  and  $GSBD$  of the *generalised  $BD$*  and  *$SBD$*  functions, respectively (see Definition 1.5 for its definition, based on properties of one-dimensional slices). Every  $GBD$  function admits a measurable symmetric approximate gradient and has a countably  $(\mathcal{H}^{n-1}, n-1)$  rectifiable jump set, so that (1) makes sense. Moreover, the compactness result [27, Theorem 11.3] requires a very mild control for sequences in  $GSBD^p$  (the space of  $GSBD$  functions with  $e(u)$   $p$ -integrable and  $\mathcal{H}^{n-1}(J_u)$  finite), namely that  $\psi_0(|u_k|)$  is bounded in  $L^1$  for some  $\psi_0$  nonnegative, continuous, increasing and unbounded. This gives compactness with respect to the convergence in measure of minimising sequences for total energies with main term (1) plus a lower order *fidelity term* of type  $\int_\Omega \psi_0(|u - g|) dx$ , for a suitable datum  $g$ , so that the displacements are not even forced to be in  $L^1$ .

Notice that, differently from the case of image reconstruction, a fidelity term in the total energy is not in general meaningful in fracture mechanics. In particular, the original formulation in [37, Section 2] considers the energy (1) only supplemented with a Dirichlet boundary condition. We remark that a Mumford-Shah-type energy, obtained from the *Mumford-Shah image segmentation functional* [46, 33] by replacing the  $L^2$  fidelity term with a Dirichlet boundary condition, describes brittle fractures in the *generalised antiplane* setting of e.g. [36].

An interesting issue is to provide  $\Gamma$ -convergence approximations, in the spirit of Ambrosio and Tortorelli [4, 5], for energies of the form (1) plus some compliance conditions on the displacement. In [4, 5] the Mumford-Shah functional is approximated by means of elliptic functionals, depending on the displacement and on a so-called *phase field* variable, whose minimisers are easier to compute. This result has been largely employed to numerically handle problems both in image reconstruction and in fracture mechanics (see for instance [10, 9, 11]). In the vector-valued case, approximations *à la* Ambrosio-Tortorelli have been proven by Chambolle [15, 16] and Iurlano [43] for the restriction of (1) (assuming  $W$  quadratic) to  $SBD^2(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$  and  $GSBD^2(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$ , respectively. A crucial point in the proof of the  $\Gamma$ -limsup inequality is to approximate, in the sense of (limit) energy, any displacement by a sequence of functions in  $SBV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  whose jump is a finite union of  $C^1$  hypersurfaces. Then it is not difficult to find a recovery sequence for regular displacement, and one concludes by a diagonal argument. Furthermore, by [24, Theorem 3.1] one may consider approximating functions whose jump is *essentially closed* and *polyhedral*, which are of class  $W^{m, \infty}$ , for every  $m \in \mathbb{N}$ , in the complement of the (closure of the) jump.

The recent works [38] and [22] prove two density results moving from the one in [43] into different directions. In [38] any integrability assumption on the displacement is removed, provided  $\Omega$  is 2-dimensional, precisely for any  $u \in GSBD^2(\Omega)$ , with  $\Omega \subset \mathbb{R}^2$ , there exists a sequence  $u_k$  in  $SBV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$  with regular jump, converging in measure to  $u$  and such that  $e(u_k) \rightarrow e(u)$  in  $L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $\mathcal{H}^{n-1}(J_{u_k} \Delta J_u) \rightarrow 0$ ,  $\Delta$  denoting the symmetric difference of sets (cf. [38, Theorem 2.5]). This follows from a piecewise Korn inequality, obtained with a careful analysis of the jump set of  $GSBD$  functions in a 2-dimensional setting. In [22] any  $u \in GSBD^p(\Omega) \cap L^p(\Omega; \mathbb{R}^n)$ , for every  $n \in \mathbb{N}$ ,  $p > 1$ , is approximated in the sense of energy (1) and in  $L^p(\Omega; \mathbb{R}^n)$ , basing on the construction of suitable interpolations of  $u$  that are piecewise affine on a decomposition of  $\Omega$

in small simplices. The setting is therein  $n$ -dimensional, but  $p$ -summability of the displacement is required.

The present paper provides a density result (Theorem 3.1) in  $GSBD^p(\Omega)$ , for a bounded open set  $\Omega \subset \mathbb{R}^n$  with finite perimeter. Here  $n \in \mathbb{N}$ ,  $p > 1$ , with no integrability assumptions on the displacement. The approximating functions  $u_k$  converge to  $u$  in measure,  $e(u_k)$  converge to  $e(u)$  in  $L^p(\Omega; \mathbb{M}_{sym}^{n \times n})$  and  $\mathcal{H}^{n-1}(J_{u_k} \Delta J_u) \rightarrow 0$ . Moreover, the difference between  $u_k$  and  $u$  is small in  $L^p$  outside a sequence of sets  $E_k \subset \Omega$  whose measure tends to 0 and, as soon as  $|u|^r \in L^1(\Omega)$  with  $r \in (0, p]$ , we have that  $|u_k - u|^r \rightarrow 0$  in  $L^1(\Omega)$ .

As in [15, 43, 22], we first prove an intermediate approximation (Theorem 2.1) which controls the measure of the jump set up to a multiplicative parameter. Then we cover a large part of the jump set  $J_u$  by suitable rectangles, that are split into two parts by  $J_u$ . This gives a partition of  $\Omega$  in subsets where the jump set has small  $\mathcal{H}^{n-1}$  measure, so that Theorem 2.1 provides here (in suitable neighbourhoods, indeed) approximating functions close in energy to the original one. A fundamental difference with respect to [15, 43, 22] is that we do not use partitions of unity neither to extend the original function in suitable neighbourhoods of the subsets of the partition nor to glue the approximating functions constructed in any subset. This is done by employing a reflection technique for vector-valued functions due to Nitsche [49] (cf. Lemma 1.8) and allows us to avoid any assumption on the integrability of  $u$ .

The proof of Theorem 2.1 is based on [17, Proposition 3] (cf. Proposition 1.9), and is close to what done in [18] to approximate a brittle fracture energy with a non-interpenetration constraint. The idea is to partition the domain into cubes of side  $k^{-1}$  and to distinguish, at any scale, the cubes where the ratio between the perimeter and the jump of  $u$  is greater than a fixed small parameter  $\theta$ . In such cubes, one may replace the original function  $u$  with a constant function, since on the one hand the new jump is less than the original jump times  $\theta^{-1}$ , and on the other hand the total volume of these cubes is small as the length scale goes to 0. In the remaining cubes, where the relative jump is small, one applies Proposition 1.9: a Korn-Poincaré-type inequality holds up to a set of small volume, and in this small exceptional set the original function may be replaced by a suitable affine function without perturbing much its energy.

We prove also an approximation property for the amplitude of the jump  $[u](x) := u^+(x) - u^-(x)$  for  $x \in J_u$ , which might be useful in cohesive fracture models. Notice that  $[u]$  is not integrable in  $J_u$  with respect to  $\mathcal{H}^{n-1}$  for a general  $u \in GSBD^p(\Omega)$ . Thus, we show that every truncation of  $u_k^\pm - u^\pm$  tends to 0 in  $L^1(J_{u_k} \cup J_u)$ , and the analogous property for the traces at the reduced boundary of  $\Omega$  (see (3.1d)). We employ a fine estimate from [6] for the truncation of the trace components in any direction, whose symmetric gradient is a bounded measure. An approximation of this type is also proven in [43], where  $u$  is integrable, and used in the  $\Gamma$ -convergence result [35] for cohesive fracture energies. We may as well consider in Theorem 3.1 smooth approximating functions in the sense of the aforementioned [24, Theorem 3.1] (which applies directly if  $\Omega$  is Lipschitz), with minor modifications in our proof.

In the last part of the work we present  $\Gamma$ -convergence results *à la* Ambrosio-Tortorelli for brittle fracture energies. First, we approximate (1) for every  $u$  which is  $GSBD^p$  in an open bounded set with finite perimeter (Theorem 4.1). Then we focus on the sum of (1) with suitable compliance terms, which prevent that the set of minimisers coincides with the constant displacements. In particular, we consider the cases of a mild fidelity term  $|u - g|^r$ , with  $r \in (0, p]$  (see Theorem 4.2), and of a Dirichlet boundary condition on a subset  $\partial_D \Omega$  of  $\partial \Omega$ , under some geometric conditions (see Theorem 4.4). In Theorem 4.2 we also prove existence of minimisers for the limit energy with fidelity term, and the convergence of quasi-minimisers for the approximating energies (existence of minimisers for the approximating energies is guaranteed if the domain is Lipschitz, see Remark 4.3). This follows from Proposition 4.5, in turn based on the argument of the compactness result [27, Theorem 11.1] (see also [43, Proposition 1]). The existence of minimisers for the (limit) Dirichlet problem has been recently shown by Friedrich and Solombrino [40, Theorem 6.2] in dimension 2, but is still unknown in dimension  $n > 2$ . By Theorem 4.4, it would be enough to prove a uniform bound for  $\psi(|u_k|)$ , for a suitable  $\psi$  as above, where  $u_k$  are quasi-minimisers of the approximate Dirichlet problems.

We conclude this introduction by mentioning some other problems for which density results as Theorem 3.1 are useful. For instance, [38, Theorem 2.5] is applied in [39] for the derivation of linearised Griffith energies from nonlinear models, while [22] is employed in [23] and [19] to prove existence of minimisers for the set function that is the strong counterpart of (1). More precisely, in [23] the setting is 2-dimensional and  $W$  may have  $p$ -growth for any  $p > 1$ , while [19] considers the case  $\Omega \subset \mathbb{R}^n$  with  $W$  quadratic (a fidelity term in the energy is required in these works). Moreover, [15, 16] are useful in other  $\Gamma$ -convergence approximations of brittle fracture energies, such as [47, 48].

The paper is organised as follows. In Section 1 we introduce notation, functional spaces, and some technical tools useful in the following, as the reflection property Lemma 1.8. In Section 2 and Section 3 we prove the rough and the main density results, respectively. Section 4 is devoted to the applications.

## 1. NOTATION AND PRELIMINARIES

For every  $x \in \mathbb{R}^n$  and  $\varrho > 0$  let  $B_\varrho(x)$  be the open ball with center  $x$  and radius  $\varrho$ . For  $x, y \in \mathbb{R}^n$ , we use the notation  $x \cdot y$  for the scalar product and  $|x|$  for the norm. We denote by  $\mathcal{L}^n$  and  $\mathcal{H}^k$  the  $n$ -dimensional Lebesgue measure and the  $k$ -dimensional Hausdorff measure. For any locally compact subset  $B$  of  $\mathbb{R}^n$ , the space of bounded  $\mathbb{R}^m$ -valued Radon measures on  $B$  is denoted by  $\mathcal{M}_b(B; \mathbb{R}^m)$ . For  $m = 1$  we write  $\mathcal{M}_b(B)$  for  $\mathcal{M}_b(B; \mathbb{R})$  and  $\mathcal{M}_b^+(B)$  for the subspace of positive measures of  $\mathcal{M}_b(B)$ . For every  $\mu \in \mathcal{M}_b(B; \mathbb{R}^m)$ , its total variation is denoted by  $|\mu|(B)$ . We denote by  $\chi_E$  the indicator function of any  $E \subset \mathbb{R}^n$ , which is 1 on  $E$  and 0 otherwise.

**Definition 1.1.** Let  $A \subset \mathbb{R}^n$ ,  $v: A \rightarrow \mathbb{R}^m$  an  $\mathcal{L}^n$ -measurable function,  $x \in \mathbb{R}^n$  such that

$$\limsup_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^n(A \cap B_\varrho(x))}{\varrho^n} > 0.$$

A vector  $a \in \mathbb{R}^m$  is the *approximate limit* of  $v$  as  $y$  tends to  $x$  if for every  $\varepsilon > 0$

$$\lim_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^n(A \cap B_\varrho(x) \cap \{|v - a| > \varepsilon\})}{\varrho^n} = 0,$$

and then we write

$$\operatorname{ap\,lim}_{y \rightarrow x} v(y) = a. \quad (1.1)$$

*Remark 1.2.* Let  $A$ ,  $v$ ,  $x$ , and  $a$  be as in Definition 1.1 and let  $\psi$  be a homeomorphism between  $\mathbb{R}^m$  and a bounded open subset of  $\mathbb{R}^m$ . Then (1.1) holds if and only if

$$\lim_{\varrho \rightarrow 0^+} \frac{1}{\varrho^n} \int_{A \cap B_\varrho(x)} |\psi(v(y)) - \psi(a)| \, dy = 0.$$

**Definition 1.3.** Let  $U \subset \mathbb{R}^n$  open, and  $v: U \rightarrow \mathbb{R}^m$  be  $\mathcal{L}^n$ -measurable. The *approximate jump set*  $J_v$  is the set of points  $x \in U$  for which there exist  $a, b \in \mathbb{R}^m$ , with  $a \neq b$ , and  $\nu \in \mathbb{S}^{n-1}$  such that

$$\operatorname{ap\,lim}_{(y-x) \cdot \nu > 0, y \rightarrow x} v(y) = a \quad \text{and} \quad \operatorname{ap\,lim}_{(y-x) \cdot \nu < 0, y \rightarrow x} v(y) = b.$$

The triplet  $(a, b, \nu)$  is uniquely determined up to a permutation of  $(a, b)$  and a change of sign of  $\nu$ , and is denoted by  $(v^+(x), v^-(x), \nu_\nu(x))$ . The jump of  $v$  is the function defined by  $[v](x) := v^+(x) - v^-(x)$  for every  $x \in J_v$ . Moreover, we define

$$J_v^1 := \{x \in J_v : |[v](x)| \geq 1\}. \quad (1.2)$$

*Remark 1.4.* By Remark 1.2,  $J_v$  and  $J_v^1$  are Borel sets and  $[v]$  is a Borel function. Moreover, by Lebesgue's differentiation theorem, it follows that  $\mathcal{L}^n(J_v) = 0$ .

**BV and BD functions.** If  $U \subset \mathbb{R}^n$  open, a function  $v \in L^1(U)$  is a *function of bounded variation* on  $U$ , and we write  $v \in BV(U)$ , if  $D_i v \in \mathcal{M}_b(U)$  for  $i = 1, \dots, n$ , where  $Dv = (D_1 v), \dots, D_n v$  is its distributional gradient. A vector-valued function  $v: U \rightarrow \mathbb{R}^m$  is  $BV(U; \mathbb{R}^m)$  if  $v_j \in BV(U)$  for every  $j = 1, \dots, m$ . The space  $BV_{\text{loc}}(U)$  is the space of  $v \in L^1_{\text{loc}}(U)$  such that  $D_i v \in \mathcal{M}_b(U)$  for  $i = 1, \dots, n$ .

A  $\mathcal{L}^n$ -measurable bounded set  $E \subset \mathbb{R}^n$  is a set of *finite perimeter* if  $\chi_E$  is a function of bounded variation. The *reduced boundary* of  $E$ , denoted by  $\partial^* E$ , is the set of points  $x \in \operatorname{supp} |D\chi_E|$  such that the limit  $\nu_E(x) := \lim_{\varrho \rightarrow 0^+} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))}$  exists and satisfies  $|\nu_E(x)| = 1$ . The reduced boundary is countably  $(\mathcal{H}^{n-1}, n-1)$  rectifiable, and the function  $\nu_E$  is called *generalised inner normal* to  $E$ .

A function  $v \in L^1(U; \mathbb{R}^n)$  belongs to the space of *functions of bounded deformation* if its distributional symmetric gradient  $Eu$  belongs to  $\mathcal{M}_b(U; \mathbb{R}^n)$ . It is well known (see [2, 51]) that for  $v \in BD(U)$ ,  $J_v$  is countably  $(\mathcal{H}^{n-1}, n-1)$  rectifiable, and that

$$Ev = E^a v + E^c v + E^j v, \quad (1.3)$$

where  $E^a v$  is absolutely continuous with respect to  $\mathcal{L}^n$ ,  $E^c v$  is singular with respect to  $\mathcal{L}^n$  and such that  $|E^c v|(B) = 0$  if  $\mathcal{H}^{n-1}(B) < \infty$ , while  $E^j v$  is concentrated on  $J_v$ . The density of  $E^a v$  with

respect to  $\mathcal{L}^n$  is denoted by  $e(u)$ , and we have that (see [2, Theorem 4.3] and recall (1.1)) for  $\mathcal{L}^n$ -a.e.  $x \in U$

$$\operatorname{ap} \lim_{y \rightarrow x} \frac{(v(y) - v(x) - e(v)(x)(y - x)) \cdot (y - x)}{|y - x|^2} = 0. \quad (1.4)$$

The space  $SBD(U)$  is the subspace of all functions  $v \in BD(U)$  such that  $E^c v = 0$ , while for  $p \in (1, \infty)$

$$SBD^p(U) := \{v \in SBD(U) : e(v) \in L^p(\Omega; \mathbb{M}_{sym}^{n \times n}), \mathcal{H}^{n-1}(J_v) < \infty\}.$$

Analogous properties hold for  $BV$ , as the countable rectifiability of the jump set and the decomposition of  $Dv$ , and the spaces  $SBV(U; \mathbb{R}^m)$  and  $SBV^p(U; \mathbb{R}^m)$  are defined similarly, with  $\nabla u$ , the density of  $D^a v$ , in place of  $e(u)$ . For a complete treatment of  $BV$ ,  $SBV$  functions and  $BD$ ,  $SBD$  functions, we refer to [3] and to [2, 8, 6, 51], respectively.

***GBD functions.*** We now recall the definition and the main properties of the space  $GBD$  of *generalised functions of bounded deformation*, introduced in [27], referring to that paper for a general treatment and more details. Since the definition of  $GBD$  is given by slicing (differently from the definition of  $GBV$ , cf. [32, 1]), we introduce before some notation for slicing.

Fixed  $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ , for any  $y \in \mathbb{R}^n$  and  $B \subset \mathbb{R}^n$  let

$$\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}, \quad B_y^\xi := \{t \in \mathbb{R} : y + t\xi \in B\},$$

and for every function  $v : B \rightarrow \mathbb{R}^n$  and  $t \in B_y^\xi$  let

$$v_y^\xi(t) := v(y + t\xi), \quad \widehat{v}_y^\xi(t) := v_y^\xi(t) \cdot \xi.$$

**Definition 1.5.** Let  $\Omega \subset \mathbb{R}^n$  be bounded and open, and  $u : \Omega \rightarrow \mathbb{R}^n$  be  $\mathcal{L}^n$ -measurable. Then  $u \in GBD(\Omega)$  if there exists  $\lambda_u \in \mathcal{M}_b^+(\Omega)$  such that the following equivalent conditions hold for every  $\xi \in \mathbb{S}^{n-1}$ :

- (a) for every  $\tau \in C^1(\mathbb{R})$  with  $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$  and  $0 \leq \tau' \leq 1$ , the partial derivative  $D_\xi(\tau(u \cdot \xi)) = D(\tau(u \cdot \xi)) \cdot \xi$  belongs to  $\mathcal{M}_b(\Omega)$ , and for every Borel set  $B \subset \Omega$

$$|D_\xi(\tau(u \cdot \xi))|(B) \leq \lambda_u(B);$$

- (b)  $\widehat{u}_y^\xi \in BV_{\text{loc}}(\Omega_y^\xi)$  for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$ , and for every Borel set  $B \subset \Omega$

$$\int_{\Pi^\xi} \left( |D\widehat{u}_y^\xi|(B_y^\xi \setminus J_{\widehat{u}_y^\xi}^1) + \mathcal{H}^0(B_y^\xi \cap J_{\widehat{u}_y^\xi}^1) \right) d\mathcal{H}^{n-1}(y) \leq \lambda_u(B). \quad (1.5)$$

The function  $u$  belongs to  $GSBD(\Omega)$  if moreover  $\widehat{u}_y^\xi \in SBV_{\text{loc}}(\Omega_y^\xi)$  for every  $\xi \in \mathbb{S}^{n-1}$  and for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$ .

$GBD(\Omega)$  and  $GSBD(\Omega)$  are vector spaces, as stated in [27, Remark 4.6], and one has the inclusions  $BD(\Omega) \subset GBD(\Omega)$ ,  $SBD(\Omega) \subset GSBD(\Omega)$ , which are in general strict (see [27, Remark 4.5 and Example 12.3]). For every  $u \in GBD(\Omega)$  the *approximate jump set*  $J_u$  is still countably  $(\mathcal{H}^{n-1}, n-1)$ -rectifiable (cf. [27, Theorem 6.2]) and can be reconstructed from the jump of the slices  $\widehat{u}_y^\xi$  ([27, Theorem 8.1]). Indeed, for every  $C^1$  manifold  $M \subset \Omega$  with unit normal  $\nu$ , it holds that for  $\mathcal{H}^{n-1}$ -a.e.  $x \in M$  there exist the *traces*  $u_M^+(x)$ ,  $u_M^-(x) \in \mathbb{R}^n$  such that

$$\operatorname{ap} \lim_{\pm(y-x) \cdot \nu(x) > 0, y \rightarrow x} u(y) = u_M^\pm(x) \quad (1.6)$$

and they can be reconstructed from the traces of the one-dimensional slices (see [27, Theorem 5.2]).

*Remark 1.6.* The trace of  $GSBD$  functions on a given  $C^1$  manifold  $M \subset \Omega$  is linear. Indeed, let us fix  $\varepsilon > 0$ ,  $\eta > 0$ ,  $x \in M$ . Then there exists  $\bar{\varrho}$  such that for  $0 < \varrho < \bar{\varrho}$

$$\mathcal{L}^n(\Omega \cap B_\varrho^+(x) \cap \{|u - u_M^+(x)| > \varepsilon/2\}), \mathcal{L}^n(\Omega \cap B_\varrho^+(x) \cap \{|v - v_M^+(x)| > \varepsilon/2\}) < \eta/2,$$

where  $B_\varrho^+(x)$  is the half ball with radius  $\varrho$  positively oriented with respect to  $\nu(x)$ . Therefore, for  $0 < \varrho < \bar{\varrho}$  it holds that

$$\mathcal{L}^n(\Omega \cap B_\varrho^+(x) \cap \{|(u+v) - (u_M^+ + v_M^+)(x)| > \varepsilon\}) < \eta,$$

so that  $(u+v)_M^+(x) = u_M^+ + v_M^+(x)$ .

Every  $u \in GBD(\Omega)$  has an *approximate symmetric gradient*  $e(u) \in L^1(\Omega; \mathbb{M}_{sym}^{n \times n})$ , characterised by (1.4) and such that for every  $\xi \in \mathbb{S}^{n-1}$  and  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$

$$e(u)_y^\xi \xi \cdot \xi = \nabla \widehat{u}_y^\xi \quad \mathcal{L}^1\text{-a.e. on } \Omega_y^\xi. \quad (1.7)$$

Using this property, we observe the following.

**Lemma 1.7.** *For any  $u \in GSBD(\Omega)$  and  $A \in M^{n \times n}$ , with  $\det A \neq 0$ , the function*

$$u_A(x) := A^T u(Ax) \quad (1.8)$$

*belongs to  $GSBD(A^{-1}(\Omega))$ , with*

$$\lambda_{u_A}(B) = \lambda_u(A(B)), \quad (1.9)$$

*for any  $B \subset A^{-1}(\Omega)$  Borel, with  $\lambda$  and  $\lambda_A$  the measures in (1.5) corresponding to  $u$  and  $u_A$ , and*

$$\begin{aligned} \mathcal{H}^{n-1}(J_{u_A}) &= \mathcal{H}^{n-1}(A^{-1}(J_u)), \\ e(u_A(x)) &= A^T e(u)(Ax) A. \end{aligned} \quad (1.10)$$

*Proof.* Let us fix  $\xi \in \mathbb{S}^{n-1}$ . A straightforward computation shows that for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi_\xi$  and  $\mathcal{L}^1$ -a.e.  $t \in (A^{-1}(\Omega))_y^\xi$  we have

$$(u_A)_y^\xi(t) \cdot \xi = u_{Ay}^{A\xi}(t) \cdot A\xi. \quad (1.11)$$

Moreover, for any  $B \subset A^{-1}(\Omega)$ , we have that

$$B_y^\xi = (A(B))_{Ay}^{A\xi}.$$

This implies that, for any Borel set  $B \subset A^{-1}(\Omega)_y^\xi$

$$\left| D(\widehat{u}_A)_y^\xi \right| \left( B_y^\xi \setminus J_{(\widehat{u}_A)_y^\xi}^1 \right) + \mathcal{H}^0 \left( B_y^\xi \cap J_{(\widehat{u}_A)_y^\xi}^1 \right) = (\widehat{\mu}_u)_{Ay}^{A\xi}(A(B)), \quad (1.12)$$

where  $(\widehat{\mu}_u)_y^\xi$  is the measure in [27, Definition 4.8] for  $u$ . By Definition 1.5, [27, Definition 4.10, Remark 4.12], and (1.12), it follows that  $u_A \in GSBD(A^{-1}(\Omega))$  and that (1.9) holds.

By definition of  $u_A$  and of jump set, one has that  $x \in J_{u_A}$  if and only if  $Ax \in J_u$ , thus

$$\mathcal{H}^{n-1}(J_{u_A}) = \mathcal{H}^{n-1}(J_u).$$

In order to show the second condition in (1.10), we can use (1.7) which allows us to reconstruct the approximate symmetric gradient from the derivatives of the slices. Thus, by taking the derivative of (1.11) with respect to  $t$ , we deduce that for any  $\xi \in \mathbb{S}^{n-1}$

$$e(u_A)(x) \xi \cdot \xi = e(u)(Ax) A\xi \cdot A\xi.$$

Being  $e(u)$  and  $e(u_A)$  symmetric matrices, by the Polarisation Identity we obtain that for any  $\xi, \eta$  in  $\mathbb{S}^{n-1}$

$$e(u_A)(x) \xi \cdot \eta = e(u)(Ax) A\xi \cdot A\eta.$$

This gives (1.10) and completes the proof.  $\square$

We now show an extension result for  $GSBD^p$  functions on rectangles, basing on [49, Lemma 1]. A similar result is stated in [40, Lemma 5.2], in dimension 2 and for  $p = 2$ , and employed in [21, Lemma 3.4], in dimension 2 and for  $SBD^p$ . Notice that the proof of [40, Lemma 5.2] employs the density result, in dimension 2 and for  $p = 2$ , that we prove in the current paper in the general framework. We follow Nitsche's argument directly for  $GSBD$  functions, without using density results.

**Lemma 1.8.** *Let  $R \subset \mathbb{R}^n$  be an open rectangle,  $R'$  be the reflection of  $R$  with respect to one face  $F$  of  $R$ , and  $\widehat{R}$  be the union of  $R$ ,  $R'$ , and  $F$ . Let  $v \in GSBD^p(R)$ . Then  $v$  may be extended by a function  $\widehat{v} \in GSBD^p(\widehat{R})$  such that*

$$\mathcal{H}^{n-1}(J_{\widehat{v}} \cap F) = 0, \quad (1.13a)$$

$$\mathcal{H}^{n-1}(J_{\widehat{v}}) \leq c \mathcal{H}^{n-1}(J_v), \quad (1.13b)$$

$$\int_{\widehat{R}} |e(\widehat{v})|^p dx \leq c \int_R |e(v)|^p dx, \quad (1.13c)$$

for a suitable  $c > 0$  independent of  $R$  and  $v$ .

*Proof.* It is not restrictive to assume that  $F \subset \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = 0\}$ . Fix any  $\mu, \nu$  such that  $0 < \mu < \nu < 1$ , and let  $q := \frac{1+\nu}{\nu-\mu}$ . We define  $v'$  on  $R'$  by

$$v' := q v_{A_\mu} + (1-q) v_{A_\nu},$$

where  $v_A$  is defined in (1.8) and  $A_\mu = \text{diag}(1, \dots, 1, -\mu)$ ,  $A_\nu = \text{diag}(1, \dots, 1, -\nu)$ , so that

$$v'_i(x) := q v_i(A_\mu x) + (1-q) v_i(A_\nu x), \quad \text{for } i = 1, \dots, n-1,$$

$$v'_n(x) := -\mu q v_n(A_\mu x) - \nu(1-q) v_n(A_\nu x).$$

Notice that

$$-\mu q - \nu(1 - q) = 1,$$

and that  $v'$  is well defined since  $F$  is a horizontal hyperplane and  $0 < \mu < \nu < 1$ , so that  $A_\mu(R'), A_\nu(R') \subset R$ . Thus the extension  $\widehat{v}$  is

$$\widehat{v} := \begin{cases} v & \text{in } R, \\ v' & \text{in } R'. \end{cases}$$

By Lemma 1.7 we have that  $\widehat{v} \in GSBD(\widehat{R})$ , and (1.13a) follows from Remark 1.6, since every component of  $v'$  is a convex combination of the same component of  $v(A_\mu x)$  and  $v(A_\nu x)$  and  $A_\mu(F) = A_\nu(F) = F$ .

The first condition in (1.10) gives that

$$\mathcal{H}^{n-1}(J_{v'}) \leq \mathcal{H}^{n-1}(A_\mu^{-1}(J_v)) + \mathcal{H}^{n-1}(A_\nu^{-1}(J_v)),$$

and (1.13b) follows. By the second condition in (1.10) we deduce (1.13c). Notice that the constant  $c$  depends on  $p, \mu,$  and  $\nu$ , but is independent on  $R$  and  $v$ .  $\square$

Let us recall the following important result, proven in [17, Proposition 3]. Notice that the result is stated in  $SBD$ , but the proof, which is based on the Fundamental Theorem of Calculus along lines, still holds for  $GSBD$ , with small adaptations.

**Proposition 1.9.** *Let  $Q = (-r, r)^n$ ,  $Q' = (-r/2, r/2)^n$ ,  $u \in GSBD(Q)$ ,  $p \in [1, \infty)$ . Then there exist a Borel set  $\omega \subset Q'$  and an affine function  $a: \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $e(a) = 0$  such that  $\mathcal{L}^n(\omega) \leq cr\mathcal{H}^{n-1}(J_u)$  such that*

$$\int_{Q' \setminus \omega} (|u - a|^p)^{1^*} dx \leq cr^{(p-1)1^*} \left( \int_Q |e(u)|^p dx \right)^{1^*}. \quad (1.14)$$

If additionally  $p > 1$ , then there is  $q > 0$  (depending on  $p$  and  $n$ ) such that, for a given mollifier  $\varphi_r \in C_c^\infty(B_{r/4})$ ,  $\varphi_r(x) = r^{-n}\varphi_1(x/r)$ , the function  $v = u\chi_{Q' \setminus \omega} + a\chi_\omega$  obeys

$$\int_{Q''} |e(v * \varphi_r) - e(u) * \varphi_r|^p dx \leq c \left( \frac{\mathcal{H}^{n-1}(J_u)}{r^{n-1}} \right)^q \int_Q |e(u)|^p dx, \quad (1.15)$$

where  $Q'' = (-r/4, r/4)^n$ . The constant in (i) depends only on  $p$  and  $n$ , the one in (ii) also on  $\varphi_1$ .

*Remark 1.10.* Condition (i) is a Korn-Poincaré-type inequality, which guarantees the existence of an affine function  $a$  such that, up to a small exceptional set,  $u - a$  is controlled in a space better than  $L^p$ . The control in the optimal space  $L^{p^*}$  is obtained only if  $p = 1$ . Even on the exceptional set, the affine function  $a$  is in some sense “close in energy” to  $u$ , as follows from (ii).

*Remark 1.11.* By Hölder inequality and (1.14) it follows that

$$\int_{Q' \setminus \omega} |u - a|^p dx \leq \mathcal{L}^n(Q' \setminus \omega)^{1/n} \left( \int_{Q' \setminus \omega} (|u - a|^p)^{1^*} dx \right)^{1/1^*} \leq cr^p \int_Q |e(u)|^p dx \quad (1.16)$$

The following lemma will be employed in zones where the jump of  $u$  is small, compared to the side of the square. It will be useful to estimate, for two cubes with nonempty intersection, the difference of the corresponding affine functions.

**Lemma 1.12.** *For every  $\alpha_i \in \{-1, 0, 1\}^n$ , with  $\alpha_0 = 0$ , let  $z_i = \frac{r}{2}\alpha_i \in \mathbb{R}^n$  and  $Q_i, Q'_i, Q''_i$  be the  $n$ -dimensional cubes of center  $z_i$  and sidelength  $2r, r, r/2$ , respectively (assume  $r < 1$ ). Let  $u \in GSBD(B(0, 6r))$  and, for  $i = 0, \dots, 3^n$ , let  $a_i$  and  $\omega_i$  be the affine function and the exceptional set given by Proposition 1.9, corresponding to  $Q_i$ . Assume that for every  $i = 0, \dots, 3^n$*

$$\mathcal{H}^{n-1}(J_u \cap Q_i) \leq \theta r^{n-1}, \quad (1.17)$$

with  $\theta$  sufficiently small (for instance  $\theta \leq 1/(16c)$ , for  $c$  as in (i) of Proposition 1.9). Then there exists a constant  $C$ , depending only on  $p$  and  $n$ , such that for each  $i \neq 0$

$$\|a_0 - a_i\|_{L^\infty(Q_0 \cap Q_i; \mathbb{R}^n)}^p \leq Cr^{-(n-p)} \int_{Q_0 \cup Q_i} |e(u)|^p dx, \quad (1.18)$$



*Proof.* By (1.17) we have that

$$\mathcal{L}^n(\omega_0 \cup \omega_i) \leq cr (\mathcal{H}^{n-1}(J_u \cap Q_0) + \mathcal{H}^{n-1}(J_u \cap Q_i)) \leq 2c\theta r^n \leq \frac{\mathcal{L}^n(Q'_0 \cap Q'_i)}{4}.$$

Therefore, following the argument of [20, Lemma 4.3] for the rectangles  $Q'_0 \cap Q'_i$  in place of  $B$  (notice that for a given  $i$  the shape of these rectangles is the same independently of  $r$ , that is the ratios between the sidelengths are independent of  $r$ ) one has that for any affine function  $a: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\mathcal{L}^n(Q'_0 \cap Q'_i) \|a\|_{L^\infty(Q'_0 \cap Q'_i; \mathbb{R}^n)} \leq \bar{c} \int_{(Q'_0 \setminus \omega_0) \cap (Q'_i \setminus \omega_i)} |a| \, dx,$$

for  $\bar{c} > 0$  depending only on  $n$  (and on  $i$ ). By Hölder's inequality we deduce that for any  $q \in [1, \infty)$

$$\mathcal{L}^n(Q'_0 \cap Q'_i) \|a\|_{L^\infty(Q'_0 \cap Q'_i; \mathbb{R}^n)}^q \leq \bar{c}^q \int_{(Q'_0 \setminus \omega_0) \cap (Q'_i \setminus \omega_i)} |a|^q \, dx.$$

For  $q = p1^*$  and  $a = a_0 - a_i$  we get

$$\mathcal{L}^n(Q'_0 \cap Q'_i) \|a_0 - a_i\|_{L^\infty(Q'_0 \cap Q'_i; \mathbb{R}^n)}^{p1^*} \leq \bar{c}^{p1^*} \int_{(Q'_0 \setminus \omega_0) \cap (Q'_i \setminus \omega_i)} |a_0 - a_i|^{p1^*} \, dx. \quad (1.19)$$

By triangle inequality and by (1.14) it follows that

$$\int_{(Q'_0 \setminus \omega_0) \cap (Q'_i \setminus \omega_i)} |a_0 - a_i|^{p1^*} \, dx \leq \int_{(Q'_0 \setminus \omega_0) \cap (Q'_i \setminus \omega_i)} (|u - a_0| + |u - a_i|)^{p1^*} \, dx \leq cr^{(p-1)1^*} \left( \int_{Q_0 \cup Q_i} |e(u)|^p \, dx \right)^{1^*}. \quad (1.20)$$

Moreover, since  $a_0 - a_i$  is an affine function, we have that

$$\|a_0 - a_i\|_{L^\infty(Q_0 \cap Q_i; \mathbb{R}^n)} \leq \bar{C} \|a_0 - a_i\|_{L^\infty(Q'_0 \cap Q'_i; \mathbb{R}^n)} \quad (1.21)$$

for a constant  $\bar{C}$  depending only on the ratio between  $\mathcal{L}^n(Q_0 \cap Q_i)$  and  $\mathcal{L}^n(Q'_0 \cap Q'_i)$ , which is independent of  $r$ .

We deduce (1.18) by collecting (1.19), (1.20), and (1.21).  $\square$

## 2. A FIRST APPROXIMATION RESULT WITH A BAD CONSTANT

As in [15, 43, 22], a first step toward the main density result consists in a rough approximation in the sense of energy. In particular, in this section we construct an approximating sequence of functions whose jumps are controlled in terms of the original jump by a multiplicative parameter. We employ this result in the next section for subdomains where the jump of the original function is very small, so that the total increase of energy will be small too.

**Theorem 2.1.** *Let  $\Omega, \tilde{\Omega}$  be bounded open subsets of  $\mathbb{R}^n$ , with  $\bar{\Omega} \subset \tilde{\Omega}$ ,  $p \in [1, \infty)$ ,  $\theta \in (0, 1)$ , and let  $u \in GSB D^p(\tilde{\Omega})$ . Then there exist  $u_k \in SBV^p(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  and  $E_k \subset \Omega$  Borel sets such that  $J_{u_k}$  is included in a finite union of  $(n-1)$ -dimensional closed cubes,  $u_k \in W^{1,\infty}(\Omega \setminus J_{u_k}; \mathbb{R}^n)$ , and the following hold:*

$$\lim_{k \rightarrow \infty} \mathcal{L}^n(E_k) = \lim_{k \rightarrow \infty} \int_{\Omega \setminus E_k} |u_k - u|^p \, dx = 0, \quad (2.1a)$$

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |e(u_k)|^p \, dx \leq \int_{\Omega} |e(u)|^p \, dx, \quad (2.1b)$$

$$\mathcal{H}^{n-1}(J_{u_k} \cap \Omega) \leq C \theta^{-1} \mathcal{H}^{n-1}(J_u \cap \tilde{\Omega}), \quad (2.1c)$$

for suitable  $q > 0$ ,  $C > 0$  independent of  $\theta$ . In particular,  $u_k$  converge to  $u$  in measure in  $\Omega$ . Moreover, if  $\int_{\Omega} \psi(|u|) \, dx$  is finite for  $\psi: [0, \infty) \rightarrow [0, \infty)$  increasing, continuous, with (for  $C_\psi > 0$ )

$$\psi(0) = 0, \quad \psi(s+t) \leq C_\psi(\psi(s) + \psi(t)), \quad \psi(s) \leq C_\psi(1 + |s|^p), \quad \lim_{s \rightarrow \infty} \psi(s) = \infty, \quad (\text{HP}\psi)$$

then

$$\lim_{k \rightarrow \infty} \int_{\Omega} \psi(|u_k - u|) \, dx = 0. \quad (2.1d)$$

The proof of the result above employs a technique introduced in [18], which is based on Proposition 1.9. The idea is to partition the domain into cubes of side  $\frac{1}{k}$  and to distinguish, at any scale, the cubes where the ratio between the perimeter and the jump of  $u$  is greater than the parameter  $\theta$ .

In such cubes, one may replace the original function  $u$  with a constant function, since on the one hand the new jump is controlled by the original jump, and on the other hand the total volume of these cubes is small as the length scale goes to 0.

In the remaining cubes, where the relative jump is small, one applies Proposition 1.9: a Korn-Poincaré-type inequality holds up to a set of small volume, and in this small exceptional set the original function may be replaced by a suitable affine function without perturbing much its energy. We need  $u$  be defined in a larger set  $\tilde{\Omega}$  since we will take convolutions of the original function.

*Proof of Theorem 2.1.* Let  $\tilde{\Omega} \subset \tilde{\Omega}$ ,  $p \in [1, \infty)$ ,  $\theta \in (0, 1)$ , and  $u \in GSBD^p(\tilde{\Omega}) \cap L^p(\tilde{\Omega}; \mathbb{R}^n)$ . Let us fix an integer  $k$  with  $k > \frac{8\sqrt{n}}{\text{dist}(\partial\tilde{\Omega}, \partial\Omega)}$ , let  $\varphi$  be a smooth radial function with compact support in the unit ball  $B(0, 1)$ , and let  $\varphi_k(x) = k^n \varphi(kx)$ .

**Good and bad nodes.** For any  $z \in (2k^{-1})\mathbb{Z}^n \cap \Omega$  consider the cubes of center  $z$

$$\begin{aligned} q_z^k &:= z + (-k^{-1}, k^{-1})^n, & \tilde{q}_z^k &:= z + (-2k^{-1}, 2k^{-1})^n, \\ Q_z^k &:= z + (-4k^{-1}, 4k^{-1})^n, & \tilde{Q}_z^k &:= z + (-8k^{-1}, 8k^{-1})^n. \end{aligned}$$

Let us define the sets of the “good” and of the “bad” nodes

$$G^k := \{z \in (2k^{-1})\mathbb{Z}^n \cap \Omega : \mathcal{H}^{n-1}(J_u \cap Q_z^k) \leq \theta k^{-(n-1)}\}, \quad B^k := (2k^{-1})\mathbb{Z}^n \cap \Omega \setminus G^k, \quad (2.2)$$

such that the amount of jump of  $u$  is small in a big neighbourhood of any  $z \in G^k$ , and the corresponding subsets of  $\tilde{\Omega}$

$$\Omega_g^k := \bigcup_{z \in G^k} q_z^k, \quad \tilde{\Omega}_b^k := \bigcup_{z \in B^k} Q_z^k.$$

Notice that  $\tilde{\Omega}_b^k$  is the union of cubes of sidelength  $8k^{-1}$ , while  $\Omega_g^k$  is the union of cubes of sidelength  $2k^{-1}$ , so that  $\tilde{\Omega} \setminus \Omega_g^k \subset \tilde{\Omega}_b^k$ . More precisely,

$$\tilde{\Omega} \setminus \Omega_g^k + B(0, k^{-1}) \subset \tilde{\Omega}_b^k \quad (2.3)$$

Indeed, by construction, a row of “boundary” cubes of  $\Omega_g^k$  belongs to  $\tilde{\Omega}_b^k$ . Moreover, by (2.2) the set  $B^k$  has at most  $\mathcal{H}^{n-1}(J_u) k^{n-1} \theta^{-1}$  elements, so that

$$\mathcal{L}^n(\tilde{\Omega}_b^k) \leq 16^n \frac{\mathcal{H}^{n-1}(J_u)}{k \theta}. \quad (2.4)$$

Let us apply Proposition 1.9 for any  $z \in G^k$ . Then there exist a set  $\omega_z \subset \tilde{q}_z^k$ , with

$$\mathcal{L}^n(\omega_z) \leq ck^{-1} \mathcal{H}^{n-1}(J_u \cap Q_z^k) \leq c\theta k^{-n}, \quad (2.5)$$

and an affine function  $a_z : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with  $e(a_z) = 0$ , such that

$$\int_{\tilde{q}_z^k \setminus \omega_z} (|u - a_z|^p)^{1^*} dx \leq ck^{-(p-1)1^*} \left( \int_{Q_z^k} |e(u)|^p dx \right)^{1^*} \quad (2.6)$$

and, letting  $v_z := u \chi_{\tilde{q}_z^k \setminus \omega_z} + a_z \chi_{\omega_z}$ ,

$$\begin{aligned} \int_{q_z^k} |e(v_z * \varphi_k) - e(u) * \varphi_k|^p dx &\leq c \left( \mathcal{H}^{n-1}(J_u \cap Q_z^k) k^{n-1} \right)^q \int_{Q_z^k} |e(u)|^p dx \\ &\leq c\theta^q \int_{Q_z^k} |e(u)|^p dx, \end{aligned} \quad (2.7)$$

for a suitable  $q > 0$  depending on  $p$  and  $n$ .

Let

$$\omega^k := \bigcup_{z \in G^k} \omega_z, \quad E_k := \tilde{\Omega}_b^k \cup \omega^k.$$

Then

$$\lim_{k \rightarrow \infty} \mathcal{L}^n(E_k) = 0, \quad (2.8)$$

by (2.4) and (2.5), which gives  $\mathcal{L}^n(\omega^k) \leq ck^{-1} \sum_{z \in G^k} \mathcal{H}^{n-1}(J_u \cap Q_z^k) \leq c\mathcal{H}^{n-1}(J_u) k^{-1}$ .

We split the set of good nodes in the two subsets

$$G_1^k := \{z \in G^k : \mathcal{H}^{n-1}(J_u \cap Q_z^k) \leq k^{-(n-\frac{1}{2})}\}, \quad G_2^k := G^k \setminus G_1^k.$$

Notice that  $G_1^k$  are the good nodes for which the condition on  $J_u$  is satisfied for  $k^{-\frac{1}{2}}$  in place of  $\theta$ . For each  $z \in G_1^k$ , we have that (2.5) and (2.7) hold with  $k^{-\frac{1}{2}}$  in place of  $\theta$ , namely

$$\mathcal{L}^n(\omega_z) \leq c k^{-(n+\frac{1}{2})}, \quad (2.9a)$$

$$\int_{q_z^k} |e(v_z * \varphi_k) - e(u) * \varphi_k|^p dx \leq c k^{-\frac{q}{2}} \int_{Q_z^k} |e(u)|^p dx. \quad (2.9b)$$

Let us introduce also

$$\begin{aligned} \tilde{G}_1^k &:= \{z \in G^k : \bar{z} \in G_1^k \text{ for each } \bar{z} \in (2k^{-1})\mathbb{Z}^n \text{ with } \|z - \bar{z}\|_\infty = 2k^{-1}\}, \\ \tilde{G}_2^k &:= \{z \in G^k : \text{there exists } \bar{z} \in G_2^k \text{ with } \|z - \bar{z}\|_\infty = 2k^{-1}\}, \end{aligned} \quad (2.10)$$

where  $\|z - \bar{z}\|_\infty := \sup_{1 \leq i \leq n} |z_i - \bar{z}_i|$  is the  $L^\infty$  norm of the vector  $z - \bar{z}$ .

Arguing as already done for  $\tilde{\Omega}_b^k$ , we get that  $G_2^k$  has at most  $\mathcal{H}^{n-1}(J_u) k^{n-\frac{1}{2}}$  elements, so  $\tilde{G}_2^k$  has at most  $(3^n - 1)\mathcal{H}^{n-1}(J_u) k^{n-\frac{1}{2}}$  elements, and

$$\mathcal{L}^n(\tilde{\Omega}_{g,2}^k) \leq C k^{-\frac{1}{2}}, \quad \text{for } \tilde{\Omega}_{g,2}^k := \bigcup_{z_j \in \tilde{G}_2^k} \tilde{Q}_{z_j}. \quad (2.11)$$

**The approximating functions.** Let  $G^k = (z_j)_{j \in J}$ , so that we order (arbitrarily) the elements of  $G^k$ , and let us define

$$\tilde{u}_k := \begin{cases} u & \text{in } \tilde{\Omega} \setminus \omega^k, \\ a_{z_j} & \text{in } \omega_{z_j} \setminus \bigcup_{i < j} \omega_{z_i}, \end{cases} \quad (2.12)$$

and

$$u_k := (\tilde{u}_k * \varphi_k) \chi_{\Omega \setminus \tilde{\Omega}_b^k}. \quad (2.13)$$

These are the approximating functions for the original  $u$ , for which we are going to prove the properties of the theorem.

By construction,  $u_k \in SBVP(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ ,  $J_{u_k} \subset \bigcup_{z \in B^k} \partial Q_z^k$ , which is a finite union of  $(n-1)$ -dimensional closed cubes, and  $u_k \in W^{1,\infty}(\Omega \setminus J_{u_k}; \mathbb{R}^n)$ .

**Proof of (2.1c).** For any  $z \in B^k$  we have that

$$\mathcal{H}^{n-1}(\partial Q_z^k) = C(n) k^{-(n-1)} \leq C(n) \theta^{-1} \mathcal{H}^{n-1}(J_u \cap Q_z^k),$$

so that (2.1c) follows by summing over  $z \in B^k$ . Notice that we use here the fact that the cubes  $Q_z^k$  are finitely overlapping; this will be done different times also in the following (also for the cubes  $\tilde{q}_z^k$ ,  $\tilde{Q}_z^k$ ).

To ease the reading, in the following we denote  $\omega_{z_j}$  by  $\omega_j$ , and the same for  $a_{z_j}$ ,  $v_{z_j}$ . We denote also  $q_{z_j}^k$  by  $q_j$ , and the same for  $\tilde{q}_{z_j}^k$ ,  $Q_{z_j}^k$ ,  $\tilde{Q}_{z_j}^k$ . Moreover, for any  $g: \Omega \rightarrow \mathbb{R}^n$ ,  $B \subset \Omega$ , and  $q \in [1, \infty]$  we write  $\|g\|_{L^q(B)}$  instead of  $\|g\|_{L^q(B; \mathbb{R}^n)}$ .

**Proof of (2.1a).** In order to prove (2.1a) let us fix  $j \in J$  such that  $q_j \subset \Omega \setminus \tilde{\Omega}_b^k$ . By triangle inequality

$$\|u_k - u\|_{L^p(q_j \setminus \omega^k)} \leq \|u_k - a_j\|_{L^p(q_j \setminus \omega^k)} + \|u - a_j\|_{L^p(q_j \setminus \omega^k)} \quad (2.14)$$

Notice that

$$u_k - a_j = \varphi_k * (\tilde{u}_k - a_j) \quad \text{in } \Omega \setminus \tilde{\Omega}_b^k,$$

by definition of  $\tilde{u}_k$  and since  $\varphi_k * a_j = a_j$ , being  $\varphi$  a radial function.

By (1.16) we get

$$\|u - a_j\|_{L^p(q_j \setminus \omega^k)} \leq \|u - a_j\|_{L^p(\tilde{q}_j \setminus \omega_j)} \leq c k^{-1} \left( \int_{Q_j} |e(u)|^p dx \right)^{1/p}. \quad (2.15)$$

We now estimate the first term on the right hand side of (2.14) as follows:

$$\begin{aligned} \|u_k - a_j\|_{L^p(q_j \setminus \omega^k)} &\leq \|u_k - a_j\|_{L^p(q_j)} = \|\varphi_k * ((\tilde{u}_k - a_j) \chi_{\tilde{q}_j})\|_{L^p(q_j)} \\ &\leq \|\varphi_k * ((u - a_j) \chi_{\tilde{q}_j \setminus \omega^k})\|_{L^p(q_j)} + \|\varphi_k * ((\tilde{u}_k - a_j) \chi_{\tilde{q}_j \cap \omega^k})\|_{L^p(q_j)} \\ &\leq \|u - a_j\|_{L^p(\tilde{q}_j \setminus \omega^k)} + \|\tilde{u}_k - a_j\|_{L^p(\tilde{q}_j \cap \omega^k)}. \end{aligned} \quad (2.16)$$

Notice that we have used the fact that  $q_j + \text{supp } \varphi_k \subset q_j + B(0, k^{-1}) \subset \tilde{q}_j$ . The first term on the right hand side of (2.16) is estimated by (2.15). As for the second one we have, by definition of  $\tilde{u}_k$ , that

$$\int_{\tilde{q}_j \cap \omega^k} |\tilde{u}_k - a_j|^p dx = \sum_{i < j} \int_{\omega_j \cap \tilde{\omega}_i} |a_i - a_j|^p dx + \sum_{i > j} \int_{\tilde{q}_j \cap \tilde{\omega}_i} |a_i - a_j|^p dx, \quad (2.17)$$

where  $\tilde{\omega}_i := \omega_i \setminus (\bigcup_{h < i} \omega_h)$ . Now, the sum above involve at most  $3^n - 1$  terms corresponding to the centers  $z_i$  with  $z_j - z_i = 2k^{-1} \alpha_i$  and  $\alpha_i \in \{-1, 0, 1\}^n$ , because for any other  $z_h$  we have  $\tilde{q}_j \cap \omega_h \subset \tilde{q}_j \cap \tilde{q}_h = \emptyset$ , and every term in (2.17) is less than

$$\begin{aligned} \int_{\tilde{q}_j \cap \tilde{\omega}_i} |a_i - a_j|^p dx &\leq \mathcal{L}^n(\omega_i) \|a_i - a_j\|_{L^\infty(Q_i \cap Q_j)}^p \leq C \theta k^{-p} \int_{Q_i \cup Q_j} |e(u)|^p dx \\ &\leq C \theta k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx, \end{aligned} \quad (2.18)$$

by using (1.18) and (2.5). Therefore

$$\|\tilde{u}_k - a_j\|_{L^p(\tilde{q}_j \cap \omega^k)} \leq C \theta^{1/p} k^{-1} \left( \int_{\tilde{Q}_j} |e(u)|^p dx \right)^{1/p}. \quad (2.19)$$

In preparation to the proof of (2.1b) and (2.1d), we remark that if  $z_j \in \tilde{G}_1^k$  then

$$\|\tilde{u}_k - a_j\|_{L^p(\tilde{q}_j \cap \omega^k)} \leq C k^{-(1+\frac{1}{2p})} \left( \int_{\tilde{Q}_j} |e(u)|^p dx \right)^{1/p}, \quad (2.20)$$

namely (2.19) holds true for  $k^{-\frac{1}{2}}$  in place of  $\theta$ . Indeed, one employs (2.9a) for every  $i$  in (2.18) ( $z_i \in G_1^k$  for any  $i$  therein, by definition of  $\tilde{G}_1^k$ ).

Collecting (2.14), (2.15), (2.16), (2.19), and summing on  $j$ , we deduce

$$\|u_k - u\|_{L^p(\Omega \setminus E_k)} \leq C k^{-1} \left( \int_{\Omega} |e(u)|^p dx \right)^{1/p},$$

which gives (2.1a) together with (2.8).

**Proof of (2.1d).** As above, let us fix  $j \in J$  such that  $q_j \subset \Omega \setminus \tilde{\Omega}_b^k$ , and let  $\psi$  as in the statement of the theorem. Then

$$\int_{q_j \cap \omega^k} \psi(|u_k - u|) dx \leq C_\psi \int_{q_j \cap \omega^k} \psi(|u_k - a_j|) dx + C_\psi \int_{q_j \cap \omega^k} \psi(|u - a_j|) dx, \quad (2.21)$$

For the first term in the right hand side above we have

$$\begin{aligned} \int_{q_j \cap \omega^k} \psi(|u_k - a_j|) dx &\leq C_\psi \mathcal{L}^n(q_j \cap \omega^k) + C_\psi \int_{q_j \cap \omega^k} |u_k - a_j|^p dx \\ &\leq C_\psi \mathcal{L}^n(q_j \cap \omega^k) + C k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx, \end{aligned} \quad (2.22)$$

by (2.15), (2.16), and (2.19) (that control  $\int_{q_j} |u_k - a_j|^p dx$ , see the first inequality in (2.16), and then

$\int_{q_j \cap \omega^k} |u_k - a_j|^p dx$ ). As for the second term in the right hand side of (2.21), it holds that

$$\int_{q_j \cap \omega^k} \psi(|u - a_j|) dx \leq C_\psi \int_{q_j \cap \omega^k} \psi(|u|) dx + C_\psi \int_{q_j \cap \omega^k} \psi(|a_j|) dx,$$

and

$$\begin{aligned} \int_{q_j \cap \omega^k} \psi(|a_j|) dx &\leq C \frac{\mathcal{L}^n(q_j \cap \omega^k)}{\mathcal{L}^n(q_j)} \int_{q_j} \psi(|a_j|) dx \leq C \theta \int_{q_j} \psi(|a_j|) dx \leq C C_\psi \theta \int_{q_j} (\psi(|u|) + \psi(|u - a_j|)) dx \\ &\leq C C_\psi \theta \left( \int_{q_j} \psi(|u|) dx + \int_{q_j \cap \omega^k} \psi(|u - a_j|) dx + C_\psi \mathcal{L}^n(q_j \setminus \omega^k) + C_\psi k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx \right), \end{aligned}$$

by (2.15). Being  $\theta$  small, by the two previous inequalities we get that

$$\int_{q_j \cap \omega^k} \psi(|u - a_j|) dx \leq C \left( \int_{q_j \cap \omega^k} \psi(|u|) dx + \theta \int_{q_j} \psi(|u|) dx + \theta \mathcal{L}^n(q_j \setminus \omega^k) + k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx \right), \quad (2.23)$$

where powers of  $C_\psi$  have been absorbed in  $C$ . We now collect (2.21), (2.22), (2.23), to get

$$\begin{aligned} \int_{q_j \cap \omega^k} \psi(|u_k - u|) dx &\leq C \left( \mathcal{L}^n(q_j \cap \omega^k) + k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx + \int_{q_j \cap \omega^k} \psi(|u|) dx + \theta \int_{q_j} \psi(|u|) dx \right. \\ &\quad \left. + \theta \mathcal{L}^n(q_j \setminus \omega^k) \right). \end{aligned}$$

Again, notice that if  $z_j \in \tilde{G}_1^k$ , then the inequality above holds for  $k^{-\frac{1}{2}}$  in place of  $\theta$  (indeed  $\frac{\mathcal{L}^n(q_j \cap \omega^k)}{\mathcal{L}^n(q_j)} \leq C k^{-\frac{1}{2}}$  in the estimate before (2.23)).

Let us sum over  $j \in J$ , distinguishing the centers in  $\tilde{G}_1^k$  and the remaining ones, that we may assume in  $\tilde{G}_2^k$ , recalling (2.3) and the definition of  $u_k$  (2.13). We deduce that

$$\begin{aligned} \int_{E_k} \psi(|u_k - u|) dx &\leq C \left( \mathcal{L}^n(E_k) + k^{-p} \int_{\tilde{\Omega}} |e(u)|^p dx + \int_{E_k} \psi(|u|) dx + \theta \int_{\tilde{\Omega}_{g,2}^k} \psi(|u|) dx \right. \\ &\quad \left. + k^{-\frac{1}{2}} \int_{\tilde{\Omega}} \psi(|u|) dx + k^{-\frac{1}{2}} \mathcal{L}^n(\tilde{\Omega}) + \theta \mathcal{L}^n \left( \bigcup_{z_j \in \tilde{G}_2^k} q_{z_j} \right) \right), \end{aligned}$$

By (2.8), (2.11), and since  $\psi(|u|) \in L^1(\tilde{\Omega})$ , it follows that

$$\lim_{k \rightarrow \infty} \int_{E_k} \psi(|u_k - u|) dx = 0. \quad (2.24)$$

Eventually, by (2.1a) and (HP $\psi$ )

$$\lim_{k \rightarrow \infty} \int_{\tilde{\Omega} \setminus E_k} \psi(|u_k - u|) dx = 0. \quad (2.25)$$

Indeed,  $\psi = \psi_1 + \psi_2$  for suitable  $0 \leq \psi_1 \leq M_\psi$ , and  $0 < \psi_2(s) \leq M_\psi |s|^p$ , with  $M_\psi > 0$ . Since  $\psi_1, \psi_2 \geq 0$ , (2.1a), (HP $\psi$ ) imply that  $v_k^i := \chi_{\tilde{\Omega} \setminus E_k} \psi_i(|u_k - u|)$  converges to 0 pointwise for  $\mathcal{L}^n$ -a.e.  $x \in \tilde{\Omega}$ , for  $i = 1, 2$ . Being  $\int_{\tilde{\Omega} \setminus E_k} \psi(|u_k - u|) dx = \int_{\tilde{\Omega}} v_k^1 dx + \int_{\tilde{\Omega}} v_k^2 dx$ , we deduce (2.25) since the two integrals go to 0, the first by Dominated Convergence Theorem and the second by (2.1a).

**Proof of (2.1b).** First we show that, for  $v_j$  as in (2.7),

$$\int_{\tilde{q}_j} |\tilde{u}_k - v_j|^p dx \leq C \theta^{1/n} k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx, \quad \text{for } j \in J, \quad (2.26)$$

and

$$\int_{\tilde{q}_j} |\tilde{u}_k - v_j|^p dx \leq C k^{-(p + \frac{1}{2n})} \int_{\tilde{Q}_j} |e(u)|^p dx, \quad \text{for } j \in J \text{ such that } z_j \in \tilde{G}_1^k, \quad (2.27)$$

Let us first consider a general  $j \in J$ . Since  $\tilde{u}_k = u = v_j$  in  $\tilde{q}_j \setminus \omega^k$  and  $v_j = a_j$  in  $\omega_j$ , it holds that

$$\begin{aligned} \int_{\tilde{q}_j} |\tilde{u}_k - v_j|^p dx &= \int_{\tilde{q}_j \cap \omega^k} |\tilde{u}_k - v_j|^p dx \leq \int_{\tilde{q}_j \cap \omega^k} |\tilde{u}_k - a_j|^p dx + \int_{\tilde{q}_j \cap \omega^k} |v_j - a_j|^p dx \\ &\leq C \theta k^{-p} \int_{\tilde{Q}_j} |e(u)|^p dx + \int_{\tilde{q}_j \cap \omega^k \setminus \omega_j} |u - a_j|^p dx, \end{aligned}$$

where in the last inequality we have used (2.19). Moreover,

$$\int_{\tilde{q}_j \cap \omega^k \setminus \omega_j} |u - a_j|^p dx = \sum_{i \neq j} \int_{\tilde{q}_j \cap \tilde{\omega}_i} |u - a_j|^p dx \quad (2.28)$$

Arguing as done for (2.17), we deduce that the sum above involve at most  $3^n - 1$  terms, each of which is bounded by

$$\int_{\tilde{q}_j \cap \tilde{\omega}_i} |u - a_j|^p dx \leq c \mathcal{L}^n(\omega_i)^{1/n} k^{-(p-1)} \int_{Q_j} |e(u)|^p dx \leq C \theta^{1/n} k^{-p} \int_{Q_j} |e(u)|^p dx,$$

by employing Hölder inequality as in (1.16), and (2.5). Thus (2.26) is proven. On the other hand, if  $j \in J$  such that  $z_j \in \tilde{G}_1^k$ , we deduce (2.27) arguing as before, employing (2.9a) and (2.20) instead of (2.5) and (2.19), respectively.

Recall now the convexity inequality

$$(a + b)^p \leq (1 + p\rho) a^p + \left(1 + \frac{p}{\rho}\right) b^p \quad (2.29)$$

for any  $\rho > 0$  small, and any positive numbers  $a, b$ .

Fix  $j \in J$ . By (2.29) with  $\rho = \theta^{\frac{1}{2n}}$  we get

$$\int_{q_j} |e(\tilde{u}_k * \varphi_k)|^p dx \leq \left(1 + p\theta^{\frac{1}{2n}}\right) \int_{q_j} |e(v_j * \varphi_k)|^p dx + \left(1 + p\theta^{-\frac{1}{2n}}\right) \int_{q_j} |e(\tilde{u}_k - v_j) * \varphi_k|^p dx. \quad (2.30)$$

By (2.26) it follows that

$$\left(1 + p\theta^{-\frac{1}{2n}}\right) \int_{q_j} |e(\tilde{u}_k - v_j) * \varphi_k|^p dx \leq C\theta^{-\frac{1}{2n}} k^p \int_{\tilde{q}_j} |\tilde{u}_k - v_j|^p dx \leq C\theta^{\frac{1}{2n}} \int_{\tilde{Q}_j} |e(u)|^p dx, \quad (2.31)$$

while by (2.7) and (2.29) (for  $\rho = \theta^{\frac{q}{2}}$ )

$$\int_{q_j} |e(v_j * \varphi_k)|^p dx \leq \left(1 + p\theta^{\frac{q}{2}}\right) \int_{q_j} |e(u) * \varphi_k|^p dx + C\theta^{\frac{q}{2}} \int_{Q_j} |e(u)|^p dx.$$

Inserting into (2.30), this gives that

$$\begin{aligned} \int_{q_j} |e(\tilde{u}_k * \varphi_k)|^p dx &\leq \int_{q_j} |e(u) * \varphi_k|^p dx + C\theta^{q'} \left( \int_{q_j} |e(u) * \varphi_k|^p dx + \int_{\tilde{Q}_j} |e(u)|^p dx \right) \\ &\leq \int_{q_j} |e(u) * \varphi_k|^p dx + C\theta^{q'} \int_{\tilde{Q}_j} |e(u)|^p dx, \end{aligned} \quad (2.32)$$

with  $q' := \min\{q/2, 1/2n\}$ . If  $j \in J$  is such that  $z_j \in \tilde{G}_1^k$ , then (2.32) holds true for  $k^{-\frac{1}{2}}$  in place of  $\theta$ , namely

$$\int_{q_j} |e(\tilde{u}_k * \varphi_k)|^p dx \leq \int_{q_j} |e(u) * \varphi_k|^p dx + C k^{-\frac{q'}{2}} \int_{\tilde{Q}_j} |e(u)|^p dx, \quad (2.33)$$

because we can argue as before, with  $\rho$  equal to  $k^{-\frac{1}{4n}}$  and  $k^{-\frac{q}{4}}$  in (2.29), and (2.27), (2.9b) instead of (2.26), (2.7), respectively. Summing for  $j \in J$  and recalling the definition of  $u_k$  we obtain (2.1b). Notice that one has to distinguish the contributions for the nodes in  $\tilde{G}_1^k$  and in  $\tilde{G}_2^k$ , and to use that

$$\lim_{k \rightarrow \infty} \int_{\tilde{\Omega}_{g,2}^k} |e(u)|^p dx = 0,$$

by (2.11) and since  $e(u)$  is in  $L^p$ . This concludes the proof.  $\square$

## 3. THE MAIN RESULT

In this section we prove the main approximation result for any  $u \in GSBD^p(\Omega)$ , through more regular functions  $u_k$  converging in measure to  $u$ . The symmetric difference between the jump sets,  $J_{u_k} \Delta J_u$ , tends to 0 in  $\mathcal{H}^{n-1}$ -measure, the deformation  $e(u)$  is approximated in the strong  $L^p$  topology, and there is also convergence for truncation of the traces on  $J_u \cup J_{u_k}$  and on the reduced boundary of the domain  $\Omega$ , which is assumed to be only a set with finite perimeter.

We apply the rough version of the result, that we have shown in Section 2, to any (neighbourhood of) set of a suitable partition on  $\Omega$ , such that the measure of the jump set of  $u$  is small in any subset.

A fundamental difference with respect to [15, 43, 22], that employ also an intermediate rough estimate, is that here we do not use partitions of unity neither to extend the original function in suitable neighbourhoods of the subsets of the partition nor to glue the approximating functions constructed in any subset. This allows us to avoid any assumption on the integrability of  $u$ .

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with finite perimeter,  $p \in [1, \infty)$ ,  $\theta \in (0, 1)$ ,  $u \in GSBD^p(\Omega)$ . Then there exist  $u_k \in SBV^p(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  and  $E_k \subset \Omega$  such that each  $J_{u_k}$  is closed in  $\Omega$  and included in a finite union of closed connected pieces of  $C^1$  curves,  $u_k \in W^{1, \infty}(\Omega \setminus J_{u_k}; \mathbb{R}^n)$ , and:*

$$\lim_{k \rightarrow \infty} \mathcal{L}^n(E_k) = \lim_{k \rightarrow \infty} \int_{\Omega \setminus E_k} |u_k - u|^p dx = 0, \quad (3.1a)$$

$$e(u_k) \rightarrow e(u) \quad \text{in } L^p(\Omega; \mathbb{M}_{sym}^{n \times n}), \quad (3.1b)$$

$$\mathcal{H}^{n-1}(J_{u_k} \Delta J_u) \rightarrow 0, \quad (3.1c)$$

$$\int_{J_{u_k} \cup J_u} \tau(|u_k^\pm - u^\pm|) d\mathcal{H}^{n-1} + \int_{\partial^* \Omega} \tau(|\text{tr}(u_k - u)|) d\mathcal{H}^{n-1} \rightarrow 0, \quad (3.1d)$$

for  $\tau \in C^1(\mathbb{R})$  with  $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ ,  $0 \leq \tau' \leq 1$ . In particular,  $u_k$  converge to  $u$  in measure in  $\Omega$ . Moreover, if  $\int_\Omega \psi(|u|) dx$  is finite for  $\psi: [0, \infty) \rightarrow [0, \infty)$  increasing, continuous, and satisfying (HP $\psi$ ) (see Theorem 2.1), then

$$\lim_{k \rightarrow \infty} \int_\Omega \psi(|u_k - u|) dx = 0. \quad (3.1e)$$

*Proof.* We split the proof into three parts. First we approximate in a suitable way  $J_u$  (and  $\partial^* \Omega$ ), in the same spirit of the beginning of the proof of [15, Theorem 2], with balls replaced by hypercubes (see also [24, Lemma 4.2]). Then we get a finite family of cubes  $Q_j$ , whose union contains almost all  $J_u$ , each of which splitted in two parts  $Q_j^1, Q_j^2$  by the jump set. This gives us a partition of  $\Omega$  up to a  $\mathcal{L}^n$ -negligible set (see (3.3) and (3.5)).

At this stage, the strategy followed in [15] and [43] is to fatten a little bit every set of the covering, defining properly a function in the fattened domain in such a way that the energy does not increase much, and to apply Theorem 2.1 for each subset. By the way, we have to be very careful both in defining the extension functions and in linking the extended domains. Indeed, for instance we cannot simply glue any approximating functions defined on each enlarged set by a suitable partition of unity subordinated to the covering, as in [15, Theorem 2] by the analogous of [15, Lemma 3.1]. The reason is that, differently from [15], we do not know a priori the strong convergence in  $L^p$  in every subdomain, since now we do not assume  $u \in L^p$ . For the same reason, even to extend the function in an enlarged domain, we cannot partition the boundary, make small outer translations and glue by a partition of unity, as in [15]. Consequently, we follow a different argument. First, we use the fact that  $\partial Q_j^1 \cap \partial Q_j^2$  is almost flat (this is the intersection of the main part of  $J_u$  with  $Q_j$ ), to apply Lemma 1.8, an extension result inspired by Nitsche [49] (see also [40]), on both sides of any cube. In such a way, we extend the original function in the direction of the outer normal to each side of  $J_u$ . Then, we take the function  $u$  itself as an extension outside  $\partial Q_j$  and apply Theorem 2.1 for each subdomain; the extensions corresponding to  $Q_j$  and to the complement of  $Q_j$  have the same value on  $\partial Q_j$ , because they are obtained from  $u$  in the same way, in particular by taking convolutions with the same kernel.

In the final part the approximating functions on  $\Omega$  are introduced, and we verify the approximation properties. The remarkable fact is that we are allowed to just sum the ‘‘local’’ approximating

functions, restricted to the original subdomains. Indeed, no additional jump is created on the relative boundaries between any square  $Q_j$  and  $B_0$ , while the relative boundary between  $Q_j^1$  and  $Q_j^2$  correspond to a jump of the original displacement  $u$ , so that here we are allowed to still have jump. A minor point is to set the approximating function as 0 in a small neighbourhood of the intersection between  $\partial Q_j$  and the small strip that contains the main jump in  $Q_j$ , in which the function is reflected.

**Approximation of  $J_u$  and  $\partial^* \Omega$ .** Since  $J_u$  is  $(\mathcal{H}^{n-1}, n-1)$ -rectifiable, there exists a sequence  $\Gamma_i$  of  $C^1$  curves such that  $\mathcal{H}^{n-1}(J_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$ . For each  $i \geq 1$ , let

$$S_i := \left\{ x \in J_u \cap \Gamma_i \setminus \bigcup_{j < i} S_j : \lim_{\varrho \rightarrow 0} \frac{\mathcal{H}^{n-1}(J_u \cap \overline{Q}(x, \varrho))}{(2\varrho)^{n-1}} = \lim_{\varrho \rightarrow 0} \frac{\mathcal{H}^{n-1}(J_u \cap \Gamma_i \cap \overline{Q}(x, \varrho))}{(2\varrho)^{n-1}} = 1 \right\},$$

where  $\overline{Q}(x, \varrho)$  is the closed cube with center  $x$ , sidelength  $2\varrho$ , and one face normal to  $\nu(x)$ , the normal to  $\Gamma_i$  at  $x$ . Thus  $\mathcal{H}^{n-1}(J_u \setminus \bigcup_{i=1}^{\infty} S_i) = 0$  and for every  $x \in S_i$

$$\lim_{\varrho \rightarrow 0^+} \frac{\mathcal{H}^{n-1}(J_u \cap \overline{Q}(x, \varrho) \setminus \Gamma_i)}{(2\varrho)^{n-1}} = 0.$$

Let us fix  $\varepsilon > 0$ . Then for every  $x \in S_i$  there exists  $\overline{\varrho}(x)$  such that for  $0 < \varrho < \overline{\varrho}(x)$

$$\mathcal{H}^{n-1}((J_u \Delta \Gamma_i) \cap \overline{Q}(x, \varrho)) < \varepsilon(2\varrho)^{n-1} < \frac{\varepsilon}{1-\varepsilon} \mathcal{H}^{n-1}(J_u \cap \overline{Q}(x, \varrho)), \quad (3.2)$$

$\overline{Q}(x, \varrho) \cap \Gamma_i$  lies (in the open region) between the hyperplanes  $T_x \pm (\varepsilon\varrho)\nu(x)$ ,

where  $T_x$  is the hyperplane normal to  $\nu(x)$  and passing through  $x$ ,

$$\mathcal{H}^{n-1}(J_u \cap \partial \overline{Q}(x, \varrho)) = 0,$$

and  $\Gamma_i$  is a graph with respect to the direction  $\nu(x)$  of Lipschitz constant less than  $\varepsilon$ .

The family  $\mathcal{V} := \{\overline{Q}(x, \varrho) : x \in J_u, 0 < \varrho < \overline{\varrho}(x)\}$  is a Vitali class of closed sets for  $J_u$ . Then, by [34, Theorem 1.10] for  $s = n-1$ , there exists a disjoint sequence  $\overline{Q}_j = \overline{Q}(x_j, \varrho_j) \subset \mathcal{V}$  such that  $\mathcal{H}^{n-1}(J_u \setminus \bigcup_{j=1}^{\infty} \overline{Q}_j) = 0$ . In particular, one face of  $\overline{Q}_j$  is normal to  $\nu_u(x_j)$ , the normal to  $J_u$  at  $x_j$ , for each  $j$  there exists  $i_j$  for which  $\Gamma_{i_j}$  separates  $Q_j$  in exactly two components  $Q_j^1$  and  $Q_j^2$  (each of the two is an open Lipschitz domain), and, for a suitable  $\bar{j} \in \mathbb{N}$ , we have

$$\mathcal{H}^{n-1}\left(J_u \setminus \bigcup_{j=1}^{\bar{j}} Q_j\right) < \varepsilon, \quad (3.3a)$$

$$\mathcal{H}^{n-1}((J_u \Delta \Gamma_{i_j}) \cap \overline{Q}_j) < \varepsilon(2\varrho_j)^{n-1} < \frac{\varepsilon}{1-\varepsilon} \mathcal{H}^{n-1}(J_u \cap \overline{Q}_j), \quad (3.3b)$$

$$Q_j \cap \Gamma_{i_j} \subset R_j := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j, \varrho_j), y_n \in (-\varepsilon\varrho_j, +\varepsilon\varrho_j) \right\}, \quad (3.3c)$$

where

$$(b_{j,i})_{i=1}^{n-1} \text{ is an orthonormal basis of } \nu_u(x_j)^\perp.$$

Moreover, we may assume that  $\overline{Q}_j \subset \Omega$  for  $j = 1, \dots, \bar{j}$ .

We can argue similarly for  $\partial^* \Omega$  in place of  $J_u$ , to find a finite set of cubes  $(Q_h^0)_{h=1}^{\bar{h}}$  of centers  $x_h^0$  and sidelength  $\varrho_h^0$ , whose closures are pairwise disjoint, such that

$$\mathcal{H}^{n-1}\left(\partial^* \Omega \setminus \bigcup_{h=1}^{\bar{h}} Q_h^0\right) < \varepsilon, \quad (3.4a)$$

$$\mathcal{H}^{n-1}(\partial^* \Omega \cap \overline{Q}_h^0) > (1-\varepsilon)(2\varrho_h^0)^{n-1}, \quad (3.4b)$$

$$\partial^* \Omega \cap Q_h^0 \subset \left\{ x_h^0 + \sum_{i=1}^{n-1} y_i b_{h,i}^0 + y_n \nu_h^0 : y_i \in (-\varrho_h^0, \varrho_h^0), y_n \in (-\varepsilon\varrho_h^0, +\varepsilon\varrho_h^0) \right\}, \quad (3.4c)$$

where  $\nu_h^0 = -\nu_\Omega(x_h^0)$  is the generalised outer normal to  $\Omega$  at  $x_h^0$  and

$$(b_{h,i}^0)_{i=1}^{n-1} \text{ is an orthonormal basis of } (\nu_h^0)^\perp.$$

Let

$$B_0 := \Omega \setminus \bigcup_{j=1}^{\bar{j}} \overline{Q}_j, \quad (3.5)$$



so that  $\Omega = B_0 \cup \bigcup_{j=1}^{\bar{j}} (Q_j^1 \cup Q_j^2)$ , up to a  $\mathcal{L}^n$ -negligible set.

**Definition and properties of the approximating functions in subdomains.** Let us fix  $j \in \{1, \dots, \bar{j}\}$ . Consider, for a given  $t > 0$ , the open rectangles

$$R_j^1 := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j, +\varrho_j), y_n \in (-3\varepsilon\varrho_j - t, -\varepsilon\varrho_j) \right\},$$

$$R_j^2 := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j, +\varrho_j), y_n \in (\varepsilon\varrho_j, 3\varepsilon\varrho_j + t) \right\},$$

and their reflections with respect to one of their faces

$$(R_j^1)' := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j, +\varrho_j), y_n \in (-\varepsilon\varrho_j, \varepsilon\varrho_j + t) \right\},$$

$$(R_j^2)' := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j, +\varrho_j), y_n \in (-\varepsilon\varrho_j - t, \varepsilon\varrho_j) \right\}.$$

Let also  $\widehat{R}_j^l$  be the union of  $R_j^l$ ,  $(R_j^l)'$ , and their common face, for  $l = 1, 2$ . We have that

$$\mathcal{L}^n(R_j^l) = \mathcal{L}^n((R_j^l)') = 2\varepsilon\varrho_j^n + t\varrho_j^{n-1},$$

and  $(R_j^1)' \cap (R_j^2)' = R_j$ .

By Lemma 1.8, we may extend the restrictions of  $u$  to  $R_j^1$  and  $R_j^2$  by two functions  $\widehat{u}_{j_1} \in GSBD^p(\widehat{R}_j^1)$  and  $\widehat{u}_{j_2} \in GSBD^p(\widehat{R}_j^2)$  such that for  $l = 1, 2$

$$\int_{\widehat{R}_j^l} |e(\widehat{u}_{j_l})|^p dx \leq c \int_{R_j^l} |e(u)|^p dx, \quad (3.6a)$$

$$\mathcal{H}^{n-1}(J_{\widehat{u}_{j_l}} \cap \widehat{R}_j^l) \leq c \mathcal{H}^{n-1}(J_u \cap R_j^l), \quad (3.6b)$$

where  $c > 0$  depends only on  $n$  and  $p$ . Recalling the definition of reflection in Lemma 1.8, it is immediate to see that if  $\psi(|u|) \in L^1(\Omega)$ , for  $\psi$  as in the statement of the theorem, then

$$\int_{\widehat{R}_j^l} \psi(|\widehat{u}_{j_l}|) dx \leq c \int_{R_j^l} \psi(|u|) dx. \quad (3.6c)$$

Let us define for  $t > 0$  small enough

$$\widetilde{Q}_j^1 := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j - t, +\varrho_j + t), y_n \in (-\varrho_j - t, \varepsilon\varrho_j + t) \right\},$$

$$\widetilde{Q}_j^2 := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-\varrho_j - t, +\varrho_j + t), y_n \in (-\varepsilon\varrho_j - t, \varrho_j + t) \right\},$$

( $t$  in particular such that  $\widetilde{Q}_j^1 \cup \widetilde{Q}_j^2$  does not intersect  $Q_i$ , for  $i \neq j$ ) and the extension of  $u \chi_{Q_j^l}$

$$u^{j_l} := u \chi_{\widetilde{Q}_j^l \setminus (R_j^l)'} + \widehat{u}_{j_l} \chi_{(R_j^l)'}. \quad (3.7)$$

By (3.6), it holds for  $t$  small enough that

$$\mathcal{L}^n(\{u^{j_l} \neq u\} \cap \widetilde{Q}_j^l) \leq \mathcal{L}^n((R_j^l)') \leq C \varepsilon \varrho_j^n, \quad (3.8a)$$

$$\int_{Q_j^l} |e(u^{j_l})|^p dx \leq \int_{Q_j^l} |e(u)|^p dx + (c-1) \int_{R_j^l} |e(u)|^p dx, \quad (3.8b)$$

$$\mathcal{H}^{n-1}(J_{u^{j_l}} \cap \widetilde{Q}_j^l) \leq \mathcal{H}^{n-1}(J_u \cap Q_j^l) + (c-1) \mathcal{H}^{n-1}(J_u \cap R_j^l) + \varepsilon. \quad (3.8c)$$

As for  $B_0$ , for  $h = 1, \dots, \bar{h}$  we consider

$$R_h^0 := \left\{ x_h^0 + \sum_{i=1}^{n-1} y_i b_{h,i}^0 + y_n \nu_h^0 : y_i \in (-\varrho_h^0, \varrho_h^0), y_n \in (-3\varepsilon\varrho_h^0 - t, -\varepsilon\varrho_h^0) \right\}, \quad (3.9a)$$

$$(R_h^0)' := \left\{ x_h^0 + \sum_{i=1}^{n-1} y_i b_{h,i}^0 + y_n \nu_h^0 : y_i \in (-\varrho_h^0, \varrho_h^0), y_n \in (-\varepsilon\varrho_h^0, \varepsilon\varrho_h^0 + t) \right\}, \quad (3.9b)$$

$\widehat{R}_h^0$  the union of  $R_h^0$ ,  $(R_h^0)'$ , and their common face, the functions  $\widehat{u}_h^0 \in GSBD^p(\widehat{R}_h^0)$ , provided by Lemma 1.8, for which

$$\int_{\widehat{R}_h^0} |e(\widehat{u}_h^0)|^p dx \leq c \int_{R_h^0} |e(u)|^p dx, \quad (3.10a)$$

$$\mathcal{H}^{n-1}(J_{\widehat{u}_h^0} \cap \widehat{R}_h^0) \leq c \mathcal{H}^{n-1}(J_u \cap R_h^0), \quad (3.10b)$$

the sets

$$\widetilde{B}_0 := B_0 + B(0, t'), \quad R'_0 := \bigcup_{h=1}^{\bar{h}} (R_h^0)',$$

for  $t' < t$  small enough, and

$$u^0 := \begin{cases} u & \text{in } \Omega \cap \widetilde{B}_0 \setminus R'_0, \\ \widehat{u}_h^0 & \text{in } (R_h^0)', \\ 0 & \text{in } \widetilde{B}_0 \setminus (\Omega \cup R'_0). \end{cases}$$

Moreover we have, for  $t'$  small enough, that

$$\mathcal{L}^n(\{u^0 \neq u\} \cap \Omega \cap \widetilde{B}_0) \leq \mathcal{L}^n(\Omega \cap \widetilde{B}_0 \cap R'_0) \leq \mathcal{L}^n(R'_0) \leq C \varepsilon \sum_{h=1}^{\bar{h}} (\varrho_h^0)^n \leq C \varepsilon, \quad (3.11a)$$

$$\int_{\widetilde{B}_0} |e(u^0)|^p dx \leq \int_{B_0} |e(u)|^p dx + (c-1) \sum_{h=1}^{\bar{h}} \int_{R_h^0} |e(u)|^p dx, \quad (3.11b)$$

$$\mathcal{H}^{n-1}(J_{u^0} \cap \widetilde{B}_0) \leq \mathcal{H}^{n-1}(J_u \cap B_0) + (c-1) \sum_{h=1}^{\bar{h}} \mathcal{H}^{n-1}(J_u \cap R_h^0) + \mathcal{H}^{n-1}(\partial^* \Omega \setminus R'_0) + \varepsilon, \quad (3.11c)$$

where in (3.11a) we have used the fact that the cubes  $Q_h^0$  are pairwise disjoint. Notice that

$$\mathcal{H}^{n-1}(\partial^* \Omega \setminus R'_0) = \mathcal{H}^{n-1}\left(\partial^* \Omega \setminus \bigcup_{h=1}^{\bar{h}} Q_h^0\right) < \varepsilon, \quad (3.12)$$

by (3.4a).

We now apply Theorem 2.1 to find, taking  $B_0, Q_j^1, Q_j^2$  as  $\Omega$  and  $u^0, u^{j1}, u^{j2}$  as  $u$  therein, functions  $u_k^0, u_k^{j1}, u_k^{j2}$  and sets  $E_k^0, E_k^{j1}, E_k^{j2}$  such that

$$\lim_{k \rightarrow \infty} \mathcal{L}^n(E_k^{ji}) = \lim_{k \rightarrow \infty} \int_{Q_j^l \setminus E_k^{ji}} |u_k^{ji} - u^{ji}|^p dx = 0, \quad (3.13a)$$

$$\limsup_{k \rightarrow \infty} \int_{Q_j^l} |e(u_k^{ji})|^p dx \leq \int_{Q_j^l} |e(u^{ji})|^p dx, \quad (3.13b)$$

$$\mathcal{H}^{n-1}(J_{u_k^{ji}} \cap Q_j^l) \leq C \theta^{-1} \mathcal{H}^{n-1}(J_{u^{ji}} \cap \widetilde{Q}_j^l), \quad (3.13c)$$

$$\lim_{k \rightarrow \infty} \int_{Q_j^l} \psi(|u_k^{ji} - u^{ji}|) dx = 0, \quad \text{if } \psi(|u|) \in L^1(\Omega), \quad (3.13d)$$

and the same for  $u_k^0, E_k^0$ , and  $B_0$  in place of  $u_k^{ji}, E_k^{ji}$ , and  $Q_j^l$ . In particular, the (internal parts of the) supports of the  $u_k^0, u_k^{j1}, u_k^{j2}$  are pairwise disjoint. Moreover, Theorem 2.1 provides us also functions  $v_k^{ji}$ , defined in a given set  $(Q_j^l)'$  with  $Q_j^l \subset (Q_j^l)' \subset \widetilde{Q}_j^l$ , such that  $u_k^{ji} = v_k^{ji}$  in  $Q_j^l$  and (3.13) hold for  $(Q_j^l)'$  in place of  $Q_j^l$ , and analogously  $v_k^0$  defined in  $(B_0)'$  with  $B_0 \subset (B_0)' \subset \widetilde{B}_0$ . In particular

$$\mathcal{H}^{n-1}(J_{v_k^{jl}} \cap (Q_j^l)') \leq C \theta^{-1} \mathcal{H}^{n-1}(J_{u^{jl}} \cap \widetilde{Q}_j^l), \quad (3.14)$$

**The approximating functions.** We set

$$u_k := u_k^0 + \sum_{j=0}^{\bar{j}} (u_k^{j1} + u_k^{j2}).$$

We are going to prove the desired approximation properties for the sequence  $u_k$ . It is immediate that  $u_k \in SBVP(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ .

**Proof of (3.1a), (3.1b), (3.1c).** In order to describe  $J_{u_k}$ , notice that

$$J_{v_k^{j_i}} \cap \partial Q_j \setminus F_j = J_{u_k} \cap \partial Q_j \setminus F_j, \quad (3.15)$$

where (recall (3.3c))

$$F_j := \{Q_z^k : Q_z^k \cap \partial Q_j \cap \partial R_j \neq \emptyset\}.$$

Indeed, for any  $x \in \partial Q_j^i \setminus F_j$  such that  $x \in Q_z^k$ , we have  $u^{j_i} = u^0 = u$  in  $Q_z^k$ , and, by construction of the approximating functions (see (2.12) and (2.13))  $v_k^0 = v_k^{j_i}$  in  $Q_z^k$ .

Since  $\mathcal{H}^{n-1}(\partial Q_j \cap \partial R_j) = 2^{n+1} \varepsilon \varrho_j^{n-1}$ , for  $k$  large we have that

$$\mathcal{H}^{n-1}(\partial Q_j \cap F_j) = C \varepsilon \varrho_j^{n-1}. \quad (3.16)$$

By (3.15), and since  $\mathcal{H}^{n-1}(J_u \cap \partial Q_j) = 0$ , we deduce that (recall the definition of  $v_k^{j_l}$  after (3.13))

$$J_{u_k} \subset (J_{u_k^0} \cap B_0) \cup \bigcup_{j=1}^{\bar{j}} \left( (J_{v_k^{j_1}} \cap (Q_j^1)') \cup (J_{v_k^{j_2}} \cap (Q_j^2)') \cup (Q_j \cap \Gamma_{i_j}) \cup (\partial Q_j \cap F_j) \right),$$

so that  $J_{u_k}$  is closed and contained in a finite union of closed connected pieces of  $C^1$  curves, and  $u_k \in W^{1,\infty}(\Omega \setminus J_{u_k}; \mathbb{R}^n)$ . Notice that we may assume that  $\bigcup_j (Q_j \cap \Gamma_{i_j}) \subset J_{u_k}$ . Indeed, we can find  $a > 0$  arbitrarily small such that  $\mathcal{H}^{n-1}(\bigcup_j (Q_j \cap \Gamma_{i_j} \cap \{x: [u](x) = a\})) = 0$  (with  $[u](x)$  the size of the jump with respect to  $\Gamma_{i_j}$  at  $x \in \Gamma_{i_j}$ ), and then we can add to  $u_k$  a perturbation with jump  $a$  on  $\bigcup_j (Q_j \cap \Gamma_{i_j})$ , smooth in  $Q_j^1$  and  $Q_j^2$  for every  $j$ , and with arbitrarily small  $W^{1,\infty}$  norm (since  $a$  is small). In particular,

$$J_{u_k} \Delta J_u \subset (J_{u_k^0} \cap B_0) \cup (J_u \cap B_0) \cup \bigcup_{j=1}^{\bar{j}} \left( (J_{v_k^{j_1}} \cap (Q_j^1)') \cup (J_{v_k^{j_2}} \cap (Q_j^2)') \cup ((J_u \Delta \Gamma_{i_j}) \cap Q_j) \cup (\partial Q_j \cap F_j) \right). \quad (3.17)$$

Let

$$\tilde{E}_k := E_k^0 \cup \bigcup_{j=1}^{\bar{j}} (E_k^{j_1} \cup E_k^{j_2}), \quad R'_k := R'_0 \cup \bigcup_{j=1}^{\bar{j}} ((R_j^1)' \cup (R_j^2)'), \quad E_k := \tilde{E}_k \cup R'_k,$$

for which, by (3.8a) (recall that  $Q_j$  are pairwise disjoint), (3.11a), and (3.13a) we have that

$$\lim_{k \rightarrow \infty} \mathcal{L}^n(\tilde{E}_k) = 0, \quad \limsup_{k \rightarrow \infty} \mathcal{L}^n(R'_k) \leq C \varepsilon. \quad (3.18)$$

It follows in particular that

$$\lim_{\varepsilon \rightarrow 0} \int_{R'_k} |e(u)|^p dx = 0. \quad (3.19)$$

Let us now put together (3.8) with (3.13) and (3.14), (3.11) with the analogous of (3.13) for  $j = 0$ , and sum over  $i = 1, 2$  and  $j = 0, \dots, \bar{j}$ . We deduce, employing also (3.12), that

$$\lim_{k \rightarrow \infty} \int_{\Omega \setminus \tilde{E}_k} |u_k - u|^p dx = 0, \quad (3.20a)$$

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |e(u_k)|^p dx \leq \int_{\Omega} |e(u)|^p dx + (c-1) \int_{R'_k} |e(u)|^p dx, \quad (3.20b)$$

$$\mathcal{H}^{n-1}(J_{u_k^0} \cup \bigcup_{j=1}^{\bar{j}} (J_{v_k^{j_1}} \cup J_{v_k^{j_2}})) \leq C \theta^{-1} \left[ \mathcal{H}^{n-1}(J_u \cap B_0) + \varepsilon + \sum_{j=1}^{\bar{j}} \mathcal{H}^{n-1}((J_u \setminus \Gamma_{i_j}) \cap \overline{Q_j}) \right]. \quad (3.20c)$$

To shorten the notation, above we have written  $J_{u_k^0} \cup \bigcup_{j=1}^{\bar{j}} (J_{v_k^{j_1}} \cup J_{v_k^{j_2}})$  in place of  $(J_{u_k^0} \cap B_0) \cup (J_{v_k^{j_1}} \cap (Q_j^1)') \cup (J_{v_k^{j_2}} \cap (Q_j^2)')$ . By (3.3a) and (3.5) it follows  $\mathcal{H}^{n-1}(J_u \cap B_0) < \varepsilon$ , while by (3.3b) that

$$\sum_{j=1}^{\bar{j}} \mathcal{H}^{n-1}((J_u \Delta \Gamma_{i_j}) \cap \overline{Q_j}) < C \varepsilon \mathcal{H}^{n-1}(J_u). \quad (3.21)$$

Therefore, collecting (3.16), (3.17), and (3.20c), we get

$$\mathcal{H}^{n-1}(J_{u_k} \Delta J_u) < C \theta^{-1} \varepsilon + C \varepsilon \mathcal{H}^{n-1}(J_u), \quad (3.22)$$

By (3.18), (3.19), (3.20a), (3.20b), (3.22), and by the arbitrariness of  $\varepsilon$ , we get (3.1a), (3.1c), and

$$\limsup_{k \rightarrow \infty} \|e(u_k)\|_{L^p(\Omega; \mathbb{M}_s^{n \times n})} \leq \|e(u)\|_{L^p(\Omega; \mathbb{M}_s^{n \times n})}. \quad (3.23)$$

Moreover, (3.1a) gives that  $u_k \rightarrow u$  in measure, and then, by [40, Remark 2.2], there exists a subsequence of  $u_k$ , not relabelled, and a nonnegative, increasing, concave function  $\bar{\psi}$  such that

$$\lim_{s \rightarrow +\infty} \bar{\psi}(s) = +\infty$$

and

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \bar{\psi}(|u_k|) dx \leq 1.$$

Therefore we can apply the Compactness Theorem for  $GSBD$  [27, Theorem 11.3], which implies that, up to a further subsequence,

$$e(u_k) \rightharpoonup e(u) \quad \text{in } L^p(\Omega; \mathbb{R}^n).$$

Therefore, by (3.23), the sequence  $u_k$  satisfies also (3.1b).

**Proof of (3.1d).** Fix  $j \in \{1, \dots, \bar{j}\}$  and consider

$$\int_{\Gamma_{i_j} \cap \bar{Q}_j} \tau(|\text{tr}(u_k^{j_1} - u)|) d\mathcal{H}^{n-1} \equiv \int_{\Gamma_{i_j} \cap \bar{Q}_j} \tau(|u_k^+ - u^+|) d\mathcal{H}^{n-1},$$

where the trace is considered from the interior side of  $\Gamma_{i_j}$  with respect to  $Q_j^1$ , and we assume by convention that this is the “positive” side of  $\Gamma_{i_j}$ . We define the rectangle

$$\widehat{Q}_j^1 := \left\{ x_j + \sum_{i=1}^{n-1} y_i b_{j,i} + y_n \nu_u(x_j) : y_i \in (-(1 - \sqrt{\varepsilon})\varrho_j, (1 - \sqrt{\varepsilon})\varrho_j), y_n \in (-\varrho_j, \varepsilon\varrho_j) \right\}$$

and call  $\xi_n$  the normal  $\nu_u(x_j)$ . Then  $\Gamma_{i_j} \cap \widehat{Q}_j^1$  is a graph in the direction  $\xi_n$  of Lipschitz constant less than  $\varepsilon$ . By [6, Lemma 3.1], there exists a universal constant  $\eta_0 > 0$  (indeed it depends decreasingly on the Lipschitz constant of the graph of  $\Gamma_{i_j} \cap \widehat{Q}_j^1$  in the direction  $\xi_n$ , which is less than  $1/2$ ) such that for any  $\xi \in \mathbb{S}^{n-1}$  with  $|\xi - \xi_n| < \eta_0$ , one has that  $\Gamma_{i_j} \cap \widehat{Q}_j^1$  is a Lipschitz graph in the direction  $\xi$ . In particular, let  $(\xi_1, \dots, \xi_{n-1}, \xi_n)$  be a basis of  $\mathbb{R}^n$  with  $|\xi_h - \xi_n| < \eta_0$ . Arguing as in [43, equations (17)–(19)],

$$\int_{\Gamma_{i_j} \cap \widehat{Q}_j^1} |\text{tr}(\tau((u_k^{j_1} - u) \cdot \xi_h))| d\mathcal{H}^{n-1} \leq C \sum_{h=1}^n \int_{\Gamma_{i_j} \cap \widehat{Q}_j^1} |\text{tr}(\tau((u_k^{j_1} - u) \cdot \xi_h))| d\mathcal{H}^{n-1}$$

for a universal constant  $C > 0$ . Since  $\tau((u_k^{j_1} - u) \cdot \xi_h) \in L^1(Q_j^1)$  and  $D_{\xi_h} \tau((u_k^{j_1} - u) \cdot \xi_h) \in \mathcal{M}_b^+(Q_j^1)$  for any  $h$ , arguing as in [6, Theorem 3.2, Steps 1 and 4] we deduce that

$$\begin{aligned} \int_{\Gamma_{i_j} \cap \widehat{Q}_j^1} |\text{tr}(\tau((u_k^{j_1} - u) \cdot \xi_h))| d\mathcal{H}^{n-1} &\leq \frac{C}{\sqrt{\varepsilon}\varrho_j} \|\tau((u_k^{j_1} - u) \cdot \xi_h)\|_{L^1(A_\varepsilon^{\xi_h})} + C \int_{A_\varepsilon^{\xi_h}} |e(u_k^{j_1} - u)| dx \\ &\quad + C \mathcal{H}^{n-1}(J_{u_k^{j_1} - u} \cap A_\varepsilon^{\xi_h}), \end{aligned}$$

for  $C > 0$  depending only on  $n$ , and  $A_\varepsilon^{\xi_h} := \{y - s\xi_h : y \in \Gamma_{i_j} \cap \widehat{Q}_j^1, 0 < s < \sqrt{\varepsilon}\varrho_j\} \subset Q_j^1$ . Being  $\tau$  bounded, by (3.8a) and (3.13a) we get that

$$\limsup_{k \rightarrow \infty} \frac{C}{\sqrt{\varepsilon}\varrho_j} \|\tau((u_k^{j_1} - u) \cdot \xi_h)\|_{L^1(A_\varepsilon^{\xi_h})} < C\sqrt{\varepsilon}\varrho_j^{n-1} < C\sqrt{\varepsilon}\mathcal{H}^{n-1}(J_u \cap \bar{Q}_j),$$

where the last inequality follows by (3.3b). By construction of  $u_k^{j_1}$  in Theorem 2.1 (in particular by (2.32)) we deduce that

$$C \int_{A_\varepsilon^{\xi_h}} |e(u_k^{j_1} - u)| dx < C \int_{A_\varepsilon^{\xi_h} + B(0, 8k^{-1})} |e(u)|^p dx, \quad (3.24)$$

by (3.13c) that

$$\mathcal{H}^{n-1}(J_{u_k^{j_1} - u} \cap A_\varepsilon^{\xi_h}) < C\theta^{-1}\mathcal{H}^{n-1}(J_u \cap Q_j^1) < C\theta^{-1}\mathcal{H}^{n-1}((J_u \setminus \Gamma_{i_j}) \cap Q_j),$$

and by definition of  $\widehat{Q}_j^1$  that

$$\mathcal{H}^{n-1}(\Gamma_{i_j} \cap Q_j \setminus \widehat{Q}_j^1) < C(\sqrt{\varepsilon}\varrho_j)^{n-1} < C\sqrt{\varepsilon}^{n-1}\mathcal{H}^{n-1}(J_u \cap \bar{Q}_j).$$

Collecting the informations above, we get (recall that  $\tau$  is bounded) that

$$\int_{\Gamma_{i_j} \cap \overline{Q_j}} \tau(|u_k^+ - u^+|) d\mathcal{H}^{n-1} < C \left( \sqrt{\varepsilon} \mathcal{H}^{n-1}(J_u \cap \overline{Q_j}) + \sup_h \int_{A_\varepsilon^{\xi h} + B(0, 8k^{-1})} |e(u)|^p dx \right. \\ \left. + \theta^{-1} \mathcal{H}^{n-1}((J_u \setminus \Gamma_{i_j}) \cap Q_j) \right).$$

We can now argue similarly in  $Q_j^2$  and sum over  $j$ . Recalling (3.21) and (3.22), and since  $\tau$  is bounded, it follows that

$$\int_{J_{u_k} \cup J_u} \tau(|u_k^\pm - u^\pm|) d\mathcal{H}^{n-1} < C \theta^{-1} \varepsilon + c \sqrt{\varepsilon} \mathcal{H}^{n-1}(J_u) + \int_{A_\varepsilon} |e(u)|^p dx, \quad (3.25)$$

where  $\mathcal{L}^n(A_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (for  $k$  much smaller than  $\varepsilon$ ). By the arbitrariness of  $\varepsilon$

$$\lim_{k \rightarrow \infty} \int_{J_{u_k} \cup J_u} \tau(|u_k^\pm - u^\pm|) d\mathcal{H}^{n-1} = 0. \quad (3.26)$$

Arguing as for (3.26), with  $u_k^0$  in place of  $u_k^{j1}, u_k^{j2}$ , we conclude (3.1d).

**Proof of (3.1e).** Assume that  $\psi(|u|) \in L^1(\Omega)$ , for  $\psi$  as in the statement of the theorem. Recalling (3.7), by (3.8a) and (3.13d) we have (sum all the contributions)

$$\lim_{k \rightarrow \infty} \int_{\Omega \setminus R'_k} \psi(|u_k - u|) dx = 0, \quad (3.27)$$

and

$$\lim_{k \rightarrow \infty} \int_{(R_j^l)'} \psi(|u_k^{jl} - \widehat{u}^{jl}|) dx = 0,$$

for every  $j$  and  $l = 1, 2$  (and also for  $u_k^0$ ). For  $R_k := R_0 \cup \bigcup_{j=1}^{\overline{j}} ((R_j^1) \cup (R_j^2))$ , by (3.18)  $\mathcal{L}^n(R_k) = \mathcal{L}^n(R'_k) \leq C\varepsilon$ , and by (3.6c)

$$\limsup_{k \rightarrow \infty} \int_{R'_k} \psi(|u_k|) dx \leq c \limsup_{k \rightarrow \infty} \int_{R_k} \psi(|u|) dx.$$

Therefore

$$\limsup_{k \rightarrow \infty} \int_{R'_k} \psi(|u_k - u|) dx \leq C_\psi \limsup_{k \rightarrow \infty} \int_{R'_k} (\psi(|u_k|) + \psi(|u|)) dx \leq c C_\psi \limsup_{k \rightarrow \infty} \int_{R_k \cup R'_k} \psi(|u|) dx,$$

which vanishes as  $\varepsilon$  tends to 0. Together with (3.27), this proves (3.1e) and completes the proof of the theorem.  $\square$

*Remark 3.2.* The construction in Theorem 3.1 may be slightly modified in the following way: apply Theorem 2.1 to suitable compact subsets of  $\Omega \setminus \left( \bigcup_{j=1}^{\overline{j}} \overline{Q_j} \cup \bigcup_{h=1}^{\overline{h}} \overline{Q_h^0} \right)$ , and reflect the smooth function obtained (so without using Lemma 1.8) on both sides of  $Q_j$  with respect to  $J_u$  (resp. on the internal part of  $Q_h^0$  with respect to  $\partial^* \Omega$ ), the further arguments being similar to what done above. Working in a compact subset of  $\Omega \setminus \left( \bigcup_{j=1}^{\overline{j}} \overline{Q_j} \cup \bigcup_{h=1}^{\overline{h}} \overline{Q_h^0} \right)$  should permit to have for free an extension of the original function to a larger domain, without employing partitions of unity. Arguing in this way, we expect that one could find alternative proofs to our density result, still without assuming that  $u$  is  $p$ -summable, using different approximation techniques, such as the one in [22].

#### 4. SOME APPLICATIONS TO THE APPROXIMATION OF BRITTLE FRACTURE ENERGIES

Here we show how the density result of Theorem 3.1 may be employed to approximate, in the sense of  $\Gamma$ -convergence, the Griffith energy for brittle fracture, under no assumption on the integrability of the displacement. This is a novelty in the vectorial case, except for  $n = 2$ , where this convergence (for quadratic bulk energy) may be proven starting from the density result [38, Theorem 2.5]. In particular, for phase field approximation *à la* Ambrosio-Tortorelli [4, 5], one needs for a density theorem of the type of Theorem 3.1 in the  $\Gamma$ -limsup inequality; the  $\Gamma$ -limit is then

determined in the subspace of  $GSBD$  in which every displacement is approximated by the density result.

On the other hand, since one is interested in the approximation of minimisers for Griffith energy, it is natural to impose some conditions to prevent that the set of minimisers coincides with the constant displacements. Two important examples are Dirichlet boundary condition and a compliance condition for the displacement with respect to a given datum  $g$  on the whole  $\Omega$ . We show how to approximate the resulting brittle fracture energy, under some geometric assumptions on the Dirichlet part of the domain in the first case, and for a very large class of compliance functions (possibly such that the displacement is not a priori forced to be even integrable) in the second case. Requiring some integrability on displacement in the density theorem forces to include lower order terms in the energy functional, in order to guarantee *a priori* such integrability.

We remark that we are able to prove the existence of minimisers for the limit problem in the case of the compliance condition on  $\Omega$ , but not for the energy with Dirichlet boundary datum. The existence of minimisers for Dirichlet problem has been proven in dimension 2 in [40, Theorem 6.2].

In the first part of the section we state the approximation results, which are proven in the second part.

Let us introduce some notation for this section. Let  $p, q > 1$ ,  $a > 0$ ,  $\varepsilon_k, \eta_k > 0$  with  $\varepsilon_k \rightarrow 0$ ,  $\eta_k \rightarrow 0$ ,  $\frac{\eta_k}{\varepsilon_k} \rightarrow 0$ , for  $k \in \mathbb{N}$ . Let  $W: \mathbb{R} \times \mathbb{M}_{sym}^{n \times n} \rightarrow [0, \infty)$  be convex in the second argument and lower semicontinuous, with

$$c_1 s |\cdot|^p \leq W(s, \cdot) \leq c_2 (1 + s |\cdot|^p) \quad \text{for every } s \in \mathbb{R} \quad (4.1)$$

for some  $0 < c_1 < c_2$ , and  $d: [0, 1] \rightarrow [0, \infty)$  continuous, decreasing, with  $d(1) = 0$ . For every bounded open set  $A \subset \mathbb{R}^n$  and measurable functions  $u: A \rightarrow \mathbb{R}^n$  and  $v: A \rightarrow [0, 1]$ , we define

$$G_k^A(u, v) := \begin{cases} \int_A \left( W(v, e(u)) + \frac{d(v)}{\varepsilon_k} + a \varepsilon_k^{q-1} |\nabla v|^q \right) dx & \text{in } W^{1,p}(A; \mathbb{R}^n) \times V_k^A, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$V_k^A := \{v \in W^{1,q}(A) : \eta_k \leq v \leq 1\}.$$

and the *generalised Griffith energy*

$$G^A(u, v) := \begin{cases} \int_A W(1, e(u)) dx + \alpha \mathcal{H}^{n-1}(J_u) & \text{in } GSBD^p(A) \times \{v = 1 \text{ } \mathcal{L}^n\text{-a.e. in } A\}, \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$\alpha := 2(q')^{\frac{1}{q'}} (aq)^{\frac{1}{q}} \int_0^1 d(s)^{\frac{1}{q'}} ds, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

**Theorem 4.1.** *Let  $A \subset \mathbb{R}^n$  be a bounded set with finite perimeter. Then  $G_k^A$   $\Gamma$ -converge to  $G^A$  with respect to the topology of the convergence in measure for  $u$  and  $v$ .*

Let  $A \subset \mathbb{R}^n$  be a bounded set and  $g: A \rightarrow \mathbb{R}^n$  be a measurable function such that  $\psi(|g|) \in L^1(A)$ , for  $\psi: [0, \infty) \rightarrow [0, \infty)$  increasing, continuous, and satisfying (HP $\psi$ ) (see Theorem 2.1). For every measurable functions  $u: A \rightarrow \mathbb{R}^n$  and  $v: A \rightarrow [0, 1]$  we define

$$F_k^A(u, v) := G_k^A + \int_A \psi(|u - g|) dx,$$

and the *generalised Griffith energy with fidelity term*

$$F^A(u, v) := G^A(u, v) + \int_A \psi(|u - g|) dx,$$

where  $F^A(u, v) = +\infty$  if  $\psi(|u - g|)$  is not in  $L^1(A)$ . Then we have the following convergence.

**Theorem 4.2.** *Let  $A \subset \mathbb{R}^n$  be a bounded set with finite perimeter. Then  $F_k^A$   $\Gamma$ -converge to  $F^A$  with respect to the topology of the convergence in measure for  $u$  and  $v$ . Moreover, if  $F_k^A(u_k, v_k) \leq \inf F_k^A + \gamma_k$  for any  $k$ , namely if  $(u_k, v_k)$  is a  $\gamma_k$ -minimiser for  $F_k^A$ , with  $\gamma_k \rightarrow 0$ , then, up to a subsequence,  $(u_k, v_k)$  converge in measure to some  $(u, 1)$ , which is a minimiser of  $F^A$ , and*

$$F_k^A(u_k, v_k) \rightarrow F^A(u, 1).$$

*Remark 4.3.* If  $A$  is a Lipschitz domain then every  $F_k^A$  admits a minimiser. First we have that

$$\left( \|\nabla u\|_{L^p(A; \mathbb{M}^{n \times n})}^p + \int_A (\psi(|u|) \wedge |u|) \, dx \right) + \|v\|_{W^{1,q}(A)} \leq C, \quad (4.2)$$

with  $C > 0$  independent of  $u, v$  such that  $F_k^A(u, v) < M$ , for a given  $M > 0$ . Indeed, (HP $\psi$ ) and Korn's Inequality, that holds since  $A$  is Lipschitz, imply (4.2) with  $\nabla(u - a)$  in place of  $\nabla u$ , for a suitable  $a: A \rightarrow \mathbb{R}^n$  affine, such that  $\int_A |u - a|^p \, dx \leq \bar{C} F_k^A(u, v)$ , for  $\bar{C} > 1$  depending on  $p, A$ , and on  $c_2$  in (4.1). In view of (HP $\psi$ ),

$$\begin{aligned} \int_A \psi(|a|) \, dx &\leq C_\psi \int_A \psi(|u|) \, dx + C_\psi \int_A \psi(|u - a|) \, dx \leq (C_\psi)^2 F_k^A(u, v) + C_\psi \int_A \psi(|g|) \\ &\quad + (C_\psi)^2 \left( |A| + \int_A |u - a|^p \, dx \right) \leq (C_\psi)^2 \bar{C} F_k^A(u, v) + \tilde{c}(|A|, C_\psi, g), \end{aligned}$$

and by [40, Lemma 2.3] this gives a bound for  $\nabla a$  in  $A$ , so that we conclude (4.2). Now, the sum  $\|\nabla u\|_{L^p(A; \mathbb{M}^{n \times n})} + \int_A (\psi(|u|) \wedge |u|) \, dx$  is a norm on  $W^{1,p}(\Omega; \mathbb{R}^n)$  equivalent to the standard norm, as one can verify by using Poincaré-Wirtinger inequality (we use  $A$  Lipschitz also here). The existence of minimisers follows now from the Direct Method of Calculus of Variations (recall the properties of  $W$  and Ioffe-Olech semicontinuity theorem, see [12, Theorem 2.3.1]).

We now consider the Dirichlet problem for the brittle fracture energy. We give some conditions on the Dirichlet part of the boundary.

Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded, connected, Lipschitz domain for which

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N,$$

with  $\partial_D\Omega$  and  $\partial_N\Omega$  relatively open,  $\partial_D\Omega \cap \partial_N\Omega = \emptyset$ ,  $\mathcal{H}^{n-1}(N) = 0$ ,  $\partial_D\Omega \neq \emptyset$ , and  $\partial(\partial_D\Omega) = \partial(\partial_N\Omega)$  with finite  $\mathcal{H}^{n-2}$  measure. Assume that  $\partial_D\Omega$  satisfies the following condition: there exist a small  $\bar{\delta}$  and  $x_0 \in \mathbb{R}^n$  such that for every  $\delta \in (0, \bar{\delta})$

$$O_{\delta, x_0}(\partial_D\Omega) \subset \Omega, \quad (4.3)$$

where  $O_{\delta, x_0}(x) := x_0 + (1 - \delta)(x - x_0)$ . Let us define, for  $u_0 \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ , the sets

$$\begin{aligned} W_{u_0}^{1,p}(\Omega; \mathbb{R}^n) &:= \{u \in W^{1,p}(\Omega; \mathbb{R}^n) : \text{tr}_\Omega u = \text{tr}_\Omega u_0 \text{ on } \partial_D\Omega\}, \\ V_k^1 &:= \{v \in V_k^\Omega : \text{tr}_\Omega v = 1 \text{ on } \partial_D\Omega\}. \end{aligned}$$

For a given  $u_0 \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$ , the *generalised Griffith energy with Dirichlet boundary condition*  $u_0$  is defined for measurable functions  $u: \Omega \rightarrow \mathbb{R}^n$  and  $v: \Omega \rightarrow [0, 1]$  by

$$D(u, v) := G^\Omega(u, v) + \alpha \mathcal{H}^{n-1}(\partial_D\Omega \cap \{\text{tr}_\Omega u \neq \text{tr}_\Omega u_0\}),$$

and its approximating energies by

$$D_k(u, v) := \begin{cases} \int_\Omega \left( W(v, e(u)) + \frac{d(v)}{\varepsilon_k} + a \varepsilon_k^{q-1} |\nabla v|^q \right) \, dx & \text{in } W_{u_0}^{1,p}(\Omega; \mathbb{R}^n) \times V_k^1, \\ +\infty & \text{otherwise,} \end{cases}$$

namely  $D_k$  is the sum of  $G_k^\Omega$  and the characteristic function of  $W_{u_0}^{1,p}(\Omega; \mathbb{R}^n) \times V_k^1$ .

**Theorem 4.4.** *Under the assumptions above,  $D_k$   $\Gamma$ -converge to  $D$  with respect to the topology of the convergence in measure for  $u$  and  $v$ .*

We now start the second part of this section, which is devoted to prove the results stated in the first part.

*Proof of Theorem 4.1.* Being the convergence in measure metrisable, by [26, Proposition 8.1] the  $\Gamma$  limit of  $G_k^A$  is characterised in terms of convergent sequences. Let us first prove the  $\Gamma$ -lim inf inequality, following the lines of the proof of [43, Theorem 8]. We show that if  $(u_k, v_k)$  converge in measure to  $(u, v)$  and  $F_k^A(u_k, v_k)$  is bounded, then  $u \in GSBD^p(A)$ ,  $v = 1$   $\mathcal{L}^n$  a.e. in  $A$ , and

$$\int_A W(1, e(u)) \, dx \leq \liminf_{k \rightarrow \infty} \int_A W(v, e(u)) \, dx, \quad (4.4a)$$

$$\alpha \mathcal{H}^{n-1}(J_u) \leq \liminf_{k \rightarrow \infty} \int_A \left( \frac{d(v)}{\varepsilon_k} + a \varepsilon_k^{q-1} |\nabla v|^q \right) \, dx. \quad (4.4b)$$

It is immediate that  $v_k \rightarrow 1$  in  $L^1(A)$ . To see (4.4a), we show that

$$(v_k)^{\frac{1}{p}} e(u_k) \rightarrow e(u) \quad \text{in } L^p(A; \mathbb{M}_{sym}^{n \times n}), \quad (4.5)$$

by proving that, for every  $\xi \in \mathbb{S}^{n-1}$  and  $w \in L^p(A)$

$$\int_A (e(u)\xi \cdot \xi - w)^p dx \leq \liminf_{k \rightarrow \infty} \int_A ((v_k)^{\frac{1}{p}} e(u_k)\xi \cdot \xi - w)^p dx. \quad (4.6)$$

This gives  $(v_k)^{\frac{1}{p}} e(u_k)\xi \cdot \xi \rightarrow e(u)\xi \cdot \xi$  in  $L^p(A)$  for every  $\xi \in \mathbb{S}^{n-1}$ , and then (4.5) by the Polarisation Identity. At this stage, (4.4a) follows by the facts that  $v_k \leq 1$ ,  $v_k \rightarrow 1$  uniformly up to a set with small measure by Egorov's Theorem, and by the Ioffe-Olech semicontinuity theorem (cf. [12, Theorem 2.3.1]).

Thus, let us fix  $\xi \in \mathbb{S}^{n-1}$ . For simplicity, we prove (4.6) in the case when  $w = 0$ , the general case being obtained by approximating every  $w \in L^p(A)$  by piecewise constant functions on a Lipschitz partition of  $A$ , for which the lower semicontinuity is then immediate. Notice that it is not restrictive to assume that the  $\liminf$  in (4.6) is a limit. Moreover, up to a subsequence, not relabelled, we have that for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$

$$(\hat{u}_k)_y^\xi \rightarrow \hat{u}_y^\xi \quad \text{in measure in } A_y^\xi, \quad (v_k)_y^\xi \rightarrow v_y^\xi \quad \text{in } L^1(A_y^\xi). \quad (4.7)$$

Indeed, a sequence  $g_k$  converges to a function  $g$  in measure if and only if  $\arctan(g_k)$  converges to  $\arctan(g)$  in  $L^1$ . Therefore, by Fubini's Theorem and the fact that  $u_k \cdot \xi \rightarrow u \cdot \xi$  in measure in  $A$ , one has

$$\int_{\Pi^\xi} \left( \int_{A_y^\xi} |\tau((\hat{u}_k)_y^\xi) - \tau(\hat{u}_y^\xi)|(t) dt \right) d\mathcal{H}^{n-1} = \int_A |\tau(u_k \cdot \xi) - \tau(u \cdot \xi)| dx \rightarrow 0,$$

for  $\tau = \arctan$ . This gives (4.7) for  $u_k$ , while the convergence for  $v_k$  follows easily from Fubini's Theorem for  $v_k \cdot \xi$ .

It is now standard to see, as in [43, (65)–(68)], that  $(\hat{u}_k)_y^\xi \in SBV^p(A_y^\xi)$  and

$$\int_{A_y^\xi} |\nabla(\hat{u}_y^\xi)|^p dt \leq \liminf_{k \rightarrow \infty} \int_{A_y^\xi} (v_k)_y^\xi |\nabla((\hat{u}_k)_y^\xi)|^p dt, \quad (4.8a)$$

$$\alpha \mathcal{H}^0(J_{\hat{u}_y^\xi}) \leq \liminf_{k \rightarrow \infty} \int_{A_y^\xi} \left( \frac{d((v_k)_y^\xi)}{\varepsilon_k} + a \varepsilon_k^{q-1} |\nabla((v_k)_y^\xi)|^q \right) dt. \quad (4.8b)$$

Moreover, we get  $u \in GSBD(A)$  and (4.6) for  $w = 0$  arguing again as in the proof of [43, Theorem 8], with the exponents 2 and  $p$  therein for  $u$  and  $v$  replaced by  $p$  and  $q$ . In particular, integrating (4.8a) over  $\Pi^\xi$  gives (4.6) for  $w = 0$ , by (1.7). In the same way, one integrates (4.8b) over  $\Pi^\xi$  and applies a localisation argument to deduce (4.4b). Notice that the analogous of the Structure Theorem [2, Theorem 4.5] holds also for  $GSBD$ , see for instance [39, Theorem 3.1]. By the discussion at the beginning of the proof, we conclude (4.4a) and the  $\Gamma$ -lim inf inequality.

The  $\Gamma$ -lim sup inequality follows from our density result. Indeed, for every  $u \in GSBD(A)$  there exist  $u_k \in SBV(A; \mathbb{R}^n) \cap L^\infty(A; \mathbb{R}^n)$  satisfying the approximation properties of Theorem 3.1. In particular,

$$G^A(u_k, 1) \rightarrow G^A(u, 1).$$

By a diagonalisation argument, it is then enough to construct a recovery sequence for  $(u, 1)$ , with  $u \in SBV(A; \mathbb{R}^n) \cap L^\infty(A; \mathbb{R}^n)$ . This is done by the same construction as in [15, 43], that was applied therein to a quadratic bulk energy in  $e(u)$  but works also for a bulk energy with  $p$ -growth.  $\square$

We now prove Theorem 4.2. The  $\Gamma$ -lim inf inequality is a trivial consequence of Theorem 4.1, while the  $\Gamma$ -lim sup inequality is easy since we have already proven (3.1e). The compactness of the minimising sequences for  $F_k^A$  is obtained arguing similarly to [27, Theorem 11.1] and [43, Proposition 1].

*Proof of Theorem 4.2.* The  $\Gamma$ -lim inf inequality follows by Theorem 4.1 (or by (4.4), which are the relevant properties here), and by Fatou lemma, that implies

$$\int_A \psi(|u - g|) dx \leq \liminf_{k \rightarrow \infty} \int_A \psi(|u_k - g|) dx,$$

when  $u_k \rightarrow u$  in measure in  $A$ .



As for the  $\Gamma$ -limsup inequality, let us fix  $u \in GSBD^p(A)$  such that  $\psi(|u - g|) \in L^1(A)$ . Since  $\psi(|g|) \in L^1(A)$ , then  $\psi(|u|) \in L^1(A)$ , and by (3.1) there exist  $u_k \in SBV(A; \mathbb{R}^n) \cap L^\infty(A; \mathbb{R}^n)$  such that

$$F^A(u_k, 1) \rightarrow F^A(u, 1).$$

Notice that we have used also (3.1e), which was not necessary for the case without fidelity term. The proof now follows as in Theorem 4.1.

It lasts to prove the sequential compactness of  $\gamma_k$ -minimisers for  $F_k^A$ . This is a consequence of [26, Corollary 7.20] and of Proposition 4.5 below.  $\square$

The following proposition employs the argument of [27, Theorem 11.1]. A similar result is proven in [43, Proposition 1], assuming  $\psi(s) = s^2$  and  $g \in L^2(A; \mathbb{R}^n)$ , and so a uniform bound for displacements in  $L^2(A; \mathbb{R}^n)$ .

**Proposition 4.5.** *Let  $(u_k, v_k)$  be a sequence such that  $F_k^A(u_k, v_k)$  is bounded. Then  $v_k \rightarrow 1$  in  $L^1(A)$  and, up to a subsequence,  $u_k$  converge in measure to a suitable  $u \in GSBD^p(A)$ , with  $\psi(|u|) \in L^1(A)$ .*

*Proof.* The first part of the proof is similar to the beginning of [43, Proposition 1].

It is immediate that  $v_k \rightarrow 1$  in  $L^1(A)$ . Let us fix  $k \in \mathbb{N}$  and  $\xi \in \mathbb{S}^{n-1}$ . For simplicity of notation, we omit to write the dependence on  $k$  and  $\xi$  of the objects introduced in the following. We still write  $u_k$  and  $v_k$  to avoid confusion with the limit functions. Let

$$\begin{aligned} \hat{A}_\lambda &:= \left\{ y \in \Pi^\xi : \int_{A_y^\xi} \left( (v_k)_y^\xi |\nabla((\hat{u}_k)_y^\xi)|^p + \frac{d((v_k)_y^\xi)}{\varepsilon_k} + a\varepsilon_k^{q-1} |\nabla((v_k)_y^\xi)|^q \right) dt \leq \lambda \right\}, \\ A_\lambda &:= \{x \in A : \Pi^\xi(x) \in \hat{A}_\lambda\}, \quad B_\lambda := A \setminus A_\lambda, \end{aligned}$$

where  $\Pi^\xi(x)$  is the projection of  $x$  on the plane  $\Pi^\xi$ . Being  $F_k(u_k, v_k)$  bounded, by Fubini's Theorem and Chebychev inequality we have

$$\mathcal{L}^n(B_\lambda) \leq c \frac{\text{diam}(A)}{\lambda}.$$

Let  $\tau_\mu(s) := -\mu \vee s \wedge \mu$ ,

$$w_\mu^\lambda := \begin{cases} \tau_\mu(u_k \cdot \xi) & \text{in } A_\lambda, \\ 0 & \text{in } B_\lambda, \end{cases}$$

and let  $g: [0, \infty) \rightarrow [0, \infty)$  be nondecreasing, continuous, subadditive, such that

$$g(0) = 0, \quad \liminf_{s \rightarrow 0^+} \frac{g(s)}{s} > 0, \quad g(s) \leq s \text{ for } s \in [0, \infty), \quad \lim_{s \rightarrow \infty} \frac{\psi(s)}{g(s)} = +\infty.$$

Therefore, following exactly [27, inequality (11.8)] we get that for every  $\delta > 0$  there exist  $\mu_\delta > 0$ ,  $\lambda_\delta > 0$  such that

$$\int_A g(|u_k \cdot \xi - w_{\mu_\delta}^\lambda|) dx < \delta,$$

and then

$$\int_A g(|\phi(v_k)(u_k \cdot \xi - w_\delta)|) dx < \delta, \quad (4.9)$$

for  $w_\delta := w_{\mu_\delta}^{\lambda_\delta}$  and  $\phi(s) := \int_0^s d(s)^{1/q'}(t) dt$ , since  $\phi(v_k) \leq c v_k \leq c$ . (It is enough to redefine  $\delta$  as  $\tilde{c}\delta$ , for a suitable  $\tilde{c}$ .) Notice that here we use the fact that  $\psi(|u_k|)$  are equibounded in  $L^1(A)$ , which follows since  $F_k(u_k, v_k)$  are equibounded.

Repeating the same computations done in [43, Proposition 1] to get (84) therein, we obtain that for every  $\delta > 0$

$$\int_{\mathbb{R}} |(\phi(v_k)w_\delta)_y^\xi(t+h) - (\phi(v_k)w_\delta)_y^\xi(t)| dt \leq c(\delta)h. \quad (4.10)$$

By (4.9) and (4.10) we are in the hypotheses of [27, Lemma 10.7], which gives that  $\phi(v_k)u_k$  converge (up to a subsequence, not relabelled) to some  $\tilde{u}$  pointwise  $\mathcal{L}^n$ -a.e. in  $A$ , or also in measure. Since  $v_k \rightarrow 1$  in  $L^1(A)$ , we obtain that  $u_k$  converge to  $u := \frac{\tilde{u}}{\phi(1)}$  in measure. As in the proof of Theorem 4.1  $u \in GSBD^p(A)$ , and by Fatou inequality  $\psi(|u|) \in L^1(A)$ .  $\square$

*Proof of Theorem 4.4.* The  $\Gamma$ -lim inf inequality follows by that one for  $G_k^A$ . Indeed, let  $\tilde{\Omega} \subset \mathbb{R}^n$  be open such that  $\Omega \subset \tilde{\Omega}$  and  $\tilde{\Omega} \cap \partial\Omega = \partial_D\Omega$ , and define for each  $u$  and  $v$  their extensions

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ u_0 & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases} \quad \tilde{v} := \begin{cases} v & \text{in } \Omega, \\ 1 & \text{in } \tilde{\Omega} \setminus \Omega. \end{cases}$$

If  $u_k$  converge in measure to some  $u \in GSBD^p(\Omega)$ , then  $\tilde{u}_k$  converge to  $\tilde{u} \in GSBD^p(\tilde{\Omega})$ . Moreover, since  $u_0 \in W^{1,p}(\Omega; \mathbb{R}^n)$ ,

$$D_k(u, v) = F_k^{\tilde{\Omega}}(\tilde{u}, \tilde{v}) - \int_{\tilde{\Omega} \setminus \Omega} W(1, e(u_0)) \, dx, \quad D(u, v) = F^{\tilde{\Omega}}(\tilde{u}, \tilde{v}) - \int_{\tilde{\Omega} \setminus \Omega} W(1, e(u_0)) \, dx.$$

Therefore Theorem 4.1 implies the  $\Gamma$ -lim inf inequality for  $D$ .

We now prove the  $\Gamma$ -lim sup inequality. Let us fix  $u \in GSBD^p(\Omega)$ . The goal is to prove that for every small  $\eta > 0$  (in no context with  $\eta_k$ ) there exists  $u^\eta \in SBV^p(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$  such that  $u^\eta = u_0$  in the intersection of  $\tilde{\Omega}$  with a  $n$ -dimensional neighbourhood of  $\partial_D\Omega$  and

$$D(u^\eta, 1) < D(u, 1) + \eta. \quad (4.11)$$

Indeed, with such  $u^\eta$  at hand, one may apply the standard construction for recovery sequences of Ambrosio-Tortorelli type (cf. for instance [43, Theorem 9]), which leaves each approximating function equal to  $u_0$  in a neighbourhood of  $\partial_D\Omega$  (in the topology of  $\tilde{\Omega}$ ), in particular with the right boundary datum. Then the  $\Gamma$ -lim sup inequality follows by a diagonal argument. Thus, let us fix  $\eta > 0$  and construct  $u^\eta$ .

Since  $\Sigma := \partial(\partial_D\Omega) = \partial(\partial_N\Omega)$  has finite  $\mathcal{H}^{n-2}$  measure, for any  $\varepsilon > 0$  (in no context with  $\varepsilon_k$ ) there exists a  $n$ -dimensional neighbourhood  $\tilde{\Sigma}$  of  $\Sigma$  with  $\mathcal{L}^n(\tilde{\Sigma}) < \varepsilon$  and

$$\mathcal{H}^{n-1}(\partial\Omega \cap \tilde{\Sigma}) < \varepsilon. \quad (4.12)$$

We now argue as done to get (3.3) and (3.4) with the role of  $J_u$  and  $\partial^*\Omega$  therein played by  $\partial_N\Omega \setminus \tilde{\Sigma}$ . For any  $\varepsilon$  we obtain a finite set of cubes  $(Q_h^N)_{h=1}^{h^N}$  of centers  $x_h^N$  and sidelength  $\varrho_h^N$ , whose closures are pairwise disjoint, such that the analogous of (3.4) hold, with the apices  $^0$  replaced by  $^N$  and  $\bar{h}$  by  $h^N$ . We introduce the rectangles  $R_h^N$ ,  $(R_h^N)'$  and  $\hat{R}_h^N$  as in (3.9), with the apices  $^0$  replaced by  $^N$ , namely for instance

$$R_h^N := \left\{ x_h^N + \sum_{i=1}^{n-1} y_i b_{h,i}^N + y_n \nu_h^N : y_i \in (-\varrho_h^N, \varrho_h^N), y_n \in (-3\varepsilon\varrho_h^N - t, -\varepsilon\varrho_h^N) \right\},$$

with  $t > 0$  small,  $\nu_h^N = -\nu_\Omega(x_h^N)$  the generalised outer normal to  $\Omega$  at  $x_h^N$ , and  $(b_{h,i}^N)_{i=1}^{n-1}$  an orthonormal basis of  $(\nu_h^N)^\perp$ . Moreover, let  $\hat{u}_h^N \in GSBD^p(\hat{R}_h^N)$  be the functions provided by Lemma 1.8 for which the analogous of (3.10) hold. Let  $\Omega_t := \Omega + B(0, t)$  and  $\tilde{u} \in GSBD^p(\Omega_t)$  be defined by

$$\tilde{u} := \begin{cases} u & \text{in } \Omega; \\ \hat{u}_h^N & \text{in } \hat{R}_h^N; \\ u_0 & \text{elsewhere in } \Omega_t. \end{cases}$$

Notice that  $\Omega_t \cap Q_h^N \subset \hat{R}_h^N$  for every  $h$ . We claim that

$$G^{\Omega_t}(\tilde{u}, 1) < F(u, 1) + \eta, \quad (4.13)$$

for  $\varepsilon$  and  $t$  small enough. Indeed, it is enough to observe that, for  $\varepsilon$  and  $t$  small enough,

$$\int_{\Omega_t \setminus \Omega} |e(u_0)|^p \, dx < \eta, \quad \sum_h \int_{\hat{R}_h^N} |e(\hat{u}_h^N)|^p \, dx \leq C \int_{R_h^N} |e(u)|^p \, dx < \eta,$$

by the absolute continuity of the integral, and

$$\mathcal{H}^{n-1}(J_{\tilde{u}}) < \mathcal{H}^{n-1}(J_u) + c \mathcal{H}^{n-1}\left(J_u \cap \bigcup_{h=1}^N R_h^N\right) + \mathcal{H}^{n-1}\left(\partial_N\Omega \setminus \bigcup_h Q_h^N\right) < \mathcal{H}^{n-1}(J_u) + \eta,$$

by Lemma 1.8, (4.12) and the analogous of (3.4a), arguing as in Theorem 3.1 to get (3.1c).

Let us consider the functions  $\tilde{u}^\delta := \tilde{u} \circ (O_{\delta, x_0})^{-1} + u_0 - u_0 \circ (O_{\delta, x_0})^{-1}$ . By (4.3) and the definition of  $\tilde{u}$ ,  $\tilde{u}^\delta = u_0$  in a neighbourhood of  $\partial_D\Omega$ . Moreover, by (4.13) and since for  $\delta$  small  $\int_{\mathbb{R}^n} |e(u_0) - e(u_0 \circ (O_{\delta, x_0})^{-1})|^p \, dx < \eta$ , we have for  $\delta$  small enough that

$$D(\tilde{u}^\delta, 1) < D(u, 1) + \eta. \quad (4.14)$$

We obtain  $u^7$  by applying the construction of Theorem 3.1 starting from a fixed  $\tilde{u}^\delta$  satisfying (4.14): since  $u_0$  does not jump, we have that the  $k$ -th approximating function for  $\tilde{u}^\delta$  is  $u_0 * \varrho_k$  in a neighbourhood of  $\partial_D \Omega$ . Then it is enough to correct it by adding  $u_0 - u_0 * \varrho_k$ , which is small in  $W^{1,p}$  norm for  $k$  large. Therefore, the approximation properties of Theorem 3.1 and (4.14) give (4.11). This concludes the proof.  $\square$

*Remark 4.6.* The main difficulty without the geometrical assumptions on  $\partial_D \Omega$  of Theorem 4.4 is to correct the boundary datum after the composition with  $(O_{\delta, x_0})^{-1}$  or after any convolution. Indeed, there could be some parts of  $\partial_D \Omega$  which are brought outside  $\Omega$  and replaced by  $u$ , so that the new trace on  $\partial_D \Omega$  may differ too much from the trace of  $u_0$  (the trace of  $u$  on strips close to  $\partial_D \Omega$  is not even in  $W^{1-1/p,p}(\partial_D \Omega)$  in general), and there is an analogous problem with the convolution. In subsets of  $\partial_D \Omega$  where the traces of  $u$  and  $u_0$  are different one could bring the jump a little bit inside  $\Omega$ , arguing as in [24, Theorem 3.1] keeping almost the same length, so almost the same energy. But as soon as there are zones where the traces of  $u$  and  $u_0$  coincide, one may increase very much the energy to fit the boundary condition.

**Acknowledgements.** Vito Crismale has been supported by a public grant as part of the *Investissement d'avenir* project, reference ANR-11-LABX-0056-LMH, LabEx LMH.

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