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Existence of pulses for the system of competition of species

Martine Marion · Vitaly Volpert

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Abstract The paper is concerned with the existence of pulses for monotone reaction-diffusion systems of two equations. For a general class of systems we prove that pulses exist if and only if the wave solutions propagate at positive speed. This result is applied to investigate the existence of pulses for the system of competition of species.

Keywords Existence of pulses · Monotone systems · Leray-Schauder method

Mathematics Subject Classification (2000) MSC 35K57 · MSC 47F05

1 Introduction

We consider the system of two equations

$$w_1'' + F_1(w_1, w_2) = 0, \quad w_2'' + F_2(w_1, w_2) = 0, \tag{1}$$

where the function $F = (F_1, F_2)$ is assumed to be sufficiently smooth and to satisfy the monotonicity condition:

$$\frac{\partial F_i}{\partial w_j}(w) > 0 \text{ for } i \neq j \text{ and } w = (w_1, w_2) \in \mathbb{R}^2_+, \tag{2}$$

together with $F_1(0,0) = F_2(0,0) = 0$.

We aim to study the existence of pulses, that is of positive solutions of system (1) with the limits at infinity $w(\pm\infty) = (0,0)$. The existence of pulses has been proved for some reaction-diffusion systems: the Gray-Scott model [1],

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[4], [13], the Gierer-Meinhardt model [6], [14] and a three component system [2]. Singular perturbation methods to study existence and stability of pulses for a system of two equations are used in [3]. In a previous work [5] we studied the particular case where the nonlinear terms in (1) take the form:

$$F_1(w_1, w_2) = f_1(w_2) - w_1, \quad F_2(w_1, w_2) = f_2(w_1) - w_2.$$

Note that in that case the graph $w_1 = f_1(w_2)$ (respectively $w_2 = f_2(w_1)$) corresponds to the zeros of F_1 (respectively F_2).

Here we aim to consider more general monotone systems. Let us state our assumptions on F. For simplicity, we consider infinitely differentiable functions. Setting $w^+ = (0, 0)$ and $w^- = (1, 1)$ we suppose that

$$\begin{gathered} F(w^\pm) = (0, 0), \\ \text{the eigenvalues of the matrices } F'(w^\pm) \text{ are negative.} \end{gathered} \tag{3}$$

Under these conditions F necessarily possesses at least one additional zero in the unit open square and we assume the uniqueness of such a zero, more precisely

$$\begin{gathered} F \text{ has three zeros } w^+, w^-, \bar{w} = (\bar{w}_1, \bar{w}_2) \text{ in } \mathbb{R}^2_+; \text{ furthermore } 0 < \bar{w}_i < 1 \\ \text{and the matrix } F'(\bar{w}) \text{ has a positive eigenvalue.} \end{gathered} \tag{4}$$

Next we assume that the zeros of F_1 lie in the graph of some function parametrized by w_2, more precisely:

$$F_1(w_1, w_2) = 0, \quad w_1, w_2 \geq 0, \quad \Leftrightarrow \quad w_1 = f_1(w_2), \tag{5}$$

where

$$f_1(0) = 0, \quad f_1(1) = 1, \quad f_1'(w_2) > 0 \text{ for } w_2 \geq 0. \tag{6}$$

Similarly, we assume that the zeros of F_2 are determined by some function depending on w_1:

$$F_2(w_1, w_2) = 0, \quad w_1, w_2 \geq 0, \quad \Leftrightarrow \quad w_2 = f_2(w_1), \tag{7}$$

with

$$f_2(0) = 0, \quad f_2(1) = 1, \quad f_2'(w_1) > 0 \text{ for } w_1 \geq 0. \tag{8}$$

Under the above conditions we aim to investigate the existence of pulses such that:

$$w(x) = w(-x), \quad w'(x) < 0 \text{ for } x > 0$$

(here and everywhere below inequalities for vectors mean that each component of the vectors satisfies this inequality). Then the problem on the whole axis reduces to looking for solutions of system (1) defined on the half-axis $\mathbb{R}_+$ such that

$$w'(0) = 0, \quad w'(x) < 0 \text{ for } x > 0, \quad w(\infty) = 0. \tag{9}$$

As already mentioned this problem was studied in [5] for specific nonlinear terms. We showed that the existence of pulses is related to the sign of the wave speed for the traveling-wave solution of the reaction-diffusion system

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + F(v). \tag{10}$$

This solution reads $v(x,t) = u(x-ct)$ where u satisfies the system of equations

$$u'' + cu' + F(u) = 0 \tag{11}$$

and has the limits at infinity

$$u(\pm\infty) = w^{\pm}. \tag{12}$$

Under the assumptions (2)-(4), it is classical that there exists a uniquely defined traveling-wave (up to some translation in space for u). Moreover, it is a monotonically decreasing vector-function (component-wise) (see [7], [8]).

The main result of this work is given by the following theorem.

Theorem 1.1. *Under the assumptions (2)-(8), Problem (1) has a solution satisfying (9) if and only if the wave speed c in (11)-(12) is positive.*

This result will be applied to investigate the existence of pulses for the system of competition of species. In order to apply the general existence result we will need to introduce some suitable perturbed system. We also discuss conditions which guarantee the positiveness of the speed.

The paper is organized as follows. In Section 2 we first recall some results concerning the functional setting for Problem (1) and some estimates of the solutions borrowed from [5]. Then the existence part in Theorem 1.1 is derived in Section 3. The proof relies on the Leray-Schauder method. We will reduce the nonlinearity to a model problem such that $F_1 = F_2$ and will look for a solution with $w_1 = w_2$. In order to obtain a priori estimates of solutions, we will have to verify that the wave speed remains positive in the process of homotopy. This condition will require some additional constructions. It is worth noting that the homotopy arguments are completely different from the ones in [5]. Finally Section 4 is devoted to the system of competition of pulses.

2 Functional spaces and estimates of solutions

Let us consider the system depending on some parameter $\tau \in [0,1]$

$$\begin{cases} w_1'' + F_1^{\tau}(w_1, w_2) = 0, \\ w_2'' + F_2^{\tau}(w_1, w_2) = 0, \end{cases} \tag{13}$$

on the half axis $x > 0$ with the boundary conditions and the conditions at infinity:

$$w_i'(0) = 0, \quad w_i(\infty) = 0, \quad i = 1, 2. \tag{14}$$

For simplicity we will suppose that the function $F^\tau = (F_1^\tau, F_2^\tau)$ is infinitely differentiable with respect to both variables $w \in \mathbb{R}^2$ and $\tau \in [0, 1]$. It satisfies the following conditions that are extensions of (2)-(8):

$$\frac{\partial F_i^\tau}{\partial w_j}(w) > 0, \quad i, j = 1, 2, \ \ i \neq j, \ \ w \in \mathbb{R}_+^2, \ \ \tau \in [0, 1], \tag{15}$$

$$F^\tau(w^\pm) = (0, 0), \quad \text{the eigenvalues of the matrices } (F^\tau)'(w^\pm) \text{ are negative}, \tag{16}$$

$$F^\tau \text{ has three zeros } w^+, w^-, \bar{w}^\tau = (\bar{w}_1^\tau, \bar{w}_2^\tau) \text{ in } \mathbb{R}_+^2; \text{ furthermore } 0 < \bar{w}_i^\tau < 1 \text{ and the matrix } (F^\tau)'(\bar{w}^\tau) \text{ has a positive eigenvalue}, \tag{17}$$

$$F_1^\tau(w_1, w_2) = 0 \Leftrightarrow w_1 = f_1^\tau(w_2), \quad f_1^{\tau\prime}(w_2) > 0 \text{ for } w_2 \geq 0, \tag{18}$$

$$F_2^\tau(w_1, w_2) = 0 \Leftrightarrow w_2 = f_2^\tau(w_1), \quad f_2^{\tau\prime}(w_1) > 0 \text{ for } w_1 \geq 0. \tag{19}$$

2.1 Operators and spaces

For the functional setting let us introduce the Hölder space $C^{k+\alpha}(\mathbb{R}_+)$ consisting of vector-functions of class C^k, which are continuous and bounded on the half-axis $\mathbb{R}_+$ together with their derivatives of order k, and such that the derivatives of order k satisfy the Hölder condition with the exponent $\alpha \in (0, 1)$. The norm in this space is the usual Hölder norm. Set

$$E^1 = \{w \in C^{2+\alpha}(\mathbb{R}_+), \ w_i'(0) = 0, \ i = 1, 2\}, \quad E^2 = C^\alpha(\mathbb{R}_+).$$

Next we introduce the weighted spaces E_μ^1 and E_μ^2 with $\mu(x) = \sqrt{1 + x^2}$. These spaces are equipped with the norms:

$$\|w\|_{E_\mu^i} = \|w\mu\|_{E^i}, \quad i = 1, 2.$$

In view of (13), let us consider the operator

$$A^\tau(w) = w'' + F^\tau(w), \tag{20}$$

acting from E_μ^1 into E_μ^2. Then the linearized operator about any function in E_μ^1 satisfies the Fredholm property and has the zero index. The nonlinear operator is proper on closed bounded sets. This means that the inverse image of a compact set is compact in any closed bounded set in E_μ^1. Finally, the topological degree can be defined for this operator. All these properties can be found in [10], [11].

Hereafter we consider solutions of (13)-(14) belonging to the space E_μ^1.

2.2 A priori estimates

We aim to derive various estimates of the solutions of (13)-(14). Note that this problem possesses the trivial solution $w \equiv w^+$.

We will be particularly interested in monotonically decreasing solutions. We start with some preliminary properties of such solutions

Lemma 2.1. Under assumptions (15)-(19), assume that w is a monotonically decreasing non trivial solution of (13)-(14). Then for $x > 0$ we have $w'(x) < 0$ and $w(x) < w^-$.

This result is a variant of similar ones in [5] (see Lemma 2.3 and the arguments in the proof of Theorem 2.1). The details are left to the reader.

Next we aim to derive estimates in E^1_μ of the monotonically decreasing solutions of (13)-(14). They will follow from the positiveness of the wave speed for the traveling wave problem

$$u'' + cu' + F^\tau(u) = 0. \tag{21}$$

In view of the assumptions (15)-(17), there exists a unique value of c denoted by c^τ such that the system (21) has a monotonically decreasing solution with the limits at infinity:

$$u(\pm\infty) = w^\pm. \tag{22}$$

The following result is proved in [5].

Theorem 2.2. *Under assumptions (15)-(19), moreover suppose that $c^\tau > 0$ for all $\tau \in [0,1]$. Then there exists some constant $R > 0$ such that for all $\tau \in [0,1]$ and for any arbitrary monotonically decreasing solution $w \in E^1_\mu$ of (13)-(14) the following estimate holds :*

$$\|w\|_{E^1_\mu} \leq R.$$

Next we state a result of separation between the monotonically decreasing solutions of (13) that will be denoted by $w^M(x)$ and the non-monotone solutions of (13) that will be denoted by $w^N(x)$.

Theorem 2.3. *Suppose that the assumptions (15)-(19) hold true. Then there exists a constant $r > 0$ such that for any monotonically decreasing solution w^M and any non-monotone solution w^N (and for all $\tau \in [0,1]$) the following estimates hold :*

$$\|w^M - w^N\|_{E^1_\mu} \geq r, \quad w^M_1(0) > r, \quad w^M_2(0) > r.$$

We refer to [5] for the proof of this result.

3 Proof of Theorem 1.1

The non-existence part in Theorem 1.1 for $c \leq 0$ can be proved as in [5] to which the reader is referred.

Hereafter we assume $c > 0$ and we aim to derive the existence of a monotonically decreasing pulse. The proof relies on the Leray-Schauder method. The key step consists in the construction of a continuous deformation (homotopy) of our problem to a model problem for which we will be able to prove that the value of the topological degree is different from zero.

The continuous deformation will take the form (13) and we will require that the conditions (15)-(19) are satisfied. Besides as τ varies we will need the speed c^τ in (21)-(22) to remain positive. Thanks to Theorem 2.2 this will yield a priori estimates in E^1_μ of the monotonically decreasing solutions.

In Section 3.1 we first describe two tools that will allow us to keep the speed c^τ positive and will be used repeatedly. Next the homotopy argument is described in Section 3.2 and we conclude the proof of the existence of a pulse in Section 3.3.

3.1 Positiveness of the wave speed

Consider the problem

$$u'' + cu' + G(u) = 0, \quad u(\pm\infty) = w^\pm. \tag{23}$$

Here G is some arbitrary function satisfying the conditions (2)-(8) (with $F = G$ and $f_i = g_i$ for the zero lines). Then there is a monotone solution of problem (23) for a unique value of the speed c. Even though we keep the same notation for the speed it is of course different than the one in (11)-(12). In particular the speed for (23) is not assumed to be positive.

We aim to modify the function G so that the conditions (2)-(8) remain satisfied, the zeros of the nonlinearities remain unchanged and the speed of the traveling wave for the new nonlinearity becomes positive.

For that purpose let us introduce infinitely differentiable and non-negative functions $\phi_i(w)$, $i = 1, 2$, whose supports are defined as follows (Figure 1).

We choose a point $w^0 = (w^0_1, w^0_2)$ such that $G_i(w^0) > 0$ for $i = 1, 2$. Due to the assumptions (2)-(8) such a point necessarily belongs to the zone between the two lines of zeros in the upper part of the square. Also note that $G_1(w^0_1, w_2) > 0$ for $w_2 \geq w^0_2$ and $G_2(w_1, w^0_2) > 0$ for $w_1 \geq w^0_1$.

Let us set

$$D_1 = \{w = (w_1, w_2), \;\; w^0_1 - \epsilon_1 \leq w_1 \leq w^0_1 + \epsilon_1, \quad w^0_2 - \epsilon_2 \leq w_2 \leq 1\},$$

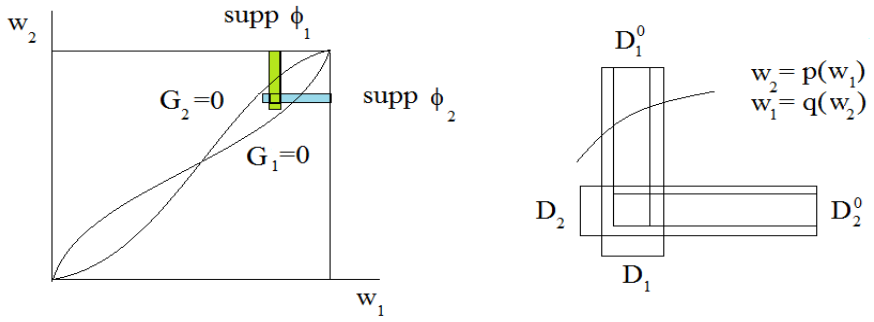

Fig. 1 Zero lines of the functions G_i and the supports of the functions ϕ_i (left). More detailed structure of the support of the functions ϕ_i (right). Inner rectangles show where these functions equal 1.

where ϵ_1 and ϵ_2 are positive numbers and sufficiently small so that $G_1(w) > 0$ for $w \in D_1$. Similarly, we introduce

$$D_2 = \{w = (w_1, w_2),\ \ w_2^0 - \delta_1 \le w_2 \le w_2^0 + \delta_1, \quad w_1^0 - \delta_2 \le w_1 \le 1\},$$

where δ_1 and δ_2 are sufficiently small positive numbers so that $G_2(w) > 0$ for $w \in D_2$.

Next, we can introduce two rectangles D_i^0, $i = 1, 2$, with common width σ located inside the rectangles D_i as it is shown in Figure 1 (right). We define the functions ϕ_i in such a way that the following conditions are satisfied:

$$\frac{\partial \phi_1}{\partial w_2} \ge 0, \quad \frac{\partial \phi_2}{\partial w_1} \ge 0, \tag{24}$$

and

$$\phi_i \ge 0, \quad \text{Supp } \ \phi_i \subset D_i, \quad \phi_i(w) = 1 \text{ for } w \in D_i^0, \quad i = 1, 2. \tag{25}$$

We now consider the nonlinearities:

$$H_i(w) = G_i(w) + k\phi_i(w), \quad i = 1, 2, \tag{26}$$

where k is some positive constant. The function H_i, $i = 1, 2$, differs from G_i in a zone where $G_i > 0$ (see Figure 1) and it is obtained by adding some positive term. Consequently the zeros of H_i coincide with the ones of G_i and $H_i(w) > 0$ at any point w such that $G_i(w) > 0$. The conditions (3)-(8) remain valid for H and the monotonicity condition

$$\frac{\partial H_i}{\partial w_j}(w) > 0, \quad i \ne j, \tag{27}$$

is satisfied.

Hence the traveling wave problem for this modified nonlinearity which reads:

$$u'' + cu' + G(u) + k\phi(u) = 0, \quad u(\pm\infty) = w^{\pm}, \tag{28}$$

where we set $\phi(u) = (\phi_1(u), \phi_2(u))$ has a monotone solution for a unique value of the speed. This value depends on k and we denote it by c_k. We aim to show that if k is large enough then $c_k > 0$.

Lemma 3.1. *Let G be a function satisfying the conditions (2)-(8) and let w^0 such that $G(w^0) > 0$. Define the rectangles D, D_i^0 (with width σ) and the function ϕ_i as above. Then if*

$$k > \frac{\|G\|_\infty}{\sigma} \tag{29}$$

the speed c_k in (28) is positive.

Proof. Let $u(x) = (u_1(x), u_2(x))$ denote the solution of (28). Since both components of the solution are monotone functions of x, we can express one coordinate as a function of the other one: either $u_2 = p(u_1)$ or $u_1 = q(u_2)$. Both of them are monotone functions of their arguments.

Next consider the solution $u(x)$ as a curve in the plane (w_1, w_2). It connects the points w^+ and w^-. Therefore it crosses the set $D^0 = D_1^0 \cup D_2^0$ (Figure 2, right). Moreover necessarily the curve crosses the whole width of either D_1^0 or D_2^0.

Without loss of generality we can suppose that the curve crosses the whole width of D_1^0. Then let us consider the function $u_2 = p(u_1)$ along the curve. The system (28) reduces to the single equation

$$u_1'' + c_k u_1' + G_1(u_1, p(u_1)) + k\phi_1(u_1, p(u_1)) = 0. \tag{30}$$

For this scalar equation it is easy to see that

$$c_k = \frac{1}{\|u_1'\|^2_{L^2(-\infty,\infty)}} \int_0^1 [G_1(s, p(s)) + k\phi_1(s, p(s))]\, ds.$$

Hence $c_k > 0$ if and only if

$$\int_0^1 [G_1(s, p(s)) + k\phi_1(s, p(s))]\, ds > 0. \tag{31}$$

Since the function ϕ_1 is non-negative and $\phi_1 = 1$ on D_1^0, we have

$$\int_0^1 \phi_1(s, p(s)) ds > \sigma,$$

where σ is the width of the rectangle D_1^0. Consequently the inequality (31) is satisfied if the stronger condition

$$\int_0^1 G_1(s, p(s)) ds + k\sigma > 0 \tag{32}$$

holds true. This last condition is clearly satisfied under the assumption (29) (where $\|G\|_\infty$ denotes the supremum norm of G on the unit square). This implies that $c_k > 0$ and concludes the proof of the lemma.

□

We will also make use of a classical result that states that the speed of the waves increases if the nonlinearity is increased (see [8]).

Theorem 3.2. *Consider the two problems*

$$u'' + cu' + G^k(u) = 0, \quad u(\pm\infty) = w^\pm, \;\; k = 1, 2, \tag{33}$$

where the functions $G^k = (G_1^k, G_2^k)$ are assumed to satisfy conditions (2) (possibly non-strict) and (3), and furthermore

$$G_i^1(u) \geq G_i^2(u) \;\text{ for }\; w_+ \leq u \leq w_- \;\text{ and }\; i = 1, 2.$$

If the two problems (33) possess monotone solutions with the values of the speed $c = c_k$, $k = 1, 2$, then we have $c_1 \geq c_2$.

3.2 Homotopy

We aim now to construct the homotopy F^τ. It is of course such that, for $\tau = 0$, $F^0 = F$ the given nonlinearity in (1) for which we recall that the wave speed c is assumed to be positive.

As already noted we need the speed c^τ for the traveling wave problem (21)-(22) to remain positive in the homotopy process. For that purpose we introduce some (arbitrary) point $w^0 \in (0, 1)^2$ such that $F(w^0) > 0$ and consider the function $\phi = (\phi_1, \phi_2)$ constructed as above in Section 3.1 (with $G = F$). We will require the homotopy to be such that $F^\tau(w^0)$ remains positive as τ varies so that Lemma 3.1 will apply with $G = F^\tau$ and ϕ defined as above.

Step 1 ($\tau \in [0, \tau_1]$, $\tau_1 = 1/4$). We start by increasing the function F and introduce:

$$F_i^\tau(w) = F_i(w) + \tau K \phi_i(w), \quad i = 1, 2, \quad 0 \leq \tau \leq \tau_1. \tag{34}$$

Here K is some arbitrary positive constant which will be specified below.

As noted in Section 3.1 the functions F_i^τ satisfy the conditions (15)-(19) and there zero lines coincide with the ones of F_i, that is $w_1 = f_1(w_2)$ and $w_2 = f_2(w_1)$.

Let c^τ denote the value of the speed for the problem

$$u'' + c^\tau u' + F^\tau(u) = 0, \quad u(\pm\infty) = w_\pm. \tag{35}$$

Since $F_i^\tau \geq F_i$, Theorem 3.2 yields that $c^\tau \geq c > 0$ for $\tau \in [0, \tau_1]$ (without any assumption on K).

Step 2 ($\tau \in [\tau_1, \tau_2]$, $\tau_2 = 1/2$). Let us introduce the functions:

$$\bar{F}_1(w) = f_1(w_2) - w_1, \quad \bar{F}_2(w) = f_2(w_1) - w_2. \tag{36}$$

The second step of the homotopy will consist in reducing the functions $F_i^{\tau_1}$ to the functions $\bar{F}_i$ while keeping K fixed. Hence we consider

$$F_i^\tau(w) = \frac{\tau - \tau_1}{\tau_2 - \tau_1}\bar{F}_i(w) + \frac{\tau_2 - \tau}{\tau_2 - \tau_1}F_i(w) + \tau_1 K\phi_i(w), \quad i = 1, 2, \quad \tau_1 \le \tau \le \tau_2. \tag{37}$$

The above functions satisfy the monotonicity condition (15) and their zero lines are still given by $w_1 = f_1(w_2)$ and $w_2 = f_2(w_1)$.

Also $F^\tau(w^0) > 0$. Hence, due to Lemma 3.1, the wave speed c^τ in (35) is positive for $\tau \in [\tau_1, \tau_2]$ if K is sufficiently large.

Step 3 ($\tau \in [\tau_2, \tau_3]$, $\tau_3 = 3/4$). At the end of the step 2 we obtained

$$F_1^{\tau_2}(w) = f_1(w_2) - w_1 + \tau_1 K\phi_1(w), \quad F_2^{\tau_2}(w) = f_2(w_1) - w_2 + \tau_1 K\phi_2(w). \tag{38}$$

We now aim to reduce the two functions f_1 and f_2 in (38) to some common value f_0 while keeping the other terms unchanged. Hence the homotopy will take the form

$$F_1^\tau(w) = f_1^\tau(w_2) - w_1 + \tau_1 K\phi_1(w), \tag{39}$$

$$F_2^\tau(w) = f_2^\tau(w_1) - w_2 + \tau_1 K\phi_2(w), \quad \tau_2 \le \tau \le \tau_3,$$

where $f_i^{\tau_2} = f_i$ and $f_1^{\tau_3} = f_2^{\tau_3} = f_0$.

In order to satisfy the conditions (15)-(17) we will require that

$$f_i^{\tau\prime}(s) > 0 \ \text{ for } \ s \ge 0, \ \ i = 1, 2. \tag{40}$$

$$\text{The system } w_1 = f_1^\tau(w_2),\ w_2 = f_2^\tau(w_1) \text{ has three solutions } w^+, w^-, \bar{w}^\tau \text{ in } \mathbb{R}_+^2, \text{ furthermore } 0 < \bar{w}_i^\tau < 1. \tag{41}$$

$$f_1^{\tau\prime}(0)f_2^{\tau\prime}(0) < 1, \quad f_1^{\tau\prime}(1)f_2^{\tau\prime}(1) < 1, \quad f_1^{\tau\prime}(\bar{w}_2^\tau)f_2^{\tau\prime}(\bar{w}_1^\tau) > 1. \tag{42}$$

These conditions yield that the graphs of the two functions f_1^τ and f_2^τ are located with respect to each other as represented in Figure 1. Besides we ask that

$$F^\tau(w^0) > 0, \tag{43}$$

so that the speed of the wave in (35) is positive if K is sufficiently large.

Finally in preparation of the step 4 below we require that the function $f_0 = f_1^{\tau_3} = f_2^{\tau_3}$ moreover satisfies

$$\text{The equation } f_0(s) = s \text{ has three solutions}: 0,\ 1,\ \mu;\ 0 < \mu < 1, \tag{44}$$

$$\int_0^1 (f_0(s) - s)ds > 0. \tag{45}$$

We claim that we can define the functions f_i^τ in such a way that all the above conditions are satisfied. Let us describe this homotopy in more details.

Recall that the system of equations $w_1 = f_1(w_2)$, $w_2 = f_2(w_1)$, has a single solution $(\bar{w}_1, \bar{w}_2)$ satisfying the inequalities $0 < \bar{w}_i < 1$, $i = 1, 2$. Let us introduce the inverse function of f_1 denoted by $\tilde{f}_1$. Then the equation $f_2(s) = \tilde{f}_1(s)$ has a single solution $\bar{w}_1$ in the interval $(0, 1)$.

Now consider a function h_2 such that

$$h_2(0) = 0, \quad h_2(\bar{w}_1) = f_2(\bar{w}_1), \quad h_2(1) = 1, \tag{46}$$

$$h_2(s) > f_2(s) \text{ for } \bar{w}_1 < s < 1; \quad h_2(s) < f_2(s) \text{ for } 0 < s < \bar{w}_1. \tag{47}$$

Next for $\theta \in [0, 1]$ introduce the function:

$$h_2^\theta = (1 - \theta) f_2 + \theta h_2.$$

Then the equation $h_2^\theta(s) = g_1(s)$ has exactly three solutions in $[0, 1]$, namely 0, 1 and $\bar{w}_1$.

An explicit form of the function h_2 can be given by:

$$h_2(s) = \begin{cases} \epsilon s \ \text{ for } 0 \le s < a, \\ \frac{1}{\epsilon}(s - \bar{w}_1) \ \text{ for } a \le s \le b, \\ \epsilon(s - 1) + 1 \text{ for } \ b \le s \le 1, \end{cases}$$

where $\epsilon > 0$ and a, b are obtained by requiring the continuity of h_2 at these points. Then if ϵ is sufficiently small the functions f_1 and h_2^θ satisfy the conditions (40)-(43) (with $f_1^\tau = f_1$ and $f_2^\tau = h_2^\theta$) except at the points where h_2^θ is not differentiable but we can use smoothing procedure to make it infinitely differentiable and satisfy the conditions everywhere.

The above procedure allowed to reduce the function f_2 to h_2 with the conditions (40)-(43) being satisfied. Next similarly, we reduce f_1 to the piecewise linear function h_1 with

$$h_1(s) = \begin{cases} \epsilon s \ \text{ for } 0 \le s < r, \\ \frac{1}{\epsilon}(s - \bar{w}_2) \ \text{ for } r \le s \le d, \\ \epsilon(s - 1) + 1 \text{ for } \ d \le s \le 1 \end{cases}$$

(up to smoothing).

At this point, in the homotopy process, the couple (f_1, f_2) became (h_1, h_2) which are piece-wise linear function depending on $\epsilon > 0$, $\bar{w}_1$ and $\bar{w}_2$ (up to smoothing). Now in the definition of h_1 and h_2 we replace the point $\bar{w}$ by by $\alpha\bar{w}$ with $\alpha \in [0, 1]$. Then if ϵ and α are sufficiently small the conditions (40)-(43) remain valid and moreover

$$\int_0^1 (h_1(s) - s) ds > 0, \quad \int_0^1 (h_2(s) - s) ds > 0.$$

Note that either $h_1 \ge h_2$ or $h_2 \ge h_1$. For sufficiently small ϵ and α let us consider the function f_0 which is the maximum of these two functions. The homotopy of the functions h_1 to f_0 (or h_2 to f_0) can be easily constructed in such a way that the conditions on the homotopy are verified.

Step 4 ($\tau \in [\tau_3, 1]$). In this last step, we eliminate the functions ϕ_i by setting:

$$F_i^\tau(w) = f_0(w_j) - w_i + \tau_1 \frac{1-\tau}{1-\tau_3} K\phi_i(w), \quad i = 1, 2 \text{ and } j \neq i, \quad \tau_3 \leq \tau \leq 1. \tag{48}$$

Note that F_i^τ decreases as τ increases. Hence by Theorem 3.2 the speed of the corresponding traveling wave decreases. However it remains larger than the one for the limit problem corresponding to $\tau = 1$. This problem reads

$$u'' + c\, u' + F^1(u) = 0, \quad u(\pm\infty) = w^\pm,$$

with

$$F^1(u) = (f_0(u_2) - u_1, f_0(u_1) - u_2).$$

It possesses a uniquely defined solution such that $u_1 = u_2$. The corresponding speed c^1 has the sign of the integral in (45) which is positive. Hence for $\tau \in [\tau_3, 1]$ we conclude that $c^\tau \geq c^1 > 0$.

3.3 Existence of solutions

We have constructed the homotopy from our original problem to the system

$$w_1'' + f_0(w_2) - w_1 = 0, \quad w_2'' + f_0(w_1) - w_2 = 0, \tag{49}$$

on the half axis $x > 0$ with the boundary condition and the condition at infinity:

$$w'(0) = 0, \quad w(\infty) = 0. \tag{50}$$

Let us look for some solution of problem (49)-(50) for which $w_1 \equiv w_2$ so that w_1 satisfies

$$w_1'' + f_0(w_1) - w_1 = 0, \;\; w_1'(0) = 0, \;\; w_1(\infty) = 0. \tag{51}$$

Due to the properties of f_0 and in particular (45) it is easily seen that (51) possesses a unique solution satisfying $w_1'(x) < 0$ for $x > 0$.

We can now prove the existence of solutions of problem (1) satisfying (9).

In Section 3.2 we constructed the homotopy F^τ. Let us now consider the spaces introduced in Section 2 and the operator A^τ given by (20).

Since $c^\tau > 0$ for $\tau \in [0, 1]$, it satisfies the conditions imposed to obtain a priori estimates of monotonically decreasing solutions (Theorem 2.2). Denote by $B \subset E^1_\mu$ a ball which contains all monotonically decreasing solutions. Since the operator A^τ is proper on closed bounded sets with respect to both variables w and τ, the set of monotonically decreasing solutions of the problem $A^\tau(w) = 0$ is compact. Since they are separated from non-monotone solutions (Theorem 2.3), then we can construct a domain $D \subset B \subset E^1_\mu$ such that all monotonically decreasing solutions (for all $\tau \in [0, 1]$) are located inside D and there are no other solutions in the closure $\bar{D}$. Indeed it is sufficient to take a union of small

balls of the radius r (Theorem 2.3) around each monotonically decreasing solution.

Let us note that only strictly monotone solutions belong to this domain. In particular, the trivial solution $w \equiv 0$ does not belong to it since the monotonically decreasing solutions are separated from the trivial solution (Theorem 2.3). This is an important remark because the index of the trivial solution equals one. If it belongs to the domain D, then the sum of the indices, which equals the value of the degree, can be zero (see below). If it was the case, then we could not affirm the existence of solutions for the original problem.

We can now determine the topological degree $\gamma(A^\tau, D)$. We have

$$\gamma(A^0, D) = \gamma(A^1, D) = \sum_j \text{ind } u^j,$$

where the sum is taken with respect to all monotonically decreasing solutions u^j of problem (49)-(50), and ind u^j denotes the index of the solution. We have ind $u^j = (-1)^{\nu_j}$, where ν_j is the number of positive eigenvalues of the linearized problem. We proved in [5] that $\nu_j = 1$ so that ind $u^j = -1$. Hence the degree is different from zero for the original problem, and the equation $A^0(w) = 0$ has a solution in D. This concludes the proof of the existence result in Theorem 1.1

Let us note that we used the existence of solutions of the model problem but not their uniqueness. The index of each monotone solution equals -1. Therefore the degree is different from zero for any number of solutions if at least one such solution exists.

4 The model of competition of species

We consider the system of equations

$$W_1'' + W_1(1 - W_1 - \alpha W_2) = 0, \quad W_2'' + \rho W_2(1 - \beta W_1 - W_2) = 0, \tag{52}$$

describing the competition of species in population dynamics. The unknowns are two population densities and the nonlinear expressions represent the rates of reproduction of the two species. All parameters are positive constants and we will suppose that $\alpha > 1$ and $\beta > 1$.

Then the system has four stationary points: $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (0, 1)$ and $P_3 = (\bar{W}_1, \bar{W}_2)$ where $0 < \bar{W}_i < 1$ satisfy $\bar{W}_1 + \alpha\bar{W}_2 = 1$ and $\beta\bar{W}_1 + \bar{W}_2 = 1$. The points P_1 and P_2 are stable while the points P_0 and P_3 are unstable.

The above system is not monotone. However it is possible to obtain a monotone system by introducing the new variable $w_1 = W_1$ and $w_2 = 1 - W_2$. The system of equations now reads:

$$w_1'' + F_1(w_1, w_2) = 0, \quad w_2'' + F_2(w_1, w_2) = 0, \tag{53}$$

where

$$F_1(w_1, w_2) = w_1(1 - w_1 - \alpha(1 - w_2)), \tag{54}$$

$$F_2(w_1, w_2) = \rho(1 - w_2)(\beta w_1 - w_2). \tag{55}$$

These functions satisfy the conditions

$$\frac{\partial F_1}{\partial w_2}(w_1, w_2) > 0, \quad \frac{\partial F_2}{\partial w_1}(w_1, w_2) > 0, \tag{56}$$

if $w_1 > 0$ and $w_2 < 1$.

We will also need to consider the system of equations

$$U_1'' + cU_1' + U_1(1 - U_1 - \alpha U_2) = 0, \quad U_2'' + cU_2' + \rho U_2(1 - \beta U_1 - U_2) = 0, \tag{57}$$

describing travelling wave solutions of the system of competition of species. We will look for its solutions with the limits at infinity

$$U_1(\infty) = 0, \;\; U_2(\infty) = 1, \quad U_1(-\infty) = 1, \;\; U_2(-\infty) = 0. \tag{58}$$

The equivalent problem in terms of the variables $u_1 = U_1$ and $u_2 = 1 - U_2$ reads

$$u_1'' + cu_1' + u_1(1 - \alpha - u_1 + \alpha u_2) = 0, \quad u_2'' + cu_2' + \rho(1 - u_2)(\beta u_1 - u_2) = 0, \tag{59}$$

with the limits :

$$u_1(\infty) = u_2(\infty) = 0, \quad u_1(-\infty) = u_2(-\infty) = 1. \tag{60}$$

Existence and uniqueness of travelling wave, that is of solutions of problem (57)-(58) (or equivalently (59)-(60)) is known (note that the inequality (56) is not strict on the closed unit square, but the result still holds, see [8]).

Hereafter we will denote by c_0 the corresponding wave speed. The wave speed c_0 can be positive, zero or negative depending on the values of parameters. If it is positive the wave propagates in the direction of the values $(U_1, U_2) = (1, 0)$ reached at $-\infty$. Some elements on the sign of the wave speed can be found in Section 4.2.

4.1 Existence of pulses for Problem (53)-(56)

4.1.1 Pulses for a perturbed system

As already mentioned the nonlinearity in (53) is such that the system is monotone. However the conditions on the zeros lines stated in (5)-(8) are not satisfied (see Figure 2 below). This leads us to introduce the following modified functions involving the parameter $\epsilon > 0$:

$$\begin{cases} F_1^\epsilon(w_1, w_2) = w_1(1 - \alpha - w_1 + \alpha w_2) + \epsilon\alpha w_2, \\ F_2^\epsilon(w_1, w_2) = \rho(1 - w_2)(\beta w_1 - w_2) - \epsilon\rho w_2. \end{cases} \tag{61}$$

For these new nonlinearities the equations $F_1^\epsilon(w_1, w_2) = 0$ and $F_2^\epsilon(w_1, w_2) = 0$ are respectively equivalent to

$$w_2 = f_1^\epsilon(w_1) \text{ and } w_1 = f_2^\epsilon(w_2),$$

where

$$f_1^\epsilon(w_1) = \frac{-w_1(1-\alpha-w_1)}{\alpha(\epsilon+w_1)}, \quad f_2^\epsilon(w_2) = \frac{(1-w_2+\epsilon)w_2}{\beta(1-w_2)}.$$

For all $\epsilon > 0$ sufficiently small the functions f_1^ϵ and f_2^ϵ are monotone functions of their respective arguments (Figure 2), and the system of equations

$$w_2 = f_1^\epsilon(w_1), \quad w_1 = f_2^\epsilon(w_2),$$

possesses three solutions. Let us denote the two exterior points by $w_\pm^\epsilon$ and the intermediate point by $\bar{w}^\epsilon$. We have $w_+^\epsilon = (0,0)$ and $w_-^\epsilon \to (1,1)$ as $\epsilon \to 0$.

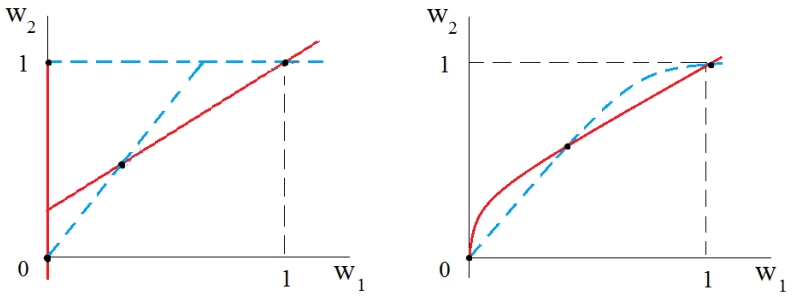

Fig. 2 Left: zeros of the functions F_1 (bold solid) and F_2 (bold dashed). Right: zeros of the functions F_1^ϵ (bold solid) and F_2^ϵ (bold dashed).

For the perturbed nonlinearities the pulse is the solution of the system:

$$w_1'' + F_1^\epsilon(w_1, w_2) = 0, \quad w_2'' + F_2^\epsilon(w_1, w_2) = 0, \quad w_i(\pm\infty) = 0, \quad i = 1, 2, \quad (62)$$

while the traveling wave problem reads:

$$u_1'' + cu_1' + F_1^\epsilon(u_1, u_2) = 0, \quad u_2'' + cu_2' + F_2^\epsilon(u_1, u_2) = 0, \quad u(\pm\infty) = w_\pm^\epsilon. \quad (63)$$

The existence of travelling waves for the perturbed system is investigated in [12] where the following result is derived.

Proposition 4.1. *For all $\epsilon > 0$ sufficiently small Problem (63) possesses a solution for a unique value of c denoted by c_ϵ. Moreover, c_ϵ is a continuous function of ϵ.*

Let us note that the limiting value c_0 in Proposition 4.1 is the speed of the travelling wave for the system (59)-(60). In particular if $c_0 > 0$, then $c_\epsilon > 0$ for

all positive and sufficiently small ϵ. This remark enables us to apply Theorem 1.1 to derive the existence of pulses for the perturbed problem for sufficiently small ϵ.

Proposition 4.2. *Assume that $c_0 > 0$. Then, for all $\epsilon > 0$ sufficiently small, Problem (62) possesses a solution w_ϵ such that*

$$w_\epsilon'(0) = 0, \quad w_\epsilon'(x) < 0 \text{ for } x > 0, \quad w_\epsilon(\infty) = 0. \tag{64}$$

4.1.2 Pulses for the original problem

The existence of some pulse for Problem (53) is related to the sign of the speed c_0 as stated in the following result.

Theorem 4.3. *System (53) has a solution satisfying*

$$w'(0) = 0, \quad w'(x) < 0 \text{ for } x > 0, \quad w(\infty) = 0$$

if and only if the speed c_0 in problem (59)-(60) is positive.

Proof. Again we refer to [5] for the proof of the non-existence part in Theorem 4.3 for $c_0 \leq 0$.

Hereafter we assume $c_0 > 0$ and we aim to derive the existence of a pulse. This will proved by taking the limit $\epsilon \to 0$ for the solutions w_ϵ provided by Proposition 4.2.

Since $0 < w_\epsilon(x) < w_-^\epsilon$ and $w_-^\epsilon \to (1,1)$ as $\epsilon \to 0$, the functions w_ϵ are bounded in $L^\infty(\mathbb{R})$ independently of ϵ sufficiently small. Hence classical regularity estimates yield that the Hölder norm in $C^{2+\alpha}(\mathbb{R}_+)$ of w_ϵ is also bounded independently of ϵ sufficiently small.

Consequently there exists a subsequence still denoted by w_ϵ locally convergent to some limiting function $\widehat{w}$. The function $\widehat{w}$ satisfies the equations (53), $\widehat{w}(x) \geq 0$ and $\widehat{w}'(x) \leq 0$ for $x \geq 0$, $\widehat{w}'(0) = 0$.

Let us first check that

$$\widehat{w}(0) \neq (0,0).$$

Arguing by contradiction suppose that $w_\epsilon(0)$ converges to the origin as $\epsilon \to 0$. Recalling the definition (61) it is easy to check the existence of some neighborhood of the origin in $\mathbb{R}_+^2$ such that for all ϵ sufficiently small and for each point $w > 0$ in this neighborhood at least one component of $F^\epsilon(w)$ is negative. Due to its convergence property, $w_\epsilon(0)$ enters this neighborhood and at least one component of $F^\epsilon(w_\epsilon(0))$ is negative. Then due to the equations the second derivative at $x = 0$ of the corresponding component of w_ϵ is positive, and this function cannot be decreasing. Similar arguments provide

$$\widehat{w}(0) \neq (0,1), \quad \widehat{w}(0) \neq (1,0).$$

Next let us verify that

$$\widehat{w}(0) > (0,0). \tag{65}$$

If this is not true then either $\widehat{w}_1(0) = 0$ and $\widehat{w}_2(0) > 0$ or $\widehat{w}_1(0) > 0$ and $\widehat{w}_2(0) = 0$. In the first case $\widehat{w}_1(x) \equiv 0$ and $\widehat{w}_2$ satisfies

$$\widehat{w}_2'' + F_2(0, w_2) = 0, \quad \widehat{w}_2 \geq 0, \quad \widehat{w}_2' \leq 0, \quad \widehat{w}_2'(0) = 0.$$

Since $F_2(0, w_2) \leq 0$ this provides readily $\widehat{w}_2' = 0$ and $\widehat{w}_2 \equiv c_2$. The constant c_2 should be such that $c_2 > 0$ and $(0, c_2)$ is a zero of F. The only possibility is $c_2 = 1$ but this is impossible since we noted that $\widehat{w}(0) \neq (0, 1)$. In the second case similar arguments provide $(\widehat{w}_1, \widehat{w}_2) \equiv (1 - \alpha, 0)$ which is also impossible since this point is not a zero of F This shows (65).

The above arguments yield the existence of some constant $\eta > 0$ such that for ϵ sufficiently small the two components of w_ϵ satisfy:

$$w_{\epsilon,1}(0) > \eta, \quad w_{\epsilon,2}(0) > \eta \tag{66}$$

(indeed otherwise there exists a sequence w_ϵ converging to some $\widehat{w}$ and at least one component of $\widehat{w}(0)$ vanishes).

In order to verify the convergence of $\widehat{w}$ to the origin as $x \to \infty$, it is sufficient to prove that the functions w_ϵ are bounded independently of $\epsilon > 0$ sufficiently small in the (weighted) norm of E^1_μ. Since they are uniformly bounded in the Hölder norm without weight, it is sufficient to prove that the norm $\sup_x |w_\epsilon(x)\mu(x)|$ is uniformly bounded. Let us recall that solutions w_ϵ converge to zero exponentially at infinity. So this norm is bounded for each solution. Suppose that these functions are not uniformly bounded with respect to ϵ in the weighted norm. Then there is a sequence of solutions w_{ϵ_k} (with $0 < w_{\epsilon_k}(x) < w_-^{\epsilon_k}$) for which

$$\sup_{x \geq 0} |w_{\epsilon_k}(x)\mu(x)| \to \infty \text{ as } k \to \infty.$$

Let $\delta > 0$ be sufficiently small so that the exponential decay of the solutions gives the estimate

$$|w_{\epsilon_k}(x)\mu(x)| \leq M,$$

for those values of x such that $|w_{\epsilon_k}(x)| \leq \delta$, with some constant $M > 0$ independent of k. Choosing $\delta < \eta$ where η is given by (66), we can select x_k so that $|w_{\epsilon_k}(x_k)| = \delta$ and

$$|w_{\epsilon_k}(x)\mu(x)| \leq M \text{ for } x \geq x_k. \tag{67}$$

If the values x_k are uniformly bounded, then the values $|w_{\epsilon_k}(x)\mu(x)|$ are uniformly bounded for $0 \leq x \leq x_k$ since $w_{\epsilon_k}(x) \leq w_{\epsilon_k}$. Together with (67), this provides the required estimate for all $x \geq 0$.

Suppose that $x_k \to \infty$. Consider the sequence of functions $u^k(x) = w_{\epsilon_k}(x + x_k)$. We can choose a subsequence converging to some limiting function $\widehat{u}$ in $C^2_{loc}(\mathbb{R})$. Then $\widehat{u}$ is a non increasing function defined on the whole axis and satisfies the equation

$$\widehat{u}'' + F(\widehat{u}) = 0.$$

The bound (67) yields that $\widehat{u}(\infty) = (0,0)$ while $\widehat{u}(-\infty) = u^-$, where u^- is such that $F(u^-) = 0$ and $u^- \neq (0,0)$ since $|\widehat{u}(0)| = \delta$. Hence we obtain a solution of system (59) with $c = 0$. Let us show that this is not possible.

Indeed, if u^- is the unstable zero of F in (w^+, w^-), then a solution of system (21) exists only for negative c [7], [8]. Let us note, that there are two unstable zeros, $w_1 = 0, w_2 = 1$ and $w_1 = (1-\alpha)/(1-\alpha\beta), w_2 = \beta w_1$. If $u^- = w^-$ (stable point), then by virtue of the assumption of Theorem 4.3, $c > 0$. Hence the function $\widehat{u}$ can not exist, and the sequence x_k is bounded.

There remains to show that $\widehat{w}'(x) < 0$ for all $x > 0$ (component-wise). Indeed, suppose that $\widehat{w}_i'(x_0) = 0$ for some i and for some $x_0 > 0$. Assuming for example $i = 1$ we differentiate the first equation of system (53). Setting $\widehat{v} = -\widehat{w}_1'$ it provides :

$$-\widehat{v}'' - a(x)\widehat{v} = b(x),$$

where

$$a(x) = \frac{\partial F_1}{\partial w_1}(\widehat{w}) \ , \quad b(x) = -\frac{\partial F_1}{\partial w_2}(\widehat{w})\ \widehat{w}_2'(x).$$

Since $b(x) \geq 0$, $\widehat{v}(x) \geq 0$ and $\widehat{v}(x_0) = 0$, the positiveness theorem yields $\widehat{v}(x) \equiv 0$ hence $\widehat{w}_1(x) \equiv 0$. As already noted this implies $\widehat{w}_2' \equiv 0$ hence $\widehat{w}_2 \equiv 0$ which is impossible.

The proof of Theorem 4.3 is completed.

□

Remark. Systems (53) and (52) are equivalent thanks to some appropriate change of variable. Hence Theorem 4.3 provides the existence of some even solution W of system (52) satisfying

$$W_1(\pm\infty) = 0, \ \ W_2(\pm\infty) = 1, \ \ W_1(x) > 0 \ \text{ and } \ W_2(x) < 1 \ \text{ for } \ x \in \mathbb{R}, \quad (68)$$

if and only if $c_0 > 0$. Similarly, if $c_0 < 0$, we obtain the existence of some even solution of (52) such that:

$$W_1(\pm\infty) = 1, \ \ W_2(\pm\infty) = 0, \ \ W_1(x) < 1 \ \text{ and } \ W_2(x) > 0 \ \text{ for } \ x \in \mathbb{R}. \quad (69)$$

Indeed let us compare these two cases. In the first one, after the change of variables $w_1 = W_1$, $w_2 = 1 - W_2$ we considered the solution of (53) with zero limits at infinity. It is proved that it exists if and only if the wave in (59) with the limits $u_i(-\infty) = 1$, $u_i(\infty) = 0$ has a positive speed. In terms of the original variables, these limits are given by (see (58)):

$$U_1(-\infty) = 1, \ U_1(\infty) = 0, \quad U_2(-\infty) = 0, \ U_2(\infty) = 1. \qquad (70)$$

If we consider a solution of (52) with limits (69), then the change of variables is different and reads: $\tilde{w}_1 = 1 - W_1$, $\tilde{w}_2 = W_2$. A study similar to the one for the first change of variable can be carried out. In particuler, the $\tilde{w}_i$ have required to have zero limits at infinity. From the analogous theorem, we conclude that such solution exists if and only if the corresponding wave with the limits

$\tilde{u}_i(-\infty) = 1$, $\tilde{u}_i(\infty) = 0$, $i = 1, 2$ has a positive speed. In terms of the original variables, these limits read:

$$U_1(-\infty) = 0,\ U_1(\infty) = 1, \quad U_2(-\infty) = 1,\ U_2(\infty) = 0. \tag{71}$$

The wave with such limits has a positive speed if and only if the speed of the wave with the limits (70) is negative. Indeed, it is sufficient to replace $U_i(x)$ by $U_i(-x)$, $i = 1, 2$. Thus, depending on the sign of the speed, either there is a solution with limits (68) or a solution with limits (69) (and no solutions if $c_0 = 0$).

4.2 The sign of the wave speed

We proved the existence of pulses when the wave speed c_0 in problem (57)-(58) (or equivalently (59)-(60)) is positive. Here we aim to present some simple conditions on the parameters ρ, α and β which guarantee that $c_0 > 0$.

We start with a case where $c_0 = 0$ that will be useful afterwards.

Lemma 4.4. *Suppose $\rho = 1$ and $\alpha = \beta$. Then the speed c_0 of the wave solution of (57)-(58) is equal to zero.*

Proof. Let us argue by contradiction and suppose $c_0 > 0$ (the case $c_0 < 0$ is similar). The functions $z_1(x) = U_2(-x)$ and $z_2(x) = U_1(-x)$ satisfy the problem

$$z_1'' + c^* z_1' + z_1(1 - z_1 - \alpha z_2) = 0, \quad z_2'' + c^* z_2' + \rho z_2(1 - \alpha z_1 - z_2) = 0, \tag{72}$$

$$z_1(\infty) = 0,\ \ z_2(\infty) = 1, \quad z_1(-\infty) = 1,\ \ z_2(-\infty) = 0, \tag{73}$$

where $c^* = -c_0 < 0$. Since the problem (72)-(73) coincides with the problem (57)-(58), then we conclude that the latter has two different waves, one with a positive speed and another one with a negative speed. We obtain a contradiction with the uniqueness of the wave

□

Our main result is the following one.

Theorem 4.5. *If $\rho = 1$ and $\beta > \alpha$, then the speed c_0 of the wave solution of (57)-(58) is positive.*

Proof. For ρ, α and β satisfying the above conditions let us set

$$F_1(u_1, u_2) = u_1(1 - u_1 - \alpha(1 - u_2)), \quad F_2(u_1, u_2) = (1 - u_2)(\beta u_1 - u_2),$$

and

$$G_1(u_1, u_2) = u_1(1 - u_1 - \alpha(1 - u_2)), \quad G_2(u_1, u_2) = (1 - u_2)(\alpha u_1 - u_2).$$

Applying Lemma 4.4 to the function G we see that the speed of the wave solution of

$$u'' + cu' + G(u) = 0, \quad u(\pm\infty) = w_\pm,$$

equals zero. Hereafter we denote by $\tilde{u}$ some solution of

$$\tilde{u}'' + G(\tilde{u}) = 0, \quad \tilde{u}(\pm\infty) = w_\pm \tag{74}$$

(the solution is unique up to space translation). Next since $\beta > \alpha$ we have

$$F_1(u) = G_1(u) \text{ and } F_2(u) > G_2(u) \text{ for } 0 < u_i < 1. \tag{75}$$

Hence we can apply Theorem 3.2 to the functions F and G and we conclude that $c_0 \geq 0$.

We now aim to show that $c_0 > 0$. Arguing by contradiction let us suppose that $c_0 = 0$. Consider the Cauchy problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + F(v), \tag{76}$$

with the initial condition $v(x, 0) = \tilde{u}(x)$, where $\tilde{u}$ is a solution of problem (74). Since

$$\tilde{u}'' + F(\tilde{u}) = F(\tilde{u}) - G(\tilde{u}) \geq 0$$

(the inequality is strict for the second component of the vector), $\tilde{u}$ is a lower function and $v(x, t)$ increases as a function of t for each x fixed. By virtue of the global asymptotic stability of travelling waves for monotone systems, $v(x, t)$ converges to a solution $u(x)$ of problem (57)-(58). Hence

$$u(x) > \tilde{u}(x), \quad \forall x \in \mathbb{R}. \tag{77}$$

For any $h \in \mathbb{R}$ the function $\tilde{u}(x - h)$ is also a solution of problem (74). If h is sufficiently large, then the inequality

$$u(x) > \tilde{u}(x - h) \tag{78}$$

does not hold for some x. Let h_0 be the supremum of the h for which (78) holds for all $x \in \mathbb{R}$. If $u(x) \geq \tilde{u}(x - h_0)$ for all x and $u(x_0) = \tilde{u}(x_0 - h_0)$ for some x_0, then we obtain a contradiction with the positiveness theorem. We will now consider the case where such finite x_0 does not exist and shows that this case also leads to some contradiction.

According to the assumptions above, there a sequence h_n such that $h_n > h_0$, $h_n \to h_0$ as $n \to \infty$ and $u(x_n) = \tilde{u}(x_n - h_n)$ for some x_n and for one of the components of the solutions. The sequence x_n can not be bounded. Consider the case where $x_n \to \infty$.

Let us take some x^* such that $F(u(x^*)) < 0$ and some $x_n > x^*$. Such x^* exists since $u(x) \sim p\ exp(-\lambda x)$ where $p > 0$ is the eigenvector of the matrix $F'(0)$ corresponding to the negative eigenvalue, λ is a positive number. We consider the boundary value problem for the equation (76) on the half-axis $x > x^*$ with the boundary condition $v(x^*, t) = u(x^*)$. It has a unique stationary solution $u(x)$. Moreover, this solution is globally asymptotically

stable. Indeed, the initial condition $v(x,0) = u(x^*)$ provides an upper solution and the initial condition $v(x,0) = 0$ a lower solution. Both of them converge to the stationary solution $u(x)$ by virtue of its uniqueness.

On the other hand, the function $\max(u(x), \tilde{u}(x - h_n))$ is a lower function. Therefore the corresponding solution is greater than $u(x)$ and it cannot converge to $u(x)$ since it increases in time. This contradiction proves that x_n cannot tend to infinity. Similarly, it can be proved that it cannot go to $-\infty$. This contradiction proves the theorem.

□

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References

1. A. Doelman, R. A. Gardner, T. J. Kaper, *Stability analysis of singular patterns in the 1-D Gray-Scott model*, Physica D, 122, 1–36 (1998).
2. A. Doelman, P. van Heijster, T. J. Kaper, *Pulse dynamics in a three-component system: existence analysis*, J. Dyn. Diff. Equat., 21, 73–115 (2009).
3. A. Doelman, F. Veerman. An explicit theory for pulses in two component, singularly perturbed, reaction-diffusion equations. Journal of Dynamics and Differential Equations, 27 no. 3, 555-595 (2015).
4. J.K. Hale, L.A. Peletier, W.C. Troy, *Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocatalysis*, SIAM J. Appl. Math., 61, 102–130 (2000).
5. M. Marion, V. Volpert, *Existence of pulses for a monotone reaction-diffusion system*, Pure and Applied Funct. Analysis, 1, 97-122 (2016).
6. Y. Nec, M.J. Ward, *The stability and slow dynamics of two-spike patterns for a class of reaction-diffusion system*, Math. Model. Nat. Phenom., 8, 206–232 (2013).
7. A.I. Volpert, V.A. Volpert, *Applications of the rotation theory of vector fields to the study of wave solutions of parabolic equations*, Trans. Moscow Math. Soc., 52, 59–108 (1990).
8. A. Volpert, Vit. Volpert, Vl. Volpert, *Travelling wave solutions of parabolic systems*, Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.
9. V. Volpert, A. Volpert, *Spectrum of elliptic operators and stability of travelling waves*, Asymptotic Analysis, 23, 111–134 (2000).
10. V. Volpert, A. Volpert, *Properness and topological degree for general elliptic operators*, Abstract and Applied Analysis, 129–182 (2003).
11. V. Volpert, *Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains*, Birkhäuser, 2011.
12. V. Volpert, *Elliptic partial differential equations. Volume 2. Reaction-diffusion equations*, Birkhäuser, 2014.
13. J. Wei, M. Winter, *Stability of spiky solutions in a reaction-diffusion system with four morphogens on the real line*, SIAM J. Math. Anal., 42, 2818–2841 (2010).
14. J. Wei, M. Winter, *Existence, classification and stability analysis of multiple-peaked solutions for the Gierer-Meinhardt system in* $\mathbb{R}^1$, Methods Appl. Anal.,14, no. 2, 119–163 (2007).