

Effective asymptotic analysis for finance

CYRIL GRUNSPAN

Léonard de Vinci Pôle Universitaire
Research Center
92916 Paris La Défense Cedex
France

Email: cyril.grunspan@devinci.fr

JORIS VAN DER HOEVEN

LIX, CNRS
École polytechnique
91128 Palaiseau Cedex
France

Email: vdhoeven@lix.polytechnique.fr

Abstract

It is known that an adaptation of Newton’s method allows for the computation of functional inverses of formal power series. We show that it is possible to successfully use a similar algorithm in a fairly general analytical framework. This is well suited for functions that are *highly tangent to identity* and that can be expanded with respect to asymptotic scales of “exp-log functions”. We next apply our algorithm to various well-known functions coming from the world of quantitative finance. In particular, we deduce asymptotic expansions for the inverses of the Gaussian and the Black–Scholes functions.

1 Introduction

One notoriously complex problem in finance is the pricing of derivative products that are frequently traded on financial markets. Practitioners have proposed various sophisticated models for the dynamics of financial assets. In particular, it has been necessary to account for the existence of U-shaped “volatility smiles” which play a central role in the pricing of so-called vanilla options. Some models seem more reasonable than others because they explain not only the volatility smile, but also have properties that are directly exploitable in practice, notably the existence of easily implementable pricing formulas involving mathematical parameters that are easy to calibrate.

Subsequently, the volatility smile has been studied in a fairly general way, with a minimum of hypotheses on the probabilistic distribution of the assets [2, 1, 21, 6, 12]. This has made it possible to isolate intrinsic behaviours that are shared by a large number of models in the study of volatility smiles.

The next step has been to study the volatility smile in a model-free setting. This ultimately leads to focusing not on the Black–Scholes formula itself but on its inverse [26, 10, 8, 32]. A notable advantage of this approach is that it simplifies pricing problems. Indeed, in the case of vanilla options, such problems usually do not admit closed form solutions (except in the Black–Scholes model), so we need to resort to approximate solutions. Different techniques have been proposed to this purpose: perturbation methods with partial or stochastic differential equations, Lie symmetry theory, Watanabe theory, heat kernel expansion theory and Minakshisundaran–Pleijel’s formula, large deviation theory, etc. [23, 22, 16, 9, 13, 5]. Most of these techniques give the asymptotics of price for large or small values of certain parameters involved in the computation of option prices. The study of the inverse function of the Black–Scholes formula then transforms vanilla option price asymptotics into implicit volatility asymptotics, which is the quantity of interest.

The problem of inverting Black–Scholes formula is challenging because of its non-analytic boundary behaviour. In fact, since the Black–Scholes model (as any other stochastic model) uses Brownian motion, it is not surprising that the asymptotics of the Black-Scholes formula involves logarithms. More precisely, after a suitable change of variables, the relation between vanilla option price and volatility can be expressed *via* an asymptotic expansion

$$y \approx x + \phi_0 + \frac{\phi_1}{x} + \frac{\phi_2}{x^2} + \dots, \quad (1)$$

where ϕ_0, ϕ_1, \dots are polynomials in $\log x$ [8, 10]. In particular, this means that

$$y = x + \phi_0 + \frac{\phi_1}{x} + \dots + \frac{\phi_n}{x^n} + \mathcal{O}(x^{-n-1/2}), \quad (2)$$

for every $n \in \mathbb{N}$. We are interested in computing a similar expansion for x in terms of y .

In computer algebra, various techniques have been developed for asymptotic expansions in general asymptotic scales. For instance, several algorithms exist for the asymptotic expansion of “exp-log” functions [30, 11, 25, 18, 31]. Such functions are built up from the rationals and an infinitely large variable $x \rightarrow \infty$ using the field operations, exponentiation and logarithm. An example of an exp-log function is $\exp(x^2 - x \log x) / \log \log(x^x + 3)$. The theory of transseries [7, 17, 19] makes it possible to cover asymptotic expansions of an even wider class of functions comprising many formal solutions to non-linear differential equations.

Several algorithms also exist for the functional inversion of exp-log functions [28, 29]. However, the right-hand side $\phi(x) := x + \phi_0 + \phi_1 x^{-1} + \phi_2 x^{-2}$ of (1) is usually not an exp-log function, so these algorithms cannot be applied directly. When considering $\phi(x)$ as a formal transseries, there are also methods for computing the formal inverse $\psi = y + \psi_0 + \psi_1 y^{-1} + \psi_2 y^{-2} + \dots$ of ϕ [17, 19]. However, *a priori*, the analytic meaning (2) is lost during such formal computations. In this paper, we will show how to invert asymptotic expansions of the form (1) from the analytic point of view.

For each $n \in \mathbb{N}$, let \mathcal{G}^n be the ring of n -fold continuously differentiable functions at infinity ($x \rightarrow \infty$). Then $\mathcal{G}^\infty := \bigcap_{n \in \mathbb{N}} \mathcal{G}^n$ is a differential ring. We recall that a *Hardy field* is a differential subfield K of \mathcal{G}^∞ . It is well-known that Hardy fields [14, 15, 3] provide a suitable setting for asymptotic analysis. In section 2, we will introduce the abstract notion of an “effective Hardy field”, which formalizes what we need in order to make this asymptotic calculus fully effective. Typical examples of effective Hardy fields are generated by exp-log functions. For instance, in Sections 2.3 and 2.4, we will show that $\mathbb{Q}(\log x, x, e^x, e^{x^2})$ is effective Hardy field. Using the aforementioned work on expansions of exp-log functions, it is possible to construct various other effective Hardy fields.

Let K be a Hardy field. We say that $\xi \in K^> \setminus \mathbb{R}$ with $\xi = \mathcal{O}(1)$ is *steep* if for any $f \in K$, there exists a $c \in \mathbb{R}$ with $f = \mathcal{O}(\xi^c)$. An element $f \in K$ is said to be *highly tangent to identity* if there exists a $c > 0$ with $(f - x)/x = \mathcal{O}(\xi^c)$. For instance, if $K = \mathbb{Q}(\log x, x)$, then $\xi = x^{-1}$ is steep and $x + \log x + 3 \log^2 x / x$ is highly tangent to identity, contrary to $x + x / \log x$. Now assume that K is an effective Hardy field. We say that a germ $f \in \mathcal{G}^\infty$ admits an effective asymptotic expansion over K if for every $n \in \mathbb{N}$ we can compute an element $\varphi_n \in K$ with $f - \varphi_n = \mathcal{O}(\xi^n)$. If φ_1 is highly tangent to identity and $f' = \mathcal{O}(1)$, then we will prove in Section 3 that f admits a functional inverse that also admits an effective asymptotic expansion over K . Applied to the case when $K = \mathbb{Q}(\log x, x)$, this gives an algorithm for inverting asymptotic expansions of the form (1). Our algorithm relies on two main ingredients: Taylor’s formula for right composition with functions that are highly tangent to identity, and Newton’s method for reducing functional inversion to functional composition.

For our application to mathematical finance, it would have sufficed to work with the particular effective Hardy field $K = \mathbb{Q}(\log x, x)$. There are several reasons why we have chosen to prove our main result for general effective Hardy fields. First of all, the more general result may be useful in other areas such as combinatorics [27]. Indeed, functional inverses frequently occur when analyzing asymptotic behavior using the saddle point method. Secondly, our general setup only requires a moderate “investment” in the terminology from Section 2. Finally, it is natural to prove the results from Section 3 in this setup; the proofs would not become substantially shorter in the special case when $K = \mathbb{Q}(\log x, x)$.

This paper contains three main contributions. As far as we are aware, the application of modern asymptotic expansion algorithms to mathematical finance is new. Secondly, we introduce the framework of effective Hardy fields which we believe to be of general interest for effective asymptotic analysis. One major advantage of this framework is that it separates the potentially difficult question of constructing a suitable effective Hardy field from its actual use. The existing literature on exp-log functions and transseries can be put to use for such constructions. But for various other problems, it suffices to assume the effective Hardy field to be given as a blackbox. The third contribution of this paper is to show that this is particularly the case for the inversion of asymptotic expansions that are “highly tangent to identity”.

Acknowledgment. We are very grateful to Martino Grasselli for his encouragement to write and submit the article and for the careful reading of our work.

2 Effective Hardy fields

2.1 Hardy fields

Consider the differential ring $\mathcal{G}^\infty := \bigcap_{n \in \mathbb{N}} \mathcal{G}^n$, where \mathcal{G}^n denotes the ring of n -fold continuously differentiable functions at infinity ($x \rightarrow \infty$) for each n . We recall that a *Hardy field* is a differential subfield K of \mathcal{G}^∞ . Since any non zero element f of Hardy fields is invertible, the sign of $f(x)$ is ultimately constant for $x \rightarrow \infty$. We define $f > 0$ if $f(x)$ is ultimately positive. It can be shown that this gives K the structure of an ordered field.

The well-known asymptotic relations \asymp , \prec , \succ and \sim can be defined in terms of the ordering on K : given $f, g \in K$, we write

$$\begin{aligned} f = \mathcal{O}(g) &\iff f \asymp g \iff \exists B \in \mathbb{Q}^{\succ}, |f| \leq B |g| \\ f = \mathcal{o}(g) &\iff f \prec g \iff \forall \varepsilon \in \mathbb{Q}^{\succ}, |f| < \varepsilon |g| \end{aligned}$$

and

$$\begin{aligned} f \succ g &\iff f \asymp g \asymp f \\ f \sim g &\iff f - g \prec g. \end{aligned}$$

The quasi-ordering \asymp is total on K^\neq : given $f, g \in K^\neq$, we have $f \asymp g \Leftrightarrow g \not\asymp f$.

Example 1. The set \mathcal{E} of *exp-log* germs at infinity is the smallest subset of \mathcal{G}^∞ that contains \mathbb{Q} and the identity function, and which is closed under $+$, $-$, \times , $/$, \exp and \log . For instance, $\exp(x^x - x \log x) / (x - 3) + \pi \log \log x \in \mathcal{E}$. In his founding work [14, 15], Hardy showed that \mathcal{E} forms a Hardy field.

Example 2. More generally, given a Hardy field K , its Liouville closure K^{Li} is the smallest subset of \mathcal{G}^∞ that contains K and that is stable under $+$, $-$, \times , $/$, \exp , \log and integration. It is well known that K^{Li} is again a Hardy field [3].

2.2 Basic properties

Let K be a Hardy field. Given $f, g \in K$, let us show that

$$f \asymp g \wedge g \not\asymp 1 \implies f' \asymp g' \tag{3}$$

$$f \prec g \wedge g \not\asymp 1 \implies f' \prec g'. \tag{4}$$

Let us first assume that $f' \not\asymp g'$, whence $g' \asymp f'$, and let $x_0 \in \mathbb{R}$ and $A > 0$ be such that $|g'(x)| \leq A |f'(x)|$ for all $x \geq x_0$. Modulo a further increase of x_0 , we may assume without loss of generality that the signs of $g'(x)$ and $f'(x)$ are constant for $x \geq x_0$. Then, for all $\sigma \geq x_0$, we have

$$\left| \int_\sigma^x g'(t) dt \right| = \int_\sigma^x |g'(t)| dt \leq A \int_\sigma^x |f'(t)| dt = A \left| \int_\sigma^x f'(t) dt \right|. \tag{5}$$

Consequently, $g + a \asymp f + b$ for suitable integration constants $a, b \in \mathbb{R}$. If $g > 1$, then this yields $g \asymp f$. If $f < 1$ and $g < 1$, then we may take $\sigma = \infty$ in (5), so that $a = b = 0$, and we again obtain $g \asymp f$. If $f \succ 1$ and $g < 1$, then we clearly have $g < 1 \asymp f$. This proves that $f' \not\asymp g' \Rightarrow f \not\asymp g \vee g \asymp 1$, which implies (4). One proves $f' \not\asymp g' \Rightarrow f \not\asymp g \vee g \succ 1$ and (3) in a similar way.

2.3 Effective Hardy fields

Let K be a Hardy field. We say that K is *effective* if its elements can be represented by instances of a concrete data structure and if we have algorithms for carrying out the basic operations $+$, $-$, \times , $/$, ∂ , as well as effective tests for the relations \leq , $<$, \preceq and \prec .

In particular, the effective inequality test for \leq yields an equality test. Inversely, if we have an algorithm to compute signs of elements in K , then this yields effective inequality tests for both \leq and $<$. Similarly, if, given $f \in K$, we have a way to test whether $f \preceq 1$ and $f \prec 1$, then this yields effective tests for the relations \preceq and \prec . Indeed, given $f \in K$ and $g \in K^\neq$, we have $f \preceq g \Leftrightarrow f/g \preceq 1$ and $f \prec g \Leftrightarrow f/g \prec 1$.

Example 3. Let us show that $K = \mathbb{Q}(x)$ is an effective Hardy field. The basic operations $+$, $-$, \times , $/$ and ∂ can clearly be carried out by algorithm, and it is also clear how to do the equality test. Now consider $f = (P_p x^p + \dots + P_0) / (Q_q x^q + \dots + Q_0) \in K^\neq$ with $P_0, \dots, P_p, Q_0, \dots, Q_q \in \mathbb{Q}$ and $P_p \neq 0, Q_q \neq 0$. Then $f \sim (P_p/Q_q) x^{p-q}$. Consequently, $\text{sign}(f) = \text{sign}(P_p/Q_q)$ and $f \preceq 1 \Leftrightarrow p \leq q$ (resp. $f \prec 1 \Leftrightarrow p < q$).

Example 4. We claim that $K = \mathbb{Q}(\log x, x)$ is an effective Hardy field. As above, the basic operations $+$, $-$, \times , $/$, ∂ and the equality test are straightforward. Now any non zero element $f \in K^\neq$ can be written as a fraction $f = (P_p x^p + \dots + P_0) / (Q_q x^q + \dots + Q_0) \in K^\neq$ with $P_0, \dots, P_p, Q_0, \dots, Q_q \in \mathbb{Q}(\log x)$ and $P_p \neq 0, Q_q \neq 0$. Similarly, we may write $P_p/Q_q = (A_a (\log x)^a + \dots + A_0) / (B_b (\log x)^b + \dots + B_0) \in K^\neq$ with $A_0, \dots, A_a, B_0, \dots, B_b \in \mathbb{Q}$ and $A_a \neq 0, B_b \neq 0$. Then $f \sim (A_a/B_b) x^{p-q} (\log x)^{b-a}$. Consequently, $\text{sign}(f) = \text{sign}(A_a/B_b)$ and $f \preceq 1 \Leftrightarrow (p, a) \leq (q, b)$ (resp. $f \prec 1 \Leftrightarrow (p, a) < (q, b)$). Here we used the lexicographical ordering on pairs: $(p, a) \leq (q, b)$ if and only if $p < q$ or $p = q$ and $a \leq b$.

Example 5. Let K be an effective Hardy field and let $\varphi \in K$ be such that $\varphi > 0$ and $\varphi \succ 1$. Then $\varphi' > 0$, whence φ is ultimately strictly increasing and invertible for composition. Let $\psi = \varphi^{\text{inv}}$ be the inverse of φ and assume that $\varphi' \circ \psi \in K$. Then $K \circ \varphi = \{f \circ \varphi : f \in K\}$ is again an effective Hardy field. Indeed, since right composition preserves the field operations and the ordering, $K \circ \varphi$ is effectively isomorphic to K as an ordered field. The derivation on $K \circ \varphi$ is given by $(f \circ \varphi)' = ((\varphi' \circ \psi) \cdot f') \circ \varphi$.

2.4 Adjunction of steep exponentials

Let $f, g \in K^\neq$ and let $\delta, \epsilon = \pm 1$ be such that $f^\delta \succ 1, g^\epsilon \succ 1$. We define the *flatness relations* \preceq , \prec and \succ by

$$\begin{aligned} f \preceq g &\iff \exists c \in \mathbb{Q}^>, |f|^\delta \preceq |g|^{\epsilon c} \\ f \prec g &\iff \forall c \in \mathbb{Q}^>, |f|^\delta \prec |g|^{\epsilon c} \\ f \succ g &\iff f \preceq g \preceq f. \end{aligned}$$

Let $f^\dagger = f'/f$ denote the logarithmic derivative of a function f . Taking logarithms, and using (3) and (4), we observe that

$$\begin{aligned} f \preceq g &\iff \log |f| \preceq \log |g| \iff f^\dagger \preceq g^\dagger \\ f \prec g &\iff \log |f| \prec \log |g| \iff f^\dagger \prec g^\dagger \\ f \succ g &\iff \log |f| \succ \log |g| \iff f^\dagger \succ g^\dagger, \end{aligned}$$

for all $f \in K^\neq$ and $g \in K^\neq = \{h \in K : h \neq 1\}$.

An element $\xi \in K^>$ is said to be *steep* if $f \preceq \xi$ (whence $f^\dagger \preceq \xi^\dagger$) for all $f \in K^\neq$. If $\xi \prec 1$, then this allows us to define a valuation with respect to ξ : we set $v_\xi(f) = \lim (f^\dagger / \xi^\dagger)$ for $f \in K^\neq$ and $v_\xi(0) = \infty$. Notice that the corresponding valuation group $\Gamma_\xi = \text{im } v_\xi$ is a subgroup of \mathbb{R} . In particular, Γ_ξ is archimedean. For $f \in K$ and $g \in K^\neq$, we notice that

$$f \preceq g \implies v_\xi(f) \geq v_\xi(g).$$

Indeed, since $f \preceq g \Leftrightarrow f/g \preceq 1$ and $v_\xi(f/g) = v_\xi(f) - v_\xi(g)$, it suffices to show this for $g = 1$. Now assume that $c := v_\xi(f) < 0$. Then $f^\dagger > \frac{c}{2} \xi^\dagger$, whence $\log |f| > \log \xi^{c/2} + C > \log \xi^{c/3}$ for some constant $C \in \mathbb{R}$. It follows that $f > \xi^{c/3} \succ 1$. If $\xi \succ x$, then we also notice that $v_\xi(\xi^\dagger) = 0$. Indeed, $\xi \succ x \Leftrightarrow \log \xi \succ \log x \Rightarrow \xi^\dagger = (\log \xi)' \succ (\log x)' \prec x^{-1}$ and $1/\xi^\dagger \prec x \Rightarrow (1/\xi^\dagger)' \prec \xi^{\dagger\dagger}/\xi^\dagger \prec 1$, whence $v_\xi(\xi^\dagger) = \lim \xi^{\dagger\dagger}/\xi^\dagger = 0$.

Two examples of steep elements are x in $\mathbb{Q}(\log x, x)$ and e^{-x^2} in $\mathbb{Q}(\log x, x, e^x, e^{x^2})$. The aim of the remainder of this section is to generalize Example 4 and prove in particular that $\mathbb{Q}(\log x, x, e^x, e^{x^2})$ is indeed an effective Hardy field.

Let K be an effective Hardy field and let $\varphi \in K^\succ = \{h \in K : h \succ 1\}$ be such that $f^\dagger \prec \varphi'$ for all $f \in K$. By what precedes, this implies that $\psi := e^\varphi \succ f$ for all $f \in K$. We claim that $L := K(\psi)$ is again an effective Hardy field. Modulo the replacement of ψ by $|\psi^{-1}|$ (and φ by $-\varphi$), we may assume without loss of generality that $\psi > 0$ and $\psi \succ 1$. We clearly have algorithms for the field operations of L . Using the rule $\psi' = \varphi \psi$, it is also straightforward to compute derivatives of elements of L .

Now consider a polynomial $P(\psi) = P_p \psi^p + \dots + P_0 \in K[\psi]$. If $P_p \neq 0$, then for each $i < p$, we have $P_i/P_p \preceq \psi$, so that $P_i \psi^i \prec P_p \psi^p$. Hence $P_p \neq 0$ implies $P(\psi) \sim P_p \psi^p$. This also shows that $P(\psi) = 0 \Leftrightarrow P_0 = \dots = P_p = 0$, which provides us with an effective zero test for $K[\psi]$, as well as for L . Given a rational function $P(\psi)/Q(\psi) = (P_p \psi^p + \dots + P_0)/(Q_q \psi^q + \dots + Q_0) \in L$ with $P_p \neq 0$ and $Q_q \neq 0$, we also have $P(\psi)/Q(\psi) \sim (P_p/Q_q) \psi^{p-q}$. Consequently, $\text{sign}(f) = \text{sign}(P_p/Q_q)$ and $f \preceq 1$ if and only if $p < q$ or $p = q$ and $P_p \preceq Q_q$. Similarly, $f \prec 1$ if and only if $p < q$ or $p = q$ and $P_p \prec Q_q$.

Example 6. Starting with $K = \mathbb{Q}(\log x, x)$ as in Example 4, applying the above argument twice shows that both $K(e^x)$ and $K(e^x, e^{x^2}) = \mathbb{Q}(\log x, x, e^x, e^{x^2})$ are effective Hardy fields. Applying Example 5 for $\varphi = \log x$, we also obtain that $\mathbb{Q}(\log \log x, \log x, x, x^{\log x})$ is an effective Hardy field.

Remark 7. In order to compute with more general exp-log germs in \mathcal{E} , one also needs to show that fields such as $\mathbb{Q}(x, e^x, e^{\sqrt{2}x})$ form effective Hardy fields. One even more difficult problem is to provide an effective zero test for exp-log constants, i.e. constants formed from the rationals, using $+$, $-$, \times , $/$, \exp and \log . Provided that Schanuel's conjecture holds, such an algorithm was given by Richardson [24]. His algorithm always returns correct results, but might not terminate if one explicitly hits a counterexample to the conjecture. Given a zero-test for exp-log constants, it can be shown that \mathcal{E} forms an effective Hardy field [18].

2.5 Limits and asymptotic scales

Let K be a Hardy field. Given $f \in K^\preceq = \{\varphi \in K : \varphi \preceq 1\}$, there exists a unique $\ell \in \mathbb{R}$ with $f - \ell \prec 1$, which is called the *limit* of f , and denoted by $\ell = \lim f$. We say that K is *closed under limits* if $\lim f \in K$ for all $f \in K$. If K is effective and $\lim : K^\preceq \rightarrow K$ is computable, then we say that K admits an *effective limit map*.

An *asymptotic scale* for K is a multiplicative subgroup $\mathfrak{M} \subseteq K^>$ such that \mathfrak{M} is totally ordered for \preceq and such that there exists a mapping $\mathfrak{d} : K^\neq \rightarrow \mathfrak{M}$ with $\mathfrak{d}(f) \preceq f$ for all $f \in K^\neq$. We call $\mathfrak{d}(f)$ the *dominant monomial* of f and notice that \mathfrak{d} is necessarily a group homomorphism. If K is effective and \mathfrak{d} is computable, then we call \mathfrak{M} an *effective asymptotic scale*.

Assume that K is closed under limits and that K also admits an asymptotic scale \mathfrak{M} . Given $f \in K^\neq$, we call $\tau(f) = (\lim f / \mathfrak{d}(f)) \mathfrak{d}(f)$ the *dominant term* of f , and notice that $f \sim \tau(f)$. If \mathfrak{d} and \lim are both computable, then the same clearly holds for τ .

Example 8. In Example 3, we have given a method for the explicit computation of an equivalent in $\mathbb{Q}^\neq x^{\mathbb{Z}} = \{c x^k : c \in \mathbb{Q}^\neq, k \in \mathbb{Z}\}$ for any $f \in \mathbb{Q}(x)^\neq$. This both shows that $\mathbb{Q}(x)$ admits an effective limit map and that it admits $x^{\mathbb{Z}}$ as an effective asymptotic scale. Similarly, Example 4 shows that the same holds for $\mathbb{Q}(\log x, x)$, in which case the asymptotic scale becomes $(\log x)^{\mathbb{Z}} x^{\mathbb{Z}}$.

More generally, let K be an effective Hardy field and let φ, ψ be as in Section 2.4. Assume that K admits an effective limit map and that \mathfrak{M} is an effective asymptotic scale. For each $f \in K(\psi)^\neq$, we have shown how to compute an equivalent $f \sim g \psi^k \sim \tau(g) \psi^k$ with $g \in K^\neq$ and $k \in \mathbb{Z}$. Since $g \ll \psi^k$ for any $g \in \mathfrak{M}$ and $k \neq 0$, the group $\mathfrak{M} \psi^{\mathbb{Z}}$ is totally ordered for \ll . This shows that $K(\psi)$ admits both an effective limit map and an effective asymptotic scale $\mathfrak{M} \psi^{\mathbb{Z}}$.

Example 9. Let K be an effective Hardy field and let φ be as in Example 5. If K admits an effective limit map, then so does $K \circ \varphi$, since $\lim f \circ \varphi = \lim f$ for all $f \in K^\neq$. If K admits an effective asymptotic scale \mathfrak{M} , then $K \circ \varphi$ admits $\mathfrak{M} \circ \varphi$ as an effective asymptotic scale, with $\mathfrak{d}(f \circ \varphi) = \mathfrak{d}(f) \circ \varphi$ for all $f \in K^\neq$.

3 Composition and functional inversion

Let K be a Hardy field which contains the identity function x , as well as a steep element $\xi \in K^{>, <} = \{\varphi \in K : \varphi < 1\}$. If $\xi \asymp x$, then also assume that $\xi = x^{-1}$.

An element $f \in K$ is said to be *highly tangent to identity* if there exists a $c > 0$ with $(f - x)/x = \mathcal{O}(\xi^c)$. Equivalently, this means that f is of the form $f = x + \delta$ with $v_\xi(\delta) > v_\xi(x)$. If $\xi = x^{-1}$, then this is the case when $\delta < \xi^\alpha$ for some $\alpha > -1$. If $\xi \asymp x$, then we rather should have $\delta < \xi^\alpha$ for some $\alpha > 0$. In particular, in both cases we have $\delta' < 1$ and even $v_\xi(\delta') > 0$. We will denote by T the subset of K of all elements that are highly tangent to identity.

Since Hardy fields are not necessarily closed under composition and functional inversion, the set T does not necessarily form a group. The main aim of this section is to show that a suitable completion of T does form a group (Theorem 20 below). Moreover, under suitable hypothesis, there are algorithms for computing asymptotic expansions of compositions and functional inverses.

3.1 First order functional inversion

Lemma 10. *Let $\delta \in T - x$. Then for any germ $\eta \in \mathcal{G}^\infty$ with $\eta \ll \delta$ and $\eta' < 1$, we have*

$$(x + \eta)^{\text{inv}} - x = \mathcal{O}(\delta).$$

Proof. Without loss of generality, we may assume that $\delta > 0$. For any $c \in \mathbb{R}$, we claim that $\delta \circ (x + c\delta) \sim \delta$. Indeed, given $\varepsilon > 0$, let x_0 be such that $\delta'(x)$ has constant sign and $|\delta'(x)| < \varepsilon$ for $x \geq x_0$. Assume also that $\delta(x + c\delta(x))$ is defined for $x \geq x_0$. Then

$$|\delta(x + c\delta(x)) - \delta(x)| \leq \left| \int_x^{x+c\delta(x)} \delta'(t) dt \right| < \varepsilon |c| \delta(x),$$

for all $x \geq x_0$. We conclude that $\delta \circ (x + c\delta) - \delta < \delta$, by letting ε tend to zero.

The assumption that $\eta' < 1$ implies that $(x + \eta)' \sim 1$, whence $\varphi(x) := x + \eta(x)$ is strictly increasing for sufficiently large x . This shows that φ indeed admits an inverse function ψ at infinity. Let $A > 0$ be such that $|\eta(x)| \leq A \delta(x)$ for sufficiently large x . Setting $l(x) = x - 2A \delta(x)$ and $r(x) = x + 2A \delta(x)$, our claim implies

$$\begin{aligned} \varphi(l(x)) &= l(x) + \eta(l(x)) \leq l(x) + A \delta(l(x)) < l(x) + 2A \delta(x) = x \\ \varphi(r(x)) &= r(x) + \eta(r(x)) \geq r(x) - A \delta(r(x)) > r(x) - 2A \delta(x) = x, \end{aligned}$$

for sufficiently large x . Since φ is strictly increasing, it follows that $l(x) < \psi(x) < r(x)$. In other words, $|\psi(x) - x| \leq 2A\delta(x)$ for sufficiently large x . \square

3.2 First order right composition

Lemma 11. *Let $f \in K$ and $g = x + \varepsilon \in T$. Then for any germs $\eta, \delta \in \mathcal{G}^\infty$ with $\eta \preceq f$ and $\delta \preceq \varepsilon$, we have*

$$\eta \circ (x + \delta) = \mathcal{O}(f).$$

Proof. Since ξ is a steep element, there exists a constant $A > 0$ with $|f^\dagger| \leq A|\xi^\dagger|$. We also notice that $\xi^\dagger \varepsilon \prec 1$. Indeed, this is immediate if $\xi = 1/x$. If $\xi \succ x$, then $\varepsilon \prec \xi^c$ for some $c > 0$ and $\xi^\dagger \varepsilon \prec \xi^\dagger \xi^c \prec (\xi^c)' \prec 1$, since $\xi^c \prec x$.

Let us first show that $f \circ g \sim f$, whenever $f \prec x$ and $g \geq x$. Since $f \prec x$ implies $f' \prec 1$, the function $|f'|$ is ultimately decreasing. For sufficiently large x , it follows that $|f'(t)| \leq |f'(x)|$ for $t \in [x, g(x)]$, whence

$$|f(g(x)) - f(x)| \leq \int_x^{g(x)} |f'(t)| dt \leq |f'(x)| \varepsilon(x) \leq A|\xi^\dagger(x)| \varepsilon(x) |f(x)|.$$

Since $\xi^\dagger \varepsilon \prec 1$, this shows that $f \circ g \sim f$.

Let us next show that we also have $f \circ g \sim f$ in the case when $f \prec x$ and $g < x$ (so that $\varepsilon < 0$). Then Lemma 10 implies $g^{\text{inv}} = x + \mathcal{O}(\varepsilon)$, whence $x < g^{\text{inv}} < x - B\varepsilon$ for some $B \in \mathbb{R}^>$. Let $\lambda > 0$. By what precedes, there exists an x_0 with $|f(x - B\varepsilon(x)) - f(x)| \leq \lambda |f(x)|$ for all $x \geq x_0$. Modulo a further increase of x_0 , we may also arrange that $f(x)$ is monotonic for $x \geq x_0$. It follows that $|f(g^{\text{inv}}(x)) - f(x)| \leq \lambda |f(x)|$, whence $f \circ g^{\text{inv}} \sim f$. Post-composing with g , we again obtain $f \circ g \sim f$.

Let us finally assume that $f \succ x$. Then the above arguments prove that $(1/f) \circ g \sim (1/f)$. Consequently, $f \circ g = ((1/f) \circ g)^{-1} \sim (1/f)^{-1} = f$.

The above arguments conclude the proof in the case when $\eta = f$ and $\delta = \varepsilon$. Let us next consider the case when we still have $\eta = f$, but $\delta \preceq \varepsilon$ is general. Let $B > 0$ be such that $|\delta| \leq B|\varepsilon|$. For sufficiently large x , it follows that $f(x + \delta(x))$ is comprised between $f(x - B|\varepsilon(x)|)$ and $f(x + B|\varepsilon(x)|)$, which are both equivalent to $f(x)$. This shows that $f \circ (x + \delta) \sim f$.

As to the general case, let $C > 0$ be such that $|\eta| \leq C|f|$. By what precedes, we have $|\eta(x + \delta(x))| \leq C|f(x + \delta(x))| \leq 2C|f(x)|$ for all sufficiently large x . This shows that $\eta \circ (x + \delta) \preceq f$. \square

3.3 General composition

Lemma 12. *Let $f \in K$ and $g \in T$. Let $\varphi, \psi \in K$ and $n \in \mathbb{N}$ be such that $x + \psi \in T$ and $f^{(n)} \cdot (g - x)^n \preceq \varphi$. Then for any $\delta, \varepsilon \in \mathcal{G}^\infty$ with $\delta \preceq \varphi$ and $\varepsilon \preceq \psi$, we have*

$$(f + \delta) \circ (g + \varepsilon) = f + f' \cdot (g - x) + \dots + \frac{1}{(n-1)!} f^{(n-1)} \cdot (g - x)^{n-1} + \mathcal{O}(\max(|\varphi|, |f' \psi|)).$$

Proof. Let us first consider the case when $\delta = \varepsilon = 0$ and consider

$$\begin{aligned} \eta &= f + f'(g - x) + \dots + \frac{1}{(n-1)!} f^{(n-1)}(g - x)^{n-1} \\ R &= f \circ g - \eta \end{aligned}$$

For sufficiently large x , Taylor's formula with integral remainder yields

$$R(x) = \int_x^{g(x)} \frac{1}{(n-1)!} f^{(n)}(t) (g(x) - t)^{n-1} dt.$$

For sufficiently large x , the function $f^{(n)}$ is also monotonic, whence

$$|R(x)| \leq \frac{1}{n!} \max(|f^{(n)}(x)|, |f^{(n)}(g(x))|) |g(x) - x|^n.$$

By Lemma 11, we have $f^{(n)} \circ g \preceq f^{(n)}$, whence $R \preceq f^{(n)} (g - x)^n \preceq \varphi$. This completes the proof in the case when $\delta = \varepsilon = 0$.

As to the general case, we have

$$|f(g(x) + \varepsilon(x)) - f(g(x))| \leq \int_{g(x)}^{g(x) + \varepsilon(x)} |f'(t)| dt \leq \max(|f'(g(x))|, |f'(g(x) + \varepsilon(x))|) |\varepsilon(x)|,$$

for all sufficiently large x . Now Lemmas 10 and 11 imply $\varphi \circ (g + \varepsilon) = \varphi \circ (x + \varepsilon \circ g^{\text{inv}}) \circ g \preceq \varphi \circ g \preceq \varphi$ and similarly $f' \circ (g + \varepsilon) \preceq f'$. Consequently,

$$\begin{aligned} |(f + \delta) \circ (g + \varepsilon) - \eta| &\leq |\delta \circ (g + \varepsilon)| + |f \circ (g + \varepsilon) - f \circ g| + |f \circ g - \eta| \\ &\preceq \max(|\varphi \circ (g + \varepsilon)|, |f' \varepsilon|, |\varphi|) \\ &\preceq \max(|f' \varepsilon|, |\varphi|). \end{aligned}$$

This concludes the proof in the general case. \square

Lemma 13. *For any $f \in K$, $g \in T$ and $\varphi \in K^\neq$, there exists an $n \in \mathbb{N}$ with $f^{(n)} \cdot (g - x)^n \preceq \varphi$.*

Proof. Let us first consider the case when $\xi = x^{-1}$, so that $v_\xi(g - x) > -1$. For any $f \in K^\neq$, we have $f' = f^\dagger f \preceq \xi^\dagger f \succ f/x$, whence $v_\xi(f') \geq v_\xi(f) + 1$. Consequently,

$$v_\xi(f^{(n)}(g - x)^n) \geq v_\xi(f) + n + n v_\xi(g - x).$$

It thus suffices to take $n > (v_\xi(\varphi) - v_\xi(f)) / (v_\xi(g - x) + 1)$ in order to ensure that $v_\xi(f^{(n)}(g - x)^n) > v_\xi(\varphi)$ and therefore $f^{(n)}(g - x)^n \preceq \varphi$.

Assume next that $\xi \succ x$, so that $v_\xi(g - x) > 0$. We again have $f' \preceq \xi^\dagger f$ for all $f \in K^\neq$, but this time, we rather obtain $v_\xi(f') \geq v_\xi(f)$, since $v_\xi(\xi^\dagger) = 0$. Therefore,

$$v_\xi(f^{(n)}(g - x)^n) \geq v_\xi(f) + n v_\xi(g - x).$$

Taking $n > (v_\xi(\varphi) - v_\xi(f)) / v_\xi(g - x)$, we again obtain the desired result. \square

If K is an effective Hardy field, then the above lemmas lead to the following algorithm for approximate composition:

Algorithm compose(f, g, φ)

INPUT: $f \in K$, $g \in T$ and $\varphi \in K^\neq$ with $v_\xi(\varphi) > v_\xi(x f')$

OUTPUT: $h \in K$ with $f \circ g = h + \mathcal{O}(\varphi)$

Moreover, for all $\delta, \varepsilon \in \mathcal{G}^\infty$ with $\delta \preceq \varphi$, $\varepsilon f' \preceq \varphi$ and $v_\xi(\varepsilon/x) > 0$, we have $(f + \delta) \circ (g + \varepsilon) = h + \mathcal{O}(\varphi)$

Let $n \in \mathbb{N}$ be minimal with $f^{(n)} \cdot (g - x)^n \preceq \varphi$

Return $f + \dots + \frac{1}{(n-1)!} f^{(n-1)} \cdot (g - x)^{n-1}$

Theorem 14. *The algorithm compose is correct.*

Proof. The existence of n is ensured by Lemma 13. Since K is effective, we have an algorithm for doing the test $f^{(n)}(g-x)^n \preceq \varphi$, which enables us to compute n . Setting $\psi = \varphi / f'$, our assumption that $v_\xi(\varphi) > v_\xi(x f')$ ensures that $x + \psi \in T$. The result now follows from Lemma 12. \square

Remark 15. In addition, by considering both cases $\xi = x^{-1}$ and $\xi \gg x$, it can be verified that $v_\xi(h) = v_\xi(f)$, that $f \in T$ implies $h \in T$, and that $v_\xi(f-1) > 0$ implies $v_\xi(h-1) > 0$.

3.4 General functional inversion

A well-known way to solve functional equations of the form $f \circ g = x$ is Newton's method [4]. We will now show that this method indeed yields a quadratic convergence in our setting.

Lemma 16. *Let $f, g \in T$ and $\varepsilon \in K$ be such that $f \circ g - x = \mathcal{O}(x\varepsilon)$ and $v_\xi(\varepsilon) > 0$. Let $\tilde{g} \in T$ be such that*

$$\tilde{g} = g - \frac{f \circ g - x}{f' \circ g} + \mathcal{O}(x\varepsilon^2).$$

Then $f \circ \tilde{g} - x = \mathcal{O}(x\varepsilon^2)$.

Proof. Since $f \sim x$, we notice that $f' \sim 1$ and $f' \circ g \sim 1$. Let $\delta = g - \tilde{g} = \frac{f \circ g - x}{f' \circ g} + \mathcal{O}(x\varepsilon^2) = \mathcal{O}(x\varepsilon)$. For all sufficiently large x , we have

$$f(\tilde{g}(x)) = f(g(x)) - f'(g(x))\delta(x) + \int_{g(x)}^{g(x)-\delta(x)} f''(t)(g(x) - \delta(x) - t) dt,$$

whence, using the ultimate monotonicity of f'' on $[g(x), \tilde{g}(x)]$,

$$f \circ \tilde{g} = f \circ g - (f' \circ g)\delta + \mathcal{O}(\max(|f'' \circ g|, |f'' \circ \tilde{g}|)\delta^2).$$

Using Lemma 11, we also have $f'' \circ g \preceq f''$ and $f'' \circ \tilde{g} \preceq f''$, whence

$$f \circ \tilde{g} = f \circ g - (f' \circ g)\delta + \mathcal{O}(f''\delta^2).$$

Consequently,

$$\begin{aligned} f \circ \tilde{g} - x &= (f \circ g - x) - (f' \circ g)\delta + \mathcal{O}(f''\delta^2) \\ &= (f' \circ g)\delta + \mathcal{O}(x\varepsilon^2) - (f' \circ g)\delta + \mathcal{O}(f''\delta^2) \\ &= \mathcal{O}(x\varepsilon^2) + \mathcal{O}(f''\delta^2). \end{aligned}$$

Now $f - x \prec x$ implies $(f - x)' \prec 1 \prec \log x$ and $f'' = (f - x)'' \prec x^{-1}$. Consequently,

$$f \circ \tilde{g} - x = \mathcal{O}(x\varepsilon^2) + \mathcal{O}(f''\delta^2) = \mathcal{O}(x\varepsilon^2) + \mathcal{O}(\delta^2/x) = \mathcal{O}(x\varepsilon^2).$$

This completes the proof. \square

If K is an effective Hardy field, then this lemma leads to the following algorithm for the computation of approximate functional inverses:

Algorithm invert(f, ε)

INPUT: $f \in T$ and $\varepsilon \in K^\neq$ with $v_\xi(\varepsilon) > 0$

OUTPUT: $g \in T$ with $f^{\text{inv}} = g + \mathcal{O}(x\varepsilon)$

Moreover, for any $\delta \in \mathcal{G}^\infty$ with $\delta \preceq x\varepsilon$ and $\delta' \prec 1$, we have $(f + \delta)^{\text{inv}} = g + \mathcal{O}(x\varepsilon)$

Let $g := x$
repeat
 Let $h := \mathbf{compose}(f, g, x\varepsilon^2)$
 If $h - x \preceq x\varepsilon$ then return g
 Let $d := \mathbf{compose}(f', g, x\varepsilon^2)$
 Let $g := g - (h - x)/d$

Theorem 17. *Let $\eta = (f - x)/x$. The algorithm **invert** is correct and terminates after at most $\lfloor \log(v_\xi(\varepsilon)/v_\xi(\eta))/\log 2 \rfloor + 1$ iterations of the main loop.*

Proof. Let us first show that $g \in T$ throughout the algorithm. This is clear at the start. At each iteration $g := g - (h - x)/d$, Remark 15 implies $v_\xi(h - x) > v_\xi(x)$ and $v_\xi(d - 1) > v_\xi(1)$, whence $v_\xi((h - x)/d) > v_\xi(x)$, so that $g - (h - x)/d \in T$.

On termination, we have $h = f \circ g + \mathcal{O}(x\varepsilon^2)$ and $h = x + \mathcal{O}(x\varepsilon)$, whence $f \circ g - x = \mathcal{O}(x\varepsilon)$. Applying Lemma 10 with $x\varepsilon$ and $f \circ g - x$ in the roles of δ and η , we obtain $g^{\text{inv}} \circ f^{\text{inv}} - x = (f \circ g)^{\text{inv}} - x = \mathcal{O}(x\varepsilon)$. Consequently, $f^{\text{inv}} = g \circ (g^{\text{inv}} \circ f^{\text{inv}}) = g + \mathcal{O}(g' \cdot (g^{\text{inv}} \circ f^{\text{inv}} - x)) = g + \mathcal{O}(x\varepsilon)$. Furthermore, $(f + \delta)^{\text{inv}} = [(x + \delta \circ f^{\text{inv}}) \circ f]^{\text{inv}} = f^{\text{inv}} \circ (x + \delta \circ f^{\text{inv}})^{\text{inv}} = f^{\text{inv}} \circ (x + \mathcal{O}(\delta))^{\text{inv}} = f^{\text{inv}} \circ (x + \mathcal{O}(x\varepsilon))^{\text{inv}} = f^{\text{inv}} \circ (x + \mathcal{O}(x\varepsilon)) = f^{\text{inv}} + \mathcal{O}((f^{\text{inv}})'x\varepsilon) = g + \mathcal{O}(x\varepsilon)$.

As to the termination, consider the quantity

$$\nu := \tilde{v}_\xi\left(\frac{f \circ g - x}{x}\right) := \sup \left\{ \alpha \in \mathbb{R} : \frac{f \circ g - x}{x} = \mathcal{O}(\xi^\alpha) \right\}.$$

At the very start, we have $\nu = \tilde{v}_\xi(\eta) = v_\xi(\eta) > 0$. At every iteration $\tilde{g} := g - (h - x)/d$, we have $\tilde{g} = g - \frac{f \circ g - x}{f' \circ g} + \mathcal{O}(x\varepsilon^2)$. Lemma 16 therefore ensures that ν doubles at least, whereas the algorithm terminates as soon as $\nu > v_\xi(\varepsilon)$. This happens after at most $\lfloor \log(v_\xi(\varepsilon)/v_\xi(\eta))/\log 2 \rfloor + 1$ iterations. \square

3.5 Effective asymptotic expansions

We now extend the definition of high tangency to identity to all germs. We say that a germ $f \in \mathcal{G}^\infty$ is *highly tangent to identity* if there exists a $c > 0$ with $f - x = \mathcal{O}(x\xi^c)$ and $f' = 1 + \mathcal{O}(1)$. We denote by \mathcal{T}^∞ the set of such germs. We say that f admits an *asymptotic expansion* over K if for every $n \in \mathbb{N}$, there exists an element $\varphi_n \in K$ with $f - \varphi_n = \mathcal{O}(\xi^n)$. If we have an algorithm for computing φ_n as a function of n , then we say that f admits an *effective asymptotic expansion* over K .

Proposition 18. *Assume that $f \in \mathcal{G}^\infty$ and $g \in \mathcal{T}^\infty$ admit effective asymptotic expansions over K . Then so does $f \circ g$. If $f \in \mathcal{T}^\infty$, then $f \circ g \in \mathcal{T}^\infty$.*

Proof. Given $n \geq 1$, we may compute $\varphi_n \in K$ and $\psi_n \in T$ with $\delta := f - \varphi_n \preceq \xi^n$ and $\varepsilon := g - \psi_n \preceq \xi^n$. Assume that there exists an $n_0 \in \mathbb{N}$ with $f \not\preceq \xi^{n_0}$. Then for all $n > n_0$, we must have $\varphi_n \succ \xi^{n_0}$ and $v_\xi(x\varphi_n') \leq n_0 < n$. Consequently, we may compute $\chi_n = \mathbf{compose}(\varphi_n, \psi_n, \xi^n)$, and $f \circ g = (\varphi_n + \delta) \circ (\psi_n + \varepsilon) = \chi_n + \mathcal{O}(\xi^n)$. If $f \preceq \xi^n$ for all $n \in \mathbb{N}$, then we also have $f \circ g \preceq \xi^n$ for all $n \in \mathbb{N}$.

If $f \in \mathcal{T}^\infty$, then we also get $\varphi_n, \psi_n, \chi_n \in T$, whence $v_\xi(f \circ g - x) = v_\xi(\chi_n - x + \mathcal{O}(\xi^n)) \geq \min(v_\xi(\chi_n - x), n) > v_\xi(x)$. Moreover, $(f \circ g)' = g' \cdot f' \circ g = (1 + \mathcal{O}(1))(1 + \mathcal{O}(1)) = 1 + \mathcal{O}(1)$, whence $f \circ g \in \mathcal{T}^\infty$. \square

Proposition 19. *Assume that $f \in \mathcal{T}^\infty$ admits an effective asymptotic expansion over K . Then so does f^{inv} and $f^{\text{inv}} \in \mathcal{T}^\infty$.*

Proof. Given $n \geq 1$, we may compute $\varphi_n \in T$ with $\delta := f - \varphi_n \preceq \xi^n$. Let $\psi_n = \mathbf{invert}(\varphi_n, \xi^n/x)$. Then $f^{\text{inv}} = (\varphi_n + \delta)^{\text{inv}} = \psi_n + \mathcal{O}(\xi^n)$. Moreover, $(f^{\text{inv}})' = (f' \circ f^{\text{inv}})^{-1} = ((1 + \mathcal{O}(1)) \circ f^{\text{inv}})^{-1} = 1 + \mathcal{O}(1)$, whence $f^{\text{inv}} \in \mathcal{F}^\infty$. \square

Combining these two propositions, we have shown the following:

Theorem 20. *The set of germs in \mathcal{F}^∞ that admit effective asymptotic expansions over K forms a group for functional composition.* \square

4 Examples and applications to finance

Example 21. (Lambert function) The Lambert function W is defined to be the inverse function of $x \mapsto x e^x$. Using our algorithm, we can compute the asymptotic expansion of the inverse function $W(e^x)$ of $x \mapsto x + \log x$. This also yields the asymptotic expansion of $W(x)$ for large x .

Example 22. (Gaussian law) Let $(\Phi_n)_{n>0}$ be defined formally by

$$\log \left(1 + \sum_{n>0} a_n X^n \right) = \sum_{n>0} \Phi_n(a_1, \dots, a_n) X^n \quad (6)$$

and let Q be the Gaussian law:

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \quad (7)$$

Then, the well-known relation

$$Q(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \left(1 + \sum_{i=1}^n \frac{(-1)^i 1 \cdot 3 \cdots (2i-1)}{x^{2i}} + \mathcal{O}\left(\frac{1}{x^n}\right) \right), \quad (8)$$

valid for any $n \in \mathbb{N}$, shows that

$$q(x) = x + \log x + \varphi_0 + \sum_{i=1}^n \frac{\varphi_i}{x^i} + \mathcal{O}\left(\frac{1}{x^n}\right), \quad (9)$$

with $x > 0$, $q(x) := -2 \log Q(\sqrt{x})$, $\varphi_0 := 2 \log(\sqrt{2\pi})$ and $\varphi_i := -\Phi_i(-1, \dots, (-1)^i 1 \cdot 3 \cdots (2i-1))$ for $i > 0$. Our algorithm now allows us to compute the asymptotic expansion of the inverse function of Gaussian law at $+\infty$. This is potentially of great interest in finance when it comes to calculate “values-at-risk”. Such computations are imposed by regulators to manage market risks, among others.

Example 23. (Incomplete Gamma function) Let $(u_k)_{k \geq 0}$ be defined by $u_0 = 1$ and, for $k > 0$, $u_k = (a-1)(a-2)\cdots(a-k)$. A well-known relation for $\Gamma(a, z)$ tells that for $a \in \mathbb{R}$ and $z > 0$,

$$\Gamma(a, z) = z^{a-1} e^{-z} \left(\sum_{k=0}^n \frac{u_k}{z^k} + \mathcal{O}\left(\frac{1}{z^n}\right) \right).$$

Taking logarithms, we get

$$-\log \Gamma(a, z) = z - (a-1) \log z + \sum_{k=1}^n \frac{\varphi_k}{z^k} + \mathcal{O}\left(\frac{1}{z^n}\right), \quad (10)$$

with $\varphi_k = -\Phi_k(u_1, \dots, u_k)$.

Example 24. (Black–Scholes formula). By definition, a call option is a contract which gives to the owner the value $\text{Max}(S_T - K, 0)$ at a future T -date (known today) called maturity of the contract, where S_T denotes the value at T -date (unknown today) of an asset (like a stock) whose initial value is S today, and K is a constant called strike (known today). The initial price of this contract is denoted by $C(S, K, T)$. In general, by no-arbitrage arguments, the option price $C(S, K, T)$ is always greater than the “intrinsic value” $(S - K)_+$ and lower than the spot value S :

$$(S - K)_+ < C(S, K, T) < S \quad (11)$$

In the Black–Scholes model, the dynamics of (S_t) is assumed to be log-normal:

$$dS_t = \sigma S_t dW_t \quad (12)$$

where $(W)_t$ is a Brownian motion and σ is a constant parameter called volatility. In this framework, the well known Black–Scholes formula gives the price of any call option. It can be shown that $C(S, K, T) = \text{BS}(S, K, T, \sigma)$ with

$$\text{BS}(S, K, T, \sigma) = SQ(d_+) - KQ(d_-) \quad (13)$$

and

$$d_{\pm} := \frac{\log\left(\frac{S}{K}\right) \pm \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \quad (14)$$

To simplify, we have assumed that the interest rate is 0. If S, K, T are fixed, then it is easy to see that the function

$$\sigma \mapsto \text{BS}(S, K, T, \sigma) \quad (15)$$

is non-decreasing and one to one from \mathbb{R}_+^* to $(\text{Max}(S - K, 0), S)$. Therefore, in an *a priori* non Black–Scholes world and for a given call option price $C \in (\text{Max}(S - K, 0), S)$ observed on the market, there is a unique solution $\sigma_{\text{BS}}(K, T)$ (or simply σ_{BS}) of the equation

$$\text{BS}(S, K, T, \sigma) = C \quad (16)$$

We call σ_{BS} the Black–Scholes implied volatility associated to K and T . For different reasons, it is interesting to invert the Black–Scholes function BS in (15) [8]. For instance, very often, and using techniques like perturbation theory, sophisticated stochastic models (in a non Black–Scholes world) give only asymptotic expansions of an option price C in terms of the maturity T , whereas we really need a formula for the implied volatility [2, 16]. Indeed, call option prices are generally quoted in term of implied volatilities (and not as prices). This can be achieved in the following manner. In the Black–Scholes model and under the conditions that $T \ll 1$ and $K \neq S$, it can be proved that the asymptotic expansion of the “time value” TV of the call price $\text{BS}(S, K, T, \sigma)$, defined by

$$\text{TV}(S, K, T, \sigma) := \text{BS}(S, K, T, \sigma) - (S - K)_+,$$

is given by

$$4\sqrt{\pi} \frac{e^{-\frac{u}{2}}}{|u|} \left(\frac{\text{TV}}{S} \right) = v^{3/2} e^{-\frac{1}{v}} \sum_{k=0}^n \frac{(-1)^k}{2^k} a_k \left(\frac{u^2}{8} \right) v^k + \mathcal{O}(v^{2n+5} e^{-\frac{1}{v}}), \quad (17)$$

with $n \in \mathbb{N}$ arbitrarily large,

$$\begin{aligned} u &:= \log\left(\frac{K}{S}\right) \\ v &:= 2\frac{\sigma^2 T}{u^2} \\ a_k(z) &:= (2k+1)!! f_k(z) \\ f_k(z) &:= \sum_{j=0}^k \frac{z^j}{j!(2j+1)!!}, \end{aligned} \tag{18}$$

and for $j \in \mathbb{Z}$, $(2j+1)!! := \prod_{l=1}^j (2l+1)$ [10]. Therefore, if we set

$$x := \frac{1}{v} = \frac{\log^2\left(\frac{K}{S}\right)}{2\sigma^2 T},$$

and

$$y := -\log\left(\frac{\text{TV}}{S}\right),$$

then, for any integer n ,

$$y = x + \frac{3}{2} \log x + \varphi_0 + \sum_{i=1}^n \frac{\varphi_i}{x^i} + \mathcal{O}\left(\frac{1}{x^n}\right), \tag{19}$$

with $\varphi_0 := -\log\left(\frac{|u|e^{\frac{u}{2}}}{4\sqrt{\pi}}\right)$ and $\varphi_i := -\Phi_i\left(-\frac{1}{2}a_1\left(\frac{u^2}{8}\right), \dots, \frac{(-1)^i}{2^i}a_i\left(\frac{u^2}{8}\right)\right)$ for $i > 0$. Hence we get an asymptotic expansion for $\sigma^2 T$ in terms of $\log\left(\frac{\text{TV}}{S}\right)$.

At the limit when $T \gg 1$, the first author previously obtained a similar result [10]. Setting this time

$$\begin{aligned} \text{CC} &= S - \text{BS}(S, K, T, \sigma) \\ x &= \frac{\sigma^2 T}{8}, \end{aligned}$$

we have

$$\sqrt{\pi} e^{-\frac{x}{2}} \frac{\text{CC}}{S} = \frac{e^{-x}}{\sqrt{x}} \sum_{k=0}^n \frac{(-1)^k}{2^k} c_k \left(\frac{u^2}{8}\right) \frac{1}{x^k} + \mathcal{O}(x^{-n-\frac{3}{2}} e^{-x}),$$

where c_k defined by

$$\begin{aligned} c_k(z) &= (2k-1)!! g_k(z) \\ g_k(z) &= \sum_{j=0}^k \frac{z^j}{j!(2j-1)!!}. \end{aligned}$$

Therefore, we get

$$y = x + \frac{1}{2} \log x + \varphi_0 + \sum_{i=1}^n \frac{\varphi_i}{x^i} + \mathcal{O}\left(\frac{1}{x^n}\right), \tag{20}$$

with $y = -\log\left(\frac{\text{CC}}{S}\right)$, and $\varphi_i := -\Phi_i\left(-\frac{1}{2}a_1\left(\frac{u^2}{8}\right), \dots, \frac{(-1)^i}{2^i}c_i\left(\frac{u^2}{8}\right)\right)$ for $i > 0$.

Example 25. We did an experimental implementation of our algorithm in the MATHEMAGIX system [20]. Each of the above examples comes down to the computation of the functional inverse of a function $y(x)$ with an asymptotic expansion of the form

$$y = x + \alpha \log x + \varphi_0 + \sum_{i=1}^n \frac{\varphi_i}{x^i} + \mathcal{O}\left(\frac{1}{x^n}\right).$$

For $n=3$, our algorithm yields:

$$\begin{aligned} x = & y - \alpha \log(y) - \varphi_0 + (\alpha^2 \log(y) + \varphi_0 \alpha - \varphi_1) \frac{1}{y} + \left(\frac{1}{2} \alpha^3 \log(y)^2 + (\varphi_0 \alpha^2 - (\varphi_1 + \alpha^2) \alpha) \log(y) + \right. \\ & \left. \frac{1}{2} \varphi_0^2 \alpha - (\varphi_0 \alpha - \varphi_1) \alpha - \varphi_1 \varphi_0 - \varphi_2\right) \frac{1}{y^2} + \left(\frac{1}{3} \alpha^4 \log(y)^3 + \left(\frac{-1}{3} (-2 \varphi_0 \alpha - \alpha^2) \alpha - \right. \right. \\ & \left. \left. \varphi_0 \alpha^2\right) \alpha - \varphi_1 \alpha^2 - \frac{7}{6} \alpha^4\right) \log(y)^2 + \left(\frac{-1}{3} (-\varphi_0^2 - \varphi_0 \alpha + \varphi_1) \alpha - (2 \varphi_0 \alpha - \alpha^2) \varphi_0\right) \alpha - \\ & \frac{1}{3} ((2 \varphi_0 \alpha - 2 \varphi_1) \alpha + 2 \varphi_0 \alpha^2) \alpha + (\varphi_1 + \alpha^2) \alpha^2 - 2 \varphi_1 \varphi_0 \alpha - \varphi_0 \alpha^3 + \varphi_1 \alpha^2 - 2 \varphi_2 \alpha) \log(y) - \\ & (-\varphi_0 \alpha - \varphi_1) \alpha - \varphi_1 \varphi_0 - \varphi_2) \alpha + \frac{1}{3} (\varphi_0^2 - \varphi_0 \alpha + \varphi_1) \varphi_0 \alpha - \frac{1}{3} (2 \varphi_0 \alpha - 2 \varphi_1) \varphi_0 \alpha - \\ & (\varphi_1 - \varphi_0 \alpha) \varphi_1 - \frac{1}{2} \varphi_0^2 \alpha^2 - \varphi_1 \varphi_0^2 - 2 \varphi_2 \varphi_0 - \varphi_3) \frac{1}{y^3} + \mathcal{O}\left(\frac{1}{y^3}\right). \end{aligned}$$

Bibliography

- [1] S. Benaim and P. Friz. Smile asymptotics. ii. models with known moment generating functions. *Journal of Applied Probability*, 45(1):16–23, 2008.
- [2] H. Berestycki, I. Florent, and J. Busca. Asymptotics and calibration of local volatility models. *Quantitative Finance*, 2(1):61–69, 2002.
- [3] N. Bourbaki. *Fonctions d’une variable réelle*. Éléments de Mathématiques (Chap. 5). Hermann, 2-nd edition, 1961.
- [4] R. P. Brent and H. T. Kung. $O((n \log n)^{3/2})$ algorithms for composition and reversion of power series. In J. F. Traub, editor, *Analytic Computational Complexity*. Pittsburg, 1975. Proc. of a symposium on analytic computational complexity held by Carnegie-Mellon University.
- [5] M. Craddock and M. Grasselli. Lie symmetry methods for local volatility models. *Ssrn.com/abstract=2836817*, 2016.
- [6] Hillairet, C. De Marco, S. and A. Jacquier. Shapes of implied volatility with positive mass at zero. Technical Report 1310.1020, Arxiv, 2013.
- [7] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Hermann, collection: Actualités mathématiques, 1992.
- [8] K. Gao and R. Lee. Asymptotics of implied volatility to arbitrary order. *Finance and Stochastics*, 18(2):349–392, 2014.
- [9] P.K.F. Gatheral, A. Gulisashvili, A. Jacquier, and J. Teichmann. *Large Deviations and Asymptotic Methods in Finance*, volume 110 of *Proceedings in Mathematics and Statistics*. Springer-Verlag, 2015.
- [10] C. Grunspan. Asymptotic expansions of the lognormal implied volatility: a model free approach. Technical Report 1112.1652, Arxiv, 2011.
- [11] D. Gruntz. *On computing limits in a symbolic manipulation system*. PhD thesis, E.T.H. Zürich, Switzerland, 1996.
- [12] A. Gulisashvili. Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes. *SIAM J. Finan. Math.*, 1(1):609–641, 2010.
- [13] Kumar, D., Lesniewski, A. Hagan, P. and D. Woodward. Managing smile risk. *Wilmott Magazine*, pages 84–108, 2002.
- [14] G. H. Hardy. *Orders of infinity*. Cambridge Univ. Press, 1910.
- [15] G. H. Hardy. Properties of logarithmico-exponential functions. *Proceedings of the London Mathematical Society*, 10(2):54–90, 1911.
- [16] P. Henry-Labordère. A general asymptotic implied volatility for stochastic volatility models. Technical Report cond-mat/050431, Arxiv, 2005.
- [17] J. van der Hoeven. *Automatic asymptotics*. PhD thesis, École polytechnique, Palaiseau, France, 1997.
- [18] J. van der Hoeven. Generic asymptotic expansions. *AAECC*, 9(1):25–44, 1998.
- [19] J. van der Hoeven. *Transseries and real differential algebra*, volume 1888 of *Lecture Notes in Mathematics*. Springer-Verlag, 2006.
- [20] J. van der Hoeven, G. Lecerf, B. Mourrain et al. Mathemagix. 2002. <http://www.mathemagix.org>.
- [21] R. Lee. The moment formula for implied volatility at extreme strikes. *Mathematical Finance*, 14(3):469–480, 2004.
- [22] Forde M. and Jacquier A. Small-time asymptotics for an uncorrelated local-stochastic volatility model. *Applied Mathematical Finance*, 18(6):517–535, 2011.

- [23] O. Osajima. The asymptotic expansion formula of implied volatility for dynamic sabr model and fx hybrid model. *Dx.doi.org/10.2139/ssrn.965265*, 2007.
- [24] D. Richardson. Zero tests for constants in simple scientific computation. *MCS*, 1(1):21–37, 2007.
- [25] D. Richardson, B. Salvy, J. Shackell, and J. van der Hoeven. Expansions of exp-log functions. In Y. N. Lakhsman, editor, *Proc. ISSAC '96*, pages 309–313. Zürich, Switzerland, July 1996.
- [26] M. Roper and M. M. Rutkowski. On the relationship between the call price surface and the implied volatility surface close to expiry. *International Journal of Theoretical and Applied Finance*, 12(4):427–441, 2009.
- [27] B. Salvy. *Asymptotique automatique et fonctions génératrices*. PhD thesis, École Polytechnique, France, 1991.
- [28] B. Salvy and J. Shackell. Asymptotic expansions of functional inverses. In P. S. Wang, editor, *Proc. ISSAC '92*, pages 130–137. New York, 1992. ACM Press.
- [29] B. Salvy and J. Shackell. Symbolic asymptotics: multiserries of inverse functions. *JSC*, 27(6):543–563, 1999.
- [30] J. Shackell. Growth estimates for exp-log functions. *JSC*, 10:611–632, 1990.
- [31] J. Shackell. *Symbolic asymptotics*, volume 12 of *Algorithms and computation in Mathematics*. Springer-Verlag, 2004.
- [32] M.R. Tehranchi. Uniform bounds for black–scholes implied volatility. *SIAM J. Finan. Math.*, 7(1):893–916, 2016.