Nonlinear Modes of Vibration of Vibro-Impact Duffing Oscillators

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Summary The objective of this research is to develop numerical methods for finding periodic solutions to autonomous conservative vibro-impact Duffing oscillators. The investigated model consists of serially connected masses the last of which collides a rigid wall. The connecting stiffness are quadratic functions of the displacements. Dedicated Ritz methods and continuation algorithms are developed in order to find families of admissible natural vibratory responses commonly named *nonlinear modes of vibration*.

MOTIVATIONS

Vibration analysis is critical for many common mechanical systems: jet engines, bridges, pistons, and acoustic resonators are some examples of systems for which the study of vibration is crucial [1]. Within a linear framework, one way to understand the behaviour of such oscillatory systems is by exploring their natural frequencies of oscillation in the vicinity of an equilibrium state. Such behaviour is called autonomous vibration and its analysis aids in predicting more complex behaviours of the investigated system. The analysis becomes much more challenging when the investigated mechanical system features smooth as well as nonsmooth nonlinearities: two typical examples are cubic nonlinearities capturing large displacements and impacts reflecting unilaterally constrained dynamics. Such a system, reduced to 2-degree-of-freedom (dof) vibro-impact Duffing oscillator, is numerically investigated in the light of vibration modal analysis.

MODELLING AND GOVERNING EQUATIONS

The system of interest, depicted in Fig. 1, consists of two nonlinear cubic springs serially connecting two masses the last of which impacts a rigid non-moving wall. Unilateral contact is mathematically expressed as a Signorini complementarity condition.

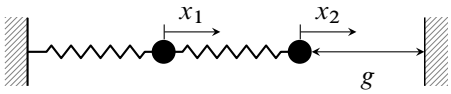

Figure 1: 2-dof vibro-impact Duffing oscillator: stiffnesses are nonlinear functions of x_1 and x_2, see Equation (1a).

For discrete systems, an additional impact law has to be incorporated in the formulation for well-posedness purposes [6]. In this work, the well-known Newton impact law with the restitution coefficient $e = 1$ is considered [3, 4]. One-impact-per-period solutions only are targeted and the complementarity conditions are advantageously replaced by a simple numerical test on the admissibility of the found solution [5, 7]. This simplified formulation tracks a subset of all possible admissible solutions. Periodic solutions with one impact per period are assumed to exist but the period T is unknown. Without loss of generality, the assumed impact occurs at $t = 0$ and the governing equations read:

- Free-flight $\forall t \in]0\,;T[$:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} (x_1 - x_2)^3 \\ (x_2 - x_1)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1a}$$

- Continuity and impact condition

$$x_i(0) = x_i(T), \quad i = 1, 2 \tag{1b}$$

$$x_2(0^+) = g \quad \text{and} \quad \dot{x}_2(0^+) = -\dot{x}_2(T^-) \tag{1c}$$

SOLUTION METHODS

When $\varepsilon = 0$, closed-form solutions to system (1) exist [5]. When the cubic springs are effective, the construction of an approximation relying on the Ritz method is implemented. Ritz methods are designed to approximate the solution using a set of known functions [1] which satisfy the boundary conditions of the formulation. In this research, the Ritz functions are defined in the time-domain and should comply with the continuity and impact conditions (1b) and (1c). The two following families

$$\phi_i(t) = \cos(i\omega(T/2 - t)) \quad \text{and} \quad \psi_i(t) = \cos(2\pi i t/T), \quad i = 1, 2, \ldots \tag{2}$$

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to approximate the displacement of the impacting mass and non-impacting mass, respectively. In Equation (2), ω is the frequency of the motion when impact is ignored: this is an unknown of the formulation and is introduced to generate a discontinuity $\dot{x}_2(0^+) = -\dot{x}_2(T^-)$ in a straightforward manner. The period of the unknown displacement (x_1, x_2) solution to (1) is T and not $2\pi/\omega$. The approximate solution is then inserted into (1a) and a Galerkin Projection [1] is performed on the family ψ_i (solely) to find the participation of the Ritz functions in the solution. As such, the formulation has always more one unknown than the number of available equations: this can be understood as a *generalized* eigenvalue problem where the period and the displacement have to be found. The solution set is intricate and organized as multiple co-existing one-parameter families of periodic orbits, some of which can be tracked through continuation strategies starting from the linear solutions.

RESULTS

Solutions were obtained for a 2-dof system but the proposed method extends to larger systems at the cost of an increased computational effort. It was concluded, from numerous tests, that the developed Ritz method converges when a sufficient number of Ritz functions is incorporated into the formulation. The admissibility of the found solution has to be continuously checked as unexpected grazing trajectories might arise when the energy of the motion increases.

The effects of the impact is indicated in Fig. 2 [left] by the first change of slope in the Frequency-energy plot. The well-known stiffening effect induced by the cubic springs is observed when ε is increased. Other branches were also found in

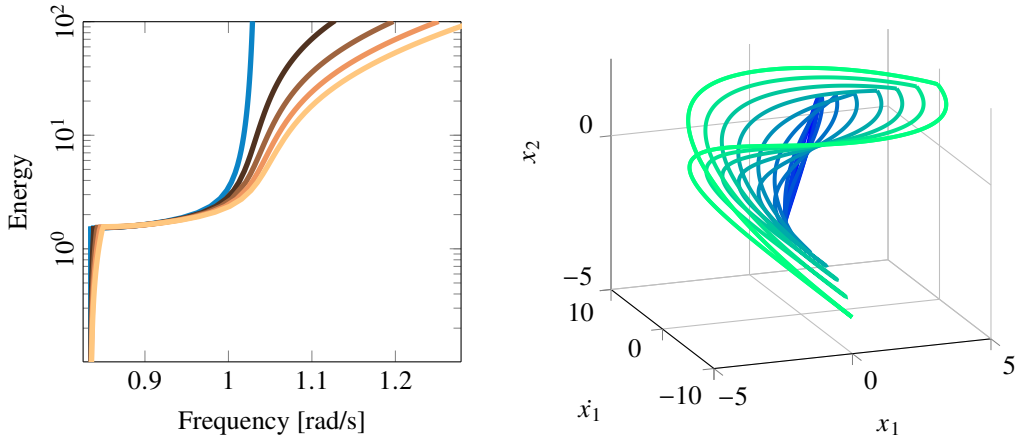

Figure 2: Left: Frequency-Energy plot of the first nonlinear vibration mode in the vicinity of the first natural frequency ω_1 with $\varepsilon = 0$ [blue curve, linear oscillator with impact] and $\varepsilon \in [0.0\,;0.2]$ [brown curves from dark to bright]. Parameters $m_1 = m_2 = k_2 = 1$, $k_1 = 3$, and $g = 2$. Right: Invariant manifold for $\varepsilon = 0.2$

the neighbourhoods of the subharmonics of the system's natural frequencies (no impact and no cubic terms in the stiffnesses). Further work is currently undertaken on the theoretical aspects of the existence of such manifolds. It should be noted that the internal resonances expected in the Duffing oscillator (with no impact) will generate unexplored difficulties when impact is introduced. Finally, finding solutions with multiple impacts per period is a direct extension to this work, yet much more challenging.

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