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Cages of small length holding convex bodies

Augustin Fruchard and Tudor Zamfirescu

August 8, 2017

Abstract

A cage G , defined as the 1-skeleton of a convex polytope, holds a compact set K disjoint from G , if K cannot be moved away without intersecting G . The main results of this paper establish the minimal length of cages holding various compact convex sets. First, planar graphs and Steiner trees are investigated. Then the notion of “almost fixing points” for planar convex bodies is introduced and studied. The last two sections treat cages holding 2-dimensional, respectively 3-dimensional, compact convex sets.

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1 Introduction

A *cage* is the 1-dimensional skeleton of a convex polytope in \mathbb{R}^3 . A cage G is said to *hold* a compact set K disjoint from G if no rigid motion can bring K in a position far away without meeting G on its way. Already in 1959, Coxeter [5] raised the problem of finding a cage of minimal total length holding the ball of radius 1 in \mathbb{R}^3 . In the following years, Besicovitch and Aberth solved Coxeter’s problem. In the present paper, we extend the investigation to other compact convex sets replacing the ball.

The space \mathbb{R}^3 is endowed with its euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product. For distinct $x, y \in \mathbb{R}^3$, let \overline{xy} be the line through x, y and xy the line-segment from x to y . The open segment $xy \setminus \{x, y\}$ is denoted by $]xy[$. Given a line \overline{xy} oriented from x to y , $(xy)^+$ denotes the open half-plane on the left of \overline{xy} .

As usual, for $M \subset \mathbb{R}^d$ with $d \geq 2$, $\text{bd}M$ denotes its boundary, $\text{relbd}M$ denotes its relative boundary, and $\text{diam}M = \sup_{x, y \in M} \|x - y\| \in \mathbb{R} \cup \{+\infty\}$. Its *convex hull* $\text{conv}M$ is the intersection of all convex subsets of \mathbb{R}^d containing M , and its *affine hull* $\text{aff}M$ is the intersection of all affine subspaces of \mathbb{R}^d containing M .

For any closed convex subset M of \mathbb{R}^3 , let $\pi_M : \mathbb{R}^3 \rightarrow M$ denote the (metric) *projection*, i.e. $\pi_M(x)$ is the unique point of M such that $\|x - \pi_M(x)\| = \inf_{y \in M} \|x - y\|$. It is known (and easy to prove) that $\langle x - \pi_M(x), y - \pi_M(x) \rangle \leq 0$ for all $y \in M$ and that π_M is 1-Lipschitz.

Here, a 2- or 3-dimensional compact convex set in \mathbb{R}^3 is called a *convex body*. Let \mathcal{K} be the space of all convex bodies in \mathbb{R}^3 . For $K \in \mathcal{K}$, we denote by $\text{wid}K$ the smallest distance between two parallel $(\dim K - 1)$ -dimensional affine subspaces H, H' of $\text{aff}K$ such that $K \subset \text{conv}(H \cup H')$. The d -dimensional unit ball (centred at $\mathbf{0}$) is denoted by \mathbb{B}_d , and $\text{relbd}\mathbb{B}_d = \mathbb{S}_{d-1}$ ($d \geq 2$). The d -dimensional regular simplex of edge length 1 is denoted by \mathbb{T}_d ($d \geq 2$). The d -dimensional cube of unit edge length is denoted by \mathbb{C}_d ($d \geq 2$).

We shall denote by λ the 1-dimensional Hausdorff measure (length).

For $n \geq 2$ and $x_1, x_2, \dots, x_n \in \mathbb{R}^2$, we put $x_1 x_2 \dots x_n = \text{conv}\{x_1, x_2, \dots, x_n\}$ and $|x_1 x_2 \dots x_n| = \text{abd}x_1 x_2 \dots x_n$. In particular $|xy| = \|x - y\|$.

Given $x_1, \dots, x_n \in \mathbb{R}^3$, the *perimeter function* is $p(x_1, \dots, x_n) = \sum_{i=1}^n |x_i x_{i+1}|$ (with $x_{n+1} = x_1$). If x_1, \dots, x_n are coplanar and *in convex position*, i.e. $x_1 \dots x_n$ is a non degenerate convex n -gon with consecutive vertices x_1, \dots, x_n , then $p(x_1, \dots, x_n) = |x_1 \dots x_n|$, but $p(x_1, x_2) = 2|x_1 x_2|$.

Let $\mathcal{G}(K)$ be the space of all cages in \mathbb{R}^3 holding the compact set K and set

$$L(K) = \inf_{G \in \mathcal{G}(K)} \lambda G.$$

We are looking for $L(K)$ for various sets K . Apart from general results, we estimate $L(K)$ for the most common convex bodies. For the unit balls we find $L(\mathbb{B}_2) = 6$, while $L(\mathbb{B}_3) = \frac{8\pi}{3} + 2\sqrt{3}$ is known, see Proposition 5.1. For the regular simplices of unit edge length, we obtain $L(\mathbb{T}_2) = \frac{3+\sqrt{3}}{2}$ and $L(\mathbb{T}_3) = 3$. For the cubes of unit edge length, we establish $L(\mathbb{C}_2) = 3\sqrt{2}$ and $L(\mathbb{C}_3) \leq 4 + 3\sqrt{2}$. We conjecture that $L(\mathbb{C}_3) = 4 + 3\sqrt{2}$.

The structure of the article is as follows. In Section 2 we relate the length of a cage with the perimeter and the length of the Steiner tree joining the projections of the vertices of the cage on some plane; this will yield lower bounds for $L(K)$ for several sets K . Section 3 deals with the notion of almost fixing points, yielding upper bounds for $L(K)$. Planar convex bodies are studied in Section 4 and nonplanar ones in Section 5.

We end this introductory section with the following result.

Theorem 1. *The function L is upper semi-continuous, but not continuous.*

Proof. Let K be a compact convex set, and $\varepsilon > 0$. Assume without loss of generality that $\mathbf{0} \in \text{int}K$. By definition, K admits a cage G holding K of length less than $L(K) + \varepsilon$. As K and G are disjoint, there exist $\nu < 1 < \mu$ such that μK and νG are disjoint too. Clearly, νG holds νK . Thus, each convex compact set K' satisfying $\nu K \subset K' \subset \mu K$ is held by νG . Hence, for each K' in a suitable neighbourhood of K , $L(K') \leq \lambda(\nu G) \leq \lambda G \leq L(K) + \varepsilon$, and thus L is upper semicontinuous.

That L is not continuous it can be seen as follows. Take the square $S = \mathbb{C}_2 = abcd$. For $0 < \eta < 1$, let $x \in ab$ satisfy $|xa| = \eta$ and choose $S_\eta = xbcd$. Then $S_\eta \rightarrow S$ if $\eta \rightarrow 0$, but we have $L(S_\eta) = 3$ for all η , as shows Theorem 7 (i) in the sequel, while $L(S) = 3\sqrt{2}$, as shows Theorem 8 (ii). \square

2 Planar graphs and Steiner trees

By a *geometric graph*, we mean a pair (G, η) , where G is a graph with vertex set $V(G)$ and edge set $E(G)$, and η is an embedding of G in a plane Π , which acts as follows. For any vertex $v \in V(G)$, $\eta(v)$ is a point in Π ; for any edge $(v, w) \in E(G)$, $\eta((v, w))$ is the line-segment $\eta(v)\eta(w) \subset \Pi$, reduced to $\{\eta(v)\}$ if $\eta(v) = \eta(w)$. The set $\cup_{e \in E(G)} \eta(e)$ will be denoted by $\eta(G)$. Observe that we may have $\eta(v) \in \eta((u, w))$ for distinct $v, u, w \in V(G)$, and $\eta((u, v)) \cap \eta((u', v'))$ may be non-void, even a line-segment, for distinct $u, v, u', v' \in V(G)$. A vertex $v \in V(G)$ is called *external* if $\eta(v) \in \text{bdconv}\eta(G)$, and an edge $e \in E(G)$ is called *external* if $\eta(e) \subset \text{bdconv}\eta(G)$.

If

- 1) $\eta(G)$ is not a line-segment,
- 2) for any point $x \in \Pi$, $\text{card}(\eta_{v(G)})^{-1}(\{x\}) \leq 2$,

3) for any side $s = ab$ of the convex polygon $\text{conv}\eta(G)$, there is at least one path (v_1, \dots, v_n) in G such that $\eta(v_1) = a$, $\eta(v_n) = b$, and $\eta((v_1, \dots, v_n)) = ab$, then (G, η) is called *convex*.

We denote by P_s one of the paths (v_1, \dots, v_n) satisfying condition 3). Notice that condition 3) implies that the union of $\eta(e)$ for all external edges e equals $\text{bd conv}\eta(G)$.

Now, for every side s of $\text{conv}\eta(G)$, delete from G the edges of P_s , to obtain a graph ∇G .

We call a geometric graph (G, η) *strongly connected* if G has at least three edges and the graph $G \setminus \{e_1, e_2\}$ is connected, for any pair of external edges $e_1, e_2 \in E(G)$.

Lemma 2.1. *If the geometric graph (G, η) is convex and strongly connected, then ∇G is connected.*

Proof. Let $v \in V(G)$ be external, let C be the connected component of v in ∇G , and assume that $C \neq G$.

Case 1. All external vertices belong to C .

Choose $w \in V(G) \setminus C$. Delete any two external edges; the resulting graph contains a path from w to some external vertex, which therefore does not belong to C , contrary to the present assumption.

Case 2. Some external vertex u does not belong to C .

In this case, at least two external edges e_1, e_2 do not belong to C . Since (G, η) is strongly connected, $G \setminus \{e_1, e_2\}$ is connected, which yields $u \in C$, and a contradiction is obtained again. \square

For the geometric graph (G, η) , we define its length $\mu(G, \eta)$ as follows.

$$\mu(G, \eta) = \lambda\eta(\nabla G) + \lambda\text{bd conv}\eta(G).$$

Consider n points $v_1, \dots, v_n \in \mathbb{R}^3$. Let $S(v_1, \dots, v_n)$ denote the length of the shortest rectifiable set containing all n points, called their *Steiner tree*.

Moreover, let

$$f(v_1, \dots, v_n) = p(v_1, \dots, v_n) + S(v_1, \dots, v_n), \quad (1)$$

where p is the perimeter function. Remember that $f(v_1, \dots, v_n) = |v_1 \dots v_n| + S(v_1, \dots, v_n)$ if $n \geq 3$ and v_1, \dots, v_n are coplanar and in convex position in the plane, but $f(v_1, v_2) = 3|v_1 v_2|$. The function f will play a central role in this article.

An immediate consequence of Lemma 2.1 is the following.

Corollary 2.2. *Let (G, η) be a convex strongly connected geometric graph, with external vertices v_1, \dots, v_n such that $\eta(v_1), \dots, \eta(v_n)$ lie in this order on $\text{bd conv}\eta(G)$. Then*

$$\mu(G, \eta) \geq f(\eta(v_1), \dots, \eta(v_n)).$$

Indeed,

$$\begin{aligned} \mu(G, \eta) &= \lambda\eta(\nabla G) + \lambda\text{bd conv}\eta(G) \geq S(\eta(v_1), \dots, \eta(v_n)) + p(\eta(v_1), \dots, \eta(v_n)) \\ &= f(\eta(v_1), \dots, \eta(v_n)). \end{aligned}$$

We shall use the following obvious fact.

Lemma 2.3. *If $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is 1-Lipschitz, then*

$$\begin{aligned} p(\varphi(v_1), \dots, \varphi(v_n)) &\leq p(v_1, \dots, v_n), \\ S(\varphi(v_1), \dots, \varphi(v_n)) &\leq S(v_1, \dots, v_n), \text{ and therefore} \\ f(\varphi(v_1), \dots, \varphi(v_n)) &\leq f(v_1, \dots, v_n). \end{aligned}$$

Theorem 2. Let G be a cage and P some plane in \mathbb{R}^3 . If v_1, \dots, v_n are the external vertices of (G, π_P) such that $\pi_P(v_1), \dots, \pi_P(v_n)$ lie in this order on $\text{bd conv}\eta(G)$, then

$$\lambda G \geq f(\pi_P(v_1), \dots, \pi_P(v_n)).$$

Proof. As a graph, the 1-dimensional skeleton of a polytope is 3-connected (i.e., every pair of vertices can be joined by three paths in the graph having only their endpoints in common), and therefore 3-edge connected (i.e., the graph minus any pair of edges is graph-theoretically connected). Then the geometric graph (G, π_P) is strongly connected. Also, it is easily verified that (G, π_P) is convex. We have $G = \nabla G \cup (\cup_s P_s)$ and $\lambda G = \lambda \nabla G + \sum_s \lambda P_s$. By successively using Lemma 2.3 and Corollary 2.2, we obtain

$$\lambda G \geq \lambda \pi_P(\nabla G) + \sum_s \lambda \pi_P(P_s) \geq \mu(G, \pi_P) \geq f(\pi_P(v_1), \dots, \pi_P(v_n)). \quad \square$$

Given $a, b, c \in \mathbb{R}^2$, let $t(a, b, c)$ be the Fermat-Torricelli point, i.e. the unique point $s \in \mathbb{R}^2$ such that $S(a, b, c) = |as| + |bs| + |cs|$. If one of the three angles of the triangle abc is greater than $\frac{2\pi}{3}$, then $t(a, b, c)$ is one of the points a, b , or c .

Now fix $a, b \in \mathbb{R}^2$ and $k > |ab|$, and consider the locus

$$\mathcal{O}(a, b, k) = \{x \in \mathbb{R}^2 ; S(a, b, x) = k\}.$$

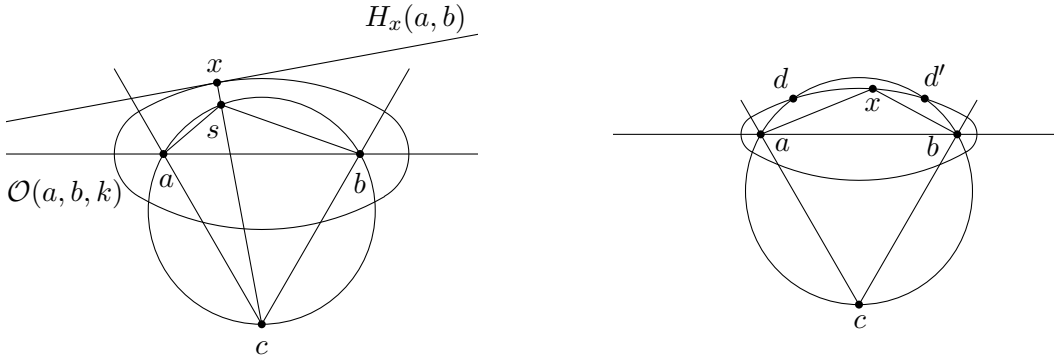


Figure 1: The locus $\mathcal{O}(a, b, k)$. Left: $k \geq \frac{2}{\sqrt{3}}|ab|$, right: $k < \frac{2}{\sqrt{3}}|ab|$.

Proposition 2.4. (i) For any $a, b \in \mathbb{R}^2$ and $k > |ab|$, the locus $\mathcal{O}(a, b, k)$ is a convex \mathcal{C}^1 curve.

(ii) For all $x \in \mathbb{R}^2$ there exists a half-plane $H_x(a, b)$ containing x on its boundary such that, for all $d \in H_x(a, b)$ we have $S(a, b, d) \geq S(a, b, x)$.

Proof. (i) Let $c \in \mathbb{R}^2$ be such that abc is an equilateral triangle, labelled clockwise. Let C be the circle circumscribed to abc . Recall that $(ab)^+$ denotes the open half-plane on the left of the line \overline{ab} oriented from a to b , i.e. here not containing c . We describe $\mathcal{O}^+(a, b, k) = \mathcal{O}(a, b, k) \cap (ab)^+$. Let $x \in \mathcal{O}^+(a, b, k)$. If $x \in (ca)^+$, then $\angle xab > \frac{2\pi}{3}$, hence the Steiner tree connecting a, b, x is the union of the segments xa and ab , and $\mathcal{O}^+(a, b, k) \cap (ca)^+$ is an arc of circle of radius $k - |ab|$ centred in a . If $x \in (ac)^+ \cap (cb)^+ \setminus \text{conv}C$, then the Fermat-Torricelli point $s = t(a, b, x)$ is on C , and the Ptolemy theorem shows that $|sa| + |sb| = |sc|$, whence $\mathcal{O}^+(a, b, k) \cap (ac)^+ \cap (cb)^+$ is an arc of circle of centre c and radius k if $k \geq \frac{2}{\sqrt{3}}|ab|$. The curve $\mathcal{O}^+(a, b, k)$ is differentiable at its point on \overline{ac} .

If $k < \frac{2}{\sqrt{3}}|ab|$, then $\mathcal{O}^+(a, b, k)$ crosses C at some points d, d' . If $x \in (ac)^+ \cap (cb)^+ \cap \text{conv}C$, then the Steiner tree connecting a, b, x is the union of the segments ax and xb , hence $\mathcal{O}^+(a, b, k) \cap$

$(ac)^+ \cap (cb)^+ \cap \text{conv}C$ is an arc of ellipse of foci a and b . At d, d' too, $\mathcal{O}^+(a, b, k)$ is differentiable. Indeed, on the one hand, the normal to the ellipse at d must bisect the angle $\angle adb$, on the other hand, denoting by ω the centre of C , we have $\angle adc = \frac{1}{2}\angle a\omega c = \frac{\pi}{3} = \angle cdb$. It follows that both normals to the ellipse and to the circle of centre c coincide at d .

Now, the convex curve $\mathcal{O}^+(a, b, k)$ is fully described via the symmetry with respect to the bisector of ab , and further $\mathcal{O}(a, b, k)$ via the symmetry with respect to \overline{ab} .

(ii) Put $k = S(a, b, x)$, let D_x be the tangent of $\mathcal{O}(a, b, k)$ at x , and choose for $H_x(a, b)$ the half-plane bounded by D_x and not containing ab . \square

Remark. One can prove that, for all integers $n \geq 3$, all $a_1, \dots, a_n \in \mathbb{R}^2$, and all $k > S(a_1, \dots, a_n)$, the locus $\{x \in \mathbb{R}^2 ; S(a_1, \dots, a_n, x) = k\}$ is a concatenation of convex \mathcal{C}^1 curves, themselves concatenations of arcs of circles and/or ellipses, with possible angular points b_1, \dots, b_r . These points b_j are those for which the Steiner tree of a_1, \dots, a_n, b_j is not unique, and where this Steiner tree combinatorially changes.

Corollary 2.5. *If $v_1, \dots, v_n \in \mathbb{R}^2$ and $a, b, c \in v_1 \dots v_n$, then $S(a, b, c) \leq S(v_1, \dots, v_n)$ and $f(a, b, c) \leq f(v_1, \dots, v_n)$.*

Proof. Since c (resp. b, a) is in the convex hull of v_1, \dots, v_n , every half-plane bounded by a straight line passing through c (resp. b, a) contains at least one point v_i (resp. v_j, v_k). Choose $c' = v_i$ in the half-plane $H_c(a, b)$ given by Proposition 2.4 (ii). Similarly, choose $b' = v_j$ in the half-plane $H_b(a, c')$, then choose $a' = v_k$ in the half-plane $H_a(b', c')$. Then we have

$$S(a, b, c) \leq S(a, b, c') \leq S(a, b', c') \leq S(a', b', c') = S(v_k, v_j, v_i) \leq S(v_1, \dots, v_n).$$

Since $|abc| \leq |v_1 \dots v_n|$, the inequality for f follows. \square

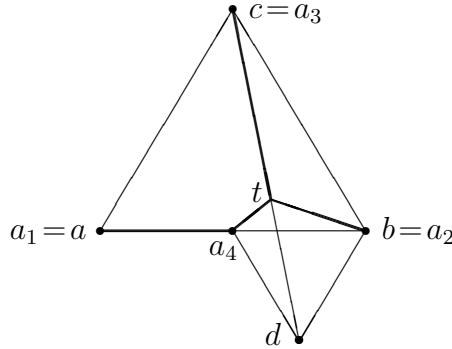


Figure 2: In bold, a Steiner tree connecting a_1, \dots, a_4 .

Remark. One could expect that, for $a_1, \dots, a_m \in v_1 \dots v_n$, we have $S(a_1, \dots, a_m) \leq S(v_1, \dots, v_n)$ as soon as $m < n$. This is however false, already for $m = 4$.

Indeed, given an equilateral triangle $\mathbb{T}_2 = abc$ of unit edge length, choose n points v_1, \dots, v_n , some of them close to a , some close to b , and others close to c , such that abc is in their convex hull. Then $S(v_1, \dots, v_n)$ is close to $S(a, b, c) = \sqrt{3}$, while for $a_1 = a, a_2 = b, a_3 = c, a_4 = \frac{1}{2}(a + b)$, one can check that one of the Steiner trees connecting a_1, \dots, a_4 is obtained by joining a_1 to a_4 , then taking the Steiner tree of a_2, a_3 and a_4 , cf. Figure 2; the other Steiner tree is symmetric with respect to the line $\overline{a_3a_4}$. If d is such that a_2, a_4, d is equilateral and if t is the Fermat-Torricelli point associated to a_2, a_3, a_4 , from $|ta_4| + |ta_2| = |td|$ one obtains

$$S(a_1, \dots, a_4) = |a_1a_4| + |a_3d| = \frac{1}{2}(1 + \sqrt{7}) > S(v_1, \dots, v_n).$$

Theorem 3. Let K be a planar convex body, $x, y, z \in \text{bd}K$, and $x', y', z' \in \mathbb{R}^2$, such that $\overline{xx'}$, $\overline{yy'}$, $\overline{zz'}$, are supporting lines of K and the vectors $x' - x$, $y' - y$, $z' - z$ point toward the direct sense on $\text{bd}K$. Assume that the order of x, y, z is also in the direct sense on $\text{bd}K$ and that the Steiner tree determined by x, y, z has a vertex v inside $\text{int}K$. Put $\alpha_x = \angle x'xy$, $\beta_x = \angle x'xv$, $\gamma_x = \angle x'xz$, and, analogously, $\alpha_y, \beta_y, \gamma_y$ and $\alpha_z, \beta_z, \gamma_z$.

If f attains a local minimum at x, y, z , then

$$\cos \alpha_x + \cos \beta_x + \cos \gamma_x = 0,$$

$$\cos \alpha_y + \cos \beta_y + \cos \gamma_y = 0,$$

$$\cos \alpha_z + \cos \beta_z + \cos \gamma_z = 0.$$

Moreover, $\text{bd}K$ is differentiable at x, y, z , and the normals at x, y, z are concurrent.

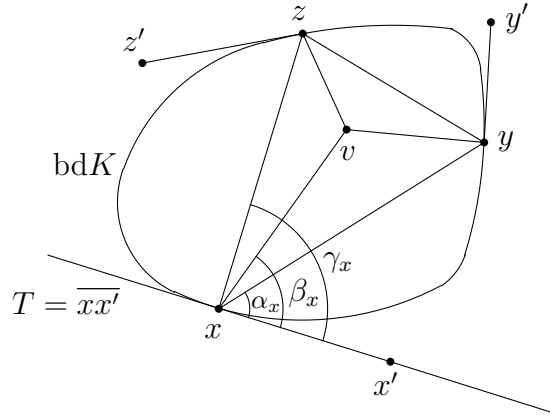


Figure 3: The law of cosines.

Proof. Consider for a moment $v = (v_1, v_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ fix, and x variable on $T = \overline{xx'}$. Put $g(x) = |xy| + |xv| + |xz|$. Again for a moment, consider T to be the real line $\mathbb{R} \times \{0\}$, with the real number x corresponding to the point x as its abscissa. We have

$$g(x) = \sqrt{(x - y_1)^2 + y_2^2} + \sqrt{(x - v_1)^2 + v_2^2} + \sqrt{(x - z_1)^2 + z_2^2},$$

whence

$$\begin{aligned} g'(x) &= \frac{x - y_1}{\sqrt{(x - y_1)^2 + y_2^2}} + \frac{x - v_1}{\sqrt{(x - v_1)^2 + v_2^2}} + \frac{x - z_1}{\sqrt{(x - z_1)^2 + z_2^2}} \\ &= \cos \alpha_x + \cos \beta_x + \cos \gamma_x. \end{aligned}$$

Now, assume

$$\cos \alpha_x + \cos \beta_x + \cos \gamma_x \neq 0.$$

Then, for some $\tilde{x} \in T$ close to x , $g(\tilde{x}) < g(x)$. This implies $|z\pi_K(\tilde{x})| + |v\pi_K(\tilde{x})| + |y\pi_K(\tilde{x})| < g(x)$, which yields $f(\tilde{x}, y, z) < f(x, y, z)$, impossible.

To prove the second part, suppose that some normals at x, y, z form a non-degenerate triangle. Then suitably and slightly turning the set $\{v, x, y, z\}$ about an interior point of that triangle brings all three points x, y, z in positions x'', y'', z'' outside K . But then

$$f(\pi_K(x''), \pi_K(y''), \pi_K(z'')) < f(x, y, z),$$

which is again impossible. \square

In the rest of this Section, we compute the infimum of f on some sets; this will be useful in Section 4.

Lemma 2.6. We have $\inf\{f(x, y, z) ; x, y, z \in \mathbb{S}_1, \mathbf{0} \in xyz\} = 6$.

Proof. Recall the perimeter function p , here defined by $p(x, y, z) = |xyz|$ if all three points are distinct and $p(x, y, y) = 2|xy|$. It is easily seen that both p and S have no local minimum with distinct x, y, z . It follows that the global minimum of both functions is attained when two of the points x, y, z collapse, i.e. when xyz is a diameter of \mathbb{S}_1 . \square

Lemma 2.7. If abc is an equilateral triangle of sides $I_1 = bc$, $I_2 = ca$, $I_3 = ab$ and side length δ , then $\inf\{f(u_1, u_2, u_3) ; u_k \in I_k\} = \frac{3+\sqrt{3}}{2}\delta$. This infimum is reached precisely at (a', b', c') , where a', b', c' are the midpoints of bc, ca, ab respectively.

Proof. We prove that both functions p and S have a global minimum on $I_1 \times I_2 \times I_3$ at (a', b', c') , and that this minimum is uniquely reached for p .

The minimality of p implies the so-called incidence/reflection law: $\angle cu_1u_2 = \angle u_3u_1b$, denoted by α , and similarly $\angle au_2u_3 = \angle u_1u_2c = \beta$ and $\angle bu_3u_1 = \angle u_2u_3a = \gamma$. Since the angles in a triangle sum to π , we have $\alpha + \beta = \beta + \gamma = \gamma + \alpha = \frac{2\pi}{3}$, yielding $\alpha = \beta = \gamma = \frac{\pi}{3}$. It follows that the minimum of p is reached only at (a', b', c') .

For S , however, there is a 2-dimensional subset of $I_1 \times I_2 \times I_3$ where S attains its global minimum: choose arbitrarily $t \in abc$ and consider $u_1 = \pi_{bc}(t)$, $u_2 = \pi_{ca}(t)$, and $u_3 = \pi_{ab}(t)$. Then t is the Fermat-Torricelli point $t(u_1, u_2, u_3)$ and $S(u_1, u_2, u_3) = \frac{\sqrt{3}}{2}\delta$. \square

Lemma 2.8. If $abcd$ is a parallelogram of sides $I_1 = ab$, $I_2 = bc$, $I_3 = cd$, $I_4 = da$, with $\delta = |ac| \leq |bd|$, then $\inf\{f(u_1, \dots, u_4) ; u_k \in I_k\} = 3\delta$.

If $abcd$ is not a rectangle, then this infimum is reached precisely at (a, c, c, a) . If $abcd$ is a rectangle, then the infimum is reached at both (a, c, c, a) and (b, b, d, d) .

Proof. As before, we prove that both functions p and S have a global minimum at the aforementioned points. Assume that $abcd$ is not a rectangle, and put $\beta = \angle abc$. We have $\beta < \frac{\pi}{2}$. Firstly, observe that the perimeter function p has no local minimum at a quadruple (u_1, \dots, u_4) with $u_i \notin \{a, b, c, d\}$ for every i . Indeed, at such four points, the minimality of p implies the incidence/reflection law, which yields $4\beta = 2\pi$, a contradiction. Hence the minimum is attained, say, for $u_1 = a$ or b . Then one easily shows that $u_4 = u_1$ in the first case and $u_2 = u_1$ in the second, and finally that the global minimum of p is attained at $u_1 = u_4 = a$, $u_2 = u_3 = c$. In

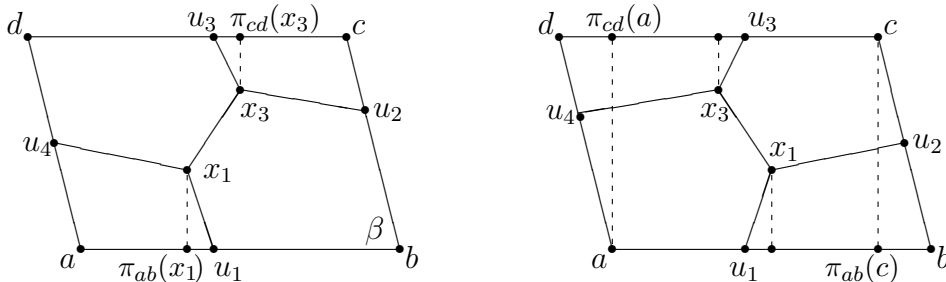


Figure 4: Left: x_1 is the neighbour of u_4 in the Steiner tree; right: x_1 is the neighbour of u_2 .

the Steiner tree connecting u_1, \dots, u_4 , let x_1 be the neighbour of u_1 , and x_3 that of u_3 . Then

$$\begin{aligned} S(u_1, \dots, u_4) &= |u_4x_1| + |u_1x_1| + |x_1x_3| + |x_3u_3| + |x_3u_2| \\ &\geq |a\pi_{ab}(x_1)| + |\pi_{ab}(x_1)x_1| + |x_1x_3| + |x_3\pi_{cd}(x_3)| + |\pi_{cd}(x_3)c| \geq |ac| \end{aligned}$$

if x_1 is the neighbour of u_4 and x_3 the neighbour of u_2 in the Steiner tree (see Fig. 4, left), and

$$\begin{aligned} S(u_1, \dots, u_4) &= |u_4x_3| + |u_3x_3| + |x_3x_1| + |x_1u_1| + |x_1u_2| \\ &\geq |\pi_{cd}(a)\pi_{cd}(x_3)| + |\pi_{cd}(x_3)x_3| + |x_3x_1| + |x_1\pi_{ab}(x_1)| + |\pi_{ab}(x_1)\pi_{ab}(c)| \\ &\geq |\pi_{cd}(a)\pi_{ab}(c)| = |ac| \end{aligned}$$

in the case x_1 is the neighbour of u_2 and x_3 that of u_4 (see Fig. 4, right). Hence, both functions p and S attain their global minimum at $u_1 = u_4 = a$, $u_2 = u_3 = c$, and this remains true for their sum. The case of a rectangle is clear. \square

3 Points almost fixing planar convex bodies

In the sense of Baire categories, for most convex bodies $K \subset \mathbb{R}^2$, the inscribed circle C , thought to be the unit circle \mathbb{S}_1 of centre $\mathbf{0}$, is unique and touches $\text{bd}K$ in exactly three points a, b, c , with $\mathbf{0} \in \text{int}abc$ [11]. Then, obviously, the points a, b, c fix K . However, there are convex bodies which cannot be fixed by any set of three points, e.g. parallelograms. A set of points is *fixing* K if any small move of K brings some point of the set in $\text{int}K$. Precisely, a_1, \dots, a_n fix K if there is a neighbourhood \mathcal{V} of the identity \mathbf{id} in the set $\text{Isom}^+\mathbb{R}^2$ of affine rotations such that, for every $f \in \mathcal{V}$ satisfying $f(K) \neq K$, at least one of the a_i belongs to $\text{int}f(K)$. This is slightly different from another commonly used definition of fixing, e.g. in [4, 3], where a subset H of $\text{bd}K$ is said to fix K if \mathbf{id} is isolated in the set $\{f \in \text{Isom}^+\mathbb{R}^2 ; H \cap f(\text{int}K) = \emptyset\}$. In particular, we consider that a rotation of a disc around its centre does not move the disc. Observe that both definitions are equivalent if K is not a disc, since in this case there is a neighbourhood \mathcal{V} of \mathbf{id} in $\text{Isom}^+\mathbb{R}^2$ such that every $f \in \mathcal{V} \setminus \{\mathbf{id}\}$ satisfies $f(K) \neq K$. We now introduce the following related notion.

The points $a_1, \dots, a_n \in \text{bd}K$ *almost fix* the convex body $K \subset \mathbb{R}^2$ if, for any neighbourhoods \mathcal{V}_i of a_i ($i = 1, \dots, n$), there are pairs of points $a'_i, a''_i \in \mathcal{V}_i \cap \text{bd}K$, such that $a'_1, a''_1, \dots, a'_n, a''_n$ fix K .

Theorem 4. *For any planar convex body K , there are two or three points almost fixing K .*

Proof. If K is a disc, then it is obviously almost fixed by two diametrically opposite boundary points. So, assume from now on that K is not a disc.

Suppose without loss of generality that $C = \mathbb{S}_1$ is an inscribed circle of K .

Case 1. There exists no $a \in C \cap \text{bd}K$ with $-a \in C \cap \text{bd}K$.

Let $a \in C \cap \text{bd}K$ and A be the component of $-a$ in $C \cap \text{int}K$. The set A is an open arc $\widetilde{a'a''} \subset C$ of length less than π . Clearly, the triangle $aa'a''$ is acute and a, a', a'' fix K .

Case 2. There exists $a \in C \cap \text{bd}K$ with $-a \in C \cap \text{bd}K$.

Let $\widetilde{a_1a_2} \subset C$ be the connected component of a in $C \cap \text{bd}K$. Then $\widetilde{a_1a_2} \neq C$ because K is not a disc. If $-a \in \widetilde{a_1a_2}$ and $\lambda\widetilde{a_1a_2} > \pi$ then K is fixed by a_1, a_2 and the midpoint a_3 of $\widetilde{a_1a_2}$. The case $\lambda\widetilde{a_1a_2} = \pi$ (hence $a_1 = \pm a$ and $a_2 = \mp a$) will be treated later on.

If $-a \notin \widetilde{a_1a_2}$, let $\widetilde{a_1^*a_2^*} \subset C$ be the component of $-a$ in $C \cap \text{bd}K$. Then $\widetilde{a_1a_2}$ and $\widetilde{a_1^*a_2^*}$ are disjoint.

Suppose $a_1, a_2 \neq a$ or $a_1^*, a_2^* \neq -a$, say $a_1^*, a_2^* \neq -a$. If $\lambda\widetilde{a_1^*a_2^*} < \pi$, then the triangle $aa_1^*a_2^*$ is acute and a, a_1^*, a_2^* fix K . If $\lambda\widetilde{a_1^*a_2^*} \geq \pi$, then $\lambda\widetilde{a_1a_2} < \pi$. Let m be the midpoint of $\widetilde{a_1a_2}$. Since $m \notin \widetilde{a_1^*a_2^*}$ and $\lambda\widetilde{a_1^*a_2^*} \geq \pi$, we have $-m \in \widetilde{a_1^*a_2^*}$. Then the triangle $(-m)a_1a_2$ is acute, and its vertices fix K .

If $a_1 = a$, $a_1^* = -a$, and $\widetilde{a_1 a_2}$, $\widetilde{a_1^* a_2^*}$ are non-degenerate and do not lie both on the same half of C , then we are in the already treated case for some point in the relative interior of $\widetilde{a_1 a_2}$ instead of a .

It remains the case that at least one of $\widetilde{a_1 a_2}$, $\widetilde{a_1^* a_2^*}$ is degenerate or they lie both on the same half of C .

Let the lines $\Lambda \ni a$, $-\Lambda \ni -a$ be orthogonal to \overline{aa} . Put $bc = \Lambda \cap \text{bd}K$, with b, c perhaps not distinct. Take small arcs $\widetilde{b'b''}$, $\widetilde{c'c''}$, $\widetilde{a'a''} \subset \text{bd}K$ containing $b, c, -a$, respectively, in their relative interior. Then $a', a'', b', b'', c', c''$ fix K , and consequently $-a, b, c$ almost fix K . This remains valid for bc degenerate ($b = c$); in that case $a, -a$ almost fix K . \square

In [7], almost necessary and sufficient conditions of first order are given for a finite collection of points to fix K , resp. almost fix K . ‘‘Of first order’’ means that these conditions make use of the left and right tangents of $\text{bd}K$ at these points. ‘‘Almost’’ means that the necessary conditions involve large inequalities, whereas the sufficient conditions involve the corresponding strict inequalities. In [4] an almost necessary and sufficient condition of *second order*, i.e. using the curvature of $\text{bd}K$, is given such that three points fix a \mathcal{C}^2 convex body K . We first introduce some notation.

Given a planar convex body K , with $\text{bd}K$ oriented counterclockwise, and $a \in \text{bd}K$, let $T_\ell(a)$, resp. $T_r(a)$, be the left, resp. right, tangent at $\text{bd}K$ in a . We orient these lines as $\text{bd}K$; thus we have $K \subset T_\ell(a)^+ \cap T_r(a)^+$. Let $N_\ell(a)$, resp. $N_r(a)$, be the left and right normals at $\text{bd}K$ in a , oriented in the directions $T_\ell(a)^+$, resp. $T_r(a)^+$, i.e. the line orthogonal to $T_\ell(a)$, resp. $T_r(a)$, containing a , and pointing (as) inward K (as possible). Let $L(a)$ be the open sector, union of the left open half-planes bounded by $N_\ell(a)$ and $N_r(a)$:

$$L(a) = N_\ell(a)^+ \cup N_r(a)^+.$$

Let $\overline{L}(a)$ be the corresponding closed sector, and let

$$\vec{L}(a) = \{x \in \mathbb{S}_1 ; a + x \in \overline{L}(a)\}$$

be the *set of directions* of $\overline{L}(a)$; it is a compact subset of \mathbb{S}_1 . Let $R(a)$ and $\overline{R}(a)$ be the analogous sectors for the right half-planes. Observe that the set of directions of $\overline{R}(a)$ is $-\vec{L}(a)$, hence will not be needed. If $\text{bd}K$ is differentiable at a , then $\overline{L}(a) = \mathbb{R}^2 \setminus R(a)$, otherwise $L(a) \cap R(a)$ is the union of two sectors of vertex a . We will also use the intersection sectors:

$$\ell(a) = N_\ell(a)^+ \cap N_r(a)^+ = \mathbb{R}^2 \setminus \overline{R}(a),$$

$r(a) = \mathbb{R}^2 \setminus \overline{L}(a)$, their corresponding closed sectors $\overline{\ell}(a)$ and $\overline{r}(a)$, and the set of directions of $\overline{\ell}(a)$: $\vec{\ell}(a) = \{x \in \mathbb{S}_1 ; a + x \in \overline{\ell}(a)\}$.

Theorem 5. [7] *Let K be a planar convex body and $a_1, \dots, a_n \in \text{bd}K$.*

- (i) *If a_1, \dots, a_n fix K , then both intersections $L(a_1) \cap \dots \cap L(a_n)$ and $R(a_1) \cap \dots \cap R(a_n)$ are empty.*
- (ii) *If the three intersections $\overline{L}(a_1) \cap \dots \cap \overline{L}(a_n)$, $\overline{R}(a_1) \cap \dots \cap \overline{R}(a_n)$, and $\vec{L}(a_1) \cap \dots \cap \vec{L}(a_n)$ are empty, then a_1, \dots, a_n fix K .*
- (iii) *If a_1, \dots, a_n almost fix K , then both intersections $\ell(a_1) \cap \dots \cap \ell(a_n)$ and $r(a_1) \cap \dots \cap r(a_n)$ are empty.*
- (iv) *If the three intersections $\overline{\ell}(a_1) \cap \dots \cap \overline{\ell}(a_n)$, $\overline{r}(a_1) \cap \dots \cap \overline{r}(a_n)$, and $\vec{\ell}(a_1) \cap \dots \cap \vec{\ell}(a_n)$ are empty, then a_1, \dots, a_n almost fix K .*

Remarks. 1. The set of directions is needed in items (ii) and (iv): If $K = [-2, 2] \times [0, 1]$ then the three points $a = (-1, 0)$, $b = (1, 0)$, $c = (0, 1)$ do not fix K although the intersections $\overline{L}(a) \cap \overline{L}(b) \cap \overline{L}(c)$ and $\overline{R}(a) \cap \overline{R}(b) \cap \overline{R}(c)$ are empty.

2. Another commonly used definition of fixing points is the following one [6]: The points $a_1, \dots, a_n \in \text{bd}K$ *weakly fix* K if, for any path $\gamma : [0, 1] \rightarrow \text{Isom}^+ \mathbb{R}^2$, $t \mapsto \gamma_t$ such that $\gamma_0(K) = K$ and $\gamma_1(K) \neq K$, there exist $i \in \{1, \dots, n\}$ and $t \in [0, 1]$ such that $a_i \in \text{int} \gamma_t(K)$. Obviously, if a_1, \dots, a_n fix K in our first sense, then they weakly fix K . Example 3 of [7] shows that the converse, however, is not true. Nevertheless Theorem 5 remains valid with this notion of weakly fixing.

3. Example 3.1 below shows that, although every planar convex body can be almost fixed by at most three points, sometimes it may be more economical to use more points to fix it, i.e. it is possible to have

$$\inf\{f(a, b, c, d) ; a, b, c, d \text{ almost fix } K\} < \inf\{f(a, b, c) ; a, b, c \text{ almost fix } K\}. \quad (2)$$

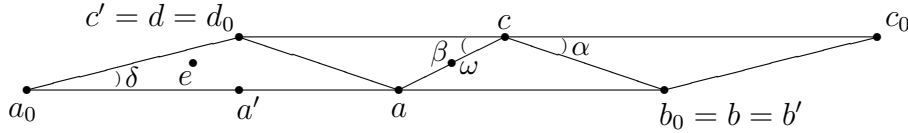


Figure 5: Example 3.1.

Example 3.1. With $\varepsilon > 0$ small enough, let K be the parallelogram $a_0b_0c_0d_0$, with $a_0 = (-2, -\varepsilon)$, $b_0 = (1, -\varepsilon)$, $c_0 = (2, \varepsilon)$, and $d_0 = (-1, \varepsilon)$. Its centre is $\omega = (0, 0)$, its ratio of side lengths is almost 3 and its angles are $\delta = \arctan 2\varepsilon$ and $\pi - \delta$.

Using Theorem 5 (i), it is rather tedious but straightforward to prove that

$$\inf\{f(a, b, c, d) ; a, b, c, d \text{ almost fix } K\}$$

is attained for $a = (-x, -\varepsilon)$, $b = b_0$, $c = (x, \varepsilon)$, and $d = d_0$, where $0 < x < 1$ is such that the angles $\alpha = \angle b_0cc_0$ and $\beta = \angle acd_0$ satisfy $2 \cos \alpha = 1 + \cos \beta$. Using $\tan \alpha = \frac{2\varepsilon}{1-x}$ and $\tan \beta = \frac{\varepsilon}{x}$, we obtain $2\alpha^2 = \beta^2 + \mathcal{O}(\varepsilon^4)$, hence $x = (1 + 2\sqrt{2})^{-1} + \mathcal{O}(\varepsilon)$.

Since the Steiner tree of a, b, c, d is the union of the three segments $da \cup ac \cup cb$ as soon as $\alpha + \beta < \frac{\pi}{6}$, we have $f(a, b_0, c, d_0) = 2|d_0a| + |ab_0| + 2|b_0c| + |cd_0| + |ac|$, hence

$$\inf\{f(a, b, c, d) ; a, b, c, d \text{ almost fix } K\} = 6 + \mathcal{O}(\varepsilon^2).$$

Besides, one easily finds that $\inf\{f(a, b, c) ; a, b, c \text{ almost fix } K\}$ is attained for $a' = (-1, -\varepsilon)$, $b' = b_0$, and $c' = d_0$, yielding

$$\inf\{f(a, b, c) ; a, b, c \text{ almost fix } K\} = 6 + (\sqrt{3} + 2)\varepsilon + \mathcal{O}(\varepsilon^2).$$

We used here that the length $S(a', b_0, d_0)$ of the Steiner tree joining a' , b_0 , and d_0 is $|eb_0| = 2 + \sqrt{3}\varepsilon + \mathcal{O}(\varepsilon^2)$, where $e = (-1 - \sqrt{3}\varepsilon, 0)$ is such that $a'd_0e$ is equilateral. As a consequence, (2) is satisfied if ε is small enough.

4 Cages for planar convex bodies in \mathbb{R}^3

Let $r(K)$ denote the inradius of the planar convex body K .

Theorem 6. *For any planar convex body K , $L(K) \geq 6r(K)$.*

Proof. We may suppose $r(K) = 1$, and \mathbb{B}_2 to be inscribed in K .

Let G be a cage holding K , and set $G_0 = G \cap \text{aff}K$. Denote by v_1, \dots, v_m the vertices of G , and write $w_i = \pi_{\text{aff}K}(v_i)$. Take the labelling so that w_1, \dots, w_n are the external vertices of $\pi_{\text{aff}K}(G)$.

Case 1. $\mathbf{0} \in \text{conv}G_0$.

By the Carathéodory Theorem, there exist $a, b, c \in G_0$ such that $\mathbf{0} \in abc$. Clearly,

$$\mathbf{0} \in \pi_{\mathbb{B}_2}(a)\pi_{\mathbb{B}_2}(b)\pi_{\mathbb{B}_2}(c),$$

too. By successively using Theorem 2, Corollary 2.5, and Lemmas 2.3 and 2.6, we obtain

$$\lambda G \geq f(w_1, \dots, w_n) \geq f(a, b, c) \geq f(\pi_{\mathbb{B}_2}(a), \pi_{\mathbb{B}_2}(b), \pi_{\mathbb{B}_2}(c)) \geq 6.$$

Case 2. $\mathbf{0} \notin \text{conv}G_0$.

We claim that $\text{diam}G > 2$, yielding the result since G is 3-connected.

Consider the arc $\mathbb{S}_1 \cap \text{conv}(\{\mathbf{0}\} \cup G_0)$, of endpoints α, β . Take $a \in \overline{\mathbf{0}\alpha} \cap G_0$, $b \in \overline{\mathbf{0}\beta} \cap G_0$, and consider the diameter $c(-c)$ of \mathbb{S}_1 , parallel to \overline{ab} . We obtain the trapezoid $ab(-c)c$. There are two half-lines starting at a and supporting K , one of which, say L_a , meets $\overline{c(-c)}$ at some point a' , with $c \in \mathbf{0}a'$. Analogously, some supporting half-line L_b of K from b meets $\overline{c(-c)}$ at b' , with $-c \in \mathbf{0}b'$. If $\|a - b\| \leq 2$ then $\|a' - b'\| \geq \|a - b\|$ and K could escape from the cage via $\text{conv}(L_a \cup L_b \cup K)$, which contradicts the assumption. Hence, indeed, $\text{diam}G \geq \text{diam}G_0 \geq \|a - b\| > 2$. \square

The next statement brings a link between points almost fixing a planar convex body and cages holding that body.

Theorem 7. *Let K be a planar convex body in \mathbb{R}^3 and $a, b, c \in \text{relbd}K$.*

(i) *If a, b almost fix K , then $L(K) \leq f(a, b)$.*

(ii) *If a, b, c almost fix K , then $L(K) \leq f(a, b, c)$.*

Remark. Example 3.1 shows that the statement of Theorem 7 would be false with four points instead of two or three. Actually, it can be shown that no cage holding the parallelogram $K = a_0b_0c_0d_0$ of Figure 5 can be approximated by the parallelogram ab_0cd_0 plus its Steiner tree $d_0a \cup ac \cup cb_0$.

Question 4.1. *Is there a planar convex body K in \mathbb{R}^3 satisfying*

$$L(K) < \inf\{f(a, b, c) ; a, b, c \text{ almost fix } K\}?$$

Proof of Theorem 7. (i) The plane $\Pi_0 = \text{aff}K$ is thought to be the horizontal plane. Let a', a'' be close to a and b', b'' close to b such that a', a'', b', b'' fix K . These points are labelled such that the quadrilateral $a'a''b'b''$ is convex. Consider the plane Π passing through $a''b'$, making an angle, say, $\frac{\pi}{4}$ with Π_0 , with $b''a'$ below it. Similarly, let Π' be the plane passing through $b''a'$, making an angle $\frac{\pi}{4}$ with Π_0 , and with $a''b'$ below it. (Since $\lambda a'a''$ and $\lambda b'b''$ are small compared with $\lambda a''b'$ and $\lambda b''a'$, these planes are almost perpendicular to each other.) The planes Π and Π'

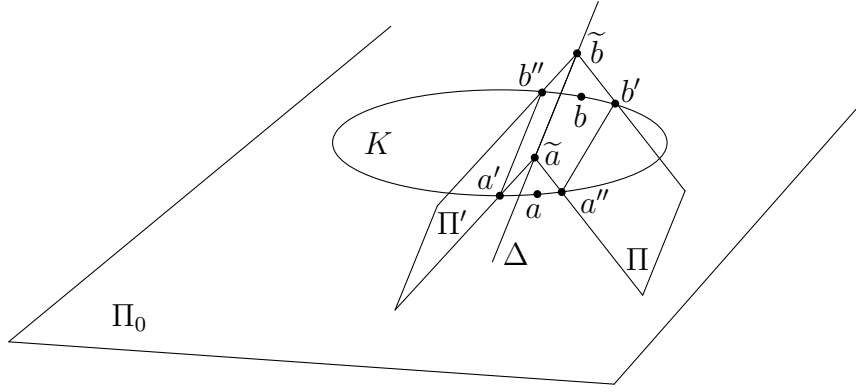


Figure 6: A cage holding K in the case (i).

cross along a straight line Δ such that the projection $\pi_{\Pi_0}(\Delta)$ crosses the quadrilateral $a'a''b'b''$ at two points $a_0 \in a'a''$ and $b_0 \in b'b''$. These points are projections of some points $\tilde{a}, \tilde{b} \in \Delta$. We now make a dilation D_γ of centre the middle of $\tilde{a}\tilde{b}$ and factor $\gamma > 1$, γ arbitrarily close to 1. Then the 1-skeleton of $D_\gamma(a'a''\tilde{a}b'b''\tilde{b})$ holds K if γ is close enough to 1, and the length of this cage is arbitrarily close to

$$|a'b''| + |a''b'| + |\tilde{a}\tilde{b}| + |a'a''| + |a''\tilde{a}| + |\tilde{a}a'| + |b'b''| + |b''\tilde{b}| + |\tilde{b}b'|.$$

Because this can be done for any a', a'', b', b'' fixing K , a', a'' close to a and b', b'' close to b , this sum can be made arbitrarily close to $f(a, b) = 3|ab|$ and the result follows.

(ii) If a, b, c are collinear with $c \in ab$, then we prove that a, b almost fix K , see Fig. 7. Actually, we have $ab \subset \text{relbd}K$. Choose the orientation such that the closure of $(ab)^+$ contains K . (Recall that $(ab)^+$ is the open half-plane on the left of the line \overline{ab} oriented from a to b .) Then \overline{ab} is the right tangent of $\text{relbd}K$ at a and the left tangent at b . Denote by A the left tangent at a and by B the right tangent at b . Since a, b, c almost fix K , we have that, either A and

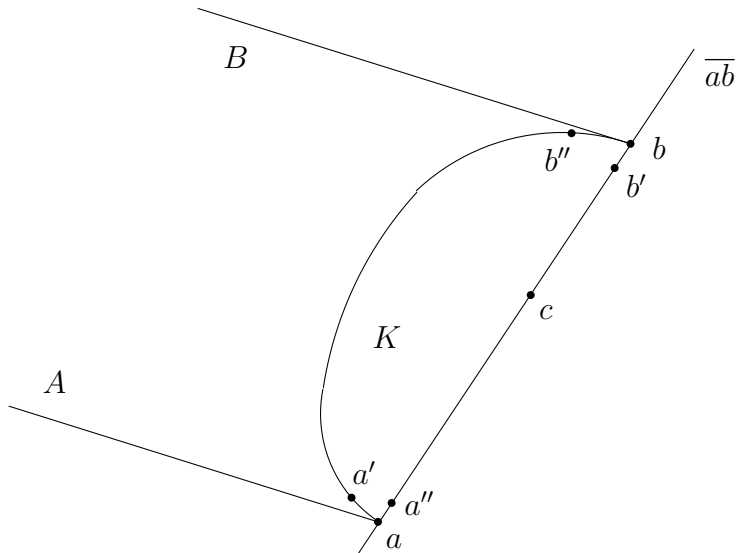


Figure 7: If a, b, c are collinear and almost fix K then a, b too.

B cross in $(ab)^+$, or they are parallel. In the latter case we also have $A \cap \text{relbd}K = \{a\}$ or $B \cap \text{relbd}K = \{b\}$ (or both), otherwise a, b, c would not almost fix K . It follows that, for any

$a' \in (ab)^+ \cap \text{relbd}K, a'' \in]ab[, a', a''$ close to a and $b' \in]ab[, b'' \in (ab)^+ \cap \text{relbd}K, b', b''$ close to b , the points a', a'', b', b'' fix K . This proves that a, b almost fix K , and case (i) applies, since $f(a, b, c) = f(a, b)$.

We now assume that a, b, c are not collinear. Let $d = t(a, b, c)$ be the Fermat-Torricelli point of a, b, c ; we first assume that d is distinct from a, b and c .

For any arbitrarily small $\varepsilon > 0$, let $a', a'', b', b'', c', c'' \in \text{relbd}K$ be six points fixing K , labelled in that cyclic order on $\text{relbd}K$, with $|aa'|, \dots, |cc''| < \varepsilon$. Let $\tilde{a} = \overline{a''b'} \cap \overline{c''a'}$. For ε small enough, we have $|\tilde{a}\tilde{c}| \leq |aa'| + |a'\tilde{a}| \leq \varepsilon + \frac{|a'a''|}{\sin(\angle a''\tilde{a}a')} \leq \frac{3\varepsilon}{\sin(\angle cab)}$ and similarly $|\tilde{b}\tilde{b}|, |\tilde{c}\tilde{c}| = \mathcal{O}(\varepsilon)$.

Let \tilde{u} be the unit vector orthogonal to $\Pi_0 = \text{aff}K$ pointing upward, and put $\tilde{d} = d + \varepsilon\tilde{u}$. Consider the tetrahedron $T = \tilde{a}\tilde{b}\tilde{c}\tilde{d}$. We chop T by three planes Π_a, Π_b, Π_c making an angle of $\frac{\pi}{4}$ with Π_0 , with \tilde{d} under them, Π_a containing $a'a''$, Π_b containing $b'b''$ and Π_c containing $c'c''$. This gives a heptahedron P , with three triangular faces $a'a''a''' \subset \Pi_a, b'b''b''' \subset \Pi_b, c'c''c''' \subset \Pi_c$, with $|aa''|, |bb''|, |cc''| = \mathcal{O}(\varepsilon)$, three pentagonal faces $\tilde{d}a'''a''b'b''$, $\tilde{d}b'''b''c'c''$ and $\tilde{d}c'''c''a'a''$, and a hexagonal face $a'a''b'b''c'c''$, see Fig. 8.

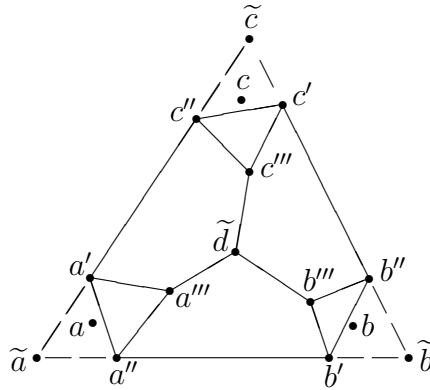


Figure 8: A cage holding K in the case (ii).

We now make a dilation $P_\gamma = D_\gamma(P)$ of P , of centre \tilde{d} and factor $\gamma > 1$, γ arbitrarily close to 1. Then, if γ is close enough to 1, the 1-skeleton of P_γ is a cage holding K and of length $(f(a, b, c) + \mathcal{O}(\varepsilon))\gamma$, hence arbitrarily close to $f(a, b, c)$.

The case where $t(a, b, c)$ is one of the points, say $t(a, b, c) = a$, goes similarly: We choose for d the point on the bisector of $\angle bac$ at the distance $\sqrt{\varepsilon}$ from a . The rest of the proof is the same. \square

We say that the convex body K is *weakly strictly convex* if it possesses two parallel supporting hyperplanes H, H' at distance $\text{wid}K$ from each other, such that $H \cap K$ is a single point.

A consequence of Theorem 7 is the following.

Corollary 4.2. *For any planar weakly strictly convex body K in \mathbb{R}^3 , we have $L(K) \leq 3\text{wid}K$.*

Proof. Take the two supporting lines H, H' given by the definition of weakly strict convexity, and the point $\{a\} = H \cap K$. Then the orthogonal projection b of a onto H' belongs to $H' \cap K$. With $\varepsilon > 0$ small, consider the two lines parallel to \overline{ab} and at distance ε from \overline{ab} . These lines cut $\text{relbd}K$ in four points a', a'', b', b'' which fix K . This proves that a, b almost fix K and Theorem 7 (i) applies. \square

An immediate consequence of Corollary 4.2 and Theorem 6 is the following.

Corollary 4.3. *Let K be a planar weakly strictly convex body. Assume that K contains a disc of diameter $\text{wid}K$. Then we have $L(K) = 3\text{wid}K$.*

In particular for the unit two-dimensional disc \mathbb{B}_2 , we have $L(\mathbb{B}_2) = 6$.

Corollary 4.3 also follows from the fact that any cage with \mathbb{S}_2 as circumscribed sphere has length more than 6. This was proven by Lillington [8], and generalized to higher dimensions by Linhart [9] and further to 1-skeletons of arbitrary convex bodies by Schneider [10].

A parallelogram will be called *acute* if, at each of its vertices, the angles between the diagonal and the sides are acute.

Theorem 8. (i) *If $T = abc$ is a triangle of sides $I_1 = bc, I_2 = ca, I_3 = ab$, then*

$$L(T) = \min\{f(x_1, x_2, x_3) ; x_k \in I_k\}.$$

In particular, for the equilateral triangle \mathbb{T}_2 of unit edge length, we have $L(\mathbb{T}_2) = \frac{3+\sqrt{3}}{2}$.

(ii) *Let K be a parallelogram of diagonal lengths δ, Δ with $\delta \leq \Delta$. Then we have $L(K) \geq 3\delta$. Moreover, if K is acute, then $L(K) = 3\delta$. In particular, for the unit square \mathbb{C}_2 , we have $L(\mathbb{C}_2) = 3\sqrt{2}$.*

Proof. (i) Since f is continuous and $I_1 \times I_2 \times I_3$ is compact, $\inf\{f(x_1, x_2, x_3) ; x_k \in I_k\}$ is reached at some point (u, v, w) . Two cases occur: Either each point is in the relative interior of its side, or two points, are at a vertex, say, $u = v = c$, and the third one w is at the foot of the corresponding height. This second case occurs for all non-acute triangles; it can be seen that it occurs also for some acute ones, if the largest angle is close enough to $\frac{\pi}{2}$.

In the first case, the three normals at u, v, w are concurrent by Theorem 3, hence the points u, v, w fix the triangle T . In the second case, we choose u', v' arbitrarily close to c such that the three normals at u', v', w are concurrent, and the points u', v', w fix T . This proves that c, w almost fix T . Then Theorem 7 applies, yielding $L(T) \leq f(u, v, w)$.

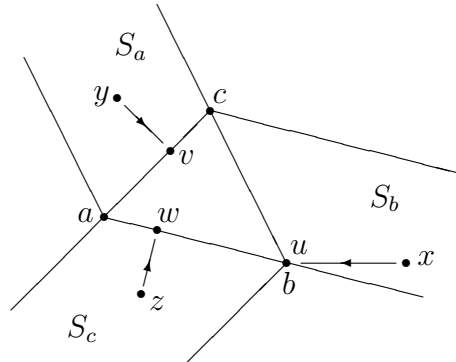


Figure 9: Proof of Theorem 8 (i).

Conversely, let G be a cage holding T . Let $P = \text{aff}T$ and consider the geometric graph $\pi_P(G)$, i.e. the projection of the cage G in the plane P . Let v_1, \dots, v_n denote the external vertices of $\pi_P(G)$.

In the plane P , let S_a denote the half-strip not containing a , bounded by the side bc and the two rays parallel to ab starting from b and from c . Then G intersects S_a at some point x , otherwise T would escape from G by a translation inside S_a .

Similarly let S_b and S_c denote the analogous half-strips for b and c . Then G intersects S_b at some point y and S_c at some point z . Now consider the projections on T : $u = \pi_T(x), y = \pi_T(y)$ and $w = \pi_T(z)$. They satisfy $u \in I_1, v \in I_2$, and $w \in I_3$. By respectively using Lemma 2.3, Corollary 2.5, and Theorem 2, we obtain

$$f(u, v, w) \leq f(x, y, z) \leq f(v_1, \dots, v_n) \leq \lambda G.$$

Lemma 2.7 proves the equality for the equilateral triangle.

(ii) Let G be a cage holding the parallelogram $K = abcd$, with $|ac| = \delta \leq |bd| = \Delta$, let $P = \text{aff}K$, and let v_1, \dots, v_n denote the external points of $\pi_P(G)$. In the plane P , let S be the half-strip not containing K determined by \overline{da} , ab , and \overline{bc} . Then G intersects S at some point m , otherwise K would escape from G by a translation inside S . Let H be the open

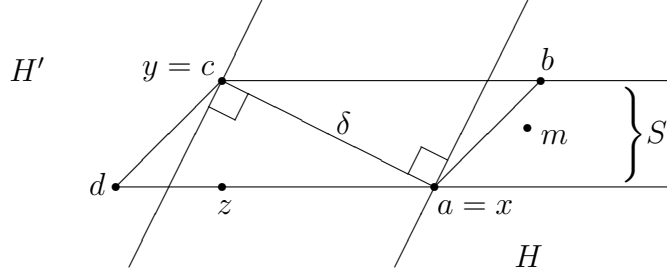


Figure 10: Proof of Theorem 8 (ii).

half-plane not containing c bounded by the straight line containing a and orthogonal to \overline{ac} , i.e. $H = \{x \in \mathbb{R}^2 ; \angle cax \text{ is obtuse} \}$. Since H contains S and since m is in the convex hull of the points v_1, \dots, v_n , one of these points, v_k , belongs to H . In the same manner, one v_l belongs to the symmetric half-plane $H' = \{x \in \mathbb{R}^2 ; \angle acx \text{ is obtuse} \}$. We then obtain

$$\lambda G \geq f(v_1, \dots, v_n) \geq f(v_k, v_l) = 3|v_k v_l| \geq 3|ac| = 3\delta.$$

For the ‘moreover’ part, if K is acute, then the angle $\angle dca$ is acute, hence the vertices a and c almost fix K , giving $L(K) = 3\delta$ by Theorem 7 (i). The case of the unit square \mathbb{C}_2 is clear. \square

Remark. In the case of a nonacute parallelogram, it is easy to prove that the vertices a and c do not almost fix K any more and that $\inf\{f(x, y, z) ; x, y, z \text{ almost fix } K\}$ is attained e.g. for $x = a$, $y = c$ and $z = \pi_{\overline{da}}(c)$, whereas $\inf\{f(x, y) ; x, y \text{ almost fix } K\}$ is attained only for $x = b$, $y = d$ and is larger. By Theorem 7 (ii), we then obtain $L(K) \leq f(a, c, \pi_{\overline{da}}(c))$. We conjecture that this is an equality.

5 Cages for convex bodies in \mathbb{R}^3

As mentioned in the Introduction, Coxeter’s problem from 1959 was completely solved, as follows.

Proposition 5.1. (Besicovitch, Aberth). *For the unit ball \mathbb{B}_3 , we have $L(\mathbb{B}_3) = \frac{8\pi}{3} + 2\sqrt{3}$.*

Denoting the number from the statement of Proposition 5.1 by γ , Besicovitch [2] provided a cage G with $\lambda G > \gamma$ arbitrarily close to γ , while Aberth [1] proved that $\lambda G \leq \gamma$ for no cage G .

Theorem 9. *For any $K \in \mathcal{K}$, we have $L(K) \leq 2\pi \text{diam}K$. If K is strictly convex, then $L(K) \leq 4(\text{diam}K + \text{wid}K)$.*

Proof. Suppose $K \in \mathcal{K}$ is strictly convex. Let Π and Π' be two parallel supporting planes of K at distance $\text{wid}K$ from each other, and let a, a' be the contact points, respectively. Consider a plane Ψ through a and a' , and let $K' = K \cap \Psi$. Approximate $\text{relbd}K'$ by a polygon P exterior to and disjoint from K' , having two sides α, α' parallel to Π , close to a, a' respectively. Take a vector v orthogonal to Ψ , and consider the prism

$$Q_v = \text{conv}((P + v) \cup (P - v)).$$

For $\|v\|$ small enough, $K \cap \text{skel}Q_v = \emptyset$.

All faces of Q_v except $(\text{conv}P) \pm v$ are rectangles. We chop Q_v using planes parallel to these faces, such that the new faces (parallel to the old) now touch K . The number of faces may decrease in the process. The new faces A and A' corresponding to the old faces of G_v containing α and α' lie in Π and Π' , respectively. If xy is the line-segment $A \cap \Psi$, then $\text{relbd}A = x_+y_+y_-x_-$, where $x_{\pm} = x \pm v$, $y_{\pm} = y \pm v$. Obviously, xy is parallel to α and contains a .

We chop again the already chopped Q_v by a plane parallel to \overline{xy} (but not necessarily to Π), close to A , to replace both edges x_+y_+ and x_-y_- by edges $x_+^*y_+^*$ and $x_-^*y_-^*$ touching K . Then, clearly, the new face $\text{conv}x_+^*y_+^*y_-^*x_-^*$ cuts K . The same procedure is applied to A' .

The resulting polytope Q'_v is not necessarily a prism any more, but $\text{skel}Q'_v$ tightly holds K . Put $P' = Q'_v \cap \Psi$.

Let $\varepsilon > 0$. By taking $\|v\|$ suitably small, we have $\lambda \text{skel}Q'_v < 2\lambda P' + \varepsilon$. But the polygon P' approximates $\text{relbd}K'$. Let R be the smallest rectangle in Ψ with two sides in Π, Π' , such that $\text{conv}R \supset K'$. This rectangle touches K' in a, a' and two further points b, b' . Two sides of R have length $|aa'| = \text{wid}K$, the other two have length at most $|bb'| \leq \text{diam}K$. Moreover, $\text{conv}R \supset P'$. Hence,

$$\lambda P' \leq \lambda R \leq 2(\text{diam}K + \text{wid}K).$$

Thus, $\lambda \text{skel}Q'_v < 4(\text{diam}K + \text{wid}K) + \varepsilon$, for every $\varepsilon > 0$, proving the second inequality of the statement.

The proof of the first inequality of the statement goes similarly, with just three modifications. Instead of starting with the points a, a' , we start with the endpoints b, b' of a diameter. Then b and b' are exposed points of K even if K is not strictly convex. One more modification consists in taking Ψ not arbitrarily through b, b' , but containing a third point, interior to K . The third modification concerns λP . It is well-known that K' is included in a (planar) convex body K'' of constant width equal to $\text{diam}K' = |bb'| = \text{diam}K$. Moreover,

$$\lambda \text{relbd}K' \leq \lambda \text{relbd}K'' = \pi \text{diam}K.$$

Farther, this leads to the first inequality. □

Consider now the regular tetrahedron $\mathbb{T}_3 = abcd$ of unit edge length, a cage G holding \mathbb{T}_3 , and the projection function $\pi_{\mathbb{T}_3}$ on \mathbb{T}_3 .

Lemma 5.2. *We have $\pi_{\mathbb{T}_3}(G) \cap (ab \cup ac) \neq \emptyset$.*

Proof. Suppose the intersection in the statement is empty. Imagine $P_0 := \text{aff}abc$ to be horizontal and d above it. Denote by P_0^- the half-space below, i.e. bounded by P_0 and not containing d . Let Π_a be the vertical plane (i.e. orthogonal to $\text{aff}abc$) which includes bc . Similarly, let Π_b and Π_c be the vertical planes which include respectively ac and ab , see Fig 11.

Denote by Π_a^+ the half space containing a and bounded by Π_a . We have

$$G \cap \Pi_b \cap \Pi_a^+ \cap P_0^- = G \cap \Pi_c \cap \Pi_a^+ \cap P_0^- = G \cap P_0 \cap \Pi_a^+ \setminus abc = \emptyset.$$

Since G holds \mathbb{T}_3 , G contains a point x below $\text{aff}abc$, in the triangular prism $\Delta = \pi_{\mathbb{T}_3}^{-1}(\text{int}abc)$. We claim that this prism also contains a *vertex* of G . To see this, take $x \in G \cap \Delta$ to be farthest from Π ; this is possible by compactness of G and because the two boundaries of Δ which are parts of Π_b and Π_c do not cross G . If x is not a vertex of G , then the side e of G to which x belongs must have an endpoint v below $\text{aff}abc$. If $v \notin \Pi^+$, then the other endpoint v' of e equals x (otherwise $\pi_{\mathbb{T}_3}(v') \in ab \cup ac$). If $v \in \Pi^+$, then both endpoints v, v' of e lie in Π^+ , and $x \in \{v, v'\}$, or e is parallel to \overline{bc} and x can be chosen in $\{v, v'\}$.

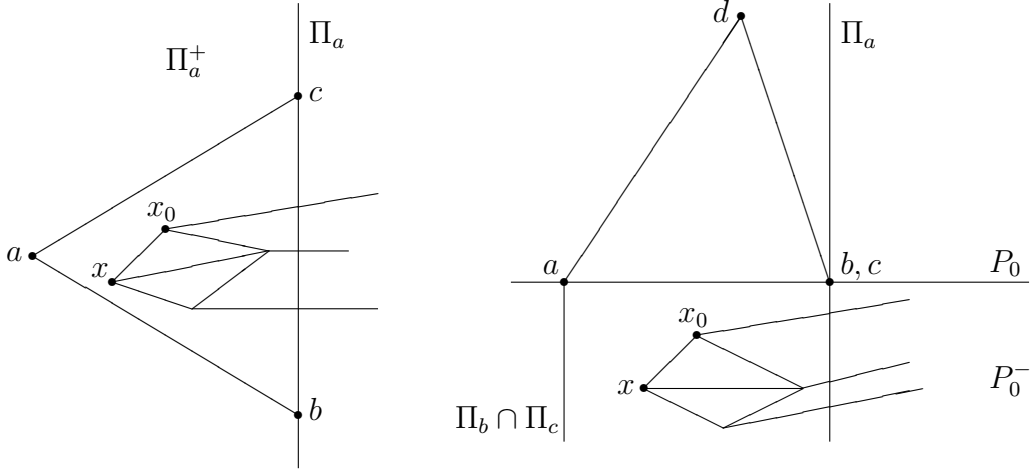


Figure 11: Proof of Lemma 5.2. Left: top view, right: front view.

Now, among all vertices of $\text{conv}G$ which are in Δ , choose one x_0 such that the angle between x_0bc and abc is minimal. Then $\text{aff}x_0bc$ separates $\text{int}\mathbb{T}_3$ from $\text{int}\text{conv}G$. Thus, G cannot hold \mathbb{T}_3 , the seeked contradiction. \square

Theorem 10. *For the regular tetrahedron \mathbb{T}_3 of unit edge length, we have $L(\mathbb{T}_3) = 3$.*

Proof. Let $\mathbb{T}_3 = abcd$. A plane parallel and close (at distance η) to ab cuts \mathbb{T}_3 along a rectangle R . Any prism P whose 1-skeleton approximates R and is disjoint from \mathbb{T}_3 , holds \mathbb{T}_3 . As $\eta \rightarrow 0$, the length of six sides of P tends to 0, while the length of each of the remaining three tends to 1. Hence, $L(\mathbb{T}_3) \leq 3$.

Conversely, let G be a cage holding \mathbb{T}_3 . If $\pi_{\mathbb{T}_3}(G)$ does not meet some side e of \mathbb{T}_3 , let e' denote the side opposite to e ; if $\pi_{\mathbb{T}_3}(G)$ meets all sides, choose opposite e, e' arbitrarily.

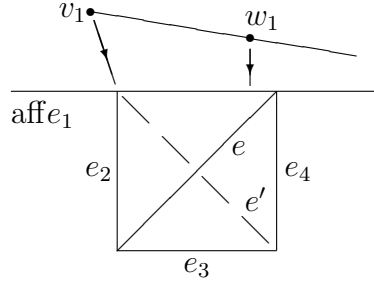


Figure 12: Proof of Theorem 10.

By Lemma 5.2, there exist $w_1, w_2, w_3, w_4 \in G$ respectively projecting via $\pi_{\mathbb{T}_3}$ into the four sides different from e and e' , which we denote by e_1, \dots, e_4 .

Consider the projection function π_P onto a plane P parallel to e and e' . Then $\pi_P(\mathbb{T}_3)$ is a square C , of side length $\frac{1}{\sqrt{2}}$ and sides I_1, \dots, I_4 , with $I_i = \pi_P(e_i)$, see Fig. 12.

Of course, $\pi_P(w_i) \notin \text{int}C$, for $i = 1, \dots, 4$, otherwise $\pi_{\mathbb{T}_3}(w_i)$ would be on a face of \mathbb{T}_3 instead of the side e_i .

We have $\pi_C(\pi_P(w_1)) \in I_1$, hence $\pi_P(w_1)$ is in the closed half-plane bounded by $\text{aff}e_1$ and not containing $\text{int}C$. Therefore there exists an external vertex v_1 of $\pi_P(G)$ in the same half-plane, hence also satisfying $\pi_C(v_1) \in I_1$. We do the same with w_2, w_3 and w_4 . This gives four external vertices v_1, \dots, v_4 of $\pi_P(G)$, with possible coincidences, such that $\pi_C(v_i) \in I_i$. By using

Theorem 2 and Lemmas 2.3 and 2.8, we then obtain

$$\lambda G \geq f(v_1, \dots, v_4) \geq f(\pi_C(v_1), \dots, \pi_C(v_4)) \geq \inf_{u_i \in I_i} f(u_1, \dots, u_4) = 3.$$

□

Theorem 11. *For the cube \mathbb{C}_3 of unit edge length, we have $L(\mathbb{C}_3) \leq 4 + 3\sqrt{2}$.*

Proof. Put $\mathbb{C}_3 = abcda^*b^*c^*d^*$, see Fig. 13. Take $\varepsilon > 0$, and let $a_b \in ab$ and $a_d \in ad$ be at distance ε^2 from \overline{ac} . Let $L \subset \text{aff}acc^*a^*$ be a line outside \mathbb{C}_3 , parallel to \overline{ac} , at distance ε from it. A plane parallel to aa^* passing through a_b, a_d cuts L at a' . Let

$$\{a_b^*\} = \overline{a'a_b} \cap \text{aff}a^*b^*c^*d^* \quad \text{and} \quad \{a_d^*\} = \overline{a'a_d} \cap \text{aff}a^*b^*c^*d^*.$$

Analogously, construct c', c_b^*, c_d^* . The 1-skeleton S_ε of the triangular prism $a'a_b^*a_d^*c'c_b^*c_d^*$ immobilises \mathbb{C}_3 .

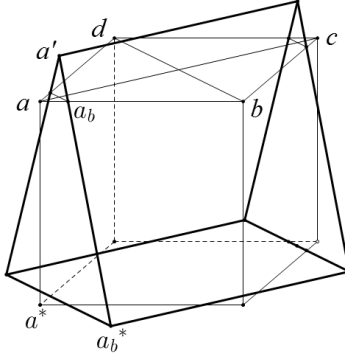


Figure 13: Proof of Theorem 11.

In order to calculate $L(\mathbb{C}_3)$, put $x = |a_b^*a_d^*|/2$ and observe that

$$\frac{\varepsilon}{1 + \varepsilon} = \frac{\varepsilon^2}{x},$$

whence $x = \varepsilon + \varepsilon^2$. Therefore, $\varepsilon \rightarrow 0$ implies $|a_b^*a_d^*| \rightarrow 0$. Moreover, it implies $|a'a_b^*| \rightarrow 1$ and

$$|a'c'| = |a_b^*c_b^*| = |a_d^*c_d^*| \rightarrow \sqrt{2}.$$

This verifies the bound from the statement. □

Compared with the bounds given by Theorem 9, namely 2π for the regular tetrahedron, and $2\pi\sqrt{3}$ for the cube, the exact value 3 in Theorem 10 and the bound $4 + 3\sqrt{2}$ in Theorem 11 are better. We even dare to formulate the following.

Conjecture. $L(\mathbb{C}_3) = 4 + 3\sqrt{2}$.

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