Local null controllability of a rigid body moving into a Boussinesq flow

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Abstract. In this paper, we study the controllability of a fluid-structure interaction system. We consider a viscous and incompressible fluid modeled by the Boussinesq system and the structure is a rigid body with arbitrary shape which satisfies Newton's laws of motion. We assume that the motion of this system is bidimensional in space. We prove the local null controllability for the velocity and temperature of the fluid and for the position and velocity of rigid body for a control acting only on the temperature equation on a fixed subset of the fluid domain.

Key words. Controllability, Fluid-structure interaction, Navier-Stokes equations, Boussinesq system, Rigid body, Carleman inequality
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1. Introduction and main result

Let Ω be a bounded, nonempty, open subset of $\mathbb{R}^2$ with C^2 boundary that contains a rigid body and a viscous incompressible fluid. The domain of the rigid body is denoted by $\mathcal{S}(t) \subset \Omega$ and it is assumed to be of class C^2, compact, simply connected and with non-empty interior. The fluid domain is denoted by $\mathcal{F}(t) = \Omega \backslash \mathcal{S}(t)$ and it is assumed to be connected. Since, we assume that the structure is a rigid solid, we can describe $\mathcal{S}(t)$ with two functions $t \mapsto h(t) \in \mathbb{R}^2$ and $t \mapsto \beta(t) \in \mathbb{R}$ through the formulas

$$\mathcal{S}(t) = \mathcal{S}_{h(t),\beta(t)}, \quad \mathcal{F}(t) = \mathcal{F}_{h(t),\beta(t)}. \tag{1.1}$$

In the above relations and in what follows, we write for any $h \in \mathbb{R}^2$ and for any $\beta \in \mathbb{R}$,

$$\mathcal{S}_{h,\beta} = h + R_\beta \mathcal{S} \quad \text{and} \quad \mathcal{F}_{h,\beta} = \Omega \backslash \mathcal{S}_{h,\beta}, \tag{1.2}$$

where $\mathcal{S}$ is a fixed subset of $\mathbb{R}^2$ of class C^2, compact, simply connected and with non-empty interior. In (1.2), R_β is the rotation matrix, defined by

$$R_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}. \tag{1.3}$$

We assume that there exist $h_0 \in \mathbb{R}^2$, $\beta_0 \in \mathbb{R}$ such that

$$\mathcal{S}_{h_0,\beta_0} \subset \Omega.$$

Without loss of generality, we can assume that the center of gravity of $\mathcal{S}$ is at the origin. In that case, $h(t)$ is the position of the centre of mass of the rigid body.

Let $\mathcal{O}$ be an open subset with $\overline{\mathcal{O}} \subset \Omega$. The fluid-rigid body system is controlled by a force field supported in $\mathcal{O}$ and we suppose that $\overline{\mathcal{O}} \subset \mathcal{F}(t)$.

We shall assume that the motion of the fluid is described by the Boussinesq approximation. The fluid is treated as incompressible when formulating the Navier-Stokes mass and momentum conservation equations and here the effect of temperature change is taken into account. The motion of the rigid body is governed by the balance equations for linear and angular momentum.

The equations of motion of fluid-structure are:

$$\frac{\partial \widehat{u}}{\partial t} + (\widehat{u}\cdot\nabla)\widehat{u} - \nu\Delta\widehat{u} + \nabla\widehat{p} = \widehat{\theta}e_2, \qquad t\in(0,T),\ x\in\mathcal{F}(t), \tag{1.4}$$

$$\text{div } \widehat{u} = 0, \qquad t\in(0,T),\ x\in\mathcal{F}(t), \tag{1.5}$$

$$\widehat{u}(t,x) = 0, \qquad t\in(0,T),\ x\in\partial\Omega, \tag{1.6}$$

$$\widehat{u}(t,x) = h'(t) + \beta'(t)(x-h(t))^{\perp}, \qquad t\in(0,T),\ y\in\partial\mathcal{S}(t), \tag{1.7}$$

$$\frac{\partial \widehat{\theta}}{\partial t} + \widehat{u}\cdot\nabla\widehat{\theta} - \mu\Delta\widehat{\theta} = w_0\mathbb{1}_{\mathcal{O}}, \qquad t\in(0,T),\ x\in\mathcal{F}(t), \tag{1.8}$$

$$\frac{\partial \widehat{\theta}}{\partial \widehat{n}}(t,x) = 0, \qquad t\in(0,T),\ x\in\partial\mathcal{F}(t), \tag{1.9}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(\widehat{u},\widehat{p})\widehat{n}d\Gamma, \qquad t\in(0,T), \tag{1.10}$$

$$J\beta''(t) = -\int_{\partial\mathcal{S}(t)} (x-h(t))^{\perp}\cdot\sigma(\widehat{u},\widehat{p})\widehat{n}d\Gamma, \qquad t\in(0,T), \tag{1.11}$$

$$\widehat{u}(0,x) = \widehat{u}_0(x), \quad \widehat{\theta}(0,x) = \widehat{\theta}_0(x), \qquad x\in\mathcal{F}(0), \tag{1.12}$$

$$h(0) = h_0,\ \beta(0) = \beta_0,\ h'(0) = \widehat{\ell_0},\ \beta'(0) = \widehat{\omega_0}. \tag{1.13}$$

In the above system, $\widehat{u}(t,y)$ is the velocity field of the fluid, $\widehat{p}(t,y)$ denotes the pressure of the fluid and $\widehat{\theta}(t,y)$ is the temperature. Here $\nu>0$ is the kinematic viscosity and $\mu>0$ is the thermal diffusivity. For all $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$, we denote by $x^{\perp}$, the vector $\begin{pmatrix}-x_2\\x_1\end{pmatrix}$. Moreover the boundaries of the rigid body and fluid domain are denoted by $\partial\mathcal{S}(t)$ and $\partial\mathcal{F}(t)$ respectively. The outward unit normal to $\partial\mathcal{F}(t)$ is denoted by $\widehat{n}(t,x)$. The constants M and J are the mass and the moment of inertia of the rigid body. For the sake of convenience, we will assume that the rigid body is homogeneous with a constant density $\rho_{\mathcal{S}}\in\mathbb{R}_+^*$ and thus we have

$$M = \rho_{\mathcal{S}}|\mathcal{S}|, \quad J = \rho_{\mathcal{S}}\int_{\mathcal{S}}|y|^2\,dy.$$

The Cauchy stress tensor is defined as:

$$\sigma(\widehat{u},\widehat{p}) = -\widehat{p}I_2 + 2\nu D(\widehat{u}),$$

where $D(\widehat{u})$ is the symmetric gradient:

$$D(\widehat{u})_{i,j} = \frac{1}{2}\left(\frac{\partial\widehat{u}_i}{\partial x_j} + \frac{\partial\widehat{u}_j}{\partial x_i}\right).$$

The state of system (1.4)-(1.13) is $(\widehat{u},\widehat{p},\widehat{\theta},h,\beta)$ and we want to emphasize the fact that the domains $\mathcal{F}(t)$ and $\mathcal{S}(t)$ are depending on the state and thus evolve through the dynamics induced by the system (1.10)-(1.11). This is one of the main difficulties in this problem: we are working on a non cylindrical domain and the spatial domain is unknown. A standard tool to

handle this difficulty consists in using a change of variables in order to rewrite the system in a cylindrical domain. We need however to take care that such a change of variables is constructed from the state and this leads to some technical estimates on the coefficients coming from this transformation.

Several studies on the existence of weak solutions or strong solutions of fluid-structure interaction system have been published in recent years, usually without the equation on the temperature. The stationary problem was studied in Serre [39] and in Galdi [23]. An existence result of strong solutions in two or three dimension was proved in Grandmont and Maday [26] under the assumption that the inertia of the rigid body is large enough with respect to the inertia of the fluid. The existence and uniqueness of strong solutions in the case of a bounded domain has been proved in [40] without the hypothesis of [26] about the inertia of the rigid body. In the case of whole space, existence and uniqueness of strong solutions in two dimensions have been proved by Takahashi and Tucsnak [41] for an infinite cylinder and a similar result has been proved in three dimension by Silvestre and Galdi [24] for a rigid body having an arbitrary form. The question of existence of weak solutions has been investigated by many authors: [11], [7], [38], [15], [14], [28] etc. We can also mention a result on existence of weak solutions of the case where the fluid motion is modeled by the Boussinesq system: in [35], Nečasová proved the existence of weak solutions in three dimension for the problem of motion of one or several rigid bodies immersed in an incompressible non-Newtonian and heat-conducting fluid.

The controllability of the Navier-Stokes system has been the objective of considerable work over the last years. In the case of the two dimensional incompressible Navier-Stokes equations with the Navier slip boundary conditions, an approximate controllability result for boundary or distributed controls was proved by Coron in [8] and local exact controllability was established by Imanuvilov in [30]. In [18] and [31] the authors obtained the local exact controllability of the $2D$ or $3D$ Navier-Stokes equations with Dirichlet boundary condition with distributed controls supported in a small subset. They established a new Carleman inequality for the linearized Navier-Stokes system, which leads to null controllability and then they deduced a local result concerning the exact controllability. Fursikov and Imanuvilov established the local exact boundary controllability to the trajectories of the N dimensional Boussinesq system with $N+1$ scalar controls acting over the whole boundary and the local exact controllability to the same trajectories with $N+1$ scalar distributed controls when Ω is a torus in [20], [21], [22] by deducing a global Carleman estimate for the adjoint system. The techniques in [18] have been adapted in [27] to obtain the local exact controllability to the trajectories of the N dimensional Boussinesq systems with $N+1$ distributed scalar controls supported in subsets of the domain. In [25], the authors also establish same result as in [27] but via a method based on applying fictitious control on the divergence equation.

Here we want to emphasize that there have been many works in the literature where the authors deal with the controllability problem of Navier-Stokes type systems via reduced number of controls. In [19], the authors show that the N dimensional Navier-Stokes and Boussinesq systems can be controlled with only $N-1$ scalar controls under some geometrical assumptions on control domains. In [9], Coron and Guerrero established the null controllability of the N dimensional Stokes system with internal controls having one vanishing component with no condition imposed on the control domain. Local null controllability of the N dimensional Navier-Stokes and Boussinesq system with $N-1$ scalar controls in an arbitrary control domain

has been obtained in [6], [5]. Here we want to mention that in [19], [5] for Boussinesq system, the authors obtained the local exact controllability result with two vanishing components of velocity control. Let us mention that in [33], Lions and Zuazua showed that three dimensional Stokes system is not necessarily null controllable with two vanishing components for the control even if the control is distributed on the entire domain. But in [10], local null controllability of the three dimensional Navier-Stokes system with a control distributed in an arbitrarily small nonempty open subset having two vanishing components has been proved by Coron and Lissy by using the return method and a Gromov method.

There are few articles in the last decade concerning the controllability results on fluid-structure interaction problem. In a paper of Raymond and Vanninathan [37], they considered a simplified model in $2D$ where the fluid equations are replaced by the Helmholtz equations and the motion of a solid represented by a harmonic oscillator. In that case, the domain is supposed to be fixed but one of the difficulties comes from the fact that there is no control in the solid part. They established exact controllability results for this model with an internal control only in the fluid part. In the work of Doubova and Fernández-Cara [12], they proved the local null controllability by boundary controls for a 1D model where point mass is immersed in a fluid which evolves in $(-1, 1)$. In that case, the domain is not fixed any more and the proof of the result is based on the global null controllability of the linearized system (by Carleman estimates) and on Kakutani's fixed point theorem. In [29], the authors established exact controllability of a 2D fluid-structure system where the body is a ball. In the paper of Boulakia and Osses [4], the authors dealt with the same problem as in [29], except that the body can have more general shape. In [3], Boulakia and Guerrero proved the local null controllability of a fluid-solid interaction problem in three dimension. Finally, in [34], the authors studied the local null controllability problem for the simplified one dimensional model considered in [12] and they managed to reduce the number of controls.

Our aim in this article is to control the fluid-structure system (1.4)-(1.13). More precisely, we want to control the position of the rigid body, the velocities of the fluid and of rigid body and the temperature of the fluid at a given time $T > 0$. Our main result can be stated as follows:

Theorem 1.1. *Assume $T > 0$, $h_T \in \mathbb{R}^2$, and $\beta_T \in \mathbb{R}$ such that*

$$\overline{\mathcal{O}} \cap \mathcal{S}_{h_T,\beta_T} = \varnothing.$$

There exists $\varepsilon > 0$ such that for every

$$(\widehat{u}_0, \widehat{\theta}_0, h_0, \widehat{\ell}_0, \beta_0, \widehat{\omega_0}) \in \mathbf{H}^1(\mathcal{F}_{h_0,\beta_0}) \times H^1(\mathcal{F}_{h_0,\beta_0}) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$$

satisfying

$$\begin{aligned} \operatorname{div} \widehat{u}_0 &= 0 \qquad \text{in } \mathcal{F}_{h_0,\beta_0}, \\ \widehat{u}_0 &= 0 \qquad \text{on } \partial\Omega, \\ \widehat{u}_0(y) &= \widehat{\ell}_0 + \widehat{\omega_0}(y - h_0)^\perp \text{ for } y \in \partial\mathcal{S}_{h_0,\beta_0} \end{aligned} \tag{1.14}$$

and

$$\|\widehat{u}_0\|_{\mathbf{H}^1(\mathcal{F}_{h_0,\beta_0})} + \|\widehat{\theta}_0\|_{H^1(\mathcal{F}_{h_0,\beta_0})} + |h_0 - h_T| + |\widehat{\ell}_0| + |\beta_0 - \beta_T| + |\widehat{\omega_0}| < \varepsilon, \tag{1.15}$$

we can find a control $w_0 \in L^2(0,T;L^2(\mathcal{O}))$ *such that the solution of* (1.4)–(1.13) *satisfies*

$$\widehat{u}(T,\cdot)=0, \quad h'(T)=0, \quad \beta'(T)=0, \tag{1.16}$$

and

$$\widehat{\theta}(T,\cdot)=0, \quad h(T)=h_T, \quad \beta(T)=\beta_T. \tag{1.17}$$

Observe that by using a translation and a rotation we can always assume that

$$h_T=0 \quad \text{and} \quad \beta_T=0, \tag{1.18}$$

and thus

$$\mathcal{S}_{h_T,\beta_T}=\mathcal{S}, \quad \mathcal{F}_{h_T,\beta_T}=\mathcal{F}.$$

Therefore in what follows, we assume (1.18).

Our main result consists of the local null controllability of a fluid-structure system in dimension two by applying a control only on the temperature equation. In our knowledge, there are no results on the controllability of fluid-structure interaction problems that deal with reduced number of controls (that is, the number of controls is less that the number of equations). We use the same change of variables and similar type fixed point argument as in [29]. But, unlike [29], we have considered the Boussinesq system and we are interested in the controllability via reduced number of controls. In [5], the author proved the local exact controllability of the N-dimensional Boussinesq system with internal controls having two vanishing components in velocity control and the main tool is to use a suitable Carleman inequality. We also prove the main result by showing a Carleman estimate. In our case, we have to incorporate some terms due to the presence of rigid body.

This paper is organized as follows. In Section 2, we give the notation used in this paper and we recall some results. In Section 3, we introduce a change of variables to rewrite the problem (1.4)-(1.13) in a fixed spatial domain. In Section 4, we study the existence and regularity of a linearized problem in a fixed domain associated to our problem. Section 5 is devoted to establish a suitable Carleman inequality of the adjoint system of the linearized problem in a fixed domain. Then, in Section 6, we first give a link between controllability properties and Carleman estimates and then prove the controllability of an auxiliary linear system associated to (1.4)-(1.13). Finally, Section 7 is devoted to the proof of Theorem 1.1 where we use a fixed point procedure to obtain a solution of the nonlinear system.

2. Notation and Preliminaries

2.1. **Notation.** We set $\mathbf{L}^2(\Omega)=L^2(\Omega;\mathbb{R}^2)$, $\mathbf{H}^1(\Omega)=H^1(\Omega;\mathbb{R}^2)$ and the same notation conventions will be used for trace spaces. We introduce the following spaces that we use frequently later on:

$$\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})=H^{\frac{1}{4}}(0,T;\mathbf{L}^2(\partial\mathcal{F}))\cap L^2(0,T;\mathbf{H}^{\frac{1}{2}}(\partial\mathcal{F})),$$
$$\mathbf{H}^{1,2}((0,T)\times\partial\mathcal{F})=H^{1}(0,T;\mathbf{L}^2(\partial\mathcal{F}))\cap L^2(0,T;\mathbf{H}^{2}(\partial\mathcal{F})),$$

with the following norms

$$\|u\|_{\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})} = \left(\|u\|^2_{H^{\frac{1}{4}}(0,T;\mathbf{L}^2(\partial\mathcal{F}))} + \|u\|^2_{L^2(0,T;\mathbf{H}^{\frac{1}{2}}(\partial\mathcal{F}))}\right)^{\frac{1}{2}},$$
$$\|u\|_{\mathbf{H}^{1,2}((0,T)\times\partial\mathcal{F})} = \left(\|u\|^2_{H^{1}(0,T;\mathbf{L}^2(\partial\mathcal{F}))} + \|u\|^2_{L^2(0,T;\mathbf{H}^{2}(\partial\mathcal{F}))}\right)^{\frac{1}{2}}.$$

We also define

$$\mathbb{H}_1 = \{u \in \mathbf{L}^2(\Omega) | \operatorname{div} u = 0 \text{ in } \Omega,\ D(u) = 0 \text{ in } \mathcal{S},\ u\cdot n = 0 \text{ on } \partial\Omega\}. \tag{2.1}$$

We recall that (see, for instance, [44, Lemma 1.1, p.18]) for any $u \in \mathbb{H}_1$, there exist $\ell_u \in \mathbb{R}^2$ and $\omega_u \in \mathbb{R}$ such that

$$u(y) = \ell_u + \omega_u y^\perp, \quad \forall\, y \in \mathcal{S}.$$

2.2. **Preliminaries.**

Lemma 2.1. *There exists a constant $C > 0$ such that* $\displaystyle\int_{\partial\mathcal{S}} |a + by_1|^2\, d\Gamma \geqslant C\left(|a|^2 + |b|^2\right)$.

Proof. Let us prove that

$$(a,b) \mapsto \left(\int_{\partial\mathcal{S}} |a + by_1|^2\, d\Gamma\right)^{\frac{1}{2}} \tag{2.2}$$

is a norm of $\mathbb{R}^2$. It is enough to show the following implication:

$$a + by_1 = 0\ (y_1 \in \partial\mathcal{S}) \implies a = 0,\ b = 0.$$

We have

$$\partial\mathcal{S} \subset \{y_1 \in \mathbb{R} \,|\, a + by_1 = 0\}.$$

If $b \neq 0$, then we obtain that $\partial\mathcal{S}$ is included in the line

$$\left\{y_1 \in \mathbb{R} \,|\, y_1 = -\frac{a}{b}\right\},$$

which is a contradiction. Thus $b = 0$, which implies $a = 0$ and consequently, (2.2) defines a norm of $\mathbb{R}^2$ and we have

$$\left(\int_{\partial\mathcal{S}} |a + by_1|^2\, d\Gamma\right)^{\frac{1}{2}} \geqslant C\left(|a|^2 + |b|^2\right)^{\frac{1}{2}}.$$

□

Lemma 2.2. *Assume $z \in \mathbb{H}_1$, with $z = \ell + \omega y^\perp$ in $\mathcal{S}$. Then there exists a constant C independent of z, ℓ, ω such that*

$$\|z\|_{L^2(\mathcal{F})} \geqslant C|\ell|.$$

If $\mathcal{S}$ is not a disk, we also have

$$\|z\|_{L^2(\mathcal{F})} \geqslant C|\omega|.$$

Proof. Using Theorem 1.2 p.9 in [43], there exists C such that

$$\|z\|_{L^2(\mathcal{F})} \geqslant C\|(\ell + \omega y^\perp)\cdot n\|_{H^{-1/2}(\partial\mathcal{S})}. \tag{2.3}$$

First let us consider the case where $\mathcal{S}$ is a disk. Then, using that the center of $\mathcal{S}$ is 0, relation (2.3) writes

$$\|z\|_{L^2(\mathcal{F})} \geqslant C\|\ell\cdot n\|_{H^{-1/2}(\partial\mathcal{S})}. \tag{2.4}$$

Let us show that

$$\ell \mapsto \|\ell\cdot n\|_{H^{-1/2}(\partial\mathcal{S})}$$

is a norm of $\mathbb{R}^2$. Indeed assume $\ell\cdot n = 0$ on ∂S.
If $\ell \neq 0$, there exists a point of $\partial\mathcal{S}$ such that $n = \ell/|\ell|$ and thus, $\ell\cdot n = |\ell| \neq 0$. Thus we conclude from (2.4) that

$$\|z\|_{L^2(\mathcal{F})} \geqslant C|\ell|.$$

If $\mathcal{S}$ is not a disk, let us prove that

$$(\ell,\omega) \mapsto \|(\ell + \omega y^\perp)\cdot n\|_{H^{-1/2}(\partial\mathcal{S})}$$

is a norm of $\mathbb{R}^3$. We want to prove the following implication:

$$(\ell + \omega y^\perp)\cdot n = 0 \quad (y\in\partial\mathcal{S}) \implies \ell = 0,\ \omega = 0.$$

This is equivalent to show

$$(a + by)\cdot\tau = 0 \quad (y\in\partial\mathcal{S}) \implies a = 0,\ b = 0.$$

Let us introduce $f(y) := a\cdot y + b\frac{|y|^2}{2}$. Then, $\frac{\partial f}{\partial\tau}(y) = (a+by)\cdot\tau$ for any $y\in\partial\mathcal{S}$. If $(a+by)\cdot\tau = 0$ for any $y\in\partial\mathcal{S}$, then it implies that there exists $c\in\mathbb{R}$ such that $f(y) + c = 0$ for any $y\in\partial\mathcal{S}$. This yields

$$\partial\mathcal{S} \subset \left\{ y\in\mathbb{R}^2 \ ;\ a\cdot y + b\frac{|y|^2}{2} + c = 0 \right\}.$$

The set in the right-hand side is either empty, a point, a line, a circle or $\mathbb{R}^2$. The last case is the only one possible and it is equivalent to $a = 0$ and $b = 0$.

Thus we conclude from (2.3) that

$$\|z\|_{L^2(\mathcal{F})} \geqslant C(|\ell| + |\omega|).$$

□

3. The change of variables

3.1. **Construction of the change of variables.** Assume $\mathcal{S}(t)$ is defined by (1.1) and $\overline{\mathcal{S}} \subset \Omega$. We also take a control region $\mathcal{O}$ such that

$$\overline{\mathcal{O}} \cap \overline{\mathcal{S}} = \varnothing. \tag{3.1}$$

The above assumptions imply that $\mathrm{dist}(\overline{\mathcal{S}}, \overline{\mathcal{O}}) \geqslant d_0$ and $\mathrm{dist}(\overline{\mathcal{S}}, \partial\Omega) \geqslant d_0$ for some $d_0 > 0$. Then we can prove the following result

Lemma 3.1. *There exists a constant c_0 such that if*

$$|h| < c_0, \quad |\beta| < c_0, \tag{3.2}$$

then $\mathrm{dist}(\mathcal{S}_{h,\beta}, \overline{\mathcal{O}}) \geqslant \frac{d_0}{2}$ *and* $\mathrm{dist}(\mathcal{S}_{h,\beta}, \partial\Omega) \geqslant \frac{d_0}{2}$.

Taking $\varepsilon < c_0$ in (1.15), we deduce that

$$\operatorname{dist}(\mathcal{S}_{h,\beta}\,,\overline{\mathcal{O}}) \geqslant \frac{d_0}{2}, \quad \operatorname{dist}(\mathcal{S}_{h,\beta}\,,\partial\Omega) \geqslant \frac{d_0}{2}.$$

We want to construct change of variables $\mathcal{X} : \Omega \to \Omega$ that transforms $\mathcal{F}$ onto $\mathcal{F}(t)$ and $\mathcal{S}$ onto $\mathcal{S}(t)$. Thus we can define

$$\mathcal{X}(t,y) = y + k(y)[h(t) + R_{\beta(t)}y - y], \quad t \in (0,T),\ y \in \Omega. \tag{3.3}$$

Here $k : \Omega \to \mathbb{R}$ is a smooth function such that

$$k(y) = \begin{cases} 1 & \text{if } \operatorname{dist}(y,\overline{\mathcal{S}}) \leqslant \frac{d_0}{16} \\ 0 & \text{if } \operatorname{dist}(y,\overline{\mathcal{S}}) \geqslant \frac{d_0}{8}. \end{cases}$$

The map $\mathcal{X}$ is a C^∞ diffeomorphism of Ω onto itself if

$$\|k\|_{W^{1,\infty}(\Omega)}(|h(t)| + |\beta(t)|) < c \tag{3.4}$$

for c small enough.

With the above choices,

- in a neighborhood of $\mathcal{S}$, $\mathcal{X}(t,y) = h(t) + R_{\beta(t)}y$, and thus $\mathcal{X}(t,\mathcal{S}) = \mathcal{S}(t)$.
- in a neighborhood of $\partial\Omega$ and of $\mathcal{O}$, $\mathcal{X}(t,y) = y$.

Let the inverse of $\mathcal{X}(t,\cdot)$ is denoted by $\mathcal{Y}(t,\cdot)$. Observe that, in a neighborhood of $\mathcal{S}(t)$, we have

$$\mathcal{Y}(t,x) = R_{-\beta(t)}(x - h(t)).$$

3.2. **The system in a cylindrical domain.** We set

$$u(t,y) = \operatorname{Cof}(\nabla\mathcal{X}(t,y))^*\widehat{u}(t,\mathcal{X}(t,y)), \tag{3.5}$$

$$p(t,y) = \widehat{p}(t,\mathcal{X}(t,y)), \tag{3.6}$$

$$\theta(t,y) = \widehat{\theta}(t,\mathcal{X}(t,y)), \tag{3.7}$$

$$\ell(t) = R_{-\beta(t)}h'(t), \quad \omega(t) = \beta'(t). \tag{3.8}$$

Here $\operatorname{Cof}(M)$ is the cofactor matrix of M, which satisfies

$$M(\operatorname{Cof}(M))^* = (\operatorname{Cof}(M))^*M = \det(M)\,\mathrm{Id}\,.$$

We transform (1.4)-(1.13) by using this change of variables. Such a calculation is already done in [1] except for the temperature equation. We give here only the part of the calculation that corresponds to the temperature equation and we refer to [1] for the calculation of the other equations. From (3.7), we have:

$$\frac{\partial\widehat{\theta}}{\partial t} = \frac{\partial\theta}{\partial t}(\mathcal{Y}) + \frac{\partial\mathcal{Y}}{\partial t}\cdot\nabla\theta(\mathcal{Y}), \tag{3.9}$$

$$\nabla\widehat{\theta} = (\nabla\mathcal{Y})^*\nabla\theta(\mathcal{Y}), \tag{3.10}$$

$$\frac{\partial^2\widehat{\theta}}{\partial x_i^2} = \sum_{k,l=1}^{2}\frac{\partial^2\theta}{\partial y_l\partial y_k}(\mathcal{Y})\frac{\partial\mathcal{Y}_k}{\partial x_i}\frac{\partial\mathcal{Y}_l}{\partial x_i} + \sum_{k=1}^{2}\frac{\partial\theta}{\partial y_k}(\mathcal{Y})\frac{\partial^2\mathcal{Y}_k}{\partial x_i^2}, \tag{3.11}$$

$$\widehat{u}\cdot\nabla\widehat{\theta} = \det(\nabla\mathcal{Y})\,(u(\mathcal{Y})\cdot\nabla\theta(\mathcal{Y}))\,. \tag{3.12}$$

In order to transform the Neumann boundary condition (1.9), we also need to rewrite the exterior normal to $\partial\mathcal{F}(t)$. Let us denote by n the exterior normal to $\partial\mathcal{F}$. Then,

$$\widehat{n} = n \quad \text{on} \quad \partial\Omega,$$

and

$$\widehat{n}(t,x) = R_{\beta(t)} n(R_{-\beta(t)}(x - h(t))) \quad x \in \partial\mathcal{S}(t).$$

In a neighborhood of $\mathcal{S}(t)$,

$$\mathcal{Y}(t,x) = (R_{-\beta(t)}(x - h(t)))$$

and in a neighborhood of $\partial\Omega$, $\mathcal{Y} = \mathrm{Id}$.

Thus on $\partial\mathcal{S}(t)$,

$$\frac{\partial\widehat{\theta}}{\partial\widehat{n}}(t,x) = (\nabla\mathcal{Y})^* \nabla\theta(\mathcal{Y}) \cdot R_{\beta(t)} n(\mathcal{Y}) = \frac{\partial\theta}{\partial n}(\mathcal{Y}), \tag{3.13}$$

and on $\partial\Omega$,

$$\frac{\partial\widehat{\theta}}{\partial n} = \frac{\partial\theta}{\partial n}(\mathcal{Y}).$$

Thus, we can rewrite the system (1.4)-(1.13) as:

$$\left[\mathcal{K}_u \frac{\partial u}{\partial t}\right] + [\mathcal{M}_u u] + [\mathcal{N}_u u] - \nu[\mathcal{L}_u u] + [\mathcal{G}_u p] = \theta e_2, \text{ in } (0,T)\times\mathcal{F}, \tag{3.14}$$

$$\text{div } u = 0, \text{ in } (0,T)\times\mathcal{F}, \tag{3.15}$$

$$u(t,y) = 0, \quad t \in (0,T),\ y \in \partial\Omega, \tag{3.16}$$

$$u(t,y) = \ell(t) + \omega(t) y^{\perp}, \quad t \in (0,T),\ y \in \partial\mathcal{S}, \tag{3.17}$$

$$\frac{\partial\theta}{\partial t} + [\mathcal{M}_\theta\theta] + [\mathcal{N}_\theta(u,\theta)] - \mu[\mathcal{L}_\theta\theta] = w_0 \mathbb{1}_{\mathcal{O}}, \text{ in } (0,T)\times\mathcal{F}, \tag{3.18}$$

$$\frac{\partial\theta}{\partial n}(t,y) = 0, t \in (0,T),\ y \in \partial\mathcal{F}, \tag{3.19}$$

$$M\ell'(t) = -\int_{\partial\mathcal{S}} \sigma(u,\, p) n \, d\Gamma \, - M\omega\ell^{\perp}, t \in (0,T), \tag{3.20}$$

$$J\omega'(t) = -\int_{\partial\mathcal{S}} y^{\perp} \cdot \sigma(u,\, p) n \, d\Gamma \, , t \in (0,T), \tag{3.21}$$

$$h'(t) = R_{\beta(t)}\ell(t) \quad t \in (0,T), \tag{3.22}$$

$$\beta'(t) = \omega(t) \quad t \in (0,T), \tag{3.23}$$

$$u(0,y) = u_0(y) \text{ and } \theta(0,y) = \theta_0(y), y \in \mathcal{F}, \tag{3.24}$$

$$h(0) = h_0,\ \ell(0) = \ell_0,\ \beta(0) = \beta_0,\ \omega(0) = \omega_0. \tag{3.25}$$

Here we want to underline the fact that the linear and nonlinear operators $[\mathcal{K}_u]$,$[\mathcal{N}_u]$, $[\mathcal{L}_u]$, $[\mathcal{G}_u]$, $[\mathcal{N}_\theta]$, $[\mathcal{L}_\theta]$ depend on h and β and the operators $[\mathcal{M}_u]$, $[\mathcal{M}_\theta]$ depend on h, β, ℓ, ω through

the change of variables $\mathcal{X}$ and its inverse $\mathcal{Y}$. The definitions of the operators are given through the following formulas:

$$[\mathcal{K}_u u] = \mathrm{Cof}((\nabla \mathcal{Y})^* \circ \mathcal{X}) u, \tag{3.26}$$

$$[\mathcal{M}_u u] = \frac{\partial}{\partial t} \mathrm{Cof}((\nabla \mathcal{Y})^* \circ \mathcal{X}) u + (\mathrm{Cof}(\nabla \mathcal{Y})^* \circ \mathcal{X})(\nabla u) \left(\frac{\partial \mathcal{Y}}{\partial t} \right) \circ \mathcal{X}, \tag{3.27}$$

$$\begin{aligned}[\mathcal{L}_u u]_i = \sum_{j,k,l,m} & \mathrm{Cof}(\nabla \mathcal{Y})_{ki}(\mathcal{X}) \frac{\partial^2 u_k}{\partial y_l \partial y_m} \frac{\partial \mathcal{Y}_l}{\partial x_j}(\mathcal{X}) \frac{\partial \mathcal{Y}_m}{\partial x_j}(\mathcal{X}) \\ & + 2 \sum_{j,k,l} \frac{\partial}{\partial x_j} \mathrm{Cof}(\nabla \mathcal{Y})_{ki}(\mathcal{X}) \frac{\partial u_k}{\partial y_l} \frac{\partial \mathcal{Y}_l}{\partial x_j}(\mathcal{X}) \\ & \sum_{j,k,l} \mathrm{Cof}(\nabla \mathcal{Y})_{ki}(\mathcal{X}) \frac{\partial u_k}{\partial y_l} \frac{\partial^2 \mathcal{Y}_l}{\partial x_j^2}(\mathcal{X}) + \sum_{j,k} \frac{\partial^2}{\partial x_j^2} \mathrm{Cof}(\nabla \mathcal{Y})_{ki}(\mathcal{X}) u_k, \end{aligned} \tag{3.28}$$

$$[\mathcal{N}_u u]_i = \sum_{j,k,r} \mathrm{Cof}(\nabla \mathcal{Y})_{kj}(\mathcal{X}) \frac{\partial}{\partial x_j} \mathrm{Cof}(\nabla \mathcal{Y})_{ri}(\mathcal{X}) u_k u_r + \sum_{k,r} \det((\nabla \mathcal{Y})(\mathcal{X}))^2 \frac{\partial \mathcal{X}_i}{\partial y_r} u_k \frac{\partial u_r}{\partial y_k}, \tag{3.29}$$

$$[\mathcal{G}_u p]_i = \sum_{k=1}^{2} \frac{\partial p}{\partial y_k} \frac{\partial \mathcal{Y}_k}{\partial x_i}(\mathcal{X}), \tag{3.30}$$

$$[\mathcal{M}_\theta \theta] = \frac{\partial \mathcal{Y}}{\partial t}(\mathcal{X}) \cdot \nabla \theta, \tag{3.31}$$

$$[\mathcal{L}_\theta \theta] = \sum_{i=1}^{2} \sum_{k=1}^{2} \left[\sum_{l=1}^{2} \frac{\partial^2 \theta}{\partial y_l \partial y_k} \frac{\partial \mathcal{Y}_l}{\partial x_i}(\mathcal{X}) \frac{\partial \mathcal{Y}_k}{\partial x_i}(\mathcal{X}) + \frac{\partial \theta}{\partial y_k} \frac{\partial^2 \mathcal{Y}_k}{\partial x_i^2}(\mathcal{X}) \right], \tag{3.32}$$

$$[\mathcal{N}_\theta(u, \theta)] = \frac{u \cdot \nabla \theta}{\det \nabla \mathcal{X}}. \tag{3.33}$$

We have set

$$u_0 := \mathrm{Cof}(\nabla \mathcal{X}(0, y))^* \widehat{u}_0(\mathcal{X}(0, y)), \quad \theta_0 := \widehat{\theta}_0(\mathcal{X}(0, y)), \tag{3.34}$$

$$\ell_0 := R_{-\beta_0} \widehat{\ell_0}, \quad \omega_0 := \widehat{\omega_0}. \tag{3.35}$$

4. Some linear systems

In this section we analyze two linear systems associated with (3.14)–(3.25):

$$\frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = \theta e_2 + \widetilde{f}, \quad \text{in } (0, T) \times \mathcal{F}, \tag{4.1}$$

$$\mathrm{div}\, u = 0, \quad \text{in } (0, T) \times \mathcal{F}, \tag{4.2}$$

$$u(t, y) = 0, \quad t \in (0, T),\ y \in \partial\Omega, \tag{4.3}$$

$$u(t, y) = \ell(t) + \omega(t) y^\perp, \quad t \in (0, T),\ y \in \partial\mathcal{S}, \tag{4.4}$$

$$\frac{\partial \theta}{\partial t} - \mu \Delta \theta = \widetilde{g} + w_0 \mathbb{1}_{\mathcal{O}}, \quad \text{in } (0,T) \times \mathcal{F}, \tag{4.5}$$

$$\frac{\partial \theta}{\partial n}(t,y) = 0, \quad t \in (0,T),\ y \in \partial \mathcal{F}, \tag{4.6}$$

$$M\ell'(t) = -\int_{\partial \mathcal{S}} \sigma(u,\, p) n \, d\Gamma + M\widetilde{h}^{(1)}, \quad t \in (0,T), \tag{4.7}$$

$$J\omega'(t) = -\int_{\partial \mathcal{S}} y^{\perp} \cdot \sigma(u,\, p) n d\Gamma + J\widetilde{h}^{(2)}, \quad t \in (0,T), \tag{4.8}$$

$$h'(t) = R_{\beta(t)} \ell(t) \quad t \in (0,T), \tag{4.9}$$

$$\beta'(t) = \omega(t) \quad t \in (0,T), \tag{4.10}$$

$$u(0,y) = u_0(y) \text{ and } \theta(0,y) = \theta_0(y), \quad y \in \mathcal{F}, \tag{4.11}$$

$$h(0) = h_0,\ \beta(0) = \beta_0, \tag{4.12}$$

$$\ell(0) = \ell_0 \in \mathbb{R}^2,\ \omega(0) = \omega_0 \in \mathbb{R}, \tag{4.13}$$

and

$$\frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = \widetilde{f}, \quad \text{in } (0,T) \times \mathcal{F}, \tag{4.14}$$

$$\operatorname{div} u = 0, \quad \text{in } (0,T) \times \mathcal{F}, \tag{4.15}$$

$$u(t,y) = 0, \quad t \in (0,T),\ y \in \partial \Omega, \tag{4.16}$$

$$u(t,y) = \ell(t) + \omega(t) y^{\perp}, \quad t \in (0,T),\ y \in \partial \mathcal{S}, \tag{4.17}$$

$$M\ell'(t) = -\int_{\partial \mathcal{S}} \sigma(u,\, p) n \, d\Gamma + M\widetilde{h}^{(1)}, \quad t \in (0,T), \tag{4.18}$$

$$J\omega'(t) = -\int_{\partial \mathcal{S}} y^{\perp} \cdot \sigma(u,\, p) n d\Gamma + J\widetilde{h}^{(2)}, \quad t \in (0,T), \tag{4.19}$$

$$u(0,y) = u_0(y) \quad y \in \mathcal{F}, \tag{4.20}$$

$$\ell(0) = \ell_0 \in \mathbb{R}^2,\ \omega(0) = \omega_0 \in \mathbb{R}, \tag{4.21}$$

For both systems, we extend u and $\widetilde{f}$ to Ω by setting:

$$u(t,y) = \ell(t) + \omega(t) y^{\perp}, \quad \forall (t,y) \in (0,T) \times \mathcal{S},$$
$$\widetilde{f}(t,y) = \widetilde{h}^{(1)} + \widetilde{h}^{(2)} y^{\perp}, \quad \forall (t,y) \in (0,T) \times \mathcal{S}.$$

In particular, u is a rigid velocity in $\mathcal{S}$, that is $D(u) = 0$ in $(0,T) \times \mathcal{S}$. We recall that $\mathbb{H}_1$ is defined by (2.1). We set

$$\mathbb{H} = \mathbb{H}_1 \times L^2(\mathcal{F}). \tag{4.22}$$

We consider the inner product on $\mathbf{L}^2(\Omega) \times L^2(\mathcal{F})$ defined by

$$\left\langle \begin{pmatrix} u \\ \theta_1 \end{pmatrix}, \begin{pmatrix} v \\ \theta_2 \end{pmatrix} \right\rangle_{\mathbf{L}^2(\Omega) \times L^2(\mathcal{F})} = \int_{\mathcal{F}} u \cdot v \, dy + \rho_{\mathcal{S}} \int_{\mathcal{S}} u \cdot v \, dy + \int_{\mathcal{F}} \theta_1 \theta_2 \, dy.$$

The corresponding norm is equivalent to the usual norm in $\mathbf{L}^2(\Omega) \times L^2(\mathcal{F})$. Moreover, if $u, v \in \mathbb{H}_1$, then we have

$$\left\langle \begin{pmatrix} u \\ \theta_1 \end{pmatrix}, \begin{pmatrix} v \\ \theta_2 \end{pmatrix} \right\rangle_{\mathbf{L}^2(\Omega)\times L^2(\mathcal{F})} = \int_{\mathcal{F}} u \cdot v \, dy + M\ell_u \cdot \ell_v + J\omega_u\omega_v + \int_{\mathcal{F}} \theta_1\theta_2 \, dy.$$

In order to work with (4.1)-(4.13), we use an approach based on semigroups. We define:

$$\mathcal{D}(A_1) = \left\{ u \in \mathbf{H}^1_0(\Omega) \,|\, u|_{\mathcal{F}} \in \mathbf{H}^2(\mathcal{F}),\ \operatorname{div} u = 0 \text{ in } \Omega,\ D(u) = 0 \text{ in } \mathcal{S} \right\}, \tag{4.23}$$

$$\mathcal{D}(A_2) = \left\{ \theta \in H^2(\mathcal{F}) \,|\, \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial\mathcal{F} \right\} \tag{4.24}$$

and

$$\mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2). \tag{4.25}$$

For all $u \in \mathcal{D}(A_1)$, we set

$$\mathcal{A}_1 u = \begin{cases} \nu\Delta u & \text{in } \mathcal{F} \\ -\dfrac{2\nu}{M} \displaystyle\int_{\partial\mathcal{S}} D(u) n d\Gamma - \left[\dfrac{2\nu}{J} \displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot D(u) n d\Gamma \right] y^\perp & \text{in } \mathcal{S} \end{cases} \tag{4.26}$$

$$A_1 u = \mathbb{P}\mathcal{A}_1 u \tag{4.27}$$

where $\mathbb{P}$ is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $\mathbb{H}_1$.

We also define for $\theta \in \mathcal{D}(A_2)$,

$$A_2\theta = \mu\Delta\theta, \quad D_0\theta = \mathbb{P}(\theta e_2 \mathbb{1}_{\mathcal{F}}).$$

Finally, we define $A : \mathcal{D}(A) \to \mathbb{H}$ by

$$A = \begin{pmatrix} A_1 & D_0 \\ 0 & A_2 \end{pmatrix}. \tag{4.28}$$

It is shown in [42, Proposition 4.2] that A_1 is a self-adjoint, maximal dissipative operator. It is also well-known that A_2 is a self-adjoint, maximal dissipative operator. Thus, using a perturbation argument (see [36, Corollary 2.2, Chapter 3, p. 81]), we deduce the following result:

Proposition 4.1. *The operator $(A, \mathcal{D}(A))$ defined by (4.28) is the generator of an analytic semigroup on $\mathbb{H}$. Its adjoint is given by $\mathcal{D}(A^*) = \mathcal{D}(A)$ and*

$$A^* = \begin{pmatrix} A_1 & 0 \\ D_0^* & A_2 \end{pmatrix}, \tag{4.29}$$

with

$$D_0^*\phi = \phi_2|_{\mathcal{F}}.$$

Observe that

$$D((-A_1)^{\frac{1}{2}}) = \left\{ u \in \mathbf{H}^1_0(\Omega) \,|\, \operatorname{div} u = 0 \text{ in } \Omega,\ D(u) = 0 \text{ in } \mathcal{S} \right\}, \tag{4.30}$$

$$D((-A)^{\frac{1}{2}}) = D((-A_1)^{\frac{1}{2}}) \times H^1(\mathcal{F}). \tag{4.31}$$

As a consequence of Proposition 4.1, and by using the isomorphism theorem (see, for instance, [2, Theorem 3.1, p. 143]), we have the following result:

Corollary 4.2. *Let $T > 0$ and $\widetilde{f} \in L^2(0,T;\mathbf{L}^2(\mathcal{F}))$, $\widetilde{g} \in L^2(0,T;L^2(\mathcal{F}))$, $w_0 \in L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{h}^{(1)} \in L^2(0,T;\mathbb{R}^2)$, $\widetilde{h}^{(2)} \in L^2(0,T;\mathbb{R})$, $u_0 \in \mathbf{H}^1(\mathcal{F})$, $\theta_0 \in H^1(\mathcal{F})$ be such that:*

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}, \quad u_0(y) = 0 \text{ on } \partial\Omega, \quad u_0(y) = \ell_0 + \omega_0 y^\perp \text{ for } y \in \partial\mathcal{S}.$$

Then the linear system (4.1)-(4.13) admits a unique solution (u,p,θ,ℓ,ω) with

$$u \in L^2(0,T;\mathbf{H}^2(\mathcal{F})) \cap H^1(0,T;\mathbf{L}^2(\mathcal{F})) \cap C([0,T];\mathbf{H}^1(\mathcal{F})),$$
$$p \in L^2(0,T;H^1(\mathcal{F})/\mathbb{R}),\ \ell \in H^1(0,T;\mathbb{R}^2),\ \omega \in H^1(0,T;\mathbb{R}),$$
$$\theta \in L^2(0,T;H^2(\mathcal{F})) \cap H^1(0,T;L^2(\mathcal{F})).$$

Moreover, the solution (u,p,θ,ℓ,ω) satisfies the following estimate:

$$\begin{aligned}
&\|u\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|p\|_{L^2(0,T;H^1(\mathcal{F}))} \\
&\qquad + \|\ell\|_{H^1(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})} + \|\theta\|_{L^2(0,T;H^2(\mathcal{F}))\cap H^1(0,T;L^2(\mathcal{F}))} \\
&\leqslant C\Bigg(\|\widetilde{f}\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\widetilde{g}\|_{L^2(0,T;L^2(\mathcal{F}))} + \|w_0\|_{L^2(0,T;L^2(\mathcal{O}))} + \|\widetilde{h}^{(1)}\|_{L^2(0,T;\mathbb{R}^2)} \\
&\qquad + \|\widetilde{h}^{(2)}\|_{L^2(0,T;\mathbb{R})} + \|u_0\|_{\mathbf{H}^1(\mathcal{F})} + \|\theta_0\|_{H^1(\mathcal{F})} + |\ell_0| + |\omega_0| \Bigg). \qquad (4.32)
\end{aligned}$$

In what follows, we also need some properties of the linear system (4.14)-(4.21) that we can write as

$$\dot{u} = A_1 u + \mathbb{P}\widetilde{f}, \quad u(0) = u_0. \qquad (4.33)$$

Corollary 4.3. *Let $T > 0$ and $\widetilde{f} \in L^2(0,T;\mathbf{L}^2(\mathcal{F}))$, $u_0 \in \mathbf{H}^1(\mathcal{F})$ such that:*

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}, \quad u_0(y) = 0 \text{ on } \partial\Omega, \quad u_0(y) = \ell_0 + \omega_0 y^\perp \text{ for } y \in \partial\mathcal{S}.$$

Then the linear system (4.14)-(4.21) admits a unique solution (u,p,ℓ,ω) with

$$u \in L^2(0,T;\mathbf{H}^2(\mathcal{F})) \cap H^1(0,T;\mathbf{L}^2(\mathcal{F})) \cap C([0,T];\mathbf{H}^1(\mathcal{F})),$$
$$p \in L^2(0,T;H^1(\mathcal{F})/\mathbb{R}),\ \ell \in H^1(0,T;\mathbb{R}^2),\ \omega \in H^1(0,T;\mathbb{R}).$$

Moreover, the solution (u,p,ℓ,ω) satisfies the following estimate:

$$\begin{aligned}
&\|u\|_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|p\|_{L^2(0,T;H^1(\mathcal{F}))} + \|\ell\|_{H^1(0,T;\mathbb{R}^2)} + \|\omega\|_{H^1(0,T;\mathbb{R})} \\
&\qquad \leqslant C\Bigg(\|\widetilde{f}\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\widetilde{h}^{(1)}\|_{L^2(0,T;\mathbb{R}^2)} + \|\widetilde{h}^{(2)}\|_{L^2(0,T;\mathbb{R})} + \|u_0\|_{\mathbf{H}^1(\Omega)} \Bigg). \qquad (4.34)
\end{aligned}$$

If

$$\mathbb{P}\widetilde{f} \in L^2(0,T;\mathcal{D}(A_1)) \cap H^1(0,T;\mathbb{H}_1)$$

and

$$u_0 \in \mathcal{D}((-A_1)^{3/2}),$$

then

$$u \in L^2(0,T;\mathcal{D}((A_1)^2)) \cap H^2(0,T;\mathbb{H}_1)$$

Moreover, there exists C such that

$$\|u\|_{L^2(0,T;\mathbf{H}^4(\mathcal{F}))\cap H^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\ell\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega\|_{H^2(0,T;\mathbb{R})} \leqslant C\left(\|\mathbb{P}\widetilde{f}\|_{L^2(0,T;\mathcal{D}(A_1))\cap H^1(0,T;\mathbb{H}_1)} + \|u_0\|_{\mathcal{D}((-A_1)^{3/2})}\right). \quad (4.35)$$

5. The Carleman Inequality

Let us introduce the adjoint system of (4.1)-(4.13):

$$\begin{cases} -\dfrac{\partial \phi}{\partial t} - \nu\Delta\phi + \nabla q = f, & \text{in } (0,T)\times\mathcal{F},\\ \text{div } \phi = 0, & \text{in } (0,T)\times\mathcal{F},\\ \phi(t,y) = 0, & t\in(0,T),\ y\in\partial\Omega,\\ \phi(t,y) = \ell_\phi(t) + \omega_\phi(t)y^\perp, & t\in(0,T),\ y\in\partial\mathcal{S},\\ -\dfrac{\partial \psi}{\partial t} - \mu\Delta\psi = g + \phi_2, & \text{in } (0,T)\times\mathcal{F},\\ \dfrac{\partial \psi}{\partial n}(t,y) = 0, & t\in(0,T),\ y\in\partial\mathcal{F},\\ -M\ell_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(\phi,\, q)n\, d\Gamma \ +\ h^{(1)}, & t\in(0,T),\\ -J\omega_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\sigma(\phi,\, q)n\, d\Gamma \ +\ h^{(2)}, & t\in(0,T),\\ \phi(T,y) = \phi^T(y) \text{ and } \psi(T,y) = \psi^T(y), & y\in\mathcal{F},\\ \ell_\phi(T) = \ell^T,\ \omega_\phi(T) = \omega^T. \end{cases} \quad (5.1)$$

In this section, our aim is to establish a suitable Carleman estimate for the adjoint system (5.1). Let us introduce the weight functions used for this estimate.

Let us consider $\eta \in C^2(\overline{\mathcal{F}})$ satisfying

$$\eta > 0 \text{ in } \mathcal{F}, \quad |\nabla\eta| \geqslant c_0 > 0 \text{ in } \mathcal{F}\backslash\mathcal{O}_0, \quad (5.2)$$

$$\eta = 0 \text{ on } \partial\mathcal{F} \quad \text{and} \quad \frac{\partial\eta}{\partial n} \leqslant -c_1 < 0 \text{ on } \partial\mathcal{F}, \quad (5.3)$$

where $\mathcal{O}_0$ be a nonempty open subset of $\mathbb{R}^2$ such that $\overline{\mathcal{O}_0} \subset \mathcal{O}$. The existence of such a function is standard (see, for instance, [21, Lemma 1.1, p. 4] or [45, Theorem 9.4.3, p. 299]).

Let $\lambda \geqslant 1$ and let us consider the following functions defined in $(0,T)\times\mathcal{F}$:

$$\alpha(t,x) = \frac{e^{2\lambda\|\eta\|_{L^\infty(\mathcal{F})}} - e^{\lambda\eta(x)}}{E(t)^8}, \quad \xi(t,x) = \frac{e^{\lambda\eta(x)}}{E(t)^8} \tag{5.4}$$

$$\alpha_M(t) = \max_{x\in\overline{\mathcal{F}}} \alpha(t,x), \quad \xi_M(t) = \max_{x\in\overline{\mathcal{F}}} \xi(t,x) \tag{5.5}$$

$$\alpha_m(t) = \min_{x\in\overline{\mathcal{F}}} \alpha(t,x), \quad \xi_m(t) = \min_{x\in\overline{\mathcal{F}}} \xi(t,x), \tag{5.6}$$

where $E \in C^\infty([0,T])$, $E>0$ in $(0,T)$, E is even, increasing in $(0,T/2)$ and satisfies $E(t)=t$ in $(0,T/4)$, $E(t)=T-t$ in $(3T/4,T)$.

Such functions are standard for Carleman estimates. Let us give some properties that are used in what follows:

$$\nabla\alpha = -\lambda\xi\nabla\eta \tag{5.7}$$

$$\nabla\xi = \lambda\xi\nabla\eta \tag{5.8}$$

$$\frac{1}{\xi_m(t)} \leqslant C \quad (t\in(0,T)), \tag{5.9}$$

$$\xi_M(t) \leqslant C\xi_m(t) \quad (t\in(0,T)), \tag{5.10}$$

$$|\xi_m'(t)| \leqslant C\xi(t)^{9/8} \quad (t\in(0,T)), \tag{5.11}$$

$$|\xi_m''(t)| \leqslant C\xi(t)^{10/8} \quad (t\in(0,T)), \tag{5.12}$$

$$|\alpha_M'(t)| \leqslant C\xi(t)^{9/8} \quad (t\in(0,T)), \tag{5.13}$$

$$|\alpha_M''(t)| \leqslant C\xi(t)^{10/8} \quad (t\in(0,T)), \tag{5.14}$$

$$\alpha_M(t) \leqslant 2\alpha_m(t) \quad (t\in(0,T)), \tag{5.15}$$

$$s^{m_1}\xi^{m_2}e^{-2s\alpha} \leqslant C \quad \text{in } (0,T)\times\mathcal{F} \quad \text{if } m_1 \leqslant m_2 \text{ and } s\geqslant 1. \tag{5.16}$$

for some positive constants C depending on T and on λ.

Now, we can state the following Carleman inequality:

Theorem 5.1. *Let $T>0$ and $\mathcal{O}$ be a nonempty open subset such that $\overline{\mathcal{O}}\subset\mathcal{F}$. Then there exists a constant $\lambda_0>0$ such that for any $\lambda\geqslant\lambda_0$ there exist constants $C(\lambda)>0$ and $s_0(\lambda)>0$ such that for all $f\in L^2(0,T;\mathbf{L}^2(\mathcal{F}))$, $g\in L^2(0,T;L^2(\mathcal{F}))$, $h^{(1)}\in L^2(0,T;\mathbb{R}^2)$, $h^{(2)}\in L^2(0,T;\mathbb{R})$ and for all $\phi^T\in\mathbb{H}_1$, $\psi^T\in L^2(\mathcal{F})$, $\ell^T\in\mathbb{R}^2$, $\omega^T\in\mathbb{R}$ satisfying $\phi^T=\ell^T+\omega^T y^\perp$ in $\mathcal{S}$, the solution of (5.1) satisfies the inequality:*

$$\begin{aligned} s^4\int_0^T\int_{\mathcal{F}} e^{-5s\alpha_M}(\xi_m)^4|\phi|^2dy\,dt + s^5\int_0^T\int_{\mathcal{F}} e^{-5s\alpha_M}(\xi_m)^5|\psi|^2\,dy\,dt + s^4\int_0^T e^{-2s\alpha_M}(\xi_m)^4(|\ell_\phi|^2+|\omega_\phi|^2)\,dt \\ \leqslant C\bigg(\int_0^T\int_{\mathcal{F}} e^{-3s\alpha_M}(|f|^2+|g|^2)dy\,dt + \int_0^T e^{-3s\alpha_M}(|h^{(1)}|^2+|h^{(2)}|^2)\,dt \\ + s^{12}\int_0^T\int_{\mathcal{O}} e^{-4s\alpha_m-s\alpha_M}(\xi_M)^{\frac{49}{4}}|\psi|^2\,dy\,dt\bigg), \end{aligned} \tag{5.17}$$

for all $s\geqslant s_0$.

Proof. In this proof, we follow similar ideas as in [9] and [5]. Throughout the proof, C stands for a positive constant depending only on $\mathcal{F}, \mathcal{O}$ and η.

First, the proof of the above estimate is done by density, for more regular solutions. More precisely, we can assume that

$$\begin{pmatrix} f \\ g \end{pmatrix} \in L^2(0,T;\mathcal{D}(A^*)) \cap H^1(0,T;\mathbb{H}) \quad \text{and} \quad \begin{pmatrix} \phi^T \\ \psi^T \end{pmatrix} \in \mathcal{D}((-A^*)^{3/2}),$$

where we have as usual extended f and ϕ^T in $\mathcal{S}$ by respectively $h^{(1)} + h^{(2)}y^\perp$ and $\ell^T + \omega^T y^\perp$. In that case, our solution satisfies

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in L^2(0,T;\mathcal{D}((A^*)^2)) \cap H^2(0,T;\mathbb{H}).$$

Step 1: decomposition of the solution of (5.1).

Let $(\phi, q, \psi, \ell_\phi, \omega_\phi)$ be the solution to (5.1). We set

$$\rho := e^{-\frac{3}{2}s\alpha_M}.$$

The function ρ is $C^\infty([0,T])$ and for any $k \in \mathbb{N}$,

$$\rho^{(k)}(0) = \rho^{(k)}(T) = 0.$$

From (5.13) and (5.14), we deduce the following relations

$$\rho' = -\frac{3}{2}s\alpha_M'\rho, \quad |\rho'| \leqslant Cs\rho(\xi)^{9/8}, \tag{5.18}$$

and

$$|\rho''| \leqslant Cs^2\rho(\xi)^{9/4}. \tag{5.19}$$

We then consider the following decomposition

$$\begin{gathered} \rho\phi = v + z, \quad \rho q = q_v + q_z, \quad \rho\psi = \widetilde{\psi}, \\ \rho\ell_\phi = \ell_v + \ell_z \quad \rho\omega_\phi = \omega_v + \omega_z, \end{gathered} \tag{5.20}$$

where $(v, p_v, \ell_v, \omega_v)$, $(z, p_z, \ell_z, \omega_z)$ and $\widetilde{\psi}$ satisfy the following systems :

$$\begin{cases} -\dfrac{\partial v}{\partial t} - \nu\Delta v + \nabla q_v = \rho f, & \text{in } (0,T)\times\mathcal{F}, \\ \text{div } v = 0, & \text{in } (0,T)\times\mathcal{F}, \\ v(t,y) = 0, & t \in (0,T),\ y \in \partial\Omega, \\ v(t,y) = \ell_v(t) + \omega_v y^\perp, & t \in (0,T),\ y \in \partial\mathcal{S}, \\ -M\ell_v'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(v,\, q_v)n d\Gamma \ + \ \rho h^{(1)}, & t \in (0,T), \\ -J\omega_v'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot \sigma(v,\, q_v)n d\Gamma \ + \ \rho h^{(2)}, & t \in (0,T), \\ v(T,y) = 0, & y \in \mathcal{F}, \\ \ell_v(T) = 0,\ \omega_v(T) = 0. \end{cases} \tag{5.21}$$

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \nu\Delta z + \nabla q_z = -\rho'\phi, & \text{in } (0,T)\times\mathcal{F}, \\ \text{div } z = 0, & \text{in } (0,T)\times\mathcal{F}, \\ z(t,y) = 0, & t\in(0,T),\ y\in\partial\Omega, \\ z(t,y) = \ell_z(t) + \omega_z y^\perp, & t\in(0,T),\ y\in\partial\mathcal{S}, \\ -M\ell_z'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(z,q_z)n d\Gamma \ - \ M\rho'\ell_\phi, & t\in(0,T), \\ -J\omega_z'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\sigma(z,q_z)n d\Gamma \ - \ J\rho'\omega_\phi, & t\in(0,T), \\ z(T,y) = 0, & y\in\mathcal{F}, \\ \ell_z(T) = 0,\ \omega_z(T) = 0. \end{cases} \tag{5.22}$$

and

$$\begin{cases} -\dfrac{\partial\widetilde{\psi}}{\partial t} - \mu\Delta\widetilde{\psi} = \rho g + \rho\phi_2 - \rho'\psi, & \text{in } (0,T)\times\mathcal{F}, \\ \dfrac{\partial\widetilde{\psi}}{\partial n}(t,y) = 0, & \text{in } (0,T)\times\partial\mathcal{F}, \\ \widetilde{\psi}(T,y) = 0, & \text{in } \mathcal{F}. \end{cases} \tag{5.23}$$

Note that since

$$\phi\in L^2(0,T;\mathcal{D}(A_1^2))\cap H^2(0,T;\mathbb{H}_1), \tag{5.24}$$

we have

$$z\in L^2(0,T;\mathcal{D}(A_1^2))\cap H^2(0,T;\mathbb{H}_1). \tag{5.25}$$

Step 2: Carleman estimates for the heat equation, the Laplace and the Gradient operators

First we apply the divergence operator to the first equation of (5.22) and we deduce that $\Delta q_z = 0$. Then we apply the operator $\nabla\Delta = (\frac{\partial}{\partial y_1}\Delta, \frac{\partial}{\partial y_2}\Delta)$ to the first equation of (5.22) satisfied by z_2 and we obtain

$$-\frac{\partial(\nabla\Delta z_2)}{\partial t} - \Delta(\nabla\Delta z_2) = \nabla(-\rho'\Delta\phi_2) \quad \text{in } (0,T)\times\mathcal{F}. \tag{5.26}$$

This means that $\nabla\Delta z_2$ satisfies a heat equation with nonhomogeneous boundary conditions. For such an equation, we have the following Carleman estimates, obtained in [32]: there exists

$C>0$, $\lambda_0>0$, $s_0>0$ such that for any $\lambda\geqslant\lambda_0$, $s\geqslant s_0$

$$\frac{1}{s}\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\frac{1}{\xi}|\nabla^2\Delta z_2|^2dy\,dt+s\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt$$
$$\leqslant C\Bigg(s\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt+s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{L^2((0,T);\mathbf{L}^2(\partial\mathcal{F}))}$$
$$+s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})}+\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}|\rho'|^2|\Delta\phi_2|^2dy\,dt\Bigg)\quad(5.27)$$

Now by using a Carleman estimate on the gradient operator (see [9, Lemma 3]) on Δz_2 there exist λ_1,s_1,C such that

$$s^3\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3|\Delta z_2|^2\,dy\,dt$$
$$\leqslant C\left(s\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt+s^3\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi^3|\Delta z_2|^2dy\,dt\right)\quad(5.28)$$

for $\lambda\geqslant\lambda_1$ and $s\geqslant s_1$.

Let $\mathcal{O}_0$, $\mathcal{O}_1$ be open subsets of $\mathcal{F}$ such that $\overline{\mathcal{O}_0}\subset\mathcal{O}_1$, $\overline{\mathcal{O}_1}\subset\mathcal{F}$. Then we can use a Carleman estimate for the Laplace operator (see for instance [3]). We recall the proof of such an estimate in the appendix (Corollary A.2).

$$s^6\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^6|z_2|^2dy\,dt+s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^4|\nabla z_2|^2dy\,dt+s^6\int_0^T e^{-2s\alpha_M}\xi_m^6\int_{\partial\mathcal{S}}|z_2|^2\,d\Gamma\,dt$$
$$\leqslant C\Bigg(s^3\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3|\Delta z_2|^2dy\,dt+s^6\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^6|z_2|^2dy\,dt$$
$$+s^4\int_0^T e^{-2s\alpha_M}\xi_m^4\int_{\partial\mathcal{S}}\left|\frac{\partial z_2}{\partial\tau}\right|^2\,d\Gamma\,dt\Bigg),\quad(5.29)$$

for $\lambda\geqslant\lambda_2$ and $s\geqslant s_2$.

On $\partial\mathcal{S}$, we have

$$z_2=\ell_z\cdot e_2+\omega_z y_1,\quad \frac{\partial z_2}{\partial\tau}=\omega_z\tau_1.$$

Using Lemma 2.1, we have

$$\int_{\partial\mathcal{S}}|z_2|^2d\Gamma\geqslant C(|\ell_z\cdot e_2|^2+|\omega_z|^2).\quad(5.30)$$

On the other hand, there exists a constant depending only on $\partial\mathcal{S}$ such that

$$\int\limits_{\partial\mathcal{S}} \left|\frac{\partial z_2}{\partial \tau}\right|^2 \, d\Gamma \leqslant C|\omega_z|^2. \tag{5.31}$$

Combining (5.9), (5.29), (5.30) and (5.31) we deduce

$$s^6 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^6 |z_2|^2 dy\, dt + s^4 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^4 |\nabla z_2|^2 dy\, dt + s^6 \int\limits_0^T e^{-2s\alpha_M}(\xi_m)^6 (|\ell_z \cdot e_2|^2 + |\omega_z|^2)\, dt$$
$$\leqslant C\left(s^3 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^3 |\Delta z_2|^2 dy\, dt + s^6 \int\limits_0^T \int\limits_{\mathcal{O}_1} e^{-2s\alpha}\xi^6 |z_2|^2 dy\, dt \right), \tag{5.32}$$

for $\lambda \geqslant \lambda_3$ and $s \geqslant s_3$.

We set

$$J(s,\widetilde{\psi}) = s \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi \left(\left|\frac{\partial \widetilde{\psi}}{\partial t}\right|^2 + |\Delta \widetilde{\psi}|^2 \right) dy\, dt + s^3 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^3 |\nabla \widetilde{\psi}|^2 dy\, dt$$
$$+ s^5 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^5 |\widetilde{\psi}|^2 dy\, dt. \tag{5.33}$$

We recall a standard Carleman estimate for equation (5.23) (see, for instance [16]). Let $\mathcal{O}_0$, $\mathcal{O}_1$ be open subsets of $\mathcal{F}$ such that $\overline{\mathcal{O}_0} \subset \mathcal{O}_1$, $\overline{\mathcal{O}_1} \subset \mathcal{F}$. Then there exist constants λ_4, κ_4, C depending only on $\mathcal{F}, \mathcal{O}_0, \mathcal{O}_1$ such that for $s \geqslant s_4$, $\lambda \geqslant \lambda_4$,

$$J(s,\widetilde{\psi}) \leqslant C\Bigg(s^2 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^2 \rho^2 (|g|^2 + |\phi_2|^2)\, dy\, dt$$
$$+ s^2 \int\limits_0^T \int\limits_{\mathcal{F}} e^{-2s\alpha}\xi^2 |\rho'|^2 |\rho|^{-2} |\widetilde{\psi}|^2\, dy\, dt + s^5 \int\limits_0^T \int\limits_{\mathcal{O}_1} e^{-2s\alpha}\xi^5 |\widetilde{\psi}|^2\, dy\, dt \Bigg). \tag{5.34}$$

Let us introduce the following quantities

$$I^{(2)}(s,z_2,\ell_z,\omega_z)=\frac{1}{s}\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\frac{1}{\xi}|\nabla^2\Delta z_2|^2dy\,dt+s\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt$$
$$+s^3\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3|\Delta z_2|^2\,dy\,dt+s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^4|\nabla z_2|^2dy\,dt$$
$$+s^6\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^6|z_2|^2dy\,dt+s^6\int_0^T e^{-2s\alpha_M}(\xi_m)^6(|\ell_z\cdot e_2|^2+|\omega_z|^2)\,dt\,, \quad (5.35)$$

$$\mathcal{B}_1=s\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt+s^3\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi^3|\Delta z_2|^2dy\,dt$$
$$+s^6\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^6|z_2|^2dy\,dt+s^5\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\,dy\,dt, \quad (5.36)$$

$$\mathcal{B}_2 = s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{L^2((0,T);\mathbf{L}^2(\partial\mathcal{F}))}+s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})}, \quad (5.37)$$

and

$$\mathcal{B}_3=\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}|\rho'|^2|\Delta\phi_2|^2dy\,dt+s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2\rho^2|\phi_2|^2\,dy\,dt$$
$$+s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2|\rho'|^2|\rho|^{-2}|\widetilde{\psi}|^2\,dy\,dt. \quad (5.38)$$

Gathering (5.27), (5.28), (5.32), (5.34) and the above definitions, we deduce

$$I^{(2)}(s,z_2,\ell_z,\omega_z)+J(s,\widetilde{\psi})\leqslant C\left(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2\rho^2|g|^2\,dy\,dt\right). \quad (5.39)$$

Step 3: recovering z_1 and $\ell_z\cdot e_1$

Using that $z=0$ on $(0,T)\times\partial\Omega$ and that the domain Ω is bounded, we can apply the Poincaré inequality

$$s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}(\xi_m)^4|z_1|^2\,dy\,dt\leqslant Cs^4\int_0^T e^{-2s\alpha_M}(\xi_m)^4\left(\int_{\mathcal{F}}\left|\frac{\partial z_1}{\partial y_1}\right|^2\,dy\right)dt.$$

Combining the above estimate with the fact that $\operatorname{div} z = 0$, we deduce

$$s^4 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha_M} (\xi_m)^4 |z_1|^2 \, dy \, dt \leqslant C s^4 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} \xi^4 |\nabla z_2|^2 \, dy \, dt. \tag{5.40}$$

Using Lemma 2.2, we have

$$s^4 \int_0^T e^{-2s\alpha_M} (\xi_m)^4 |\ell_z|^2 \, dt \leqslant C s^4 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha_M} (\xi_m)^4 |z|^2 \, dy \, dt. \tag{5.41}$$

Step 4: estimate of $\mathcal{B}_3$

Here (5.18) and (5.20) allow us to write

$$\begin{aligned} \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} |\rho'|^2 |\Delta \phi_2|^2 dy \, dt &= \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} |\rho'|^2 |\rho|^{-2} |\Delta(\rho \phi_2)|^2 dy \, dt \\ &\leqslant C s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} (\xi)^{9/4} |\Delta z_2|^2 dy \, dt + C s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} (\xi)^{9/4} |\Delta v_2|^2 dy \, dt. \end{aligned} \tag{5.42}$$

By applying Corollary 4.3 on system (5.21), we have

$$\begin{aligned} &\|v\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F})) \cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\ell_v\|^2_{H^1(0,T;\mathbb{R}^2)} + \|\omega_v\|^2_{H^1(0,T;\mathbb{R})} \\ &\leqslant C \left(\|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})} \right). \end{aligned} \tag{5.43}$$

Using (5.16) and applying estimate (5.43), we deduce

$$\begin{aligned} s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} (\xi)^{9/4} |\Delta v_2|^2 dy \, dt &\leqslant C \int_0^T \int_{\mathcal{F}} |\Delta v_2|^2 dy \, dt \\ &\leqslant C \left(\int_0^T \int_{\mathcal{F}} |\rho f|^2 dy \, dt + \int_0^T (|\rho h^{(1)}|^2 + |\rho h^{(2)}|^2) \, dt \right). \end{aligned}$$

From the above estimate, (5.9) and (5.42), we obtain

$$\begin{aligned} &\int_0^T \int_{\mathcal{F}} e^{-2s\alpha} |\rho'|^2 |\Delta \phi_2|^2 dy \, dt \\ &\leqslant C s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha} \xi^3 |\Delta z_2|^2 dy \, dt + C \left(\int_0^T \int_{\mathcal{F}} |\rho f|^2 dy \, dt + \int_0^T \left(|\rho h^{(1)}|^2 + |\rho h^{(2)}|^2 \right) dt \right). \end{aligned} \tag{5.44}$$

Similarly, by using (5.9), (5.18), (5.20) and (5.43)

$$s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2\rho^2|\phi_2|^2\,dy\,dt + s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2|\rho'|^2|\rho|^{-2}|\widetilde{\psi}|^2\,dy\,dt$$
$$\leqslant Cs^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2|z_2|^2dy\,dt + C\left(\int_0^T\int_{\mathcal{F}}|\rho f|^2\,dy\,dt + \int_0^T\left(|\rho h^{(1)}|^2+|\rho h^{(2)}|^2\right)\,dt\right)$$
$$+ Cs^4\int_0^T\int_{\mathcal{F}}\xi^{17/4}e^{-2s\alpha}|\widetilde{\psi}|^2\,dy\,dt. \quad (5.45)$$

Adding (5.44) and (5.45), we deduce

$$|\mathcal{B}_3| \leqslant Cs^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3|\Delta z_2|^2\,dy\,dt + Cs^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^2|z_2|^2dy\,dt$$
$$+ C\left(\int_0^T\int_{\mathcal{F}}|\rho f|^2\,dy\,dt + \int_0^T\left(|\rho h^{(1)}|^2+|\rho h^{(2)}|^2\right)\,dt\right) + Cs^4\int_0^T\int_{\mathcal{F}}\xi^{17/4}e^{-2s\alpha}|\widetilde{\psi}|^2\,dy\,dt. \quad (5.46)$$

Step 5: estimate of $\mathcal{B}_1$

We recall here a technical lemma that is obtained in [6, Step 3, Section 2.1]:

Lemma 5.2. *Let $\mathcal{O}_0$, $\mathcal{O}_1$ be open subsets of $\mathcal{F}$ such that $\overline{\mathcal{O}_0}\subset\mathcal{O}_1$, $\overline{\mathcal{O}_1}\subset\mathcal{F}$. There exist constants λ_5, s_5 and C depending on $\mathcal{F},\mathcal{O}_0,\mathcal{O}_1$ such that for every $s\geqslant s_5$, $\lambda\geqslant\lambda_5$, $\varepsilon>0$*

$$s\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt + s^3\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi^3|\Delta z_2|^2dy\,dt$$
$$\leqslant\varepsilon\left(\frac{1}{s}\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\frac{1}{\xi}|\nabla^2\Delta z_2|^2dy\,dt + s\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2dy\,dt + s^3\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^3|\Delta z_2|^2dy\,dt\right)$$
$$+ Cs^7\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^7|z_2|^2dy\,dt.$$

Let us introduce

$$\begin{aligned} I(s,z,\ell_z,\omega_z) =& \frac{1}{s}\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\frac{1}{\xi}|\nabla^2\Delta z_2|^2 dy\,dt + s\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi|\nabla\Delta z_2|^2 dy\,dt \\ &+ s^3\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3|\Delta z_2|^2\,dy\,dt + s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^4|\nabla z_2|^2 dy\,dt \\ &+ s^6\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^6|z_2|^2 dy\,dt + s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}\xi_m^4|z_1|^2\,dy\,dt \\ &+ s^4\int_0^T e^{-2s\alpha_M}\xi_m^4(|\ell_z|^2+|\omega_z|^2)\,dt. \end{aligned}$$

Thus by using (5.16), (5.39), (5.40), (5.41), (5.44), (5.46) and Lemma 5.2, we obtain that for $\lambda \geqslant \lambda_6$, $s \geqslant s_6$

$$\begin{aligned} I(s,z,\ell_z,\omega_z) + J(s,\widetilde{\psi}) \leqslant C\Bigg(&\int_0^T\int_{\mathcal{F}} \left(|\rho f|^2 + |\rho g|^2\right)\,dy\,dt + \int_0^T \left(|\rho h^{(1)}|^2 + |\rho h^{(2)}|^2\right)\,dt \\ &+ s^5\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\,dy\,dt + s^7\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^7|z_2|^2\,dy\,dt + \mathcal{B}_2\Bigg). \end{aligned} \tag{5.47}$$

Step 6: estimate of $\mathcal{B}_2$

In order to estimate the first term of $\mathcal{B}_2$, we use a trace theorem and an interpolation result:

$$\begin{aligned} \|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{\mathbf{L}^2(\partial\mathcal{F})} &\leqslant C\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{\mathbf{H}^{\frac{1}{2}}(\mathcal{F})} \\ &\leqslant C\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|_{\mathbf{L}^2(\mathcal{F})}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|_{\mathbf{H}^1(\mathcal{F})} \\ &= C\|e^{-s\alpha_M}s^{\frac{1}{2}}(\xi_m)^{\frac{1}{4}}\nabla\Delta z_2\|_{\mathbf{L}^2(\mathcal{F})}\|e^{-s\alpha_M}s^{-\frac{1}{2}}(\xi_m)^{-\frac{1}{2}}\nabla\Delta z_2\|_{\mathbf{H}^1(\mathcal{F})} \\ &\leqslant C\left(\|e^{-s\alpha_M}s^{\frac{1}{2}}(\xi_m)^{\frac{1}{4}}\nabla\Delta z_2\|^2_{\mathbf{L}^2(\mathcal{F})} + \|e^{-s\alpha_M}s^{-\frac{1}{2}}(\xi_m)^{-\frac{1}{2}}\nabla\Delta z_2\|^2_{\mathbf{H}^1(\mathcal{F})}\right). \end{aligned}$$

Now integrating both sides in $(0,T)$ and using (5.9) we obtain

$$\begin{aligned} &s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{L^2((0,T);\mathbf{L}^2(\partial\mathcal{F}))} \\ &\qquad\leqslant Cs^{-\frac{1}{2}}\left(s\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}\xi_m|\nabla\Delta z_2|^2\,dy\,dt + \frac{1}{s}\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}\frac{1}{\xi_m}|\nabla^2\Delta z_2|^2\,dy\,dt\right) \\ &\qquad\leqslant Cs^{-\frac{1}{2}}I(s,z,\ell_z,\omega_z). \end{aligned} \tag{5.48}$$

In order to estimate the second term of $\mathcal{B}_2$, we use that

$$L^2(0,T;H^2(\mathcal{F}))\cap H^1(0,T;L^2(\mathcal{F}))\subset H^{1/4}(0,T;H^{3/2}(\mathcal{F}))$$

with continuous embedding. In particular, combining this and the trace theorem, we find

$$\begin{aligned}
s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{H^{\frac{1}{4}}(0,T;\mathbf{L}^2(\partial\mathcal{F}))} &\leqslant Cs^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{H^{\frac{1}{4}}(0,T;\mathbf{H}^{1/2}(\mathcal{F}))}\\
&\leqslant Cs^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\Delta z_2\|^2_{H^{\frac{1}{4}}(0,T;H^{3/2}(\mathcal{F}))}\\
&\leqslant Cs^{-\frac{1}{2}}\left(\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{L^2(0,T;H^4(\mathcal{F}))}+\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{H^1(0,T;H^2(\mathcal{F}))}\right). \quad (5.49)
\end{aligned}$$

On the other hand, by using the trace theorem,

$$s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{L^2(0,T;\mathbf{H}^{1/2}(\partial\mathcal{F}))}\leqslant Cs^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{L^2(0,T;H^4(\mathcal{F}))}.$$

Combining the above estimate and (5.49), we deduce

$$\begin{aligned}
&s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})}\\
&\qquad\leqslant Cs^{-\frac{1}{2}}\left(\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{L^2(0,T;H^4(\mathcal{F}))}+\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{H^1(0,T;H^2(\mathcal{F}))}\right). \quad (5.50)
\end{aligned}$$

We now estimate the right-hand side of (5.50). Let us write

$$\widehat{z}=e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z,\qquad \widehat{q_z}=e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}q_z; \quad (5.51)$$

$$\widehat{\ell_z}=e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\ell_z,\qquad \widehat{\omega_z}=e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\omega_z. \quad (5.52)$$

Since (z,q_z,ℓ_z,ω_z) satisfies (5.22), $(\widehat{z},\widehat{q_z},\widehat{\ell_z},\widehat{\omega_z})$ is the solution of the following system

$$\begin{cases}
-\dfrac{\partial\widehat{z}}{\partial t}-\nu\Delta\widehat{z}+\nabla\widehat{q_z}=F^{(4)}, & \text{in } (0,T)\times\mathcal{F},\\
\text{div }\widehat{z}=0, & \text{in } (0,T)\times\mathcal{F},\\
\widehat{z}(t,y)=0, & t\in(0,T),\ y\in\partial\Omega,\\
\widehat{z}(t,y)=\widehat{\ell_z}(t)+\widehat{\omega_z}y^{\perp}, & t\in(0,T),\ y\in\partial\mathcal{S},\\
-M\widehat{\ell_z}'(t)=-\displaystyle\int_{\partial\mathcal{S}}\sigma(\widehat{z},\widehat{q_z})n d\Gamma\ +\ F^{(5)}, & t\in(0,T),\\
-J\widehat{\omega_z}'(t)=-\displaystyle\int_{\partial\mathcal{S}}y^{\perp}\cdot\sigma(\widehat{z},\widehat{q_z})n d\Gamma\ +\ F^{(6)}, & t\in(0,T),\\
\widehat{z}(T,y)=0, & y\in\mathcal{F},\\
\widehat{\ell_z}(T)=0,\ \widehat{\omega_z}(T)=0,
\end{cases} \quad (5.53)$$

where

$$F^{(4)} = -e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\rho'\phi - \frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)z,$$
$$F^{(5)} = -M\rho' e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\ell_\phi - M\frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)\ell_z,$$
$$F^{(6)} = -J\rho' e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\omega_\phi - J\frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)\omega_z.$$

Note that if we extend $F^{(4)}$ by $F^{(5)} + F^{(6)}y^\perp$ for $y \in \mathcal{S}$, we have from (5.24) and (5.25) that

$$F^{(4)} \in L^2(0,T;\mathcal{D}(A_1)) \cap H^1(0,T;\mathbb{H}_1).$$

We can thus apply Corollary 4.3 and we have the following estimate

$$\begin{aligned}&\|\widehat{z}\|^2_{L^2(0,T;\mathbf{H}^4(\mathcal{F}))\cap H^1(0,T;\mathbf{H}^2(\mathcal{F}))} + \|\widehat{\ell_z}\|^2_{H^2(0,T;\mathbb{R}^2)} + \|\widehat{\omega_z}\|^2_{H^2(0,T;\mathbb{R})}\\ &\leqslant C\left(\|F^{(4)}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|F^{(5)}\|^2_{H^1(0,T;\mathbb{R}^2)} + \|F^{(6)}\|^2_{H^1(0,T;\mathbb{R})}\right).\end{aligned} \tag{5.54}$$

Now

$$\begin{aligned}\|F^{(4)}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} &\leqslant \|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\rho'\phi\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\quad + \left\|\frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)z\right\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}.\end{aligned} \tag{5.55}$$

Since $|\rho'| \leqslant Cs(\xi_m)^{9/8}\rho$ and by using (5.9)-(5.16) we obtain

$$\begin{aligned}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\rho'\phi\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} &\leqslant C\Big(\|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}(z+v)\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\quad + \|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|v\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\Big).\end{aligned} \tag{5.56}$$

With the help of (5.9)-(5.16), the second term in right hand side of (5.55) becomes

$$\begin{aligned}\left\|\frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)z\right\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} &\leqslant C\Big(\|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}z\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\quad + \|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\Big).\end{aligned} \tag{5.57}$$

Thus, in order to estimate $\|F^{(4)}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}$, we have to find an estimate on $\|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}z\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}$. Let us define

$$\begin{aligned}\breve{z} &= se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}z, \qquad \breve{q_z} = se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}q_z;\\ \breve{\ell_z} &= se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\ell_z, \qquad \breve{\omega_z} = se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\omega_z.\end{aligned} \tag{5.58}$$

From (5.22), we deduce that $(\breve{z}, \breve{q}_z, \breve{\ell}_z, \breve{\omega}_z)$ satisfies the following system

$$\begin{cases} -\dfrac{\partial \breve{z}}{\partial t} - \nu\Delta\breve{z} + \nabla\breve{q}_z = F^{(1)}, & \text{in } (0,T)\times\mathcal{F}, \\ \text{div } \breve{z} = 0, & \text{in } (0,T)\times\mathcal{F}, \\ \breve{z}(t,y) = 0, & t\in(0,T),\ y\in\partial\Omega, \\ \breve{z}(t,y) = \breve{\ell}_z(t) + \breve{\omega}_z(t)y^{\perp}, & t\in(0,T),\ y\in\partial\mathcal{S}, \\ -M\breve{\ell}_z'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(\breve{z},\breve{q}_z)n d\Gamma \ + \ F^{(2)}, & t\in(0,T), \\ -J\breve{\omega}_z'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^{\perp}\cdot\sigma(\breve{z},\breve{q}_z)n d\Gamma \ + \ F^{(3)}, & t\in(0,T), \\ \breve{z}(T,y) = 0, & y\in\mathcal{F}, \\ \breve{\ell}_z(T) = 0,\ \breve{\omega}_z(T) = 0, \end{cases} \tag{5.59}$$

where

$$\begin{aligned} F^{(1)} &= -se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\rho'\phi - \frac{d}{dt}(se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}})z, \\ F^{(2)} &= -Mse^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\rho'\ell_\phi - M\frac{d}{dt}(se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}})\ell_z, \\ F^{(3)} &= -Jse^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\rho'\omega_\phi - J\frac{d}{dt}(se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}})\omega_z. \end{aligned}$$

By applying Corollary 4.3 on system (5.59), we have

$$\begin{aligned} &\|\breve{z}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\breve{\ell}_z\|^2_{H^1(0,T;\mathbb{R}^2)} + \|\breve{\omega}_z\|^2_{H^1(0,T;\mathbb{R})} \\ &\leqslant C\left(\|F^{(1)}\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|F^{(2)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|F^{(3)}\|^2_{L^2(0,T;\mathbb{R})}\right). \end{aligned} \tag{5.60}$$

Now we are going to estimate the quantities in the right-hand side of (5.60). Using (5.18), (5.20) and (5.16), we deduce

$$\begin{aligned} \|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}\rho'\phi\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} &\leqslant C\|s^2e^{-s\alpha_M}\xi_m^2(z+v)\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \\ &\leqslant C\left(\|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|v\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\right). \end{aligned} \tag{5.61}$$

Using (5.13) and (5.11)

$$\begin{aligned} &\left\|\frac{d}{dt}(se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}})z\right\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \\ &\qquad\leqslant C\left(\left\|s^2e^{-s\alpha_M}\xi_m^2 z\right\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \left\|se^{-s\alpha_M}\xi_m z\right\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\right). \end{aligned}$$

Gathering the above estimate with (5.61) and (5.43), we deduce

$$\|F^{(1)}\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leqslant C\bigg(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\bigg). \tag{5.62}$$

Similarly, we obtain

$$\|F^{(2)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|F^{(3)}\|^2_{L^2(0,T;\mathbb{R})} \leqslant C\bigg(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\bigg). \tag{5.63}$$

Thus from (5.60), (5.62) and (5.63), we get

$$\begin{aligned}&\|\breve{z}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\breve{\ell}_z\|^2_{H^1(0,T;\mathbb{R}^2)} + \|\widetilde{\omega_z}\|^2_{H^1(0,T;\mathbb{R})}\\ &\leqslant C\bigg(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\bigg).\end{aligned} \tag{5.64}$$

Now we go back to (5.56) and by applying (5.43), (5.64) with (5.9), we obtain

$$\begin{aligned}&\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\rho'\phi\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\leqslant C\Big(\|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}(z+v)\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|v\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\Big)\\ &\leqslant C\bigg(\|\breve{z}\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|v\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\bigg)\\ &\leqslant C\bigg(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\bigg).\end{aligned} \tag{5.65}$$

Now we look at (5.57) and the second term in right hand side of (5.55) becomes

$$\begin{aligned}&\left\|\frac{d}{dt}\left(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\right)z\right\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\leqslant C\Big(\|se^{-s\alpha_M}(\xi_m)^{\frac{7}{8}}z\|^2_{L^2(0,T;\mathbf{H}^2(\mathcal{F}))\cap H^1(0,T;\mathbf{L}^2(\mathcal{F}))} + \|s^2e^{-s\alpha_M}\xi_m^2 z\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\Big)\\ &\leqslant C\bigg(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\bigg).\end{aligned} \tag{5.66}$$

Similarly we obtain

$$\begin{aligned}\|F^{(5)}\|^2_{H^1(0,T;\mathbb{R}^2)} + \|F^{(6)}\|^2_{H^1(0,T;\mathbb{R})}&\\ \leqslant \|M\rho' e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\ell_\phi\|^2_{H^1(0,T;\mathbb{R}^2)} + \|M(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}})_t\ell_z\|^2_{H^1(0,T;\mathbb{R}^2)}&\\ + \|J\rho' e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\omega_\phi\|^2_{H^1(0,T;\mathbb{R})} + \|M(e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}})_t\omega_z\|^2_{H^1(0,T;\mathbb{R})}&\\ \leqslant C\left(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\right).&\quad (5.67)\end{aligned}$$

Thus by using (5.65), (5.66) and (5.67), inequality (5.54) becomes

$$\begin{aligned}&\|\widehat{z}\|^2_{L^2(0,T;\mathbf{H}^4(\mathcal{F}))\cap H^1(0,T;\mathbf{H}^2(\mathcal{F}))} + \|\widehat{\ell_z}\|^2_{H^2(0,T;\mathbb{R}^2)} + \|\widehat{\omega_z}\|^2_{H^2(0,T;\mathbb{R})}\\ &\leqslant C\left(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\right).\end{aligned}\quad (5.68)$$

By definition (5.51) of $\widehat{z}$, the above estimate yields

$$\begin{aligned}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{L^2(0,T;H^4(\mathcal{F}))} + \|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}z_2\|^2_{H^1(0,T;H^2(\mathcal{F}))}&\\ \leqslant C\left(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\right).&\end{aligned}$$

Hence by above estimate and (5.48), (5.50), we get

$$\begin{aligned}\mathcal{B}_2 = s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{8}}\nabla\Delta z_2\|^2_{L^2(0,T;\mathbf{L}^2(\partial\mathcal{F}))} + s^{-\frac{1}{2}}\|e^{-s\alpha_M}(\xi_m)^{-\frac{1}{4}}\nabla\Delta z_2\|^2_{\mathbf{H}^{\frac{1}{4},\frac{1}{2}}((0,T)\times\partial\mathcal{F})}&\\ \leqslant Cs^{-\frac{1}{2}}\left(I(s,z,\ell_z,\omega_z) + \|\rho f\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\rho h^{(1)}\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\rho h^{(2)}\|^2_{L^2(0,T;\mathbb{R})}\right).&\end{aligned}$$

Step 7: going back to $\phi, \ell_\phi, \omega_\phi$

By taking s large enough, from (5.47) we can conclude that:

$$\begin{aligned}I(s,z,\ell_z,\omega_z) + J(s,\widetilde{\psi}) \leqslant C\Bigg(\int_0^T\int_{\mathcal{F}}\left(|\rho f|^2 + |\rho g|^2\right) + \int_0^T\left(|\rho h^{(1)}|^2 + |\rho h^{(2)}|^2\right)&\\ + s^5\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\,dy\,dt + s^7\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^7|\rho\phi_2|^2\,dy\,dt\Bigg).&\quad (5.69)\end{aligned}$$

Let us introduce

$$\widetilde{I}(s,\rho\phi,\rho\ell_\phi,\rho\omega_\phi) = s^3\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^3\rho^2|\Delta\phi_2|^2\,dy\,dt + s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^4\rho^2|\nabla\phi_2|^2 dy\,dt$$
$$+ s^6\int_0^T\int_{\mathcal{F}} e^{-2s\alpha}\xi^6\rho^2|\phi_2|^2 dy\,dt + s^4\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}(\xi_m)^4\rho^2|\phi_1|^2\,dy\,dt$$
$$+ s^4\int_0^T e^{-2s\alpha_M}(\xi_m)^4\left(|\rho\ell_\phi|^2+|\rho\omega_\phi|^2\right)\,dt.$$

Again by using (5.16), (5.20), (5.43), (5.58), (5.64) and (5.69), for all $\lambda \geqslant \lambda_7$, $s \geqslant s_7$, we have

$$s^2\int_0^T\int_{\mathcal{F}} e^{-2s\alpha_M}\xi^{7/4}\rho^2\left|\frac{\partial\phi_2}{\partial t}\right|^2 dy\,dt + \widetilde{I}(s,\rho\phi,\rho\ell_\phi,\rho\omega_\phi) + J(s,\widetilde{\psi})$$
$$\leqslant C\Bigg(\int_0^T\int_{\mathcal{F}}(|\rho f|^2+|\rho g|^2) + \int_0^T(|\rho h^{(1)}|^2+|\rho h^{(2)}|^2)$$
$$+ s^5\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\,dy\,dt + s^7\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^7|\rho\phi_2|^2\,dy\,dt\Bigg), \quad (5.70)$$

Step 8: removing the local term in ϕ_2

We are going to estimate the last term of inequality (5.70) by following the same approach as in [5]:
Let $\overline{\mathcal{O}_1} \subset \mathcal{O}$. Consider a non-negative function $\chi \in C_c^2(\mathcal{O})$ such that $\chi = 1$ in $\mathcal{O}_1$. Now by using equation (5.23), we get

$$s^7\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^7|\rho\phi_2|^2\,dy\,dt \leqslant Cs^7\int_0^T\int_{\mathcal{O}} \chi e^{-2s\alpha}\xi^7|\rho\phi_2|^2\,dy\,dt$$
$$= Cs^7\int_0^T\int_{\mathcal{O}} \chi e^{-2s\alpha}\xi^7\rho\phi_2\left(-\frac{\partial\widetilde{\psi}}{\partial t} - \Delta\widetilde{\psi} - \rho g + \rho'\psi\right)\,dy\,dt.$$

Our main aim is to estimate the local integrals of $\widetilde{\psi}$ and g. Then via integration by parts and Young's inequality, we obtain that for any $\varepsilon > 0$, there exists $C > 0$ such that

$$s^7 \int_0^T \int_{\mathcal{O}_1} e^{-2s\alpha} \xi^7 |\rho\phi_2|^2 \, dy \, dt \leqslant \varepsilon \left(s^2 \int_0^T \int_{\mathcal{O}} e^{-2s\alpha_M} (\xi_m)^{7/4} \rho^2 |\phi_2'|^2 \, dy \, dt + \widetilde{I}(s, \rho\phi, \rho\ell_\phi, \rho\omega_\phi) \right)$$
$$+ C \left(s^{12} \int_0^T \int_{\mathcal{O}} e^{-4s\alpha + 2s\alpha_M} \xi^{\frac{49}{4}} |\widetilde{\psi}|^2 \, dx \, dt + s^8 \int_0^T \int_{\mathcal{O}} e^{-2s\alpha} \xi^8 |\rho g|^2 \right). \quad (5.71)$$

Thus finally from (5.70) and (5.71), we get

$$s^2 \int_0^T \int_{\mathcal{F}} e^{-2s\alpha_M} \xi^{7/4} \rho^2 \left| \frac{\partial \phi_2}{\partial t} \right|^2 \, dy \, dt + \widetilde{I}(s, \rho\phi, \ell_z, \omega_z) + J(s, \widetilde{\psi})$$
$$\leqslant C \left(\int_0^T \int_{\mathcal{F}} \left(|\rho f|^2 + |\rho g|^2 \right) + \int_0^T \left(|\rho h^{(1)}|^2 + |\rho h^{(2)}|^2 \right) \right.$$
$$\left. + s^{12} \int_0^T \int_{\mathcal{O}} e^{-4s\alpha + 2s\alpha_M} \xi^{\frac{49}{4}} |\widetilde{\psi}|^2 \, dx \, dt + s^8 \int_0^T \int_{\mathcal{O}} e^{-2s\alpha} \xi^8 |\rho g|^2 \right). \quad (5.72)$$

We have finished the proof of Proposition 5.1. □

6. Null controllability of the linearized system

In this section, we use the Carleman estimate obtained in Theorem 5.1 to deduce the null controllability of a linear system associated with (3.14)–(3.25). We recall that $\mathbb{H}$ is defined in (4.22) and the operator A is defined in (4.23)-(4.28). We define the control operator $B \in \mathcal{L}(L^2(\mathcal{O}), \mathbb{H})$ as

$$Bw_0 = (0, \, w_0 \mathbb{1}_{\mathcal{O}}),$$

and the operator $C \in \mathcal{L}(\mathbb{H}, \mathbb{R}^3)$ is defined as

$$C(u, \theta) = (\ell_u, \omega_u), \text{ if } u = \ell_u + \omega_u y^\perp \text{ in } \mathcal{S}.$$

If we set $Z = \begin{pmatrix} u \\ \theta \end{pmatrix}$, $d = \begin{pmatrix} h \\ \beta \end{pmatrix}$ and $Z_0 = \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix}$, $d_0 = \begin{pmatrix} h_0 \\ \beta_0 \end{pmatrix}$, then the linear system (4.1)-(4.13) can be written as

$$\begin{cases} \dot{Z}(t) = AZ(t) + Bw_0(t) + F(t), \\ \dot{d}(t) = CZ(t), \\ Z(0) = Z_0 \in \mathbb{H}, \\ d(0) = d_0 \in \mathbb{R}^3, \end{cases} \quad (6.1)$$

with

$$F = \begin{pmatrix} \mathbb{P} f_1 \\ \widetilde{g} \end{pmatrix},$$

where

$$f_1 = \begin{cases} \widetilde{f} & \text{in} \quad \mathcal{F}, \\ \widetilde{h}^{(1)} + \widetilde{h}^{(2)} y^{\perp} & \text{in} \quad \mathcal{S}. \end{cases}$$

The adjoint system of (6.1) is given by:

$$\begin{cases} -\dot{\Phi}(t) = A^*\Phi(t) + \gamma^1(t) + C^*\gamma^2, \\ \Phi(T) = 0, \end{cases} \tag{6.2}$$

where $(\gamma^1,\, \gamma^2) \in L^2(0,T;\mathbb{H}) \times \mathbb{R}^3$.
Let us fix $s \geqslant s_0$, $\lambda \geqslant \lambda_0$ as in Theorem 5.1 and consider ρ_i for $i \in \{1,2,3\}$ and $\widetilde{\rho}$ in the following way

$$\rho_1(t) = \begin{cases} s^2 e^{-\frac{5}{2}s\alpha_M(T/2)}(\xi_m(T/2))^2 & \text{if } t \in (0,T/2), \\ s^2 e^{-\frac{5}{2}s\alpha_M(t)}(\xi_m(t))^2 & \text{if } t \in (T/2,T); \end{cases} \tag{6.3}$$

$$\rho_2(t) = \begin{cases} e^{-\frac{3}{2}s\alpha_M(T/2)} & \text{if } t \in (0,T/2), \\ e^{-\frac{3}{2}s\alpha_M(t)} & \text{if } t \in (T/2,T); \end{cases} \tag{6.4}$$

$$\rho_3(t) = \begin{cases} s^6 e^{-2s\alpha_m(T/2)-\frac{s}{2}\alpha_M(T/2)}(\xi_M(T/2))^{\frac{49}{8}} & \text{if } t \in (0,T/2), \\ s^6 e^{-2s\alpha_m(t)-\frac{s}{2}\alpha_M(t)}(\xi_M(t))^{\frac{49}{8}} & \text{if } t \in (T/2,T), \end{cases} \tag{6.5}$$

and

$$\widetilde{\rho}(t) = \begin{cases} e^{-\frac{11}{8}s\alpha_M(T/2)} & \text{if } t \in (0,T/2), \\ e^{-\frac{11}{8}s\alpha_M(t)} & \text{if } t \in (T/2,T). \end{cases} \tag{6.6}$$

Thus ρ_i and $\widetilde{\rho}$ are continuous functions such that

$$\begin{aligned} \rho_i(T) = 0 \quad \text{and} \quad \rho_i > 0 \quad \text{in} \quad [0,T), \\ \widetilde{\rho}(T) = 0 \quad \text{and} \quad \widetilde{\rho} > 0 \quad \text{in} \quad [0,T). \end{aligned}$$

We define the following spaces

$$\begin{aligned} \mathfrak{F} &= \left\{ F \in L^2(0,T;\mathbb{H});\ \frac{F}{\rho_1} \in L^2(0,T;\mathbb{H}) \right\}, \\ \mathfrak{Z} &= \left\{ Z \in L^2(0,T;\mathbb{H});\ \frac{Z}{\rho_2} \in L^2(0,T;\mathbb{H}) \right\}, \\ \mathfrak{U} &= \left\{ w_0 \in L^2(0,T;L^2(\mathcal{O}));\ \frac{w_0}{\rho_3} \in L^2(0,T;L^2(\mathcal{O})) \right\}. \end{aligned}$$

Our main result here is the following

Theorem 6.1. *There exists a linear bounded operator*

$$E_T : \mathbb{H} \times \mathbb{R}^3 \times \mathfrak{F} \to \mathfrak{U}$$

such that for any $(Z_0,\, d_0,\, F) \in \mathbb{H} \times \mathbb{R}^3 \times \mathfrak{F}$, *the control* $w_0 = E_T((Z_0,\, d_0,\, F))$ *is such that the solution* (Z,d) *to equation* (6.1) *satisfy* $Z \in \mathfrak{Z}$ *and* $d(T) = 0$.

Moreover, if we assume that $Z_0 \in D((-A)^{\frac{1}{2}})$*, then we have*

$$\frac{Z}{\widetilde{\rho}} \in L^2(0,T;D(A)) \cap C([0,T];D((-A)^{\frac{1}{2}})) \cap H^1(0,T;\mathbb{H}), \tag{6.7}$$

and we have the following estimate:

$$\left\| \frac{Z}{\widetilde{\rho}} \right\|_{L^2(0,T;D(A))\cap C([0,T];D((-A)^{\frac{1}{2}}))\cap H^1(0,T;\mathbb{H})} \leqslant C\left(\|F\|_{\mathfrak{F}} + \|d_0\|_{\mathbb{R}^3} + \|Z_0\|_{D((-A)^{\frac{1}{2}})} \right). \tag{6.8}$$

Proof. We use [29, Theorem 4.1]: the existence of E_T is obtained from the following observability inequality for adjoint equation (6.2):

$$\|\gamma^2\|^2_{\mathbb{R}^3} + \|\Phi(0)\|^2_{\mathbb{H}} + \int_0^T \|\rho_1 \Phi\|^2_{\mathbb{H}}\, dt \leqslant C\left(\int_0^T \|\rho_2 \gamma^1\|^2_{\mathbb{H}}\, dt + \int_0^T \|\rho_3 B^* \Phi\|^2_{L^2(\mathcal{O})}\, dt \right). \tag{6.9}$$

We thus prove the above estimate and this gives us the existence of E_T and the second part of the theorem. Indeed, using [29, Corollary 4.3], this second part comes from the following relations

$$\frac{(\widetilde{\rho})'\rho_2}{(\widetilde{\rho})^2} \in L^\infty(0,T) \quad \text{and} \quad \frac{\rho_i}{\widetilde{\rho}} \in L^\infty(0,T),\ \forall i \in \{1,3\} \tag{6.10}$$

that can be obtained from the definition of functions (6.3)-(6.6) and from the relations (5.9)-(5.16).

It remains to prove (6.9). First, we notice that (6.2) can be written in the following form:

$$\begin{cases} -\dfrac{\partial \phi}{\partial t} - \nu\Delta\phi + \nabla q = \gamma_1^1, & \text{in } (0,T)\times\mathcal{F}, \\ \text{div } \phi = 0, & \text{in } (0,T)\times\mathcal{F}, \\ \phi(t,y) = 0, & t\in(0,T),\ y\in\partial\Omega, \\ \phi(t,y) = \ell_\phi(t) + \omega_\phi(t) y^\perp, & t\in(0,T),\ y\in\partial\mathcal{S}, \\ -\dfrac{\partial \psi}{\partial t} - \mu\Delta\psi = \gamma_2^1 + \phi_2, & \text{in } (0,T)\times\mathcal{F}, \\ \dfrac{\partial \psi}{\partial n}(t,y) = 0, & t\in(0,T),\ y\in\partial\mathcal{F} \\ -M\ell_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(\phi,\, q) n\, d\Gamma + M\ell_{\gamma^1} + M\ell_{\gamma^2}, & t\in(0,T), \\ -J\omega_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^\perp\cdot\sigma(\phi,\, q) n\, d\Gamma + J\omega_{\gamma^1} + J\omega_{\gamma^2}, & t\in(0,T), \\ \phi(T,y) = 0 \text{ and } \psi(T,y) = 0, & y\in\mathcal{F}, \\ \ell_\phi(T) = 0,\ \omega_\phi(T) = 0, & \end{cases} \tag{6.11}$$

where $\gamma^1 = (\gamma_1^1, \gamma_2^1) \in \mathbb{H}_1 \times L^2(\mathcal{F})$ and $\gamma^2 = (\ell_{\gamma^2}, \omega_{\gamma^2}) \in \mathbb{R}^3$. In particular, we have

$$\gamma^1(t,y) = \ell_{\gamma^1}(t) + \omega_{\gamma^1}(t) y^\perp \quad t\in(0,T),\ y\in\mathcal{S}.$$

With the above notation, the condition (6.9) can be rewritten as

$$|\gamma^2|^2 + \|\phi(0)\|^2_{\mathbf{L}^2(\mathcal{F})} + \|\psi(0)\|^2_{L^2(\mathcal{F})} + \int_0^T \|\rho_1\phi\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T \|\rho_1\psi\|^2_{L^2(\mathcal{F})}\,dt \\ \leqslant C\left(\int_0^T \|\rho_2\gamma^1\|^2_{\mathbb{H}}\,dt + \int_0^T\int_{\mathcal{O}} |\rho_3\psi|^2\,dy\,dt\right). \quad (6.12)$$

The proof of (6.12) is based on Theorem 5.1. We set

$$\rho_i^*(t) = \begin{cases} \rho_i(T-t) & \text{if } t\in(0,T/2),\\ \rho_i(t) & \text{if } t\in(T/2,T)\,. \end{cases}$$

and then, (5.17) implies that

$$\int_0^T \|\rho_1^*\phi\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T \|\rho_1^*\psi\|^2_{L^2(\mathcal{F})}\,dt \leqslant C\left(\int_0^T \|\rho_2^*(\gamma^1 + C^*\gamma^2)\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T\int_{\mathcal{O}} |\rho_3^*\psi|^2\,dy\,dt\right). \quad (6.13)$$

Then by following similar steps as in [5, Lemma 3.2] (using in particular the energy estimates), we can deduce from the above estimate

$$\|\phi(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\psi(0)\|^2_{L^2(\mathcal{F})} + \int_0^T \|\rho_1\phi\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T \|\rho_1\psi\|^2_{L^2(\mathcal{F})}\,dt \\ \leqslant C\left(\int_0^T \|\rho_2(\gamma^1 + C^*\gamma^2)\|^2_{\mathbb{H}}\,dt + \int_0^T\int_{\mathcal{O}} |\rho_3\psi|^2\,dy\,dt\right). \quad (6.14)$$

In order to prove (6.12) from the above estimate, it is sufficient to show the following inequality:

$$|\gamma^2|^2 \leqslant C\left(\int_0^T \|\rho_2\gamma_1^1\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T \|\rho_2\gamma_2^1\|^2_{L^2(\Omega)}\,dt + \int_0^T\int_{\mathcal{O}} |\rho_3\psi|^2\,dy\,dt\right). \quad (6.15)$$

We argue by contradiction: assume that (6.15) is false. Then there exists a sequence

$$(\gamma_n^2,\ \gamma_{1,n}^1,\ \gamma_{2,n}^1,\ \phi_n,\ \psi_n)$$

such that (6.11) holds and such that

$$\int_0^T \|\rho_2\gamma_{1,n}^1\|^2_{\mathbf{L}^2(\Omega)}\,dt + \int_0^T \|\rho_2\gamma_{2,n}^1\|^2_{L^2(\Omega)}\,dt + \int_0^T\int_{\mathcal{O}} |\rho_3\psi_n|^2\,dy\,dt \to 0, \quad |\gamma_n^2|^2 = 1. \quad (6.16)$$

Writing $\Phi_n = (\phi_n, \psi_n)$, we have

$$\begin{cases} -\dot\Phi_n(t) = A^*\Phi_n(t) + \gamma_n^1(t) + C^*\gamma_n^2,\\ \Phi_n(T) = 0. \end{cases} \quad (6.17)$$

Let us fix $\varepsilon > 0$. From (6.16), we deduce, up to a subsequence, $\gamma_n^2 \to \gamma^2$ in $\mathbb{R}^3$ with $|\gamma^2| = 1$ and

$$\gamma_n^1 \to 0 \quad \text{in} \quad L^2(0, T-\varepsilon; \mathbb{H}).$$

From inequality (6.14), we also have that $\|(\rho_1\phi_n, \rho_1\psi_n)\|_{L^2(0,T;\mathbb{H})}$ is bounded. In particular, up to a subsequence,

$$(\phi_n, \psi_n) \to (\phi, \psi) \quad \text{weakly in} \quad L^2(0, T-\varepsilon; \mathbb{H}),$$

where (ϕ, ψ) satisfies the following system

$$\begin{cases} -\dfrac{\partial \phi}{\partial t} - \nu\Delta\phi + \nabla q = 0, & \text{in } (0, T-\varepsilon) \times \mathcal{F}, \\ \operatorname{div} \phi = 0, & \text{in } (0, T-\varepsilon) \times \mathcal{F}, \\ \phi(t, y) = 0, & t \in (0, T-\varepsilon),\ y \in \partial\Omega, \\ -\dfrac{\partial \psi}{\partial t} - \mu\Delta\psi = \phi_2, & \text{in } (0, T-\varepsilon) \times \mathcal{F}, \\ \psi(t, y) = 0, & t \in (0, T-\varepsilon),\ y \in \partial\Omega, \\ \phi(t, y) = \ell_\phi(t) + \omega_\phi(t) y^\perp, & t \in (0, T-\varepsilon),\ y \in \partial\mathcal{S}, \\ \psi(t, y) = 0, & t \in (0, T-\varepsilon),\ y \in \partial\mathcal{S}, \\ -M\ell_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} \sigma(\phi, q) n \, d\Gamma + M\ell_{\gamma^2}, & t \in (0, T-\varepsilon), \\ -J\omega_\phi'(t) = -\displaystyle\int_{\partial\mathcal{S}} y^\perp \cdot \sigma(\phi, q) n \, d\Gamma + J\omega_{\gamma^2}, & t \in (0, T-\varepsilon), \end{cases} \tag{6.18}$$

with $(\ell_{\gamma^2}, \omega_{\gamma^2}) = \gamma^2$.

On the other hand, we have from (6.16)

$$\psi = 0 \quad \text{in } (0, T-\varepsilon) \times \mathcal{O}. \tag{6.19}$$

Thus from (6.18) and (6.19), we obtain

$$\phi_2 = 0 \quad \text{in } (0, T-\varepsilon) \times \mathcal{O}. \tag{6.20}$$

Now, combining $\operatorname{div} \phi = 0$ and $\phi_2 = 0$ in $(0, T-\varepsilon) \times \mathcal{O}$, we deduce

$$\frac{\partial \phi_1}{\partial x_1} = 0 \text{ in } (0,\ T-\varepsilon) \times \mathcal{O}. \tag{6.21}$$

On the other hand, $\frac{\partial \phi}{\partial x_1}$ satisfies the system

$$-\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x_1}\right) - \nu\Delta\left(\frac{\partial \phi}{\partial x_1}\right) + \nabla\left(\frac{\partial q}{\partial x_1}\right) = 0, \qquad \text{in } (0, T-\varepsilon) \times \mathcal{F}, \tag{6.22}$$

$$\operatorname{div}\left(\frac{\partial \phi}{\partial x_1}\right) = 0, \qquad \text{in } (0, T-\varepsilon) \times \mathcal{F}, \tag{6.23}$$

$$\frac{\partial \phi}{\partial x_1} = 0 \qquad \text{in } (0, T-\varepsilon) \times \mathcal{O}. \tag{6.24}$$

Thus, by using unique continuation property of the Stokes system ([13]), we obtain that

$$\frac{\partial \phi}{\partial x_1} = 0 \quad \text{in } (0, T-\varepsilon) \times \mathcal{F}. \tag{6.25}$$

By applying the Poincaré inequality, the above relation yields

$$\phi = 0 \quad \text{in } (0, T-\varepsilon) \times \mathcal{F}. \tag{6.26}$$

In particular, $(\ell_\phi, \omega_\phi) = (0,0)$ in $(0, T-\varepsilon)$ and from last two equations of (6.18), we find

$$\gamma^2 = (\ell_{\gamma^2}, \omega_{\gamma^2}) = (0,0), \tag{6.27}$$

which contradicts the fact that $|\gamma^2| = 1$.

Thus we have established inequality (6.15) and combining this inequality with (6.14), we have proven (6.12). □

7. The Nonlinear Problem

This section is devoted to the proof of the main result.

7.1. **Estimates of the nonlinear terms.** In this section, we give some estimates on the coefficients appearing in the system (3.14)-(3.25).

We assume here that h and β satisfy

$$h(T) = 0, \quad \beta(T) = 0, \quad \frac{(h', \beta')}{\widetilde{\rho}} \in L^2(0,T).$$

With our choice of $\widetilde{\rho}$ (see (6.6) and (5.5)), we deduce in particular that

$$|h(t)| + |\beta(t)| \leqslant T^{1/2} \widetilde{\rho}(t) \left(\left\| \frac{h'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{\beta'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right).$$

Following the proofs of [1, Proposition 12] and [1, Lemma 31], we obtain the following estimates

Lemma 7.1. *Assume (3.4). Then, for any $(\overline{u}, \overline{p}, \overline{\theta}) \in \mathbf{H}^2(\mathcal{F}) \times H^1(\mathcal{F}) \times H^2(\mathcal{F})$, the following relations holds for a.e. $t \in (0,T)$:*

$$\|(\mathcal{K}_u - \mathrm{I}_2)\overline{u}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{h'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{\beta'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{L}^2(\mathcal{F})},$$

$$\|(\mathcal{L}_u - \Delta)\overline{u}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{h'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{\beta'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{H}^2(\mathcal{F})},$$

$$\|\mathcal{N}_u \overline{u}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C \left(1 + \left\| \frac{h'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{\beta'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})} \|\overline{u}\|_{\mathbf{H}^2(\mathcal{F})},$$

$$\|\mathcal{M}_u \overline{u}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{h'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{\beta'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})},$$

$$\|(\mathcal{G}_u-\nabla)\overline{p}\|_{\mathbf{L}^2(\mathcal{F})}\leqslant C\widetilde{\rho}(t)\left(\left\|\frac{h'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{\beta'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\nabla\overline{p}\|_{\mathbf{L}^2(\mathcal{F})},$$

$$\|(\mathcal{L}_\theta-\Delta)\overline{\theta}\|_{L^2(\mathcal{F})}\leqslant C\widetilde{\rho}(t)\left(\left\|\frac{h'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{\beta'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{\theta}\|_{H^2(\mathcal{F})},$$

$$\|\mathcal{N}_\theta(\overline{u},\overline{\theta})\|_{L^2(\Omega)}\leqslant C\left(1+\left\|\frac{h'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{\beta'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})}\|\overline{\theta}\|_{H^2(\mathcal{F})},$$

$$\|\mathcal{M}_\theta\overline{\theta}\|_{L^2(\mathcal{F})}\leqslant C\widetilde{\rho}(t)\left(\left\|\frac{h'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{\beta'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{\theta}\|_{H^1(\mathcal{F})}.$$

Since we will use the Banach fixed point theorem, we also need to estimate the differences of coefficients. More precisely, let us consider, for $i=1,2$, $h^{(i)}$ and $\beta^{(i)}$ that satisfy

$$h^{(i)}(T)=0,\quad \beta^{(i)}(T)=0,\quad \frac{((h^{(i)})',(\beta^{(i)})')}{\widetilde{\rho}}\in L^2(0,T).$$

With our choice of $\widetilde{\rho}$ (see (6.6) and (5.5)), we deduce in particular that

$$|h^{(1)}(t)-h^{(2)}(t)|+|\beta^{(1)}(t)-\beta^{(2)}(t)|\\ \leqslant T^{1/2}\widetilde{\rho}(t)\left(\left\|\frac{(h^{(1)})'-h^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{(\beta^{(1)})'-\beta^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right).$$

We assume that for all i, $h^{(i)}$ and $\beta^{(i)}$ satisfy (3.4). In particular we can define the change of variables $\mathcal{X}^{(i)}$, $\mathcal{Y}^{(i)}$, and the operators

$$\mathcal{K}_u^{(i)},\ \mathcal{L}_u^{(i)},\ \mathcal{N}_u^{(i)},\ \mathcal{G}_u^{(i)},\ \mathcal{L}_\theta^{(i)},\ \mathcal{N}_\theta^{(i)},\ \mathcal{M}_u^{(i)},\ \mathcal{M}_\theta^{(i)}$$

defined by (3.26)–(3.33).

Following the proof of [1, Lemma 33], we obtain the following estimates of the difference of coefficients:

Lemma 7.2. *For any $(\overline{u},\overline{p},\overline{\theta})\in\mathbf{H}^2(\mathcal{F})\times H^1(\mathcal{F})\times H^2(\mathcal{F})$, the following relations hold for a.e. $t\in(0,T)$:*

$$\|(\mathcal{K}_u^{(1)}-\mathcal{K}_u^{(2)})\overline{u}\|_{\mathbf{L}^2(\mathcal{F})}\leqslant C\widetilde{\rho}(t)\left(\left\|\frac{(h^{(1)})'-h^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{(\beta^{(1)})'-\beta^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{u}\|_{\mathbf{L}^2(\mathcal{F})},$$

$$\|(\mathcal{L}_u^{(1)}-\mathcal{L}_u^{(2)})\overline{u}\|_{\mathbf{L}^2(\mathcal{F})}\leqslant C\widetilde{\rho}(t)\left(\left\|\frac{(h^{(1)})'-h^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{(\beta^{(1)})'-\beta^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{u}\|_{\mathbf{H}^2(\mathcal{F})},$$

$$\|\left(\mathcal{N}_u^{(1)}-\mathcal{N}_u^{(2)}\right)\overline{u}\|_{\mathbf{L}^2(\mathcal{F})}\\ \leqslant C\left(\left\|\frac{(h^{(1)})'-h^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}+\left\|\frac{(\beta^{(1)})'-\beta^{(2)})'}{\widetilde{\rho}}\right\|_{L^2(0,T)}\right)\|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})}\|\overline{u}\|_{\mathbf{H}^2(\mathcal{F})},$$

$$\| \left(\mathcal{M}_u^{(1)} - \mathcal{M}_u^{(2)}\right) \overline{u}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{(h^{(1)})' - h^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{(\beta^{(1)})' - \beta^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})},$$

$$\| \left(\mathcal{G}_u^{(1)} - \mathcal{G}_u^{(2)}\right) \overline{p}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{(h^{(1)})' - h^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{(\beta^{(1)})' - \beta^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\nabla \overline{p}\|_{\mathbf{L}^2(\mathcal{F})},$$

$$\| \left(\mathcal{L}_\theta^{(1)} - \mathcal{L}_\theta^{(2)}\right) \overline{\theta}\|_{L^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{(h^{(1)})' - h^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{(\beta^{(1)})' - \beta^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{\theta}\|_{H^2(\mathcal{F})},$$

$$\begin{aligned} \| \left(\mathcal{N}_\theta^{(1)} - \mathcal{N}_\theta^{(2)}\right) (\overline{u}, \overline{\theta})\|_{L^2(\Omega)} \\ \leqslant C \left(\left\| \frac{(h^{(1)})' - h^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{(\beta^{(1)})' - \beta^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{u}\|_{\mathbf{H}^1(\mathcal{F})} \|\overline{\theta}\|_{H^2(\mathcal{F})}, \end{aligned}$$

$$\| \left(\mathcal{M}_\theta^{(1)} - \mathcal{M}_\theta^{(2)}\right) \overline{\theta}\|_{L^2(\mathcal{F})} \leqslant C\widetilde{\rho}(t) \left(\left\| \frac{(h^{(1)})' - h^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} + \left\| \frac{(\beta^{(1)})' - \beta^{(2)})'}{\widetilde{\rho}} \right\|_{L^2(0,T)} \right) \|\overline{\theta}\|_{H^1(\mathcal{F})}.$$

7.2. **The fixed point argument.** We are now in position to prove the main result.

Proof of Theorem 1.1. First, we assume that (h_0, β_0) satisfies (3.2) and (3.4) so that we can consider $\mathcal{X}(0,.)$, $\mathcal{Y}(0,.)$ and define $(u_0, \theta_0, \ell_0, \omega_0)$ by (3.34)-(3.35).
From (1.14), (4.31) and the properties of $\mathcal{X}$ and $\mathcal{Y}$ (Section 3), we can check that

$$(u_0, \theta_0) \in D((-A)^{1/2}) \quad \text{and} \quad \|(u_0, \theta_0)\|_{D((-A)^{1/2})} \leqslant C\|(\widehat{u_0}, \widehat{\theta_0})\|_{\mathbf{H}^1(\mathcal{F}_{h_0,\beta_0}) \times H^1(\mathcal{F}_{h_0,\beta_0})}, \tag{7.1}$$

where A is defined by (4.23)-(4.25).

The proof of Theorem 1.1 is based on a fixed point argument. If we set $Z = \begin{pmatrix} u \\ \theta \end{pmatrix}$, $d = \begin{pmatrix} h \\ \beta \end{pmatrix}$ and $Z_0 = \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix}$, $d_0 = \begin{pmatrix} h_0 \\ \beta_0 \end{pmatrix}$ then we can write (3.14)–(3.25) as

$$\begin{cases} \dot{Z}(t) = AZ(t) + Bw_0(t) + \widetilde{F}(Z, d), \\ \dot{d}(t) = CZ(t), \\ Z(0) = Z_0 \in \mathbb{H}, \\ d(0) = d_0 \in \mathbb{R}^3, \end{cases} \tag{7.2}$$

with

$$\widetilde{F}(Z, d) = \begin{pmatrix} \mathbb{P} f_2 \\ -[\mathcal{M}_\theta \theta] - [\mathcal{N}_\theta(u, \theta)] + \mu[(\mathcal{L}_\theta - \Delta)\theta] \end{pmatrix}, \tag{7.3}$$

where

$$f_2 = \begin{cases} -[(\mathcal{K}_u - \mathrm{I}_2)\frac{\partial u}{\partial t}] - [\mathcal{M}_u u] - [\mathcal{N}_u u] + \nu[(\mathcal{L}_u - \Delta)u] + [(\nabla - \mathcal{G}_u)p] \quad \text{in} \quad \mathcal{F}, \\ -\omega \ell^\perp \quad \text{in} \quad \mathcal{S}. \end{cases} \tag{7.4}$$

Now from Theorem 6.1, we know there exists a control $w_0 = E_T(Z_0, d_0, F)$ such that the solution of

$$\begin{cases} \dot{Z}(t) = AZ(t) + Bw_0(t) + F(t), \\ \dot{d}(t) = CZ(t), \\ Z(0) = Z_0 \in \mathbb{H}, \\ d(0) = d_0 \in \mathbb{R}^3, \end{cases} \tag{7.5}$$

satisfies (6.7) and (6.8). Let us consider $r > 0$ (that is fixed later) and let us set

$$\mathcal{K}_r = \left\{ F \in L^2(0,T;\mathbb{H}) \ ; \ \left\| \frac{F}{\rho_1} \right\|_{L^2(0,T;\mathbb{H})} \leqslant r \right\}. \tag{7.6}$$

If

$$\|Z_0\|_{D((-A)^{1/2})} \leqslant r \quad \text{and} \quad \|d_0\|_{\mathbb{R}^3} \leqslant r, \quad F \in \mathcal{K}_r, \tag{7.7}$$

then we deduce

$$\left\| \frac{Z}{\widetilde{\rho}} \right\|_{L^2(0,T;D(A)) \cap C([0,T];D((-A)^{\frac{1}{2}})) \cap H^1(0,T;\mathbb{H})} \leqslant Cr. \tag{7.8}$$

We take r small enough so that (3.2) and (3.4) holds true and we can construct the change of variables $\mathcal{X}$ and $\mathcal{Y}$ as in Section 3. We can thus define

$$\mathcal{T}(F) = \widetilde{F}(Z, d), \tag{7.9}$$

where $\widetilde{F}(Z,d)$ is given by (7.3)-(7.4). By using Lemma 7.1, (7.8) and $\frac{\widetilde{\rho}^2}{\rho_1} \in L^\infty(0,T)$, we can verify that

$$\mathcal{T} : \mathcal{K}_r \to \mathcal{K}, \tag{7.10}$$

and

$$\left\| \frac{\mathcal{T}(F)}{\rho_1} \right\|_{L^2(0,T;\mathbb{H})} \leqslant Cr^2. \tag{7.11}$$

In particular for r small enough, $\mathcal{T}$ maps $\mathcal{K}_r$ to $\mathcal{K}_r$. Similarly, by using Lemma 7.2 and (7.8), we deduce that

$$\left\| \frac{\mathcal{T}(F_1) - \mathcal{T}(F_2)}{\rho_1} \right\|_{L^2(0,T;\mathbb{H})} \leqslant Cr \left\| \frac{F_1 - F_2}{\rho_1} \right\|_{L^2(0,T;\mathbb{H})} \tag{7.12}$$

and thus for r small enough, $\mathcal{T}$ admits a unique fixed point F in $\mathcal{K}_r$. The corresponding solution of (7.5) is the solution of (7.2) and satisfies

$$\left\| \frac{Z}{\widetilde{\rho}} \right\|_{L^2(0,T;D(A)) \cap C([0,T];D((-A)^{\frac{1}{2}})) \cap H^1(0,T;\mathbb{H})} \leqslant C \left(\|\widetilde{F}(Z,d)\|_{\mathfrak{F}} + \|d_0\|_{\mathbb{R}^3} + \|Z_0\|_{D((-A)^{\frac{1}{2}})} \right), \tag{7.13}$$

and $d(T) = 0$. In particular, we obtain (1.16)-(1.17).

□

Appendix A. Carleman estimates for the Laplace operator

In this section, we recall a Carleman estimate for the Laplace equation. We give the proof of such an estimate for completeness.

Proposition A.1. *Let $\lambda > 1$ and $\zeta = \exp(\lambda\eta)$ with η given by (5.2)-(5.3). Assume $\mathcal{O}_0$, $\mathcal{O}_1$ are open subsets of $\mathcal{F}$ such that $\overline{\mathcal{O}_0} \subset \mathcal{O}_1$, $\overline{\mathcal{O}_1} \subset \mathcal{F}$. Then there exist constants λ_1, κ_1, C depending only on $\mathcal{F}, \mathcal{O}_0, \mathcal{O}_1$ such that, for any $\lambda \geqslant \lambda_1$, $\kappa \geqslant \kappa_1$ and $u \in H^2(\mathcal{F})$, the following inequality holds:*

$$\begin{aligned}
&\lambda^8\kappa^6 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^6|u|^2dx \; + \lambda^6\kappa^4 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^4|\nabla u|^2dx \; + \lambda^7\kappa^6 \int_{\partial\mathcal{F}} e^{2\kappa}|u|^2d\Gamma \\
&\leqslant C\left(\lambda^4\kappa^3 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^3|\Delta u|^2dx \; + \lambda^8\kappa^6 \int_{\mathcal{O}_1} e^{2\kappa\zeta}\zeta^6|u|^2dx \; + \lambda^5\kappa^4 \int_{\partial\mathcal{F}} e^{2\kappa}\left|\frac{\partial u}{\partial\tau}\right|^2\right). \qquad \text{(A.1)}
\end{aligned}$$

Proof. We follow the same steps as [21] and [17] but here we incorporate the boundary terms. Let us set $f = -\Delta u$ and

$$\sigma = e^{\kappa\zeta}u \qquad \text{and} \qquad g = e^{\kappa\zeta}f.$$

Then we obtain

$$M\sigma + N\sigma = g_{\kappa,\lambda}, \qquad \text{(A.2)}$$

where

$$\begin{aligned}
M_1\sigma &= 5\lambda^4\kappa^{\frac{5}{2}}\zeta^{\frac{5}{2}}|\nabla\eta|^2\sigma, \quad M_2\sigma = 2\lambda^3\kappa^{\frac{5}{2}}\zeta^{\frac{5}{2}}\nabla\eta\cdot\nabla\sigma\,; \\
N_1\sigma &= -\lambda^4\kappa^{\frac{7}{2}}\zeta^{\frac{7}{2}}|\nabla\eta|^2\sigma, \quad N_2\sigma = -\lambda^2\kappa^{\frac{3}{2}}\zeta^{\frac{3}{2}}\Delta\sigma\,; \\
M\sigma &= M_1\sigma + M_2\sigma, \quad N\sigma = N_1\sigma + N_2\sigma\,; \qquad \text{(A.3)} \\
g_{\kappa,\lambda} &= \lambda^2\kappa^{\frac{3}{2}}\zeta^{\frac{3}{2}}g - \lambda^3\kappa^{\frac{5}{2}}\zeta^{\frac{5}{2}}\Delta\eta\,\sigma + 4\lambda^4\kappa^{\frac{5}{2}}\zeta^{\frac{5}{2}}|\nabla\eta|^2\sigma.
\end{aligned}$$

We have from (A.2)

$$\|M\sigma\|^2_{L^2(\mathcal{F})} + \|N\sigma\|^2_{L^2(\mathcal{F})} + 2\sum_{i,j=1}^{2} I_{ij} = \|g_{\kappa,\lambda}\|^2_{L^2(\mathcal{F})}, \qquad \text{(A.4)}$$

where $I_{ij} = \displaystyle\int_{\mathcal{F}} (M_i\sigma)\,(N_j\sigma)\,dx$. First, we have

$$I_{11} = -5\lambda^8\kappa^6 \int_{\mathcal{F}} \zeta^6|\nabla\eta|^4|\sigma|^2\,dx\,. \qquad \text{(A.5)}$$

Then

$$
\begin{aligned}
I_{21} &= -2\lambda^7\kappa^6 \int_{\mathcal{F}} \zeta^6 |\nabla\eta|^2 (\nabla\eta\cdot\nabla\sigma)\sigma\, dx \\
&= 6\lambda^8\kappa^6 \int_{\mathcal{F}} \zeta^6 |\nabla\eta|^4 |\sigma|^2 dx \; + \lambda^7\kappa^6 \int_{\mathcal{F}} \zeta^6 \Delta\eta |\nabla\eta|^2 |\sigma|^2 dx \\
&+ 2\lambda^7\kappa^6 \sum_{i,j=1}^{2} \int_{\mathcal{F}} \zeta^6 \partial_i\eta\, \partial_{ij}\eta\, \partial_j\eta\, |\sigma|^2 dx \; - \lambda^7\kappa^6 \int_{\partial\mathcal{F}} \zeta^6 |\nabla\eta|^2 \frac{\partial\eta}{\partial n}\sigma^2 d\Gamma \\
&= A_1 + A_2 + A_3 + A_4. \qquad \text{(A.6)}
\end{aligned}
$$

Due to the property (5.2) of η, we have

$$
\begin{aligned}
I_{11} + A_1 &= \lambda^8\kappa^6 \int_{\mathcal{F}} \zeta^6 |\nabla\eta|^4 |\sigma|^2\, dx \\
&\geqslant c_0^4\lambda^8\kappa^6 \int_{\mathcal{F}} \zeta^6 |\sigma|^2 dx \; - c_0^4\lambda^8\kappa^6 \int_{\mathcal{O}_0} \zeta^6 |\sigma|^2 dx\,. \qquad \text{(A.7)}
\end{aligned}
$$

A_2 and A_3 satisfy:

$$
|A_2| + |A_3| \leqslant C\lambda^7\kappa^6 \int_{\mathcal{F}} \zeta^6 \sigma^2 dx\,. \qquad \text{(A.8)}
$$

Since $\eta = 0$ on $\partial\mathcal{F}$, we have $\nabla\eta = \left(\frac{\partial\eta}{\partial n}\right) n$ and $\zeta(x) = 1\ \forall x \in \partial\mathcal{F}$. Moreover, $\frac{\partial\eta}{\partial n} < 0$ on $\partial\mathcal{F}$, so $\left|\frac{\partial\eta}{\partial n}\right| = -\frac{\partial\eta}{\partial n}$. Therefore,

$$
A_4 = \lambda^7\kappa^6 \int_{\partial\mathcal{F}} \left|\frac{\partial\eta}{\partial n}\right|^3 \sigma^2 d\Gamma. \qquad \text{(A.9)}
$$

The next term of (A.4) that we estimate is:

$$
\begin{aligned}
I_{12} &= -5\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\eta|^2 (\Delta\sigma)\, \sigma dx \\
&= 5\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\eta|^2 |\nabla\sigma|^2 dx \; + 10\lambda^6\kappa^4 \sum_{i,j=1}^{2} \int_{\mathcal{F}} \zeta^4 \partial_i\eta\, \partial_{ij}\eta\, (\partial_j\sigma)\, \sigma dx \\
&+ 20\lambda^7\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\eta|^2 (\nabla\eta\cdot\nabla\sigma)\sigma dx \; - 5\lambda^6\kappa^4 \int_{\partial\mathcal{F}} \zeta^4 |\nabla\eta|^2 \frac{\partial\sigma}{\partial n}\sigma d\Gamma \\
&= C_1 + C_2 + C_3 + C_4 \qquad \text{(A.10)}
\end{aligned}
$$

We first estimate the quantities C_2 and C_3:

$$
|C_2| \leqslant C\lambda^8\kappa^5 \int_{\mathcal{F}} \zeta^5 |\sigma|^2\, dx + C\lambda^4\kappa^3 \int_{\mathcal{F}} \zeta^3 |\nabla\sigma|^2\, dx \qquad \text{(A.11)}
$$

and

$$|C_3| \leqslant C\lambda^8\kappa^5 \int_{\mathcal{F}} \zeta^5 |\sigma|^2 \, dx + C\lambda^6\kappa^3 \int_{\mathcal{F}} \zeta^3 |\nabla\sigma|^2 \, dx. \tag{A.12}$$

Since $\lambda > 1$,

$$\begin{aligned} I_{12} = C_1 + C_2 + C_3 + C_4 \geqslant 5\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\eta|^2 |\nabla\sigma|^2 dx - C\lambda^8\kappa^5 \int_{\mathcal{F}} \zeta^5 |\sigma|^2 \, dx \\ - C\lambda^6\kappa^3 \int_{\mathcal{F}} \zeta^3 |\nabla\sigma|^2 \, dx + C_4. \end{aligned} \tag{A.13}$$

We have the following estimate on C_4:

$$\begin{aligned} -C_4 = 5\lambda^6\kappa^4 \int_{\partial\mathcal{F}} \zeta^4 |\nabla\eta|^2 \frac{\partial\sigma}{\partial n} \sigma d\Gamma &= 5\lambda^6\kappa^4 \int_{\partial\mathcal{F}} \left|\frac{\partial\eta}{\partial n}\right|^2 \frac{\partial\sigma}{\partial n} \sigma d\Gamma \\ &= \int_{\partial\mathcal{F}} \left[\lambda^{\frac{7}{2}}\kappa^3 \left|\frac{\partial\eta}{\partial n}\right|^{\frac{3}{2}} \sigma\right] \cdot \left[5\lambda^{\frac{5}{2}}\kappa \left|\frac{\partial\eta}{\partial n}\right|^{\frac{1}{2}} \frac{\partial\sigma}{\partial n}\right] d\Gamma \\ &\leqslant \int_{\partial\mathcal{F}} \frac{\lambda^7\kappa^6}{2} \left|\frac{\partial\eta}{\partial n}\right|^3 \sigma^2 d\Gamma + \int_{\partial\mathcal{F}} \frac{25\lambda^5\kappa^2}{2} \left|\frac{\partial\eta}{\partial n}\right| \left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma \,. \end{aligned} \tag{A.14}$$

Finally, we estimate I_{22}:

$$\begin{aligned} I_{22} &= -2\lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4 (\nabla\eta \cdot \nabla\sigma) \Delta\sigma dx \\ &= -2\lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n} \left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma + 2\lambda^5\kappa^4 \sum_{i,j=1}^{2} \int_{\mathcal{F}} \partial_{ij}\eta \, \zeta^4 \, \partial_i\sigma \partial_j\sigma dx \\ &+ 8\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\eta \cdot \nabla\sigma|^2 dx + \lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4 \nabla\eta \cdot \nabla|\nabla\sigma|^2 dx \\ &= D_1 + D_2 + D_3 + D_4 \,. \end{aligned} \tag{A.15}$$

We have

$$|D_2| \leqslant C\lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4 |\nabla\sigma|^2 dx \tag{A.16}$$

and

$$
\begin{aligned}
D_4 &= \lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4 \nabla\eta\cdot\nabla|\nabla\sigma|^2 dx \\
&= \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma + \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma - 4\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4|\nabla\eta|^2|\nabla\sigma|^2\, dx \\
&- \lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4\, \Delta\eta\, |\nabla\sigma|^2\, dx. \qquad \text{(A.17)}
\end{aligned}
$$

We obtain :

$$
\begin{aligned}
I_{22} = D_1 + D_2 + D_3 + D_4 \geqslant & -4\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4|\nabla\eta|^2|\nabla\sigma|^2\, dx - C\lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4\, |\nabla\sigma|^2\, dx \\
& - \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma + \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma \\
& = -4\lambda^6\kappa^4 \int_{\mathcal{F}} \zeta^4|\nabla\eta|^2|\nabla\sigma|^2\, dx - C\lambda^5\kappa^4 \int_{\mathcal{F}} \zeta^4\, |\nabla\sigma|^2\, dx + B_1 + B_2\,. \qquad \text{(A.18)}
\end{aligned}
$$

Now we look at the boundary terms appearing in the estimates of $I_{i,j}$, $i,j \in \{1,2\}$ and by using (A.9), (A.14) and (A.18), we deduce:

$$
\begin{aligned}
B &= A_4 + C_4 + B_1 + B_2 \\
&\geqslant \lambda^7\kappa^6 \int_{\partial\mathcal{F}} \left|\frac{\partial\eta}{\partial n}\right|^3 \sigma^2 d\Gamma - \frac{\lambda^7\kappa^6}{2} \int_{\partial\mathcal{F}} \left|\frac{\partial\eta}{\partial n}\right|^3 \sigma^2 d\Gamma - \frac{25\lambda^5\kappa^2}{2} \int_{\partial\mathcal{F}} \left|\frac{\partial\eta}{\partial n}\right| \left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma \\
&\quad - \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial n}\right|^2 d\Gamma + \lambda^5\kappa^4 \int_{\partial\mathcal{F}} \frac{\partial\eta}{\partial n}\left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\,.
\end{aligned}
$$

There exists κ_2 such that for any $\kappa \geqslant \kappa_2$, we have

$$
B \geqslant C\lambda^7\kappa^6 \int_{\partial\mathcal{F}} \sigma^2 d\Gamma - C\lambda^5\kappa^4 \int_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\,. \qquad \text{(A.19)}
$$

Gathering (A.7),(A.8), (A.13), (A.18), (A.19), we deduce the existence of $\lambda_3 \geqslant 1, \kappa_3 \geqslant \kappa_2$ such that for $\lambda \geqslant \lambda_3, \kappa \geqslant \kappa_3$:

$$
\begin{aligned}
\sum_{i,j=1}^{2} I_{ij} \geqslant & C \int_{\mathcal{F}} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \; - C \int_{\mathcal{O}_0} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \\
& + C\lambda^7\kappa^6 \int_{\partial\mathcal{F}} \sigma^2 d\Gamma - C\lambda^5\kappa^4 \int_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\,. \qquad \text{(A.20)}
\end{aligned}
$$

Combining (A.3), (A.4) and the above estimate, we obtain:

$$\begin{aligned}
&\|N\sigma\|^2_{L^2(\mathcal{F})} + \int\limits_{\mathcal{F}} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \ + \lambda^7\kappa^6 \int\limits_{\partial\mathcal{F}} \sigma^2 d\Gamma \\
&\leqslant C\left(\|g_{\kappa,\lambda}\|^2 + \int\limits_{\mathcal{O}_0} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx + \lambda^5\kappa^4 \int\limits_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\right) \\
&\leqslant C\Bigg(\lambda^4\kappa^3 \int\limits_{\mathcal{F}} \zeta^3|g|^2\, dx + \lambda^8\kappa^5 \int\limits_{\mathcal{F}} \zeta^5|\sigma|^2\, dx + \int\limits_{\mathcal{O}_0} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \\
&+ \lambda^5\kappa^4 \int\limits_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\Bigg).
\end{aligned} \tag{A.21}$$

The above relation yields the existence of $\lambda_4 \geqslant \lambda_3, \kappa_4 \geqslant \kappa_3$ such that for $\lambda \geqslant \lambda_4$, $\kappa \geqslant \kappa_4$:

$$\begin{aligned}
&\|N\sigma\|^2_{L^2(\mathcal{F})} + \int\limits_{\mathcal{F}} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \ + \lambda^7\kappa^6 \int\limits_{\partial\mathcal{F}} \sigma^2 d\Gamma \\
&\leqslant C\Bigg(\lambda^4\kappa^3 \int\limits_{\mathcal{F}} \zeta^3|g|^2\, dx + \int\limits_{\mathcal{O}_0} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx + \lambda^5\kappa^4 \int\limits_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\Bigg).
\end{aligned} \tag{A.22}$$

On the other hand, from the definition (A.3) of N, we can check the following relation for $\kappa \geqslant 1$:

$$\lambda^4\kappa^2 \int\limits_{\mathcal{F}} \zeta^2|\Delta\sigma|^2\, dx \leqslant C\left(\|N\sigma\|^2_{L^2(\mathcal{F})} + \lambda^8\kappa^6 \int\limits_{\mathcal{F}} \zeta^6|\sigma|^2\, dx\right). \tag{A.23}$$

Thus we deduce from (A.22)-(A.23) that for $\lambda \geqslant \lambda_4$, $\kappa \geqslant \kappa_4$:

$$\begin{aligned}
&\lambda^4\kappa^2 \int\limits_{\mathcal{F}} \zeta^2|\Delta\sigma|^2\, dx + \int\limits_{\mathcal{F}} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx \ + \lambda^7\kappa^6 \int\limits_{\partial\mathcal{F}} \sigma^2 d\Gamma \\
&\leqslant C\Bigg(\lambda^4\kappa^3 \int\limits_{\mathcal{F}} \zeta^3|g|^2\, dx + \int\limits_{\mathcal{O}_0} \left(\lambda^8\kappa^6\zeta^6|\sigma|^2 + \lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right) dx + \lambda^5\kappa^4 \int\limits_{\partial\mathcal{F}} \left|\frac{\partial\sigma}{\partial\tau}\right|^2 d\Gamma\Bigg).
\end{aligned} \tag{A.24}$$

To eliminate the term involving $\nabla\sigma$ in the right hand side of (A.24), let us introduce a function $\chi \in C_c^2(\mathcal{O}_1)$, with $\chi = 1$ in $\mathcal{O}_0$, $0 \leqslant \chi \leqslant 1$ and $\overline{\mathcal{O}_0} \subset \mathcal{O}_1$, $\overline{\mathcal{O}_1} \subset \mathcal{F}$. Then for any $\epsilon > 0$, there

exists $C>0$ such that

$$\lambda^6\kappa^4\int_{\mathcal{O}_0}\zeta^4|\nabla\sigma|^2\,dx\leqslant\lambda^6\kappa^4\int_{\mathcal{O}_1}\chi\zeta^4|\nabla\sigma|^2\,dx$$
$$=-\lambda^6\kappa^4\int_{\mathcal{O}_1}\chi\zeta^4\left(\Delta\sigma\right)\sigma\,dx-\lambda^6\kappa^4\int_{\mathcal{O}_1}\zeta^4(\nabla\sigma\cdot\nabla\chi)\sigma\,dx-4\lambda^7\kappa^4\int_{\mathcal{O}_1}\chi\zeta^4(\nabla\sigma\cdot\nabla\eta)\sigma\,dx$$
$$\leqslant\epsilon\lambda^4\kappa^2\int_{\mathcal{O}_1}\zeta^2|\Delta\sigma|^2\,dx+C\left(\lambda^8\kappa^6\int_{\mathcal{O}_1}\zeta^6|\sigma|^2\,dx\right).\tag{A.25}$$

Combining (A.24) and (A.25) yields:

$$\lambda^4\kappa^2\int_{\mathcal{F}}\zeta^2|\Delta\sigma|^2\,dx+\int_{\mathcal{F}}\left(\lambda^8\kappa^6\zeta^6|\sigma|^2+\lambda^6\kappa^4\zeta^4|\nabla\sigma|^2\right)dx\;+\lambda^7\kappa^6\int_{\partial\mathcal{F}}\sigma^2d\Gamma$$
$$\leqslant C\left(\lambda^4\kappa^3\int_{\mathcal{F}}\zeta^3|g|^2\,dx+\int_{\mathcal{O}_1}\lambda^8\kappa^6\zeta^6|\sigma|^2dx+\lambda^5\kappa^4\int_{\partial\mathcal{F}}\left|\frac{\partial\sigma}{\partial\tau}\right|^2d\Gamma\right).\tag{A.26}$$

Now we can go back to our original function $u=e^{-\kappa\zeta}\sigma$. Observe that

$$\nabla u=e^{-\kappa\zeta}\left(\nabla\sigma-\kappa\lambda\nabla\eta\,\zeta\,\sigma\right).\tag{A.27}$$

Hence,

$$\lambda^6\kappa^4\int_{\mathcal{F}}e^{2\kappa\zeta}\zeta^4|\nabla u|^2\,dx\leqslant C\lambda^6\kappa^4\int_{\mathcal{F}}\zeta^4|\nabla\sigma|^2\,dx+C\lambda^8\kappa^6\int_{\mathcal{F}}\zeta^6|\sigma|^2\,dx\tag{A.28}$$

and

$$\int_{\mathcal{F}}\lambda^8\kappa^6\zeta^6e^{2\kappa\zeta}|u|^2\,dx=\int_{\mathcal{F}}\lambda^8\kappa^6\zeta^6|\sigma|^2\,dx.\tag{A.29}$$

Thus, we have obtained the estimate (A.1). □

From the above proposition, we deduce the following result.

Corollary A.2. *Let $\mathcal{O}_0$, $\mathcal{O}_1$ be open subsets of $\mathcal{F}$ such that $\overline{\mathcal{O}_0}\subset\mathcal{O}_1$, $\overline{\mathcal{O}_1}\subset\mathcal{F}$. Then there exist constants s_0,C depending only on $\lambda,\mathcal{F},\mathcal{O}_0,\mathcal{O}_1$ such that, for any $s\geqslant s_0$ and $u\in L^2(0,T;H^2(\mathcal{F}))$, the following inequality holds:*

$$s^6\int_0^T\int_{\mathcal{F}}e^{-2s\alpha}\xi^6|u|^2dy\,dt+s^4\int_0^T\int_{\mathcal{F}}e^{-2s\alpha}\xi^4|\nabla u|^2dy\,dt+s^6\int_0^T\int_{\partial\mathcal{F}}e^{-2s\alpha_M}(\xi_m)^6|u|^2d\Gamma\,dt$$
$$\leqslant C\left(s^3\int_0^T\int_{\mathcal{F}}e^{-2s\alpha}\xi^3|\Delta u|^2dy\,dt+s^6\int_0^T\int_{\mathcal{O}_1}e^{-2s\alpha}\xi^6|u|^2dy\,dt+s^4\int_0^T\int_{\partial\mathcal{F}}e^{-2s\alpha_M}(\xi_m)^4\left|\frac{\partial u}{\partial\tau}\right|^2d\Gamma\right).\tag{A.30}$$

Proof. We take $\lambda \geqslant \lambda_1$ in Proposition A.1. Then there exists $C = C(\lambda)$ such that

$$\kappa^6 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^6|u|^2dx \ + \kappa^4 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^4|\nabla u|^2dx \ + \kappa^6 \int_{\partial\mathcal{F}} e^{2\kappa}|u|^2d\Gamma$$

$$\leqslant C\left(\kappa^3 \int_{\mathcal{F}} e^{2\kappa\zeta}\zeta^3|\Delta u|^2dx \ + \kappa^6 \int_{\mathcal{O}_1} e^{2\kappa\zeta}\zeta^6|u|^2dx \ + \kappa^4 \int_{\partial\mathcal{F}} e^{2\kappa}\left|\frac{\partial u}{\partial\tau}\right|^2 d\Gamma\right),$$

for every $\kappa \geqslant \kappa_1$. We take $\kappa = \frac{s}{E(t)^8}$. For s large enough, we have $\kappa \geqslant \kappa_1$. This gives :

$$s^6 \int_{\mathcal{F}} e^{2s\xi}\xi^6|u|^2dx \ + s^4 \int_{\mathcal{F}} e^{2s\xi}\xi^4|\nabla u|^2dx \ + s^6 \int_{\partial\mathcal{F}} e^{2s\xi_m}(\xi_m)^6|u|^2d\Gamma$$

$$\leqslant C\left(s^3 \int_{\mathcal{F}} e^{2s\xi}\xi^3|\Delta u|^2dx \ + s^6 \int_{\mathcal{O}_1} e^{2s\xi}\xi^6|u|^2dx \ + s^4 \int_{\partial\mathcal{F}} e^{2s\xi_m}(\xi_m)^4\left|\frac{\partial u}{\partial\tau}\right|^2 d\Gamma\right), \tag{A.31}$$

Now if we multiply the above inequality by $\exp\left(-2s\frac{e^{2\lambda\|\eta\|_{L^\infty}}}{E(t)^8}\right)$ and integrate from 0 to T, we obtain (A.30). □

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