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On some properties of the hazard rate function for compound distributions

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Abstract

The hazard rate function is an important quantity characterizing several random phenomena. In this paper, we investigate some crucial properties satisfied by the hazard rate function for a general power series class of distributions, as monotonicity properties, sharp bounds and convexity properties. We highlight how these properties are related to those of the baseline distribution and the corresponding probability-generating function of discrete power series distributions.

Keywords: Compound distributions, Hazard rate function, Power series distributions.

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1. Introduction

Several aspects of an absolutely continuous distribution can be seen more clearly from the hazard rate function (hrf) than from either the distribution or density functions. The hrf is an important quantity characterizing life phenomena. Let X be a random variable with probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$. The hrf of X is defined by $f(x)/\bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$ is the survival function. The hrf may be increase, decrease, constant, upside-down bathtub (unimodal), bathtub-shaped or indicate a more complicated process. In many applications there is a qualitative information about the hrf shape, which can help in selecting a specified model.

Several classes of distributions were proposed by compounding some useful lifetime and discrete distributions in the last few years. The compounding procedure follows the pioneering work of Marshall and Olkin (1997). Chakandi and Ganjali (2009) defined the exponential power series (EPS) class of distributions, which contains as special cases involving the exponential geometric (Adamidis and Loukas, 1998), exponential Poisson (Kus, 2007) and exponential logarithmic (Tamasbi and Rezaei, 2008) distributions. Morais and Barreto-Souza (2011) defined the Weibull power series (WPS) class which includes the EPS distribution as a sub-model. The WPS distributions can have increasing, decreasing and upside down bathtub hrf. The generalized exponential power series distributions were proposed by Mahmoudi and Jafari (2012). Silva et al. (2013) studied the extended Weibull power series family of distributions, which includes as special models the EPS and WPS classes of distributions. Bourguignon et al. (2014) and Silva and Cordeiro (2014) proposed the Birnbaum-Saunders power series (BSPS) and Burr XII power series (BXIIPS) classes of distributions, respectively. Recently, Silva et al. (2016) proposed the generalized gamma power series (GGPS) distributions and defines 76 sub-models as special cases. The GGPS hrf can be constant, decreasing, increasing, upside-down bathtub or bathtub-shaped.

In this paper, we consider a general power series class of distributions characterized by the distributions of the random variables $X = \min(X_1, \dots, X_N)$, where X_1, X_2, \dots, X_N are independent and identically distributed (i.i.d.) continuous random variables and N is a discrete random variable having a power series distributions (including translated Poisson distribution, truncated Poisson distribution, geometric distribution, binomial distribution, and many more). For example, X can represent the lifetime of a system with N components where the lifetime of the i -th component is represented by X_i .

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Further details on this general power series class can be found in Tahir and Cordeiro (2016), and the reference therein. In this study, we focus our attention on the nature of the hrf of X . We investigate some of its properties according to the properties of the distributions of X_1 and N . In particular, we prove that if the hrf of X_1 is decreasing, the hrf of X is decreasing too, without extra assumptions on the distribution of N ; the power series family of distributions preserve the decreasing hrf property of the baseline distribution. We determine sharp and simple lower and upper bounds for the hrf of X depending only of the hrf of X_1 and the expectation of N . An immediate consequence is the preservation of the discontinuities for these hrfs. We also provide a differential equation where the hrf of X is the solution, with some consequences on its monotonicity properties. Additional properties are given for three specific distributions on N : the translated Poisson distribution, the truncated Poisson distribution and the geometric distribution.

The rest of the paper is organized as follows. In Section 2, we introduce the power series family of distributions and present our main results. Section 3 provided more contributions for three specific class of distributions.

2. The power series class of distributions

Let N be a discrete random variable having a power series distribution (truncated at zero) with probability mass function

$$p_n = P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots, \quad (1)$$

where the support \mathbb{S} of N is a subset of the non-negative integers, $a_n \geq 0$ depends only on n and there is $r > 0$ such that $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ is finite for all $\theta \in (0, r)$ (r can be ∞). Although we will always consider θ as a value in $(0, r)$. Note that $C(\theta)$ has derivatives of all orders in $(0, r)$ and those derivatives can be obtained by differentiating the power series term to term. Also, because $a_n \geq 0$ for all n , $C(\theta)$ and all its derivatives will be positive in $(0, r)$.

The moments of the random variable N can be easily obtained from the probability-generating function $K(s) = E(s^N)$. By noticing that $K(s) = C(s\theta)/C(\theta)$, they also can be determined using $C(\theta)$. By setting $G(\theta) = \log[C(\theta)]$, $G'(\theta) = dG(\theta)/d\theta$ and $G''(\theta) = d^2 G(\theta)/d\theta^2$, the expectation of N is given by $E(N) = \theta G'(\theta)$ and the variance is given by $\text{Var}(N) = \theta^2 G''(\theta) + \theta G'(\theta)$.

Table 1 presents well-known power series distributions and some useful quantities (to determine moments, quantile ...): $C(\theta)$, $C'(\theta)$, $C''(\theta)$ and $C^{-1}(\theta)$.

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	Θ
Trunc. Poisson	$n!^{-1}$	$e^\theta - 1$	e^θ	e^θ	$\log(\theta + 1)$	$\theta \in (0, +\infty)$
Logarithmic	n^{-1}	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	$\theta \in (0, 1)$
Geometric	1	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(\theta + 1)^{-1}$	$\theta \in (0, 1)$
Binomial	$\binom{m}{n}$	$(\theta + 1)^m - 1$	$m(\theta + 1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta - 1)^{1/m} - 1$	$\theta \in (0, 1)$

Table 1: Useful quantities for some power series distributions.

Let us now present the considered general power series (GPS) class of distributions. Let ξ be a parameter vector. Let X_1, X_2, \dots be i.i.d. random variables with cdf $G(x; \xi)$ (baseline distribution), pdf $g(x; \xi)$ and hrf $r(x; \xi) = g(x; \xi)/(1 - G(x; \xi))$. We set $X = \min(X_1, \dots, X_N)$. The conditional cdf of $X|N = n$ is given by $P(X \leq x | N = n) = 1 - [1 - G(x; \xi)]^n$. The GPS class of distributions is characterized by the cdf:

$$F_X(x; \theta, \xi) = \sum_{n=1}^{+\infty} P(X \leq x | N = n)P(N = n) = 1 - \frac{C(\theta \bar{G}(x; \xi))}{C(\theta)}, \quad (2)$$

where $\bar{G}(x; \xi) = 1 - G(x; \xi)$. Using the derivative of $C(\theta)$ according to θ , i.e. $C'(\theta) = \sum_{n=1}^{+\infty} a_n n \theta^{n-1}$, the pdf is given by

$$f_X(x; \theta, \xi) = \theta g(x; \xi) \frac{C'(\theta \bar{G}(x; \xi))}{C(\theta)}.$$

The hrf of GPS class of distributions is given by

$$h_X(x; \theta, \boldsymbol{\xi}) = \theta g(x; \boldsymbol{\xi}) \frac{C'(\theta \bar{G}(x; \boldsymbol{\xi}))}{C(\theta \bar{G}(x; \boldsymbol{\xi}))}.$$

The following result shows a sufficient condition to ensure that $h_X(x; \theta, \boldsymbol{\xi})$ decreases.

Proposition 1. *If $r(x; \boldsymbol{\xi})$ is a decreasing function of x , then $h_X(x; \theta, \boldsymbol{\xi})$ is a decreasing function of x .*

Proof of Proposition 1. Let us introduce the function $M(s)$ defined by

$$M(s) = \frac{E(Ns^N)}{E(s^N)}, \quad s \in (0, 1].$$

It is of interest because it allows us to write $h_X(x; \theta, \boldsymbol{\xi})$ in the following form:

$$h_X(x; \theta, \boldsymbol{\xi}) = \frac{g(x; \boldsymbol{\xi})}{\bar{G}(x; \boldsymbol{\xi})} \frac{C'(\theta \bar{G}(x; \boldsymbol{\xi}))}{C(\theta \bar{G}(x; \boldsymbol{\xi}))} \theta \bar{G}(x; \boldsymbol{\xi}) = r(x; \boldsymbol{\xi}) M(\bar{G}(x; \boldsymbol{\xi})) \quad (3)$$

and, since $r(x; \boldsymbol{\xi})$ is positive and is supposed to be a decreasing function of x , and $\bar{G}(x; \boldsymbol{\xi})$ is a decreasing function of x , it is enough to prove that $M(s)$ is an increasing function of s with $s \in (0, 1]$. The standard derivative ratio formula with $[E(Ns^N)]' = E(N^2 s^{N-1})$ and $[E(s^N)]' = E(Ns^{N-1})$ gives

$$M'(s) = \frac{E(N^2 s^{N-1})E(s^N) - E(Ns^N)E(Ns^{N-1})}{[E(s^N)]^2} = \frac{E(N^2 s^{N-1})E(s^N) - s [E(Ns^{N-1})]^2}{[E(s^N)]^2}. \quad (4)$$

Clearly, $[E(s^N)]^2 > 0$. The Cauchy-Schwarz inequality gives

$$s [E(Ns^{N-1})]^2 = s \left[E(Ns^{\frac{N}{2}-1} s^{\frac{N}{2}}) \right]^2 \leq s E(N^2 s^{N-2}) E(s^N) = E(N^2 s^{N-1}) E(s^N),$$

implying that $E(N^2 s^{N-1})E(s^N) - s [E(Ns^{N-1})]^2 \geq 0$. Thus $M'(s) \geq 0$, so $M(s)$ is an increasing function of s , implying that $h_X(x; \theta, \boldsymbol{\xi})$ is a decreasing function of x . This ends the proof. ■

Thus, Proposition 1 ensures that the power series family of distributions preserve the decreasing hrf property of the baseline distribution.

The following result presents lower and upper bounds for $h_X(x; \theta, \boldsymbol{\xi})$ depending on $r(x; \boldsymbol{\xi})$ and $E(N)$ (without assumptions on the monotony of $r(x; \boldsymbol{\xi})$).

Proposition 2. *For any $x \in \mathbb{R}$, we have*

$$r(x; \boldsymbol{\xi}) \leq h_X(x; \theta, \boldsymbol{\xi}) \leq E(N)r(x; \boldsymbol{\xi}).$$

Proof of Proposition 2. We use the expression (3). Owing to $N \geq 1$ and $s^N > 0$ for $s \in (0, 1]$, we have

$$M(s) = \frac{E(Ns^N)}{E(s^N)} \geq \frac{E(s^N)}{E(s^N)} = 1.$$

Since $r(x; \boldsymbol{\xi})$ is positive, we obtain $r(x; \boldsymbol{\xi}) \leq h_X(x; \theta, \boldsymbol{\xi})$. Now using the fact that $M(s)$ is increasing in s , $\bar{G}(x; \boldsymbol{\xi}) \in [0, 1]$ and $M(1) = E(N)$, we have

$$h_X(x; \theta, \boldsymbol{\xi}) = r(x; \boldsymbol{\xi}) M(\bar{G}(x; \boldsymbol{\xi})) \leq r(x; \boldsymbol{\xi}) M(1) = E(N)r(x; \boldsymbol{\xi}).$$

The proof of Proposition 2 is complete. ■

Owing to Proposition 2, we have $\lim_{x \rightarrow x_*} r(x; \boldsymbol{\xi}) = +\infty$ if and only if $\lim_{x \rightarrow x_*} h_X(x; \theta, \boldsymbol{\xi}) = +\infty$. The discontinuities of $r(x; \boldsymbol{\xi})$ are preserved in $h_X(x; \theta, \boldsymbol{\xi})$.

Proposition 3 below show a differential equation where $h_X(x; \theta, \boldsymbol{\xi})$ is the solution.

Proposition 3. *The hrf $h_X(x; \theta, \xi)$ satisfies the non-linear equation:*

$$\frac{h'_X(x; \theta, \xi)}{h_X(x; \theta, \xi)} = \frac{g'(x; \xi)}{g(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi) + r(x; \xi),$$

where $M'(s)$ is given by (4).

Proof of Proposition 3. Using the expression (3), we have

$$h'_X(x; \theta, \xi) = [r(x; \xi)M(\overline{G}(x; \xi))] = r'(x; \xi)M(\overline{G}(x; \xi)) - r(x; \xi)M'(\overline{G}(x; \xi))g(x; \xi).$$

Hence, using again (3),

$$\frac{h'_X(x; \theta, \xi)}{h_X(x; \theta, \xi)} = \frac{r'(x; \xi)}{r(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi).$$

It is well-known that a hrf as $r(x; \xi)$ satisfies the non-linear differential equation:

$$\frac{r'(x; \xi)}{r(x; \xi)} = \frac{g'(x; \xi)}{g(x; \xi)} + r(x; \xi).$$

Therefore

$$\frac{h'_X(x; \theta, \xi)}{h_X(x; \theta, \xi)} = \frac{g'(x; \xi)}{g(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi) + r(x; \xi).$$

Proposition 3 is proved. ■

An important consequence of Proposition 3 is that, since $h_X(x; \theta, \xi) > 0$, the sign of $h'_X(x; \theta, \xi)$ is the same as the sign of $\frac{g'(x; \xi)}{g(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi) + r(x; \xi)$ (or $\frac{r'(x; \xi)}{r(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi)$). This can provide important informations on the monotonicity properties of $h_X(x; \theta, \xi)$ according to the complexity of the functions considered (discontinuities, ...).

Also, remark that we can rediscover the result of Proposition 1: in the proof of Proposition 1, it is shown that $M'(s) \geq 0$. Since $h_X(x; \theta, \xi) > 0$, $g(x; \xi) > 0$, $M(\overline{G}(x; \xi)) > 0$ and $r(x; \xi) > 0$, if $r'(x; \xi)$ is decreasing, so $r'(x; \xi) < 0$, then $\frac{r'(x; \xi)}{r(x; \xi)} - \frac{M'(\overline{G}(x; \xi))}{M(\overline{G}(x; \xi))} g(x; \xi) < 0$ implying that $h'_X(x; \theta, \xi) < 0$, thus $h_X(x; \theta, \xi)$ is a decreasing function of x .

The next section presents more properties of $h_X(x; \theta, \xi)$ by considering specific distributions for N , namely the translated Poisson distribution, the truncated Poisson distribution and the geometric distribution.

3. Study of three examples

3.1. Translated Poisson distribution case

Here we supposed that N follows the translated Poisson distribution of parameter $\theta > 0$, i.e. $P(N = n) = e^{-\theta} \frac{\theta^{n-1}}{(n-1)!}$, $n = 1, 2, \dots$. Then one can apply Propositions 1, 2 (with $E(N) = 1 + \theta$) and 3. However, since we have the analytic expression of $h_X(x; \theta, \xi)$, some properties can be refined. The result below provides some equalities in this case.

Proposition 4. *If N follows the translated Poisson distribution with parameter $\theta > 0$, then we have*

$$h_X(x; \theta, \xi) = r(x; \xi) [1 + \theta \overline{G}(x; \xi)], \quad h'_X(x; \theta, \xi) = r'(x; \xi) + \theta g'(x; \xi).$$

Proof of Proposition 4. We have $E(s^N) = s e^{\theta(s-1)}$ and $E(N s^N) = [E(s^N)]'s = (1 + \theta s)s e^{\theta(s-1)}$. We now use the expression (3). Since $M(s) = \frac{E(N s^N)}{E(s^N)} = 1 + \theta s$, we have

$$h_X(x; \theta, \xi) = r(x; \xi)M(\overline{G}(x; \xi)) = r(x; \xi) [1 + \theta \overline{G}(x; \xi)].$$

Moreover

$$\begin{aligned} h'_X(x; \theta, \boldsymbol{\xi}) &= r'(x; \boldsymbol{\xi}) [1 + \theta \overline{G}(x; \boldsymbol{\xi})] - \theta g(x; \boldsymbol{\xi}) r(x; \boldsymbol{\xi}) \\ &= r'(x; \boldsymbol{\xi}) + \theta [r'(x; \boldsymbol{\xi}) \overline{G}(x; \boldsymbol{\xi}) - g(x; \boldsymbol{\xi}) r(x; \boldsymbol{\xi})]. \end{aligned}$$

We end the proof by observing that $r(x; \boldsymbol{\xi}) \overline{G}(x; \boldsymbol{\xi}) = g(x; \boldsymbol{\xi})$ and

$$r'(x; \boldsymbol{\xi}) \overline{G}(x; \boldsymbol{\xi}) - g(x; \boldsymbol{\xi}) r(x; \boldsymbol{\xi}) = (r(x; \boldsymbol{\xi}) \overline{G}(x; \boldsymbol{\xi}))' = g'(x; \boldsymbol{\xi}).$$

Proposition 4 is proved. ■

Some additional consequences of Proposition 4 are described below. Note that the extrema of $h_X(x; \theta, \boldsymbol{\xi})$ are given by the solution x of the equation $h'_X(x; \theta, \boldsymbol{\xi}) = 0$, i.e. $r'(x; \boldsymbol{\xi}) + \theta g'(x; \boldsymbol{\xi}) = 0$.

As ever shown in Proposition 1, if $r(x; \boldsymbol{\xi})$ is a decreasing function of x , we have $r'(x; \boldsymbol{\xi}) \leq 0$, implying that $g'(x; \boldsymbol{\xi}) \leq 0$, so $h'_X(x; \theta, \boldsymbol{\xi}) = r'(x; \boldsymbol{\xi}) + \theta g'(x; \boldsymbol{\xi}) \leq 0$, hence $h_X(x; \theta, \boldsymbol{\xi})$ is a decreasing function of x .

On the other hand, if $r(x; \boldsymbol{\xi})$ and $g(x; \boldsymbol{\xi})$ are increasing functions of x , then $h'_X(x; \theta, \boldsymbol{\xi}) = r'(x; \boldsymbol{\xi}) + \theta g'(x; \boldsymbol{\xi}) \geq 0$, so $h_X(x; \theta, \boldsymbol{\xi})$ is an increasing functions of x .

Also note that if $r(x; \boldsymbol{\xi})$ and $g(x; \boldsymbol{\xi})$ are convex (resp concave), then $h''_X(x; \theta, \boldsymbol{\xi}) = r''(x; \boldsymbol{\xi}) + \theta g''(x; \boldsymbol{\xi}) \geq 0$ (resp $h''_X(x; \theta, \boldsymbol{\xi}) \leq 0$), then $h''_X(x; \theta, \boldsymbol{\xi})$ is convex (resp concave).

3.2. Truncated Poisson distribution case

Here we supposed that N follows the truncated Poisson distribution of parameter $\theta > 0$, i.e. $P(N = n) = \frac{1}{e^\theta - 1} \frac{\theta^n}{n!}$, $n = 1, 2, \dots$. This case is standard in the literature. See, for instance, the truncated-exponential skew-symmetric distributions (Nadarajah et al., 2014), and the references therein. In this case, one can apply Propositions 1, 2 (with $E(N) = \frac{\theta e^\theta}{e^\theta - 1}$) and 3. The analytic expression of $h_X(x; \theta, \boldsymbol{\xi})$ allows us to prove some new properties. The result below provides some equalities in this case.

Proposition 5. *If N follows the truncated Poisson distribution with parameter $\theta > 0$, then we have*

$$h_X(x; \theta, \boldsymbol{\xi}) = r(x; \boldsymbol{\xi}) \frac{\theta \overline{G}(x; \boldsymbol{\xi})}{1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})}}.$$

Let us now present two expressions of $h'_X(x; \theta, \boldsymbol{\xi})$.

First expression:

$$h'_X(x; \theta, \boldsymbol{\xi}) = r'(x; \boldsymbol{\xi}) \frac{\theta \overline{G}(x; \boldsymbol{\xi})}{1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})}} - \theta g(x; \boldsymbol{\xi}) r(x; \boldsymbol{\xi}) e^{-\theta \overline{G}(x; \boldsymbol{\xi})} w(x),$$

where

$$w(x) = \frac{e^{\theta \overline{G}(x; \boldsymbol{\xi})} - 1 - \theta \overline{G}(x; \boldsymbol{\xi})}{(1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})})^2}.$$

Second expression:

$$h'_X(x; \theta, \boldsymbol{\xi}) = \frac{\theta}{1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})}} g'(x; \boldsymbol{\xi}) + \theta^2 [g(x; \boldsymbol{\xi})]^2 \frac{e^{-\theta \overline{G}(x; \boldsymbol{\xi})}}{(1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})})^2}.$$

Proof of Proposition 5. We have $E(s^N) = \frac{1}{e^\theta - 1} (e^{\theta s} - 1)$ and $E(N s^N) = [E(s^N)]' s = \frac{1}{e^\theta - 1} \theta s e^{\theta s}$. We now use the expression (3). Since $M(s) = \frac{E(N s^N)}{E(s^N)} = \frac{\theta s}{1 - e^{-\theta s}}$, we have

$$h_X(x; \theta, \boldsymbol{\xi}) = r(x; \boldsymbol{\xi}) M(\overline{G}(x; \boldsymbol{\xi})) = r(x; \boldsymbol{\xi}) \frac{\theta \overline{G}(x; \boldsymbol{\xi})}{1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})}}.$$

Now remark that $M'(s) = \theta e^{-\theta s} \frac{e^{\theta s} - 1 - \theta s}{(1 - e^{-\theta s})^2}$. So

$$\begin{aligned} h'_X(x; \theta, \boldsymbol{\xi}) &= r'(x; \boldsymbol{\xi}) M(\overline{G}(x; \boldsymbol{\xi})) - r(x; \boldsymbol{\xi}) M'(\overline{G}(x; \boldsymbol{\xi})) g(x; \boldsymbol{\xi}) \\ &= r'(x; \boldsymbol{\xi}) \frac{\theta \overline{G}(x; \boldsymbol{\xi})}{1 - e^{-\theta \overline{G}(x; \boldsymbol{\xi})}} - \theta g(x; \boldsymbol{\xi}) r(x; \boldsymbol{\xi}) e^{-\theta \overline{G}(x; \boldsymbol{\xi})} w(x). \end{aligned}$$

This first expression is proved.

On the other hand, one can also express $M'(s)$ as $M'(s) = \theta \left(\frac{1}{1-e^{-\theta s}} - \theta s \frac{e^{-\theta s}}{(1-e^{-\theta s})^2} \right)$. So

$$\begin{aligned} h'_X(x; \theta, \xi) &= r'(x; \xi)M(\overline{G}(x; \xi)) - r(x; \xi)M'(\overline{G}(x; \xi))g(x; \xi) \\ &= r'(x; \xi) \frac{\theta \overline{G}(x; \xi)}{1 - e^{-\theta \overline{G}(x; \xi)}} - \theta g(x; \xi)r(x; \xi) \left[\frac{1}{1 - e^{-\theta \overline{G}(x; \xi)}} - \theta \overline{G}(x; \xi) \frac{e^{-\theta \overline{G}(x; \xi)}}{(1 - e^{-\theta \overline{G}(x; \xi)})^2} \right] \\ &= \frac{\theta}{1 - e^{-\theta \overline{G}(x; \xi)}} [r'(x; \xi)\overline{G}(x; \xi) - g(x; \xi)r(x; \xi)] \\ &\quad + \theta^2 g(x; \xi)r(x; \xi)\overline{G}(x; \xi) \frac{e^{-\theta \overline{G}(x; \xi)}}{(1 - e^{-\theta \overline{G}(x; \xi)})^2}. \end{aligned}$$

Let us now observe that $r(x; \xi)\overline{G}(x; \xi) = g(x; \xi)$ and

$$r'(x; \xi)\overline{G}(x; \xi) - g(x; \xi)r(x; \xi) = (r(x; \xi)\overline{G}(x; \xi))' = g'(x; \xi).$$

Therefore

$$h'_X(x; \theta, \xi) = \frac{\theta}{1 - e^{-\theta \overline{G}(x; \xi)}} g'(x; \xi) + \theta^2 [g(x; \xi)]^2 \frac{e^{-\theta \overline{G}(x; \xi)}}{(1 - e^{-\theta \overline{G}(x; \xi)})^2}.$$

The second expression is proved; Proposition 5 is proved. ■

Some additional consequences of Proposition 5 are described below. Note that the extreme of $h_X(x; \theta, \xi)$ are given by the solution x of the equation $h'_X(x; \theta, \xi) = 0$. One can consider the most simple expression for $h'_X(x; \theta, \xi)$ between the first one and the second one.

Let us now show a fact ever proved in Proposition 1 by considering the first expression of $h'_X(x; \theta, \xi)$. If $r(x; \xi)$ is a decreasing function of x , we have $r'(x; \xi) \leq 0$. Since $e^x - 1 - x \geq 0$, we have $w(x) \geq 0$, implying that $h'_X(x; \theta, \xi) \leq 0$, so $h_X(x; \theta, \xi)$ is a decreasing function of x .

On the other hand, if $g(x; \xi)$ are increasing functions of x , then $g'(x; \xi) \geq 0$ and it follows from the second expression of $h'_X(x; \theta, \xi)$ that $h'_X(x; \theta, \xi) \geq 0$, so $h_X(x; \theta, \xi)$ is an increasing functions of x .

3.3. Geometric distribution case

Here we supposed that N follows the geometric distribution of parameter $\theta \in (0, 1)$, i. e. $P(N = n) = (1 - \theta)\theta^{n-1}$, $n = 1, 2, \dots$. For an application, we refer the reader to Marshall and Olkin (1997). Naturally, we can apply Propositions 1, 2 (with $E(N) = \frac{1}{1-\theta}$) and 3. However, since we have access to the expression of $h_X(x; \theta, \xi)$, some properties can be refined. The result below provides some equalities in this case.

Proposition 6. *If N follows the geometric distribution with parameter $\theta \in (0, 1)$, then we have*

$$h_X(x; \theta, \xi) = \frac{r(x; \xi)}{1 - \theta \overline{G}(x; \xi)}, \quad h'_X(x; \theta, \xi) = h_X(x; \theta, \xi)w(x),$$

where

$$w(x) = \frac{r'(x; \xi)}{r(x; \xi)} - \frac{\theta g(x; \xi)}{1 - \theta \overline{G}(x; \xi)}.$$

Proof of Proposition 6. We have $E(s^N) = \frac{(1-\theta)s}{1-\theta s}$ and $E(N s^N) = [E(s^N)]'s = \frac{(1-\theta)s}{(1-\theta s)^2}$. Let us now use the expression (3). Since $M(s) = \frac{E(N s^N)}{E(s^N)} = \frac{1}{1-\theta s}$, we have

$$h_X(x; \theta, \xi) = r(x; \xi)M(\overline{G}(x; \xi)) = \frac{r(x; \xi)}{1 - \theta \overline{G}(x; \xi)}.$$

Moreover, observe that $M'(s) = \frac{\theta}{(1-\theta s)^2} = \theta [M(s)]^2$. Hence

$$\begin{aligned} h'_X(x; \theta, \xi) &= r'(x; \xi)M(\overline{G}(x; \xi)) - r(x; \xi)M'(\overline{G}(x; \xi))g(x; \xi) \\ &= \frac{r'(x; \xi)}{1 - \theta \overline{G}(x; \xi)} - \frac{\theta r(x; \xi)g(x; \xi)}{[1 - \theta \overline{G}(x; \xi)]^2} \\ &= h_X(x; \theta, \xi) \left[\frac{r'(x; \xi)}{r(x; \xi)} - \frac{\theta g(x; \xi)}{1 - \theta \overline{G}(x; \xi)} \right] = h_X(x; \theta, \xi)w(x). \end{aligned}$$

Proposition 6 is proved. ■

From the second equality, we see that the sign of $h'_X(x; \theta, \xi)$ is the same as the sign of $w(x)$ since $h_X(x; \theta, \xi) \geq 0$. As ever shown in Proposition 1, if $r(x; \xi)$ is a decreasing function of x , we have $r'(x; \xi) \leq 0$, so $w(x) \leq 0$, hence $h_X(x; \theta, \xi)$ is a decreasing function of x .

References

- Adamidis K., Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics & Probability Letters*, **39**, 35–42.
- Bourguignon, M., Silva, R.B., Cordeiro, G.M. (2014). A new class of fatigue life distributions. *J. Stat. Comput. Simul.*, **12**, 2619–2635.
- Chahkandi, M., Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics & Data Analysis*, **53**, 4433–4440.
- Kus, C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis*, **51**, 4497–4509.
- Mahmoudi, E., Jafari, A.A. (2012). Generalized exponential?power series distributions. *Computational Statistics and Data Analysis*, **56**, 4047–4066.
- Marshall, N.A., Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika*, **84**, 641–652.
- Morais, A.L., Barreto-Souza, W. (2011). A Compound Class of Weibull and Power Series Distributions. *Computational Statistics & Data Analysis*, **55**, 1410–1425.
- Nadaraajah, S., Nassiri, V., Mohammadpour, A. (2014). Truncated-exponential skew-symmetric distributions. *Statistics*, **48**, 872–895.
- Silva, R.B., Bourguignon, M., Dias, C.R.B., Cordeiro, G.M. (2013). The compound family of extended Weibull power series distributions. *Comput. Statist. Data Anal.*, **58**, 352–367.
- Silva, R.B, G.M. Cordeiro, G.M. (2014). The Burr XII power series distributions: A new compounding family. *Braz. J. Probab. Stat.*, **29**, 565–589.
- Silva, R.B, Bourguignon, M, Cordeiro, G.M (2016). A new compounding family of distributions: The generalized gamma power series distributions. *J. Comput. Appl. Math.*, **303**, 119–139.
- Tahir, M. H. and Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. *Journal of Statistical Distributions and Applications*, **3**, 13, 1–35.
- Tahmasbi, R., Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics & Data Analysis*, **52**, 3889–3901.