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Exact solutions of infinite dimensional total-variation regularized problems.

Axel Flinth & Pierre Weiss

August 5, 2017

Abstract

We study the solutions of infinite dimensional linear inverse problems over Banach spaces. The regularizer is defined as the total variation of a linear mapping of the function to recover, while the data fitting term is a near arbitrary convex function. The first contribution is about the solution's structure: we show that under suitable assumptions, there always exist an m -sparse solution, where m is the number of linear measurements of the signal. Our second contribution is about the computation of the solution. While most existing works first discretize the problem, we show that exact solutions of the infinite dimensional problem can be obtained by solving two consecutive finite dimensional convex programs. These results extend recent advances in the understanding of total-variation regularized problems.

1 Introduction

Let $u \in \mathcal{B}$ be a signal in some vector space \mathcal{B} and assume that it is probed indirectly, with m corrupted linear measurements:

$$b = P(A^*u),$$

where $A^* : \mathcal{B} \rightarrow \mathbb{R}^m$ is a measurement operator defined by $(A^*u)_i = \langle a_i, u \rangle$, each a_i being an element in \mathcal{B}^* , the dual of \mathcal{B} . The mapping $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ denotes a perturbation of the measurements, such as quantization, additional Gaussian or Poisson noise, or any other common degradation operator. Inverse problems consist of estimating u from the measurements b . Assuming that $\dim(\mathcal{B}) > m$, it is clearly impossible to recover u knowing b only. Hence, various regularization techniques have been proposed to stabilize the recovery.

Probably the most well known and used example is Tikhonov regularization [18], which consists in minimizing quadratic cost functions. The regularizers are particularly appreciated for their ease of analysis and implementation. Over the last 20 years, sparsity promoting regularizers have proved increasingly useful, especially when the signals to recover have some underlying sparsity structure. Sparse regularization can be divided into two categories: the analysis formulation or the synthesis formulation.

The analysis formulation consists of solving optimization problems of form

$$\inf_{u \in \mathcal{B}} J(u) := f_b(A^*u) + \|Lu\|_{TV}, \quad (1)$$

where $f_b : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is an application dependent data fidelity term and $L : \mathcal{B} \rightarrow \mathcal{E}$ is a linear operator, mapping \mathcal{B} to some space \mathcal{E} such as \mathbb{R}^n , the space of sequences in ℓ_1 or the space of Radon measures \mathcal{M} . The total variation norm $\|\cdot\|_{TV}$ coincides with the ℓ_1 -norm when \mathcal{E} is discrete, but it is more general since it also applies to measures. The synthesis formulation on its side consists in minimizing

$$\inf_{\mu \in \mathcal{E}} f_b(A^* D\mu) + \|\mu\|_{TV}, \quad (2)$$

where $D : \mathcal{E} \rightarrow \mathcal{B}$ is the linear synthesis operator, also called dictionary. The estimate of u in that case reads $\hat{u} = D\hat{\mu}$, where $\hat{\mu}$ is a solution of (2).

Problems (1) and (2) triggered a massive interest from both theoretical and practical perspectives. Among the most impressive theoretical results, one can cite the field of compressed sensing [7], which certifies that under suitable assumptions, the minimizers of (1) or (2) coincide with the true signal u , or the theory of super-resolution [6, 14].

Most of the studies in this field are confined to the case where both \mathcal{B} and \mathcal{E} are finite dimensional [7, 11, 16, 17]. In the last few years, some efforts have been provided to get a better understanding of (1) and (2) where \mathcal{B} and \mathcal{E} are sequence spaces [1, 2, 25, 24]. Finally, a different route, which will be followed in this paper, is the case where $\mathcal{E} = \mathcal{M}$, the space of Radon measures on a continuous domain. In that case, problems (1) and (2) are infinite dimensional problems over measure spaces. One instance in that class is that of total variation minimization (in the PDE sense [3], that is the total variation of the distributional derivative), which became extremely popular in the field of imaging since its introduction in [20]. There has been surge of interest in understanding the fine properties of the solutions in this setting, with many significant results [5, 6, 23, 14, 8, 26]. The aim of this paper is to continue these efforts by bringing new insights in a general setting.

Contributions and related works The main contributions are twofold: one is about the structure of the solutions of (1), while the other is about how to numerically solve this problem without discretization. The results directly apply to problem (2) since, with regards to our concerns, the synthesis problem (2) is a special case of the analysis problem (1). It indeed suffices to take $L = \text{Id}$ and $\mathcal{B} = \mathcal{M}$ for (2) to be an instance of (1). Notice however that in general, the two approaches should be studied separately [16].

On the theoretical side, we provide a theorem characterizing the structure of the solutions of problem (1) under certain assumptions on the operator L . Roughly speaking, this theorem states that there always exist m -sparse solutions. The precise meaning of this claim will be clarified in Theorem 1. This result is strongly related and was actually motivated by [26]. In there, the authors restrict their study to certain stationary operators L over spaces of functions defined on $\Omega = \mathbb{R}^d$. Their main result states that in that case, generalized splines with m knots actually describe the whole set of solutions. Our result is a bit weaker in the sense that we only prove that a subset of the solutions have a certain structure, but on the other hand, it holds for more general classes of operators L , spaces \mathcal{B} and domains Ω . Furthermore, the proof technique is different from [26]: it is constructive and presumably applicable to wider settings.

On the numerical side, let us first emphasize that in an overwhelming number of works, problem (1) is solved by first discretizing the problem to make it finite dimensional and then approximate solutions are found with standard procedures from convex programming. Theories such as Γ -convergence [4] then sometimes allow showing that as the discretization parameter goes to 0, solutions of the discretized problem converge (in a weak sense) to the solutions of the continuous problem. In this paper, we show that under some assumptions on the measurement vectors (a_i) , the infinite dimensional problem (1) can be attacked directly without discretization, thanks to the first representation theorem: the resolution of two consecutive finite dimensional convex programs allows recovering exact solutions to problem (1) or (2). The structure of the convex programs depend on the structure of measurement vectors. Once again, this result is strongly related to recent advances. For instance, it is shown in [6, 23] that a specific instance of (2) with $L = \text{Id}$ can be solved exactly thanks to semi-definite relaxation or Prony type methods when the signal domain is the torus $\Omega = \mathbb{T}$. Similar results were obtained in [10] for more general semi-algebraic domains using Lasserre hierarchies [19]. Once again, the value of our paper lies in the fact that it holds for near arbitrary convex functions f_b and for a large class of operators L such as the derivative. To the best of our knowledge, the only case considered until now was $L = \text{Id}$.

2 Main results

2.1 Notation

In all of the paper, $\Omega \subseteq \mathbb{R}^d$ denotes an open subset either bounded or unbounded. We let $\mathcal{M}(\Omega)$ denote the set of Radon measures on Ω , i.e. the dual $\mathbb{C}_0(\Omega)^*$ of $\mathbb{C}_0(\Omega)$, the space of continuous functions on Ω vanishing at infinity:

$$\mathbb{C}_0(\Omega) = \{f : \Omega \rightarrow \mathbb{R}, f \text{ continuous}, \\ \forall \epsilon > 0, \exists \omega \subset \Omega \text{ compact}, \forall x \notin \omega, |f(x)| < \epsilon\}.$$

We will throughout the whole paper view $(\mathcal{M}(\Omega), \|\cdot\|_{TV})$ as a Banach space, and not, as often is done, as a locally convex space equipped with the weak-* topology. When we do this, $\mathbb{C}_0(\Omega)$ is a true subset of, and not the whole of, the dual \mathcal{M}^* of \mathcal{M} (as it would have been if we have viewed \mathcal{M} as a locally convex space).

Let $J : \mathcal{B} \rightarrow \mathbb{R} \cup \{+\infty\}$ denote a convex lower-semicontinuous function. We let J^* denote its Fenchel conjugate and $\partial J(u)$ denote its subdifferential at $u \in \mathcal{B}$. Let $X \subset \mathcal{E}$ be a subset of some vector space \mathcal{E} . The indicator function of X is defined for all $e \in \mathcal{E}$ by:

$$\iota_X(e) = \begin{cases} 0 & \text{if } e \in X \\ +\infty & \text{otherwise.} \end{cases}$$

We refer the reader to [15] for more insight on convex analysis in vector spaces.

Remark 1. *All the results in our paper hold when Ω is a separable, locally compact topological space. The proofs require minor technical amendments, but we chose to keep things simpler.*

2.2 Assumptions

The main results in this paper will be proven the following assumptions.

Assumption 1 (Assumptions on L).

- The operator L is continuous on \mathcal{B} .
- The kernel of L has finite dimension: $\dim(\ker L) = r < +\infty$.
- The range of L is closed, and has a complementary subspace, i.e. a closed subspace W such that $\text{ran}L \oplus W = \mathcal{M}$.

An important special case of operators satisfying the assumption 1 are *Fredholm operators*. A Fredholm operator for which the space W complementary to $\text{ran}L$ is finite-dimensional. Many important differential operators are Fredholm.

Under assumption (1), there exists a closed subspace V of $\mathcal{B}(\Omega)$ complementary to $\ker L$, i.e. a space so that $V \oplus \ker L = \mathcal{B}(\Omega)$ (see e.g. [21, Lemma 4.21]). The restriction $L|_V$ of $L : V \rightarrow \text{ran}L$ is bijective operator, and therefore has a continuous inverse $(L|_V)^{-1}$ by the continuous inverse theorem. With the help of this inverse, we can define a *pseudoinverse* $L^+ : \mathcal{M} \rightarrow \mathcal{B}$ through

$$L^+ = j_V(L|_V)^{-1}\Pi_{\text{ran}L},$$

where j_V denotes the injection $V \hookrightarrow \mathcal{B}$ and $\Pi_{\text{ran}L}$ the projection $M \rightarrow \text{ran}L$. Both of these as well as $(L|_V)^{-1}$ are continuous, so that L^+ is continuous.

We will also need the following assumptions on the data fidelity term.

Assumption 2 (Assumptions on f_b).

- Function $f_b : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous.
- Function f_b is lower-bounded: there exists $\gamma \in \mathbb{R}$ such that $f_b(x) \geq \gamma$ for all $x \in \mathbb{R}^m$.
- Function f_b is coercive: $\lim_{\|x\|_2 \rightarrow +\infty} f_b(x) = +\infty$.

Assumption 2 is quite light and covers many common data fidelity terms as exemplified below.

Equality constraints This case corresponds to

$$f_b(x) = \iota_{\{0\}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases} \quad (3)$$

This data fidelity term is commonly used when the data is not corrupted. The two super-resolution papers [6, 23] use this assumption.

Quadratic The case $f_b(x) = \frac{\lambda}{2}\|x\|_2^2$, where $\lambda > 0$ is a data fit parameter, is commonly used when the data suffers from additive Gaussian noise.

ℓ_1 -norm When data suffers from outliers, it is common to set $f_b(x) = \lambda\|x\|_1$, with $\lambda > 0$.

Box constraints When the data is quantized, a natural data fidelity term is a box constraint of the following type

$$f_b(x) = \begin{cases} 0 & \text{if } \|Cx\|_\infty \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

where $C \in \mathbb{R}^{m \times m}$ is a diagonal matrix with positive entries.

Finally, we will have to restrict the functionals a_i used to probe the signals slightly.

Assumption 3 (Assumptions on a_i). *The functionals $a_i \in \mathcal{B}^*(\Omega)$ have the property that $(L^+)^* a_i \in \mathcal{C}_0(\Omega)$. That is, there exist functions $\rho_i \in \mathcal{C}_0(\Omega)$ with*

$$\forall \mu \in \mathcal{M} : \langle (L^+)^* a_i, \mu \rangle = \langle \rho_i, \mu \rangle.$$

This assumption may seem a bit artificial, but we will see that it is crucial, both in the more theoretical first part of the paper, as well as in the second one dealing with the numerical resolution of the problem. Furthermore, it is equivalent to an assumption in the main result of [26], as will be made explicit in the sequel.

2.3 Structure of the solutions

We are now ready to state the first important result of this paper.

Theorem 1. *Under assumptions 1, 2 and 3, problem (1) has a solution of the form*

$$\hat{u} = u_K + \sum_{k=1}^p d_k L^+ \delta_{x_k},$$

with $p \leq m$, $u_K \in \ker L$, $d = (d_k)_{1 \leq k \leq p}$ in \mathbb{R}^p and $X = (x_k)_{1 \leq k \leq p}$ in Ω^p .

Before going further, let us show some consequences of this theorem.

2.3.1 Example 1: $L = \text{Id}$ and the space \mathcal{M}

Probably the easiest case consists in choosing an arbitrary open domain $\Omega \subseteq \mathbb{R}^d$, to set $\mathcal{B} = \mathcal{M}(\Omega)$ and $L = \text{Id}$. In this case, all the assumptions 1 on L are trivially met. In fact, we even have $\text{ran Id} = \mathcal{M}(\Omega)$, $\ker \text{Id} = \{0\}$ and $\text{Id}^+ = \text{Id}$. Therefore, Theorem 1 in this specific case guarantees the existence of a minimizer of (1) of the form

$$\hat{\mu} = \sum_{k=1}^p d_k \delta_{x_k},$$

with $p \leq m$. The assumption 3 in this case simply means that the functionals a_i can be identified with continuous operators vanishing at infinity.

Note that the synthesis formulation (2) can be seen as a subcase of this setting. The structure of the minimizing measure in Theorem 1 implies that the signal estimate \hat{u} has the following form

$$\hat{u} = D\hat{\mu} = \sum_{k=1}^p d_k D\delta_{x_k}.$$

The vectors $(D\delta_x)_{x \in \Omega}$ can naturally be interpreted as the atoms of a dictionary. Hence, Theorem 1 says that there will always exist at least one estimate from the synthesis formulation which is sparsely representable in the dictionary $(D\delta_x)_{x \in \Omega}$.

2.3.2 Example 2: Spline-admissible operators and their native spaces

The authors of [26] consider a generic operator L defined on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ and mapping into $\mathcal{M}(\mathbb{R}^d)$, which is

- Shift-invariant,
- for which there exists a function ρ_L (a generalized spline) of polynomial growth, say

$$\operatorname{esssup}_{x \in \mathbb{R}^d} |\rho_L(x)| (1 + \|x\|)^{-r} < +\infty, \quad (4)$$

obeying $L\rho_L = \delta_0$.

- The space of functions in the kernel of L obeying the growth estimate (4) is finite dimensional.

The authors call such operators *spline-admissible*. For each such operator L , they define a space $\mathcal{M}_L(\mathbb{R}^d)$ as the set of functions f obeying the growth estimate (4) while still having the property $Lf \in \mathcal{M}(\mathbb{R}^d)$. A typical example is the distributional derivative D on $\Omega = \mathbb{R}$.

They go on to prove that $\mathcal{M}_L(\mathbb{R}^d)$ is a Banach space, and has a predual $\mathcal{C}_L(\mathbb{R}^d)$, and (in our notation) assume that the functionals $a_i \in M_L^*(\mathbb{R}^d)$ can be identified with elements of $\mathcal{C}_L(\mathbb{R}^d)$.

It turns out that using this construction, the operator L and functionals (a_i) obey the assumptions 1 and 3, respectively.

Proposition 1.

- The operator $L : \mathcal{M}_L(\mathbb{R}^d) \rightarrow \mathcal{M}(\mathbb{R}^d)$ is Fredholm. In fact, $\operatorname{ran}L$ is even equal to $\mathcal{M}(\mathbb{R}^d)$.
- The functionals $a_i \in M_L^*(\mathbb{R}^d)$ obey assumption 3. In fact, we even have

$$(L^+)^*a \in \mathcal{C}_0(\mathbb{R}^d) \iff a \in \mathcal{C}_L(\mathbb{R}^d).$$

Hence, the assumptions in [26] are a special case of the ones used in this paper.

2.3.3 Example 3: $L = D$ and the space $BV(]0, 1[)$

We set $\Omega =]0, 1[$. The space $BV(\Omega)$ of bounded variation functions is defined by (see [3]):

$$BV(\Omega) = \{u \in L^1(\Omega), Du \text{ is a Radon measure, } \|Du\|_{TV} < +\infty\}, \quad (5)$$

where D is the distributional derivative. Using our notations, it amounts to taking $L = D$ and $\mathcal{B} = BV(\Omega)$. For this space, we have $\ker L = \operatorname{span}(1)$, the vector space of constant functions on Ω .

Lemma 1. *With this choice, $\text{ran}L = \mathcal{M}$, and for all $\mu \in \mathcal{M}$ and all $s \in [0, 1]$,*

$$(L^+\mu)(s) = \mu([0, s]) - \int_0^1 \mu([0, t]) dt. \quad (6)$$

In addition, for a functional $\xi \in BV(]0, 1[)^$ of the form*

$$\langle \xi, u \rangle = \int_0^1 \xi(t)u(t)dt,$$

with $\xi \in L^1(\Omega)$, we have $(L^+)^\xi \in \mathcal{C}_0(\Omega)$ and the formula*

$$((L^+)^*\xi)(s) = \left(\int_s^1 \xi(t)dt \right) - \left(\int_0^1 \xi(t)dt \right) \cdot (1 - s)$$

holds.

As can be seen, L^+ is simply a primitive operator. The elementary functions $L^+\delta_x$ are simply Heavyside functions translated at a distance x of the origin. Hence, Theorem 1 states that there always exist total variation minimizers in 1D that can be written as staircase functions with at most m jumps. Note that in this case, the Heavyside functions coincide with the general splines introduced in [26].

2.3.4 An uncovered case: $L = \nabla$ and the space $BV(]0, 1[^2)$

It is very tempting to use Theorem 1 on the space $\mathcal{B} = BV(]0, 1[^2)$. As mentioned in the introduction, this space is crucial in image processing since its introduction in [20]. Unfortunately, this case is not covered by Theorem 1, since $L\mathcal{B}$ is then a space of vector valued Radon measures, and our assumptions only cover the case of scalar measures.

2.4 Numerical resolution

In this section, we show how problem (1) can be expressed as a finite dimensional convex problem under certain assumptions on the measurement functions (a_i) .

2.4.1 The dual problem and its relationship to the primal

The first step to express (1) as a finite dimensional problem is to use duality. This is a now standard approach, used for instance for super-resolution purposes in [6].

Proposition 2 (Dual of problem (1)). *Define $h : \mathcal{M}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ by $h(\mu) = \|\mu\|_{TV} + \iota_{\text{ran}L}(\mu)$. Then, the following duality relationship holds:*

$$\min_{u \in \mathcal{B}} J(u) = \sup_{q \in \mathbb{R}^m, Aq \in \text{ran}L^*} -h^*((L^+)^*Aq) - f_b^*(q). \quad (7)$$

In the special case $\text{ran}L = \mathcal{M}$, this yields

$$\min_{u \in \mathcal{B}} J(u) = \sup_{q \in \mathbb{R}^m, Aq \in \text{ran}L^*, \|(L^+)^*Aq\|_\infty \leq 1} -f_b^*(q). \quad (8)$$

In addition, let (\hat{u}, \hat{q}) denote any primal-dual pair of problem (7). The following duality relationships hold:

$$\begin{aligned} A\hat{q} &\in L^* \partial(\|\cdot\|_{TV})(L\hat{u}) \\ -\hat{q} &\in \partial f_b(A^*\hat{u}). \end{aligned} \quad (9)$$

For a general operator L , computing h^* may be out of reach, since the conjugate of a sum cannot be easily deduced from the conjugates of each function in the sum. Hence, we now focus on problem (8) corresponding to the case $\text{ran}L = \mathcal{M}$. Notice that this covers at least the two important cases $L = \text{Id}$ and $L = D$, as shown in examples 2.3.1 and 2.3.3.

Solving the dual problem (8) does not directly provide a solution for the primal problem (1). Let us now discuss how to go from a dual solution \hat{q} to a primal solution \hat{u} . The following proposition will come handy for that.

Proposition 3. *Assume that $\text{ran}L = \mathcal{M}$ and let (\hat{u}, \hat{q}) denote a primal-dual pair of problem (8). Let $I(\hat{q}) = \{x \in \Omega, |(L^+)^*(A\hat{q})|(x) = 1\}$. Then $\text{supp}(L\hat{u}) \subseteq I$. In particular, if $I(\hat{q}) = \{x_1, \dots, x_p\}$, then \hat{u} can be written as:*

$$\hat{u} = u_K + \sum_{k=1}^p d_k L^+ \delta_{x_k}. \quad (10)$$

If problem (1) admits a unique solution, then $p \leq m$ and \hat{u} is the solution in Theorem 1.

In the case where $I(\hat{q})$ is a finite set, Proposition 3 provides a constructive way to recover a primal solution \hat{u} from \hat{q} . Letting $(\lambda_i)_{1 \leq i \leq r}$ denote a basis of $\ker L$, we can inject the specific structure (10) into (1). Define the matrix

$$M = [(\langle a_i, \lambda_k \rangle)_{1 \leq i \leq m, 1 \leq k \leq r}, (\langle a_i, L^+ \delta_{x_j} \rangle)_{1 \leq i \leq m, 1 \leq j \leq p}] \quad (11)$$

Then problem (1) becomes a finite dimensional convex program which can be solved with off-the-shelf algorithms:

$$\min_{c \in \mathbb{R}^r, d \in \mathbb{R}^p} f_b \left(M \begin{bmatrix} c \\ d \end{bmatrix} \right) + \|d\|_1. \quad (12)$$

2.4.2 Solving the dual

Problem (8) is finite dimensional but involves two infinite dimensional sets

$$\mathcal{Q}_1 = \{q \in \mathbb{R}^m, Aq \in \text{ran}L^*\} \quad (13)$$

and

$$\mathcal{Q}_2 = \{q \in \mathbb{R}^m, \|(L^+)^* Aq\|_\infty \leq 1\}. \quad (14)$$

In what follows, we let $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$. This set is convex as an intersection of a subspace with and the pre-image by a linear transform of a convex set.

The following lemma states that the constraint set \mathcal{Q}_1 is just a finite dimensional linear constraint. It can be handled with arbitrary precision with a computer by using numerical integration procedures.

Lemma 2. *Let $r = \dim(\ker(L))$ and $(\lambda_i)_{1 \leq i \leq r}$ denote a basis of $\ker L$. The set \mathcal{Q}_1 can be rewritten as*

$$\mathcal{Q}_1 = \{q \in \mathbb{R}^m, \forall 1 \leq i \leq r, \langle q, A^* \lambda_i \rangle = 0\}.$$

Handling the set \mathcal{Q}_2 is much more involved from a numerical point of view. For general measurement functions (a_i) , it seems impossible at first sight to compute the solutions of (8), since \mathcal{Q}_2 could be an arbitrarily complex convex body. This problem actually received considerable attention lately (see for instance [22]) and finding general conditions on (a_i) making the problem attackable with a computer is still an open issue.

In what follows, we provide three specific assumptions making (8) solvable. The important thing is to control the structure of the functions

$$\rho_i := (L^+)^* a_i.$$

Compactly supported functions with disjoint supports Let us begin by considering an academic example. Assume the functions $(\rho_i)_{1 \leq i \leq m}$ are continuous functions with compact and disjoint supports on an open set $\Omega \subset \mathbb{R}^d$. Also assume that each $|\rho_i|$ attains its maximal value only on a discrete set of finite cardinality J_i . Then it is clear that $\|(L^+)^* Aq\|_\infty$ attains its maximal value in $\cup_{i=1}^m J_i$. This value is less than or equal to 1 if and only if $|q_i| \|\rho_i\|_\infty \leq 1$ for all i . This means that:

$$\mathcal{Q}_2 = \{q \in \mathbb{R}^m, \forall 1 \leq i \leq m, |q_i| \|\rho_i\|_\infty \leq 1\},$$

which simply is a box constraint. Hence problem (8) can be solved with standard procedures.

By Proposition 3, the minimizers obtained by solving (8) and (10) all can be written as:

$$\hat{u} = u_K + L^+ \left(\sum_{i=1}^m \sum_{x \in J_i} d_{i,x} \delta_x \right),$$

hence we know a priori, where the Dirac masses can be located.

Piecewise linear functions Now, assume that each ρ_i is a piecewise linear function, with finitely many polyhedral pieces. Then, whatever $q \in \mathbb{R}^m$, the function $(L^+)^* Aq$ is still a piecewise linear function with finitely many polyhedral pieces. The maximum of the function has to be attained in at least one of finitely many vertices $(v_j)_{j \in J}$ of the pieces. We let $f_{i,j} = \rho_i(v_j)$. Then, the constraint set \mathcal{Q}_2 can be written as:

$$\mathcal{Q}_2 = \left\{ q \in \mathbb{R}^m, \forall j \in J, \left| \sum_{i=1}^m f_{i,j} q_i \right| \leq 1 \right\},$$

hence, the constraint set in (8) can be expressed as the intersection of a finite number of half spaces and be handled numerically.

In this case however, we cannot guarantee the set I in Proposition 3 to be finite, since the maximum could be obtained on a whole face of one of the pieces.

Trigonometric polynomials In this section, we assume that $\Omega = [0, 1]$ ¹. For $j \in \mathbb{N}$, let $p_j(t) = \exp(-2\iota\pi jt)$. Assume that the functions ρ_i are real trigonometric polynomials:

$$\rho_i = \sum_{j=-K}^K \gamma_{j,i} p_j,$$

with $\gamma_{j,i} = \gamma_{-j,i}^*$.

The following lemma is a simple variation of [13, Thm 4.24]. It was used already for super-resolution purposes [6].

Lemma 3. *The set \mathcal{Q}_2 can be rewritten as follows:*

$$\mathcal{Q}_2 = \left\{ \alpha \in \mathbb{R}^m, \exists Q \in \mathbb{C}^{(2K+1) \times (2K+1)}, \begin{bmatrix} Q & \Gamma\alpha \\ (\Gamma\alpha)^* & 1 \end{bmatrix} \succeq 0, \right. \\ \left. \sum_{i=1}^{2K+2-j} Q_{i,i+j} = \begin{cases} 1, & j = 1, \\ 0, & 2 \leq j \leq 2K + 1. \end{cases} \right\}.$$

With Lemmas 2 and 3 at hand, the dual problem (8) becomes a semidefinite program that can be solved with a variety of approaches, such as interior point methods [27].

The case of trigonometric polynomials makes Proposition 3 particularly helpful. In that case, either the trigonometric polynomial is zero and the solution \hat{u} lives in the kernel of L , or the set I is finite with cardinality at most $2K$, since $|(L^+)^* Aq|^2 - 1$ is a negative trigonometric polynomial of degree $4K + 2$. In both cases, note that it is possible to solve (12) to retrieve a primal solution with $p \leq 2K$.

2.4.3 Summary

Let us now summarize how exact solutions of (1) can be obtained by solving two finite dimensional problems.

1. Check whether $\text{ran}L = \mathcal{M}$. If it is not the case, then the conjugate h^* should be evaluated, which might or might not be possible.
2. Check whether the structure of the functions $\rho_i := (L^+)^* a_i$ make it possible to express \mathcal{Q}_2 in a form adapted to computers. Two useful examples are piecewise linear functions on polyhedral pieces in dimension $d \geq 1$ or trigonometric polynomials in dimension $d = 1$. Note that these functions can approximate sufficiently regular functions with an arbitrary precision.
3. Use numerical integration procedures to evaluate the r integrals $\langle a_i, \lambda_i \rangle$, where $(\lambda_i)_{1 \leq i \leq r}$ is a basis of $\ker L$. This allows expressing \mathcal{Q}_1 as r linear equality constraints.
4. Solve problem (8) with an arbitrary convex programming method to retrieve a solution $\hat{q} \in \mathbb{R}^m$.

¹Formally, we should work on the torus. This is feasible, see remark (1).

5. Find the set $I = \{x \in \Omega, |(L^+)^* Aq(x)| = 1\}$. The ability to do this once again depends on the structure of the functions (ρ_i) . In the case of piecewise linear functions with polyhedral pieces, it can be achieved by simply evaluating the function on a finite number of vertices. For trigonometric polynomials, it amounts to finding the roots of the trigonometric polynomial $|(L^+)^* Aq|^2 - 1$, which can be expressed as an eigenvalue evaluation problem [9].
6. If I is a discrete set with finite dimensionality $p = |I|$ (it will always be the case with trigonometric polynomials), then a solution can be written as (10) from Proposition 3. Form the matrix M in (11). Otherwise, the primal solution cannot be evaluated directly from \hat{q} , but $L\hat{u} = I$.
7. Solve problem (12) to evaluate a solution (\hat{c}, \hat{d}) of (12). A solution \hat{u} of (1) can then be written as:

$$\hat{u} = \sum_{i=1}^r \hat{c}_i \lambda_i + L^+ \left(\sum_{k=1}^p \hat{d}_k \delta_{x_k} \right).$$

3 Proofs

In this section, we include all proofs left out in the main text.

3.1 Structure of solutions

The heart of the proof of Theorem 1 is a limit argument. To anchor that, we will need the fact that in finite dimensions, there exists solutions of exactly the same structure we are trying to derive about the solutions in infinite dimensions. This statement is well known, see e.g. [25, Theorem 6]. We provide a proof for completeness. It has a geometrical flavour, and is inspired by the work of [12].

Lemma 4. *Let $H^* \in \mathbb{R}^{m,n}$ for some $n \in \mathbb{N}$ and $m \leq n$. Then a problem of the form*

$$\min_{u \in \mathbb{R}^n} f_b(H^*u) + \|u\|_1. \quad (15)$$

has a solution \hat{u} of (1) of the form

$$\hat{u} = \sum_{k=1}^p c_k e_{i_k},$$

with $(c_k)_{k=1}^m$ some real scalars and $p \leq m$.

Proof. We may restrict ourselves to the case that f is the indicator function of a closed convex set \mathcal{C} . To see this, note that if \hat{u} solves (15), it will also solve

$$\min_u \|u\|_1 \text{ subject to } H^*u \in \{v \mid f_b(v) \leq f_b(H^*\hat{u})\}.$$

(if there is a \tilde{u} with $\|\tilde{u}\|_1 < \|\hat{u}\|_1$ and $f_b(H^*\tilde{u}) \leq f_b(H^*\hat{u})$, \tilde{u} surely has a smaller value of the objective function of the problem (1)).

So consider a problem of the form

$$\min_u \|u\|_1 \text{ subject to } H^*u \in \mathcal{C}, \quad (16)$$

and a solution \hat{u} of it (its existence easily follows from the coercivity of the 1-norm and the closedness of \mathcal{C}). The image $\hat{b} = H^*\hat{u}$ then lies on the boundary of the polytope $P = H^*\{u \mid \|u\|_1 \leq \|\hat{u}\|_1\}$ – if it did not, \hat{b} is of the form $H^*\tilde{u}$ with $\|\tilde{u}\|_1 < \|\hat{u}\|_1$. Then \tilde{u} is a feasible point (16) with smaller objective value than \hat{u} , which is a contradiction to the optimality of \hat{u} .

The boundary of $P \subseteq \mathbb{R}^m$ consists of faces of dimension at most $m - 1$. Hence, \hat{b} lies on one of those faces, say F . Concretely, $\hat{b} \in \text{conv}(\text{vert}(F))$, where $\text{vert}(F)$ denotes the set of vertices of face F . The vertices of F are the images by H^* of a subset of the ℓ_1 -ball's vertices, so they can be written as $\|\hat{u}\|_1 \epsilon_k H^*e_i$, for some $i \in \{1, \dots, n\}$ and for $\epsilon_k \in \{-1, 1\}$. Caratheodory's theorem applied in the $(m - 1)$ -dimensional space $\text{aff}F$ implies that \hat{b} can be written as

$$\hat{b} = \sum_{k=1}^m \theta_k \|\hat{u}\|_1 \epsilon_k H^*e_{i_k}$$

with $\sum_{k=1}^m \theta_k = 1$ and $\epsilon \in \{\pm 1\}^m$. The vector $\|\hat{u}\|_1 \sum_{k=1}^m \theta_k \epsilon_k e_{i_k}$ is a solution of (15) of the stated form. \square

The strategy will now be to discretize the problem on finer and finer grids, use the previous lemma and pass to the limit. Let us define a canonical scheme of such discretization and record a crucial property of it.

Lemma 5. *Define a sequence of discretizations $(\Omega_n)_{n \in \mathbb{N}}$ of Ω as*

$$\Omega_n = \left([-2^n, 2^n]^d \cap \frac{\mathbb{Z}^d}{2^n} \right) \cap \Omega. \quad (17)$$

For $k \in \Omega_n$, define ω_n^k to be the hypercube of center k and side-length 2^{-n} intersected with Ω . Let $\mu \in \mathcal{M}(\Omega)$ denote a measure and define the sequence:

$$\nu_n = \sum_{k \in \Omega_n} \mu(\omega_n^k) \delta_k. \quad (18)$$

Then $\nu_n \xrightarrow{*} \mu$ and $\|\nu_n\|_{TV} \leq \|\mu\|_{TV}$.

Proof. First, it follows directly from the definition of the total variation that

$$\|\nu_n\|_{TV} = \sum_{k \in \Omega_n} |\mu(\omega_n^k)| \leq \|\mu\|_{TV}. \quad (19)$$

We now need to prove that for each $\phi \in \mathcal{M}^*$, $\langle \nu_n, \phi \rangle \rightarrow \langle \mu, \phi \rangle$. So fix ϕ and let $\epsilon > 0$. Since $\phi \in \mathcal{M}^*$, there exists a compact set K with the property $|\phi(x)| < \epsilon$ for $x \notin K$. Since ϕ is equicontinuous on K , there exists a $\delta > 0$ so that if $\|x - x'\|_\infty < \delta$, $|\phi(x) - \phi(x')| < \epsilon$. If we choose n so large so that $2^{-n} < \delta$, we

will have

$$\begin{aligned}
|\langle \mu - \nu_n, \phi \rangle| &\leq \int_{\Omega \setminus K} |\phi| d(|\mu| + |\nu_n|) + \left| \int_K \phi d\mu - \int_K \phi d\nu_n \right| \\
&\leq \epsilon (\|\mu\|_{TV} + \|\nu_n\|_{TV}) + \left| \sum_{k \in \Omega_n} \int_{\omega_k} \phi d\mu - \phi(k) \mu(\omega_k) \right| \\
&\leq 2\epsilon \|\mu\|_{TV} + \sum_{k \in \Omega_n} \int_{\omega_k} |\phi(\ell) - \phi(k)| d\mu(\ell) \\
&\leq 2\epsilon \|\mu\|_{TV} + \epsilon \sum_{k \in \Omega_n} |\mu(\omega_k)| \\
&\leq 3\epsilon \|\mu\|_{TV}.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the claim follows. \square

Now let us prove that the optimal value of the problem (1) can be found by solving the discretized problem.

Lemma 6. *Consider for $n \in \mathbb{N}$ the problem*

$$\hat{J}_n := \min_{c \in \mathbb{R}^{|\Omega_n|}, u \in \ker L} J \left(u + L^+ \left(\sum_{k \in \Omega_n} c_k \delta_k \right) \right). \quad (\mathcal{P}_n)$$

Then $\lim_{n \rightarrow \infty} \hat{J}_n = \hat{J}$, where \hat{J} is the optimal value in (1).

Proof. First, notice that elements of the form $u + L^+ \left(\sum_{k \in \Omega_n} c_k \delta_k \right)$ with $c \in \mathbb{R}^{|\Omega_n|}$ and $u \in \ker L$ are feasible points of (1). Hence $\hat{J}_n \geq \hat{J}$.

Now, let \hat{u} be the solution of (1) and let $\hat{\mu} = L\hat{u}$. Write $\hat{u} = L^+L\hat{u} + \hat{u}_K$ with $\hat{u}_K \in \ker L$. According to Lemma 5, there exists a sequence of measures μ_n of the form

$$\mu_n = \sum_{k \in \Omega_n} c_k \delta_k$$

with $\mu_n \xrightarrow{*} \hat{\mu}$ and $\|\mu_n\|_{TV} \leq \|\hat{\mu}\|_{TV}$ for each n .

Now, $A^*L^+\mu_n \rightarrow A^*L^+\hat{\mu}$. To see this, we simply need to note that $\mu_n \xrightarrow{*} \hat{\mu}$ and assumption 3 implies that

$$\langle a_i, L^+\mu_n \rangle = \langle (L^+)^* a_i, \mu_n \rangle = \langle \mu_n, \rho_i \rangle \rightarrow \langle \mu, \rho_i \rangle = \langle a_i, L^+\mu \rangle = \langle (L^+)^* a_i, \mu \rangle. \quad (20)$$

We conclude $f_b(A^*\hat{u}) = \lim_{n \rightarrow \infty} f_b(A^*\hat{u}_K + A^*L^+\mu_n)$ and hence

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \hat{J}_n &= \liminf_{n \rightarrow \infty} f_b(A^*L^+\mu_n + A^*\hat{u}_K) + \|\mu_n\|_{TV} \\
&\leq \liminf_{n \rightarrow \infty} f_b(A^*L^+\mu_n + A^*\hat{u}_K) + \|\hat{\mu}\|_{TV} \\
&= f_b(A^*\hat{u}) + \|L\hat{u}\|_{TV} = \hat{J}.
\end{aligned}$$

\square

We may now prove the main result of this section.

Proof of Theorem 1. Consider the programs \mathcal{P}_n as defined in the proof of the last lemma. Hence, by Lemma 4 we can construct a sequence $(\hat{u}_{K,n}, \hat{c}_n)$ of solutions, with \hat{c}_n containing $p_n \leq m$ nonzero components for $n \geq m$. We may hence write

$$\sum_{k \in \Omega_n} \hat{c}_{n,k} \delta_k = \sum_{\ell=1}^m d_{n,\ell} \delta_{x_{n,\ell}},$$

for some $d_n \in \mathbb{R}^m$ and $X_n = (x_{n,\ell})_{\ell} \in \Omega^m$. In case $p_n < m$, we may repeat positions in the vector X_n .

Since f_b is convex and coercive, it is bounded from below. Without loss of generality, we may assume that f_b is nonnegative, since adding the minimum value of f_b does not change the minimizers. Under this assumption, $\|d_n\|_1 \leq \hat{J}_n \leq \hat{J}_1$ for each n , hence d_n is bounded. This implies that there exists a subsequence, which we do not rename, such that d_n is converging to $d^* \in \mathbb{R}^m$. By possibly considering a subsequence of this subsequence, we may assume that X_n converges in $\bar{\Omega}^\times$, where $\bar{\Omega}^\times$ denotes the one-point-compactification Ω . This means that each of the component sequences $(x_{n,\ell})_n$ either converges to a point x_ℓ^* in Ω , or diverges to ∞ , meaning that it escapes every compact subset of Ω .

Consequently, the subsequence $\mu_n = \sum_{\ell=1}^m d_{n,\ell} \delta_{x_{n,\ell}} \xrightarrow{*} \sum_{\ell=1}^m d_\ell^* \delta_{x_\ell^*} =: \mu^*$, where we identify δ_∞ with the zero measure (note that if $x_{n,\ell} \rightarrow \infty$, then $\delta_{x_{n,\ell}} \xrightarrow{*} 0$). Also, $A^*L^+(\mu_n)$ converges to $A^*(L^+\mu^*)$ (see the proof of the previous lemma), and in particular is bounded.

Due to $f_b(A^*(\hat{u}_{K,n} + L^+\mu_n)) \leq \hat{J}_n$ and the coercivity of f_b , $A^*(\hat{u}_{K,n} + L^+\mu_n)$ is bounded. The boundedness of $A^*(L^+\mu_n)$ now implies the same property for $A^*\hat{u}_{K,n}$. Now assume that $\ker(A^*) \cap \ker(L) = \{0\}$. This implies that there exists $\alpha > 0$ such that $\|A^*u\|_2 \geq \alpha \|u\|_{\mathcal{B}}$ and so $\hat{u}_{K,n}$ is bounded. Therefore it has a subsequence which converges to some element $u_K^* \in \ker L$ since $\dim(\ker L) < +\infty$. Lower semi-continuity of f_b as well as weak lower semi-continuity of the TV -norm implies

$$\begin{aligned} & f_b \left(Au_K^* + AL^+ \sum_{\ell=1}^m d_\ell^* \delta_{x_\ell^*} \right) + \left\| \sum_{\ell=1}^m d_\ell^* \delta_{x_\ell^*} \right\|_{TV} \\ & \leq \liminf_{n \rightarrow \infty} f_b \left(A\hat{u}_{K,n} + AL^+ \sum_{\ell=1}^p d_{n,\ell} \delta_{x_{n,\ell}} \right) + \left\| \sum_{\ell=1}^p d_{n,\ell} \delta_{x_{n,\ell}} \right\|_{TV} \\ & = \liminf_{n \rightarrow \infty} \hat{J}_n = \hat{J}, \end{aligned}$$

where we used Lemma 5 in the final step. Hence, $u_K^* + L^+ \sum_{\ell=1}^m d_\ell^* \delta_{x_\ell^*}$ is a solution of (1), which was exactly what was needed to be proven. (Note that any $x_\ell^* = \infty$ will only cause the linear combination of δ -peaks to be shorter).

If $\ker(A^*) \cap \ker(L) \neq \{0\}$, then the argumentation above still holds by decomposing $\ker(L) = (\ker(A^*) \cap \ker(L)) \oplus U$, where U is the complement of $\ker(A^*) \cap \ker(L)$ in $\ker(L)$ and choosing $\hat{u}_{K,n} \in U$ \square

3.2 Numerical Resolution

In this section, we prove the propositions stated in Section 2.4. We begin with the one describing the dual problem of (1).

Proof of Proposition 2. Define $g : \mathcal{B} \rightarrow \mathcal{M}$ with $g(u) := \|Lu\|_{TV}$. Then $J(u) = f_b(A^*u) + g(u)$. Standard duality arguments [15, p.60] yield:

$$\min_{u \in \mathcal{B}} J(u) = \sup_{q \in \mathbb{R}^m} -g^*(-Aq) - f_b^*(q). \quad (21)$$

Now, we have:

$$\begin{aligned} g^*(z) &= \sup_{u \in \mathcal{B}} \langle z, u \rangle - g(u) \\ &= \sup_{u \in \mathcal{B}} \langle z, u \rangle - \|Lu\|_{TV} \\ &= \sup_{v \in V, u_K \in \ker L} \langle z, v + u_K \rangle - \|Lv\|_{TV} \\ &= \begin{cases} \sup_{v \in V} \langle z, v \rangle - \|Lv\|_{TV} & \text{if } z \in (\ker L)^\perp, \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} \sup_{v \in V} \langle z, L^+Lv \rangle - \|Lv\|_{TV} & \text{if } z \in \text{ran}L^*, \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} \sup_{w \in \text{ran}L} \langle (L^+)^*z, w \rangle - \|w\|_{TV} & \text{if } z \in \text{ran}L^*, \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} h^*((L^+)^*z) & \text{if } z \in \text{ran}L^*, \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

We used the closed range theorem, which in particular implies that $\text{ran}L^* = (\ker L)^\perp$ for an operator L with closed range.

For the special case of $\text{ran}L = \mathcal{M}$, we note that

$$h^*(\phi) = \sup_{\mu \in \mathcal{M}} \langle \phi, \mu \rangle - \|\mu\|_{TV} = \begin{cases} 0 & \text{if } \|\phi\|_\infty \leq 1. \\ +\infty & \text{otherwise,} \end{cases}$$

Note that the subdifferential of g at every $u \in \mathcal{B}$ reads $\partial g(u) = L^* \partial(\|\cdot\|_{TV})(Lu)$ (see e.g. [15, Prop.5.7]). The duality relations also follows from standard arguments, see e.g. [15, p.60]. \square

Next, we prove the proposition describing how to construct a primal solution from a dual one in the case that $\text{ran}L = \mathcal{M}$.

Proof of Proposition 3. We have for any operator L obeying assumption 1

$$(L^+)^*L^* = (LL^+)^* = \Pi_{\text{ran}L}^* = \text{Id}, \quad L^*(L^+)^* = (L^+L)^* = j_V^*.$$

By construction, $A\hat{q}$ and $L^* \partial(\|\cdot\|_{TV})(L\hat{u})$ are elements of $\text{ran}L^*$. Due to the closed range theorem, $\text{ran}L^*$ is isomorphic to the annihilator $(\ker L)^\perp$. On that space, j_V^* is injective. Hence, the inclusion (9) is equivalent to

$$(L^+)^*(A\hat{q}) \in \partial(\|\cdot\|_{TV})(L\hat{u}). \quad (22)$$

Now, it is well known (see for instance [14]), that for all $\mu \in \mathcal{M}$,

$$\partial(\|\cdot\|_{TV})(\mu) = \left\{ \eta \in \mathcal{M}^*, \|\eta\|_\infty \leq 1, \int_\Omega \eta(t) d\mu(t) = \|\mu\|_{TV} \right\}. \quad (23)$$

Consequently, (22) tells us that the by assumption continuous function $(L^+)^*(A\hat{q})$ has modulus 1 $L\hat{u}$ -almost everywhere on $\text{supp}(L\hat{u})$. This means that $(L\hat{u})(I(\hat{q}) \setminus \text{supp}(L\hat{u})) = 0$. In particular, if the set I only consists of isolated points, we get $\text{supp}(L\hat{u}) \subseteq I$. Hence, there exists $(d_k)_{1 \leq k \leq p}$ with

$$L\hat{u} = \sum_{k=1}^p d_k \delta_{x_k} \implies \hat{u} = u_K + \sum_{k=1}^p d_k L^+ \delta_{x_k} \quad (24)$$

for some $u_K \in \ker L$. \square

3.3 Miscellaneous

Here, the rest of the left out proofs are given, starting with the one including spline-admissible operators in our framework.

Proof of Lemma 1. 1. The finite-dimensionality of $\ker L$ is simply assumption 3 of Theorem 1 of [26]. Theorem 4 and 5 of [26] proves that L has a right inverse L_{Φ}^{-1} . This implies that

$$\text{ran} L \subseteq \text{ran} L L_{\Phi}^{-1} = \text{ran} \text{Id} = \mathcal{M}.$$

2. The space \mathcal{C}_L as defined in Theorem 6 of [26] is defined as

$$\mathcal{C}_L = L^*(\mathbb{C}_0(\mathbb{R}^d)) + (\vec{\phi}_i)_{i=1}^r,$$

where ϕ_i is a system of functionals which restricted to $\ker L$ becomes a of the dual of $\ker L$. Without loss of generality, we can assume that $\phi_i|_V = 0$ for each i (if not, we could instead consider the operators $\tilde{\phi}_i = \phi_i \Pi_{\ker L}$).

Then if $a \in \mathcal{C}_L$, we have

$$(L^+)^* a = (L^+)^* L^* \rho + \sum_{i=1}^r \gamma_i (L^+)^* \phi_i$$

for some $\rho \in \mathcal{C}_0(\mathbb{R}^d)$ and γ_i . Now $(L^+)^* L^* = (L L^+)^* = \Pi_{\text{ran} L}^* = \text{Id}$ and $(L^+)^* \phi_i = 0$, so that $(L^+)^* a = \rho \in \mathcal{C}_0(\mathbb{R}^d)$.

If on the other $(L^+)^* a \in \mathcal{C}_0(\mathbb{R}^d)$, we have

$$L^* \mathcal{C}_0(\mathbb{R}^d) \ni L^* (L^+)^* a = (L^+ L)^* a = \Pi_V^* a.$$

Since each functional $a \in \mathcal{M}_L^*$ can be written as $\Pi_V^* a + \Pi_{\ker L}^* a$, and $\Pi_{\ker L}^* a \in (\vec{\phi}_i)_{i=1}^r$, $a \in \mathcal{C}_L$. \square

Next, we discuss the case of L being the differential operator on $BV((0, 1))$.

Proof of Lemma 1. Note that we have $\ker L = \text{span}(1)$, the vector space of constant functions on Ω , hence the space V can be identified with the space of functions with zero mean:

$$V = \left\{ u \in BV(\Omega), \int_{\Omega} u(t) dt = 0 \right\}.$$

For $\mu \in \mathcal{M}$, consider the mapping $I : \mu \mapsto u$ defined for $s \in [0, 1]$ by $u(s) = \mu([0, s])$. We only need to prove that $DI(\mu) = \mu$ in the distributional sense. Let $\phi \in C_c^\infty(\Omega)$:

$$\begin{aligned} \langle I(\mu), \phi' \rangle &= \int_0^1 \mu([0, t]) \phi'(t) dt \\ &= \int_0^1 \int_0^1 1_{[0, t]}(s) d\mu(s) \phi'(t) dt \\ &= \int_0^1 \int_0^1 1_{[s, 1]}(t) \phi'(t) dt d\mu(s) \\ &= \int_0^1 -\phi(s) d\mu(s) = -\langle \mu, \phi \rangle. \end{aligned}$$

This proves the surjectivity of L . We see that the proposed form of L^+ is the right one, since $s \mapsto \mu([0, s]) - \int_0^1 \mu([0, s]) ds$ is a function of zero mean.

We now calculate

$$\begin{aligned} \langle (L^+)^* \xi, \mu \rangle &= \langle \xi, L^+ \mu \rangle \\ &= \int_0^1 \xi(t) \left(\int_0^1 1_{[0, t]}(s) d\mu(s) - \int_0^1 \mu([0, r]) dr \right) dt \\ &= \int_0^1 \left(\int_0^1 1_{[s, 1]}(t) \xi(t) dt \right) d\mu(s) - \int_0^1 \xi(t) dt \cdot \int_0^1 1_{[0, r]}(s) d\mu(s) dr \\ &= \int_0^1 \left(\int_s^1 \xi(t) dt \right) d\mu(s) - \int_0^1 \xi(t) dt \cdot \int_0^1 (1-s) d\mu(s) \end{aligned}$$

In particular, the action of $(L^+)^* \xi$ is given by a continuous function, which is vanishing on the boundary of $(0, 1)$ \square

We now show how the set \mathcal{Q}_1 can be expressed as a finite set of linear equality constraints.

Proof of Lemma 2. Since $\text{ran} L^* = (\ker L)^\perp$ (by the closed range theorem), $Aq \in \text{ran} L^*$ if and only if $\forall 1 \leq i \leq r, \langle Aq, \lambda_i \rangle = 0$. \square

Finally, we provide the argument that the constraint of the dual problem can be rewritten as an inequality on the space of Hermitian matrices in the case of the functions ρ_i being trigonometric polynomials.

Proof of Lemma 3. Note that $|\sum_{i=1}^m \alpha_i \rho_i| \leq 1$ is equivalent to

$$1 \geq \left| \sum_{i=1}^m \alpha_i \sum_{j=-K}^K \gamma_{i,j} p_j \right| = \left| \sum_{j=-K}^K \sum_{i=1}^m \alpha_i \gamma_{i,j} p_j \right| = \left| \sum_{j=-K}^K (\Gamma \alpha)_j p_j \right| = \left| p_{-K} \sum_{j=-K}^K (\Gamma \alpha)_j p_j \right|.$$

The function $f = p_{-K} \sum_{j=-K}^K (\Gamma \alpha)_j p_j$ is a causal trigonometric polynomial. We know from [13, Cor.4.27] that it obeys the constraint $\|f\|_\infty \leq 1$ if and only if there exists a positive semi-definite matrix $Q \in \mathbb{C}^{(2K+1) \times (2K+1)}$ such that

$$\begin{bmatrix} Q & \Gamma \alpha \\ (\Gamma \alpha)^* & 1 \end{bmatrix} \succeq 0 \text{ and } \sum_{i=1}^{2K+2-j} Q_{i,i+j} = \begin{cases} 1, & j = 1, \\ 0, & 2 \leq j \leq 2K+1. \end{cases} \quad (25)$$

\square

4 Conclusion & Outlook

In this paper we have studied the properties of total variation regularized problems, where total-variation should be understood as a term of form $\|Lu\|_{TV}$, with L a linear operator. We have shown that under a convexity assumption on the data-fit term, some of the solutions \hat{u} of total-variation regularized inverse problems are m -sparse, where m denotes the number of measurements. This precisely means that $L\hat{u}$ is an atomic measure supported on at most m points. This result extends recent advances [26], by relaxing some hypotheses on the linear operator L and on the domain of the functions.

The second contribution of this paper is to show that solutions of this infinite dimensional problem can be obtained by solving two consecutive finite dimensional problems, given that the measurements belong to some function spaces such as the trigonometric polynomials or the set of piecewise linear functions on polyhedral domains. Once again, this result extends significantly recent results on super-resolution [6, 23].

As an outlook, we want to stress out that the hypotheses formulated on the linear operator L rule out a number of interesting applications, such as total variation regularization in image processing. As an outlook, we plan to study how the results and the proof techniques in this paper could apply to more general cases.

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