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A non-commutative algorithm for multiplying (7×7) matrices using 250 multiplications

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Abstract

We present a non-commutative algorithm for multiplying (7×7) matrices using 250 multiplications and a non-commutative algorithm for multiplying (9×9) matrices using 520 multiplications. These algorithms are obtained using the same divide-and-conquer technique that could be applied to any suitable matrix sizes.

1 Introduction

The main tool of this note could be summarised in the following proposition:

Proposition 1. — Denoting by $\langle u, v, w \rangle$ the number of multiplications necessary to multiply an $(u \times v)$ matrix with an $(v \times w)$ matrix to yield an $(u \times w)$ product matrix, the following relation holds:

$$\langle u + v, u + v, u + v \rangle \leq \langle u, u, u \rangle + 3 \langle u, u, v \rangle + 3 \langle v, v, u \rangle \text{ when } u > v. \quad (1)$$

□

For $(u, v) = (4, 3)$, by selecting already known matrix multiplication algorithms and applying this proposition, we obtain a new upper bounds 250 and the explicit corresponding algorithm [\(7 × 7 × 7; 250\)](#).

In fact, we use the Strassen's matrix multiplication algorithm [13] to *divide* the (7×7) matrix multiplication problem into smaller sub-problems; the use of three Smirnov's rectangular matrix multiplication algorithms [11, 12] allows to *conquer* new upper bounds on the number of necessary non-commutative multiplications.

To illustrate this point, we first present a scheme that evaluate the product $P = N \cdot M$:

$$N = \begin{pmatrix} n_{11} & \cdots & n_{17} \\ \vdots & & \vdots \\ n_{71} & \cdots & n_{77} \end{pmatrix}, M = \begin{pmatrix} m_{11} & \cdots & m_{17} \\ \vdots & & \vdots \\ m_{71} & \cdots & m_{77} \end{pmatrix}, P = \begin{pmatrix} p_{11} & \cdots & p_{17} \\ \vdots & & \vdots \\ p_{71} & \cdots & p_{77} \end{pmatrix}, \quad (2)$$

using 250 multiplications. This algorithm improves slightly the previous known upper bound 258 presented in [2] and likely obtained with the same kind of techniques presented in this note.

In the last section of this work, we stress the main limitation of our approach by constructing a (9×9) matrix multiplication algorithm using 520 multiplications (that is only two multiplications less than the corresponding result in [2] but 6 multiplications more than the algorithm $(9 \times 9 \times 9; 514)$ cited in Table A—that summarise what we know about 286 matrix multiplication algorithms and was obtained automatically during the elaboration of this note—see [10] for a more complete list with all details). This shows that, as the approach presented here is based on the knowledge of fast matrix multiplication algorithms for rectangular matrices, it is limited by the restricted knowledge we have on these algorithms.

2 Divide

For any (2×2) matrices:

$$A = (a_{ij})_{1 \leq i, j \leq 2}, \quad B = (b_{ij})_{1 \leq i, j \leq 2} \quad \text{and} \quad C = (c_{ij})_{1 \leq i, j \leq 2}, \quad (3)$$

V. Strassen shows in [13] that the matrix product $C = A \cdot B$ could be computed by performing the following operations:

$$\begin{aligned} t_1 &= (a_{11} + a_{22})(b_{11} + b_{22}), & t_2 &= (a_{12} - a_{22})(b_{21} + b_{22}), \\ t_3 &= (-a_{11} + a_{21})(b_{11} + b_{12}), & t_4 &= (a_{11} + a_{12})b_{22}, \\ t_5 &= a_{11}(b_{12} - b_{22}), & t_6 &= a_{22}(-b_{11} + b_{21}), & t_7 &= (a_{21} + a_{22})b_{11}, \end{aligned} \quad (4)$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} t_1 + t_2 - t_4 + t_6 & t_6 + t_7, \\ t_4 + t_5 & t_1 + t_3 + t_5 - t_7 \end{pmatrix},$$

in the considered non-necessarily commutative coefficients algebra.

To construct our algorithm, we are going to work with the algebra of (4×4) matrices and thus, we have to adapt our inputs P, N and M (2) to that end. So we rewrite these matrices (2) in the following equivalent form:

$$X = \left(\begin{array}{cccc|ccc} x_{11} & x_{12} & x_{13} & 0 & x_{14} & \cdots & x_{17} \\ x_{21} & x_{22} & x_{23} & 0 & x_{24} & \cdots & x_{27} \\ x_{31} & x_{32} & x_{33} & 0 & x_{34} & \cdots & x_{37} \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_{41} & x_{42} & x_{43} & 0 & x_{44} & \cdots & x_{47} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_{71} & x_{72} & x_{73} & 0 & x_{74} & \cdots & x_{77} \end{array} \right), \quad X \in \{P, M, N\}, \quad (5)$$

in which we have just added a line and a column of zeros. After that padding, the product $P = N \cdot M$ is unchanged.

Notations 2. — Hence, in the sequel P (resp. M, N) designates the (8×8) matrices defined in (5) and P_{ij} (resp. M_{ij}, N_{ij}) designates (4×4) matrices (e.g. P_{11} stands for the upper left submatrix of P , etc). \square

Remark 3. — The process of peeling the result (removing rows and columns) of our computations might be better understood if we use—even implicitly—the tensor interpretation of matrix multiplication algorithms. Using this framework it

appears that the bilinear application $\mathcal{B} : \mathbb{K}^{(8 \times 8)} \times \mathbb{K}^{(8 \times 8)} \mapsto \mathbb{K}^{(8 \times 8)}$ with indeterminates N and M that defines the matrix multiplication $\mathcal{B}(N, M) = N \cdot M = P$ is completely equivalent to the trilinear form $\mathbb{K}^{(8 \times 8)} \times \mathbb{K}^{(8 \times 8)} \times \mathbb{K}^{(8 \times 8)} \mapsto \mathbb{K}$ with indeterminates N, M and P defined by $\text{Trace}(N \cdot M \cdot {}^\top P)$ (e.g. see [6, §4.6.4, page 506], [8, § 2.5.2], [1, § 2.2] or [3] for a complete description of this equivalence). \square

Computationally, this equivalence induces that the following relation holds:

$$\text{Trace}(N \cdot M \cdot {}^\top P) = \text{Trace}((N_{11} + N_{22}) \cdot (M_{11} + M_{22}) \cdot {}^\top (P_{11} + P_{22})) \quad (6a)$$

$$+ \text{Trace}((N_{12} - N_{22}) \cdot (M_{21} + M_{22}) \cdot {}^\top P_{11}) \quad (6b)$$

$$+ \text{Trace}((N_{21} - N_{11}) \cdot (M_{11} + M_{12}) \cdot {}^\top P_{22}) \quad (6c)$$

$$+ \text{Trace}((N_{11} + N_{12}) \cdot M_{22} \cdot {}^\top (P_{21} - P_{11})) \quad (6d)$$

$$+ \text{Trace}(N_{11} \cdot (M_{12} - M_{22}) \cdot {}^\top (P_{21} + P_{22})) \quad (6e)$$

$$+ \text{Trace}(N_{22} \cdot (M_{21} - M_{11}) \cdot {}^\top (P_{11} + P_{12})) \quad (6f)$$

$$+ \text{Trace}((N_{21} + N_{22}) \cdot M_{11} \cdot {}^\top (P_{12} - P_{22})). \quad (6g)$$

As the bilinear application and the trilinear form are equivalent, one could retrieve directly the algorithm from this last form. Our original problem is now divided in 7 lower dimensional subproblems encoded by trilinear forms. In the next section, we enumerate the algorithms used to compute the matrix products (6a-6g).

3 Conquer

The first summand (6a) involves unstructured (4×4) matrix multiplication that could be computed using Strassen's algorithm and could be done with 7^2 multiplications. Before studying the other summands, we emphasise the following trivial remarks:

Remarks 4. — *As we consider some matrices with zero last column and/or row, recall that the product of two (4×4) matrices $(x_{ij})_{1 \leq i, j \leq 4}$ and $(y_{ij})_{1 \leq i, j \leq 4}$ is equivalent to the product of the:*

- (4×3) matrix $(x_{ij})_{1 \leq i \leq 4, 1 \leq j \leq 3}$ with the (3×4) matrix $(y_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 4}$ when for $1 \leq i \leq 4$ we have $y_{i4} = 0$ (zero last row);
- (3×4) matrix $(x_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 4}$ with the (4×3) matrix $(y_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 4}$ when for $1 \leq j \leq 4$ we have $y_{4j} = 0$ (zero last column).

\square

Let us now review the matrices involved in the summands (6b-6g).

Facts 5. — *We notice that by construction:*

- the last row and column of X_{11} are only composed by zeros;
- X_{22} is a (4×4) matrix;
- the last column of $X_{21} - X_{11}$ is only composed by zeros;

- the last line of $X_{11} + X_{12}$ is only composed by zeros;
- $X_{12} - X_{22}$ and $X_{21} - X_{22}$ are (4×4) matrices without zero row or column.

□

Hence, taking into account Remarks 4 and Facts 5, we have a better description of the sub-problems considered in this section:

Remarks 6. — *The summand:*

- (6b) involves an (3×4) times (4×3) times (3×3) matrices product;
- (6c) involves an (4×3) times (3×4) times (4×4) matrices product;
- (6d) involves an (3×4) times (4×4) times (4×3) matrices product;
- (6e) involves an (3×3) times (3×4) times (4×3) matrices product;
- (6f) involves an (4×4) times (4×3) times (3×4) matrices product;
- (6g) involves an (4×3) times (3×3) times (3×4) matrices product.

□

Remark 7. — *We also rely our construction on the representation of matrix multiplication algorithm by trilinear forms and the underlying tensor representation (see Remark 3) because, as quoted in [6, § 4.6.4 p. 507]:*

“[...] a normal scheme for evaluating an $(m \times n)$ times $(n \times s)$ matrix product implies the existence of a normal scheme to evaluate an $(n \times s)$ times $(s \times m)$ matrix product using the same number of chain multiplications.”

□

This is exactly what we are using in the sequel. To do so and in order to express complexity of the summands (6b-6g), we (re)introduce more precisely the following notations (already used in Proposition 1):

Notation 8. — *For matrices U, V and W of size $(u \times v)$, $(v \times w)$ and $(w \times u)$, we denote by $\langle u, v, w \rangle$ the known supremum on the multiplication necessary for computing $\text{Trace}(U \cdot V \cdot {}^tW)$ (that is the number of multiplication used by the best known algorithm allowing to compute $U \cdot V = W$ (a.k.a. tensor rank)).* □

Remarks 9. — *Using this notation, we see that Remarks 6 can be restated as follow:*

- (6b) can be computed using $\langle 3, 4, 3 \rangle$ multiplications;
- (6c) can be computed using $\langle 4, 3, 4 \rangle$ multiplications;
- (6d) can be computed using $\langle 3, 4, 4 \rangle$ multiplications;
- (6e) can be computed using $\langle 3, 3, 4 \rangle$ multiplications;
- (6f) can be computed using $\langle 4, 4, 3 \rangle$ multiplications;

- (6g) can be computed using $\langle 4, 3, 3 \rangle$ multiplications.

□

We are going to see that we need only two algorithms to perform all these computations. In fact, the Remark 7 is a direct consequence of the following Trace properties:

$$\begin{aligned} \text{Trace}(U \cdot V \cdot W) &= \text{Trace}(W \cdot U \cdot V) &&= \text{Trace}(V \cdot W \cdot U), \\ &= \text{Trace}(\tau(U \cdot V \cdot W)) &&= \text{Trace}(\tau W \cdot \tau V \cdot \tau U). \end{aligned} \quad (7)$$

To be more precise, Relations (7) imply the following well-known result:

Lemma 10. — *The following relations hold:*

$$\langle u, v, w \rangle = \langle w, u, v \rangle = \langle v, w, u \rangle = \langle v, u, w \rangle = \langle w, v, u \rangle = \langle u, w, v \rangle. \quad (8)$$

□

This allows us to state that algorithm (6b-6g) requires:

$$\langle 4, 4, 4 \rangle + 3 \langle 3, 3, 4 \rangle + 3 \langle 3, 4, 4 \rangle \quad (9)$$

multiplications. As A. V. Smirnov states that $\langle 3, 3, 4 \rangle = 29$ and $\langle 3, 4, 4 \rangle = 38$ in [11, Table 1, N^o 13 and 21], we conclude that our algorithm required 250 multiplications ($7^2 + 3 \cdot 29 + 3 \cdot 38$). Furthermore, Smirnov provides in [12] the explicit description of these algorithms, allowing us to do the same with the first algorithm constructed in this note at url:

<http://fmm.univ-lille.fr/7x7x7.html>.

In the next section, we show how to apply the very same manipulations to the product of two (9×9) matrices.

4 An algorithm for multiplying (9×9) matrices

As in the previous section, we are going to pad our (9×9) matrices:

$$N = (n_{ij})_{1 \leq i, j \leq 9}, \quad M = (M_{ij})_{1 \leq i, j \leq 9} \quad \text{and} \quad P = (P_{ij})_{1 \leq i, j \leq 9}, \quad (10)$$

in order to work this time with equivalent (12×12) matrices:

$$Y = \left(\begin{array}{cccccc|ccc} y_{11} & y_{12} & y_{13} & 0 & 0 & 0 & y_{14} & \cdots & y_{19} \\ y_{21} & y_{22} & y_{23} & 0 & 0 & 0 & y_{24} & \cdots & y_{29} \\ y_{31} & y_{32} & y_{33} & 0 & 0 & 0 & y_{34} & \cdots & y_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & y_{44} & \cdots & y_{49} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_{91} & y_{92} & y_{93} & 0 & 0 & 0 & y_{94} & \cdots & y_{99} \end{array} \right), \quad Y \in \{P, M, N\}, \quad (11)$$

After that padding, the product $P = N \cdot M$ is unchanged.

Notations 11. — In the sequel P (resp. M, N) designates the (12×12) matrices defined in (11) and P_{ij} (resp. M_{ij}, N_{ij}) designates (6×6) matrices (e.g. P_{11} stands for the upper left submatrix of P , etc). \square

The process described in Section 3 remains—mutatis mutandis—exactly the same and we obtain the following special case of Proposition 1:

Lemma 12. — With the Notations 8, there exists an algorithm that multiply two (9×9) matrices using $\langle 6, 6, 6 \rangle + 3 \langle 6, 6, 3 \rangle + 3 \langle 6, 3, 3 \rangle$ multiplications. \square

Again, we found in [11, Table 1, N^s 27] that $\langle 3, 3, 6 \rangle = 40$ (an explicit form of this algorithm could be found in the source code of [1] or in the collection [10]). But this time, we do not have found in the literature any *fast* matrix multiplication algorithm for $\langle 6, 6, 3 \rangle$ and we have to provide an algorithm for $\langle 6, 6, 6 \rangle$. Nevertheless, as done implicitly in Section 2, we are going to use again the following constructive result on tensor's Kronecker product.

Lemma 13. — Given an algorithm computing $(u_1 \times v_1)$ times $(v_1 \times w_1)$ matrix product using $\langle u_1, v_1, w_1 \rangle$ multiplications and an algorithm computing $(u_2 \times v_2)$ times $(v_2 \times w_2)$ matrix product using $\langle u_2, v_2, w_2 \rangle$ multiplications, one can construct an algorithm computing $(u_1 u_2 \times v_1 v_2)$ by $(v_1 v_2 \times w_1 w_2)$ matrix multiplication using $\langle u_1, v_1, w_1 \rangle \cdot \langle u_2, v_2, w_2 \rangle$ multiplications (a.k.a. the tensor's Kronecker product of the two previous algorithms). \square

Hence, as we know that trivially $\langle 1, 2, 2 \rangle = 4$, we conclude that $\langle 6, 6, 6 \rangle$ is equal to $\langle 6, 3, 3 \rangle \cdot \langle 1, 2, 2 \rangle$ (that is 160) and that $\langle 6, 6, 3 \rangle = 80$. So, the algorithm constructed in this section requires 520 multiplications ($4 \cdot 40 + 3 \cdot 80 + 3 \cdot 40$).

5 Concluding remarks

The complexity of our algorithm for multiplying (9×9) matrices could likely be improved by finding a *better* algorithm for $\langle 6, 6, 3 \rangle$ and $\langle 6, 6, 6 \rangle$ then those used above (algorithms obtained by tensor Kronecker product are not always optimal as shown by the fact that $\langle 3, 3, 3 \rangle \cdot \langle 3, 3, 3 \rangle$ is equal to 529 while computations summarised in Table A show that $\langle 9, 9, 9 \rangle$ is now 514).

By combining tensor's based description of matrix multiplication algorithms with rectangular algorithms found by numerical computer search (see [1, § 2.3.2] and [11]), it is possible—as already shown in [2]—to improve the theoretical complexity of small size matrix products (see Table A in appendix in which new results obtained with the method presented in this note are in bold face).

Remark 14. — Furthermore, by considering the case of algorithm :

$$(4 \times 7 \times 10; 207) = 3(2 \times 3 \times 5; 25) + 4(2 \times 4 \times 5; 33), \quad (12)$$

one can remark that the complexity of the resulting algorithm could be smaller than the complexity of the algorithms used to construct it:

$$C(4, 7, 10) = 3 \frac{\ln(207)}{\ln(4 \cdot 7 \cdot 10)} = 2.839175463 < 2.839184673 = C(2, 3, 5), \quad (13)$$

and

$$C(4, 7, 10) < 2.843552579 = C(2, 4, 5). \quad (14)$$

\square

The author thinks that some symmetry-based geometrical methods could reduce further the upper bounds presented in this note.

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A Summary of known results

We gather below a summary of some known results up to $\langle 12, 12, 12 \rangle$.

$\langle 1, x, y \rangle$	$x \cdot y$	$x \cdot y$	trivial algorithms
$\langle 2, 2, 2 \rangle$	7	8	Strassen [13]
$\langle 2, 3, 3 \rangle$	15	18	Hopcroft & Kerr [4]
$\langle 2, 3, 4 \rangle$	20	24	Hopcroft & Kerr [4]
$\langle 2, 3, 5 \rangle$	25	30	Hopcroft & Kerr [4]
$\langle 2, 3, 6 \rangle$	30	36	$\langle 2, 3, 3 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 2, 3, 7 \rangle$	35	42	$\langle 2, 3, 3 \rangle + \langle 2, 3, 4 \rangle$
$\langle 2, 3, 8 \rangle$	40	48	$\langle 2, 3, 4 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 2, 3, 9 \rangle$	45	54	$\langle 2, 3, 3 \rangle \cdot \langle 1, 1, 3 \rangle$
$\langle 2, 3, 10 \rangle$	50	60	$\langle 2, 3, 4 \rangle + \langle 2, 3, 6 \rangle$
$\langle 2, 3, 11 \rangle$	55	66	$\langle 2, 3, 7 \rangle + \langle 2, 3, 4 \rangle$
$\langle 2, 3, 12 \rangle$	60	72	$\langle 1, 1, 2 \rangle \cdot \langle 2, 3, 6 \rangle$
$\langle 2, 4, 4 \rangle$	26	32	Hopcroft & Kerr [4]
$\langle 2, 4, 5 \rangle$	33	40	Hopcroft & Kerr [4]
$\langle 2, 4, 6 \rangle$	39	48	Hopcroft & Kerr [4]
$\langle 2, 4, 7 \rangle$	46	56	$\langle 2, 4, 3 \rangle + \langle 2, 4, 4 \rangle$
$\langle 2, 4, 8 \rangle$	52	64	$\langle 2, 4, 4 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 2, 4, 9 \rangle$	59	72	$\langle 2, 4, 4 \rangle + \langle 2, 4, 5 \rangle$
$\langle 2, 4, 10 \rangle$	65	80	$\langle 2, 4, 4 \rangle + \langle 2, 4, 6 \rangle$
$\langle 2, 4, 11 \rangle$	72	88	$\langle 2, 4, 7 \rangle + \langle 2, 4, 4 \rangle$
$\langle 2, 4, 12 \rangle$	78	96	$\langle 1, 1, 3 \rangle \cdot \langle 2, 4, 4 \rangle$
$\langle 2, 5, 5 \rangle$	40	50	Hopcroft & Kerr [4]
$\langle 2, 5, 6 \rangle$	48	60	Hopcroft & Kerr [4]
$\langle 2, 5, 7 \rangle$	56	70	Hopcroft & Kerr [4]
$\langle 2, 5, 8 \rangle$	64	80	Hopcroft & Kerr [4]
$\langle 2, 5, 9 \rangle$	72	90	Hopcroft & Kerr [4]
$\langle 2, 5, 10 \rangle$	80	100	$\langle 2, 5, 5 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 2, 5, 11 \rangle$	88	110	$\langle 2, 5, 6 \rangle + \langle 2, 5, 5 \rangle$
$\langle 2, 5, 12 \rangle$	96	120	$\langle 2, 5, 6 \rangle + \langle 2, 5, 6 \rangle$
$\langle 2, 6, 6 \rangle$	57	72	Hopcroft & Kerr [4]
$\langle 2, 6, 7 \rangle$	67	84	Hopcroft & Kerr [4]
$\langle 2, 6, 8 \rangle$	76	96	Hopcroft & Kerr [4]
$\langle 2, 6, 9 \rangle$	86	108	Hopcroft & Kerr [4]
$\langle 2, 6, 10 \rangle$	95	120	Hopcroft & Kerr [4]
$\langle 2, 6, 11 \rangle$	105	132	$\langle 2, 6, 5 \rangle + \langle 2, 6, 6 \rangle$
$\langle 2, 6, 12 \rangle$	114	144	$\langle 1, 1, 2 \rangle \cdot \langle 2, 6, 6 \rangle$
$\langle 2, 7, 7 \rangle$	77	98	Hopcroft & Kerr [4]
$\langle 2, 7, 8 \rangle$	88	112	Hopcroft & Kerr [4]
$\langle 2, 7, 9 \rangle$	99	126	Hopcroft & Kerr [4]

$\langle 2, 7, 10 \rangle$	110	140	Hopcroft & Kerr [4]
$\langle 2, 7, 11 \rangle$	121	154	Hopcroft & Kerr [4]
$\langle 2, 7, 12 \rangle$	132	168	Hopcroft & Kerr [4]
$\langle 2, 8, 8 \rangle$	100	128	Hopcroft & Kerr [4]
$\langle 2, 8, 9 \rangle$	113	144	Hopcroft & Kerr [4]
$\langle 2, 8, 10 \rangle$	125	160	$\langle 1, 1, 2 \rangle \cdot \langle 2, 8, 5 \rangle$
$\langle 2, 8, 11 \rangle$	138	176	$\langle 2, 8, 3 \rangle + \langle 2, 8, 8 \rangle$
$\langle 2, 8, 12 \rangle$	150	192	Hopcroft & Kerr [4]
$\langle 2, 9, 9 \rangle$	126	162	Hopcroft & Kerr [4]
$\langle 2, 9, 10 \rangle$	140	180	$\langle 1, 1, 2 \rangle \cdot \langle 2, 9, 5 \rangle$
$\langle 2, 9, 11 \rangle$	154	198	$\langle 2, 9, 2 \rangle + \langle 2, 9, 9 \rangle$
$\langle 2, 9, 12 \rangle$	168	216	Hopcroft & Kerr [4]
$\langle 2, 10, 10 \rangle$	155	200	Hopcroft & Kerr [4]
$\langle 2, 10, 11 \rangle$	171	220	$\langle 2, 10, 1 \rangle + \langle 2, 10, 10 \rangle$
$\langle 2, 10, 12 \rangle$	186	240	$\langle 2, 10, 2 \rangle + \langle 2, 10, 10 \rangle$
$\langle 2, 11, 11 \rangle$	187	242	Hopcroft & Kerr [4]
$\langle 2, 11, 12 \rangle$	204	264	$\langle 2, 11, 1 \rangle + \langle 2, 11, 11 \rangle$
$\langle 2, 12, 12 \rangle$	222	288	Hopcroft & Kerr [4]
$\langle 3, 3, 3 \rangle$	23	27	Laderman [7]
$\langle 3, 3, 4 \rangle$	29	36	Smirnov [11]
$\langle 3, 3, 5 \rangle$	36	45	Smirnov [11]
$\langle 3, 3, 6 \rangle$	40	54	Smirnov [11]
$\langle 3, 3, 7 \rangle$	49	63	$\langle 3, 3, 1 \rangle + \langle 3, 3, 6 \rangle$
$\langle 3, 3, 8 \rangle$	55	72	$\langle 3, 3, 6 \rangle + \langle 3, 3, 2 \rangle$
$\langle 3, 3, 9 \rangle$	63	81	$\langle 3, 3, 3 \rangle + \langle 3, 3, 6 \rangle$
$\langle 3, 3, 10 \rangle$	69	90	$\langle 3, 3, 4 \rangle + \langle 3, 3, 6 \rangle$
$\langle 3, 3, 11 \rangle$	76	99	$\langle 3, 3, 5 \rangle + \langle 3, 3, 6 \rangle$
$\langle 3, 3, 12 \rangle$	80	108	$\langle 1, 1, 2 \rangle \cdot \langle 3, 3, 6 \rangle$
$\langle 3, 4, 4 \rangle$	38	48	Smirnov [11]
$\langle 3, 4, 5 \rangle$	48	60	Smirnov [11]
$\langle 3, 4, 6 \rangle$	58	72	$\langle 3, 4, 3 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 3, 4, 7 \rangle$	67	84	$\langle 3, 4, 3 \rangle + \langle 3, 4, 4 \rangle$
$\langle 3, 4, 8 \rangle$	76	96	$\langle 3, 4, 4 \rangle \cdot \langle 1, 1, 2 \rangle$
$\langle 3, 4, 9 \rangle$	86	108	$\langle 3, 4, 4 \rangle + \langle 3, 4, 5 \rangle$
$\langle 3, 4, 10 \rangle$	96	120	$\langle 3, 4, 2 \rangle + \langle 3, 4, 8 \rangle$
$\langle 3, 4, 11 \rangle$	105	132	$\langle 3, 4, 3 \rangle + \langle 3, 4, 8 \rangle$
$\langle 3, 4, 12 \rangle$	114	144	$\langle 1, 1, 3 \rangle \cdot \langle 3, 4, 4 \rangle$
$\langle 3, 5, 5 \rangle$	61	75	$\langle 3, 5, 2 \rangle + \langle 3, 5, 3 \rangle$
$\langle 3, 5, 6 \rangle$	70	90	$\langle 3, 2, 6 \rangle + \langle 3, 3, 6 \rangle$
$\langle 3, 5, 7 \rangle$	84	105	$\langle 3, 2, 7 \rangle + \langle 3, 3, 7 \rangle$
$\langle 3, 5, 8 \rangle$	95	120	$\langle 3, 5, 6 \rangle + \langle 3, 5, 2 \rangle$
$\langle 3, 5, 9 \rangle$	106	135	$\langle 3, 5, 3 \rangle + \langle 3, 5, 6 \rangle$
$\langle 3, 5, 10 \rangle$	118	150	$\langle 3, 5, 4 \rangle + \langle 3, 5, 6 \rangle$
$\langle 3, 5, 11 \rangle$	130	165	$2 \langle 2, 3, 5 \rangle + 2 \langle 3, 3, 6 \rangle$
$\langle 3, 5, 12 \rangle$	140	180	$\langle 3, 3, 12 \rangle + \langle 3, 2, 12 \rangle$
$\langle 3, 6, 6 \rangle$	80	108	$\langle 3, 3, 6 \rangle \cdot \langle 1, 2, 1 \rangle$
$\langle 3, 6, 7 \rangle$	98	126	$\langle 3, 3, 7 \rangle \cdot \langle 1, 2, 1 \rangle$
$\langle 3, 6, 8 \rangle$	110	144	$\langle 1, 2, 1 \rangle \cdot \langle 3, 3, 8 \rangle$
$\langle 3, 6, 9 \rangle$	120	162	$\langle 3, 6, 3 \rangle \cdot \langle 1, 1, 3 \rangle$
$\langle 3, 6, 10 \rangle$	138	180	$\langle 3, 6, 1 \rangle + \langle 3, 6, 9 \rangle$

$\langle 3, 6, 11 \rangle$	150	198	$\langle 3, 6, 2 \rangle + \langle 3, 6, 9 \rangle$
$\langle 3, 6, 12 \rangle$	160	216	$\langle 1, 1, 2 \rangle \cdot \langle 3, 6, 6 \rangle$
$\langle 3, 7, 7 \rangle$	116	147	$\langle 3, 7, 3 \rangle + \langle 3, 7, 4 \rangle$
$\langle 3, 7, 8 \rangle$	131	168	$\langle 3, 3, 8 \rangle + \langle 3, 4, 8 \rangle$
$\langle 3, 7, 9 \rangle$	147	189	$\langle 3, 7, 3 \rangle \cdot \langle 1, 1, 3 \rangle$
$\langle 3, 7, 10 \rangle$	165	210	$\langle 3, 7, 4 \rangle + \langle 3, 7, 6 \rangle$
$\langle 3, 7, 11 \rangle$	180	231	$\langle 3, 7, 3 \rangle + \langle 3, 7, 8 \rangle$
$\langle 3, 7, 12 \rangle$	194	252	$\langle 3, 3, 12 \rangle + \langle 3, 4, 12 \rangle$
$\langle 3, 8, 8 \rangle$	150	192	$\langle 3, 8, 2 \rangle + \langle 3, 8, 6 \rangle$
$\langle 3, 8, 9 \rangle$	165	216	$\langle 1, 1, 3 \rangle \cdot \langle 3, 8, 3 \rangle$
$\langle 3, 8, 10 \rangle$	186	240	$\langle 3, 8, 4 \rangle + \langle 3, 8, 6 \rangle$
$\langle 3, 8, 11 \rangle$	205	264	$\langle 3, 8, 2 \rangle + \langle 3, 8, 9 \rangle$
$\langle 3, 8, 12 \rangle$	220	288	$\langle 1, 1, 4 \rangle \cdot \langle 3, 8, 3 \rangle$
$\langle 3, 9, 9 \rangle$	183	243	$\langle 3, 9, 3 \rangle + \langle 3, 9, 6 \rangle$
$\langle 3, 9, 10 \rangle$	206	270	$\langle 3, 9, 4 \rangle + \langle 3, 9, 6 \rangle$
$\langle 3, 9, 11 \rangle$	226	297	$\langle 3, 9, 5 \rangle + \langle 3, 9, 6 \rangle$
$\langle 3, 9, 12 \rangle$	240	324	$\langle 1, 1, 2 \rangle \cdot \langle 3, 9, 6 \rangle$
$\langle 3, 10, 10 \rangle$	234	300	$\langle 3, 10, 4 \rangle + \langle 3, 10, 6 \rangle$
$\langle 3, 10, 11 \rangle$	255	330	$\langle 3, 10, 3 \rangle + \langle 3, 10, 8 \rangle$
$\langle 3, 10, 12 \rangle$	274	360	$\langle 3, 4, 12 \rangle + \langle 3, 6, 12 \rangle$
$\langle 3, 11, 11 \rangle$	280	363	$\langle 3, 11, 5 \rangle + \langle 3, 11, 6 \rangle$
$\langle 3, 11, 12 \rangle$	300	396	$\langle 3, 11, 6 \rangle + \langle 3, 11, 6 \rangle$
$\langle 3, 12, 12 \rangle$	320	432	$\langle 1, 2, 2 \rangle \cdot \langle 3, 6, 6 \rangle$
$\langle 4, 4, 4 \rangle$	49	64	$\langle 2, 2, 2 \rangle \cdot \langle 2, 2, 2 \rangle$
$\langle 4, 4, 5 \rangle$	64	80	$\langle 4, 4, 2 \rangle + \langle 4, 4, 3 \rangle$
$\langle 4, 4, 6 \rangle$	75	96	$\langle 4, 4, 4 \rangle + \langle 4, 4, 2 \rangle$
$\langle 4, 4, 7 \rangle$	87	112	$\langle 4, 4, 4 \rangle + \langle 4, 4, 3 \rangle$
$\langle 4, 4, 8 \rangle$	98	128	$\langle 1, 1, 2 \rangle \cdot \langle 4, 4, 4 \rangle$
$\langle 4, 4, 9 \rangle$	113	144	$\langle 4, 4, 2 \rangle + \langle 4, 4, 7 \rangle$
$\langle 4, 4, 10 \rangle$	124	160	$\langle 4, 4, 2 \rangle + \langle 4, 4, 8 \rangle$
$\langle 4, 4, 11 \rangle$	136	176	$\langle 4, 4, 3 \rangle + \langle 4, 4, 8 \rangle$
$\langle 4, 4, 12 \rangle$	147	192	$\langle 1, 1, 3 \rangle \cdot \langle 4, 4, 4 \rangle$
$\langle 4, 5, 5 \rangle$	80	100	$\langle 2, 5, 5 \rangle \cdot \langle 2, 1, 1 \rangle$
$\langle 4, 5, 6 \rangle$	93	120	$3 \langle 2, 2, 3 \rangle + 4 \langle 2, 3, 3 \rangle$
$\langle 4, 5, 7 \rangle$	109	140	$\langle 2, 2, 3 \rangle + 2 \langle 2, 2, 4 \rangle + 2 \langle 2, 3, 3 \rangle + 2 \langle 2, 3, 4 \rangle$
$\langle 4, 5, 8 \rangle$	122	160	$3 \langle 2, 2, 4 \rangle + 4 \langle 2, 3, 4 \rangle$
$\langle 4, 5, 9 \rangle$	140	180	$\langle 2, 2, 4 \rangle + 2 \langle 2, 2, 5 \rangle + 2 \langle 2, 3, 4 \rangle + 2 \langle 2, 3, 5 \rangle$
$\langle 4, 5, 10 \rangle$	154	200	$3 \langle 2, 2, 5 \rangle + 4 \langle 2, 3, 5 \rangle$
$\langle 4, 5, 11 \rangle$	170	220	$\langle 4, 5, 8 \rangle + \langle 4, 5, 3 \rangle$
$\langle 4, 5, 12 \rangle$	183	240	$3 \langle 2, 2, 6 \rangle + 4 \langle 2, 3, 6 \rangle$
$\langle 4, 6, 6 \rangle$	105	144	$\langle 2, 2, 2 \rangle \cdot \langle 2, 3, 3 \rangle$
$\langle 4, 6, 7 \rangle$	125	168	$3 \langle 2, 3, 3 \rangle + 4 \langle 2, 3, 4 \rangle$
$\langle 4, 6, 8 \rangle$	140	192	$\langle 2, 2, 2 \rangle \cdot \langle 2, 3, 4 \rangle$
$\langle 4, 6, 9 \rangle$	160	216	$3 \langle 2, 3, 4 \rangle + 4 \langle 2, 3, 5 \rangle$
$\langle 4, 6, 10 \rangle$	175	240	$\langle 2, 2, 2 \rangle \cdot \langle 2, 3, 5 \rangle$
$\langle 4, 6, 11 \rangle$	195	264	$3 \langle 2, 3, 5 \rangle + 4 \langle 2, 3, 6 \rangle$
$\langle 4, 6, 12 \rangle$	210	288	$\langle 1, 1, 2 \rangle \cdot \langle 4, 6, 6 \rangle$
$\langle 4, 7, 7 \rangle$	147	196	$\langle 2, 3, 3 \rangle + 2 \langle 2, 4, 4 \rangle + 4 \langle 2, 3, 4 \rangle$
$\langle 4, 7, 8 \rangle$	164	224	$3 \langle 2, 3, 4 \rangle + 4 \langle 2, 4, 4 \rangle$
$\langle 4, 7, 9 \rangle$	188	252	$\langle 2, 3, 4 \rangle + 2 \langle 2, 3, 5 \rangle + 2 \langle 2, 4, 4 \rangle + 2 \langle 2, 4, 5 \rangle$

$\langle 4, 7, 10 \rangle$	207	280	$3 \langle 2, 3, 5 \rangle + 4 \langle 2, 4, 5 \rangle$
$\langle 4, 7, 11 \rangle$	229	308	$\langle 2, 3, 5 \rangle + 2 \langle 2, 3, 6 \rangle + 2 \langle 2, 4, 5 \rangle + 2 \langle 2, 4, 6 \rangle$
$\langle 4, 7, 12 \rangle$	246	336	$3 \langle 2, 3, 6 \rangle + 4 \langle 2, 4, 6 \rangle$
$\langle 4, 8, 8 \rangle$	182	256	$\langle 2, 2, 2 \rangle \cdot \langle 2, 4, 4 \rangle$
$\langle 4, 8, 9 \rangle$	210	288	$3 \langle 2, 4, 4 \rangle + 4 \langle 2, 4, 5 \rangle$
$\langle 4, 8, 10 \rangle$	231	320	$\langle 2, 2, 2 \rangle \cdot \langle 2, 4, 5 \rangle$
$\langle 4, 8, 11 \rangle$	255	352	$3 \langle 2, 4, 5 \rangle + 4 \langle 2, 4, 6 \rangle$
$\langle 4, 8, 12 \rangle$	273	384	$\langle 2, 2, 2 \rangle \cdot \langle 2, 4, 6 \rangle$
$\langle 4, 9, 9 \rangle$	225	324	$\langle 2, 3, 3 \rangle \cdot \langle 2, 3, 3 \rangle$
$\langle 4, 9, 10 \rangle$	259	360	$3 \langle 2, 4, 5 \rangle + 4 \langle 2, 5, 5 \rangle$
$\langle 4, 9, 11 \rangle$	284	396	$\langle 4, 9, 2 \rangle + \langle 4, 9, 9 \rangle$
$\langle 4, 9, 12 \rangle$	300	432	$\langle 2, 3, 3 \rangle \cdot \langle 2, 3, 4 \rangle$
$\langle 4, 10, 10 \rangle$	280	400	$\langle 2, 2, 2 \rangle \cdot \langle 2, 5, 5 \rangle$
$\langle 4, 10, 11 \rangle$	312	440	$3 \langle 2, 5, 5 \rangle + 4 \langle 2, 5, 6 \rangle$
$\langle 4, 10, 12 \rangle$	336	480	$\langle 2, 2, 2 \rangle \cdot \langle 2, 5, 6 \rangle$
$\langle 4, 11, 11 \rangle$	346	484	$\langle 2, 5, 5 \rangle + 2 \langle 2, 6, 6 \rangle + 4 \langle 2, 5, 6 \rangle$
$\langle 4, 11, 12 \rangle$	372	528	$3 \langle 2, 5, 6 \rangle + 4 \langle 2, 6, 6 \rangle$
$\langle 4, 12, 12 \rangle$	390	576	$\langle 2, 3, 3 \rangle \cdot \langle 2, 4, 4 \rangle$
$\langle 5, 5, 5 \rangle$	99	125	Sedoglavic [9]
$\langle 5, 5, 6 \rangle$	117	150	$\langle 2, 2, 3 \rangle + 2 \langle 3, 3, 3 \rangle + 4 \langle 2, 3, 3 \rangle$
$\langle 5, 5, 7 \rangle$	136	175	$\langle 2, 2, 4 \rangle + \langle 3, 3, 3 \rangle + \langle 3, 3, 4 \rangle + 2 \langle 2, 3, 3 \rangle + 2 \langle 2, 3, 4 \rangle$
$\langle 5, 5, 8 \rangle$	152	200	$\langle 2, 2, 4 \rangle + 2 \langle 3, 3, 4 \rangle + 4 \langle 2, 3, 4 \rangle$
$\langle 5, 5, 9 \rangle$	173	225	$\langle 2, 2, 5 \rangle + \langle 3, 3, 4 \rangle + \langle 3, 3, 5 \rangle + 2 \langle 2, 3, 4 \rangle + 2 \langle 2, 3, 5 \rangle$
$\langle 5, 5, 10 \rangle$	190	250	$\langle 2, 2, 5 \rangle + 2 \langle 3, 3, 5 \rangle + 4 \langle 2, 3, 5 \rangle$
$\langle 5, 5, 11 \rangle$	206	275	$\langle 2, 2, 6 \rangle + \langle 2, 3, 6 \rangle + 2 \langle 3, 3, 6 \rangle + 3 \langle 2, 3, 5 \rangle$
$\langle 5, 5, 12 \rangle$	221	300	$\langle 2, 2, 6 \rangle + 2 \langle 3, 3, 6 \rangle + 4 \langle 2, 3, 6 \rangle$
$\langle 5, 6, 6 \rangle$	137	180	$\langle 3, 6, 6 \rangle + \langle 2, 6, 6 \rangle$
$\langle 5, 6, 7 \rangle$	159	210	$\langle 2, 3, 3 \rangle + 2 \langle 2, 3, 4 \rangle + 2 \langle 3, 3, 3 \rangle + 2 \langle 3, 3, 4 \rangle$
$\langle 5, 6, 8 \rangle$	176	240	$3 \langle 2, 3, 4 \rangle + 4 \langle 3, 3, 4 \rangle$
$\langle 5, 6, 9 \rangle$	200	270	$\langle 2, 3, 4 \rangle + 2 \langle 2, 3, 5 \rangle + 2 \langle 3, 3, 4 \rangle + 2 \langle 3, 3, 5 \rangle$
$\langle 5, 6, 10 \rangle$	218	300	$\langle 2, 3, 4 \rangle + 2 \langle 2, 3, 6 \rangle + 2 \langle 3, 3, 4 \rangle + 2 \langle 3, 3, 6 \rangle$
$\langle 5, 6, 11 \rangle$	236	330	$\langle 2, 3, 6 \rangle + \langle 3, 3, 5 \rangle + 2 \langle 2, 3, 5 \rangle + 3 \langle 3, 3, 6 \rangle$
$\langle 5, 6, 12 \rangle$	250	360	$3 \langle 2, 3, 6 \rangle + 4 \langle 3, 3, 6 \rangle$
$\langle 5, 7, 7 \rangle$	185	245	$\langle 2, 4, 4 \rangle + \langle 3, 3, 3 \rangle + \langle 3, 4, 4 \rangle + 2 \langle 2, 3, 4 \rangle + 2 \langle 3, 3, 4 \rangle$
$\langle 5, 7, 8 \rangle$	206	280	$\langle 2, 3, 4 \rangle + 2 \langle 2, 4, 4 \rangle + 2 \langle 3, 3, 4 \rangle + 2 \langle 3, 4, 4 \rangle$
$\langle 5, 7, 9 \rangle$	235	315	$\langle 2, 3, 5 \rangle + \langle 2, 4, 4 \rangle + \langle 2, 4, 5 \rangle + \langle 3, 3, 4 \rangle + \langle 3, 3, 5 \rangle + \langle 3, 4, 4 \rangle + \langle 3, 4, 5 \rangle$
$\langle 5, 7, 10 \rangle$	259	350	$\langle 2, 3, 5 \rangle + 2 \langle 2, 4, 5 \rangle + 2 \langle 3, 3, 5 \rangle + 2 \langle 3, 4, 5 \rangle$
$\langle 5, 7, 11 \rangle$	283	385	$\langle 2, 3, 5 \rangle + \langle 2, 4, 5 \rangle + \langle 2, 4, 6 \rangle + \langle 3, 4, 5 \rangle + \langle 3, 4, 6 \rangle + 2 \langle 3, 3, 6 \rangle$
$\langle 5, 7, 12 \rangle$	304	420	$\langle 2, 3, 6 \rangle + 2 \langle 2, 4, 6 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 4, 6 \rangle$
$\langle 5, 8, 8 \rangle$	230	320	$3 \langle 2, 4, 4 \rangle + 4 \langle 3, 4, 4 \rangle$
$\langle 5, 8, 9 \rangle$	264	360	$\langle 2, 4, 4 \rangle + 2 \langle 2, 4, 5 \rangle + 2 \langle 3, 4, 4 \rangle + 2 \langle 3, 4, 5 \rangle$
$\langle 5, 8, 10 \rangle$	291	400	$3 \langle 2, 4, 5 \rangle + 4 \langle 3, 4, 5 \rangle$
$\langle 5, 8, 11 \rangle$	323	440	$\langle 2, 4, 5 \rangle + 2 \langle 2, 4, 6 \rangle + 2 \langle 3, 4, 5 \rangle + 2 \langle 3, 4, 6 \rangle$
$\langle 5, 8, 12 \rangle$	346	480	$\langle 5, 2, 12 \rangle + \langle 5, 6, 12 \rangle$
$\langle 5, 9, 9 \rangle$	300	405	$\langle 2, 6, 6 \rangle + \langle 3, 3, 3 \rangle + \langle 3, 6, 6 \rangle + 2 \langle 2, 3, 6 \rangle + 2 \langle 3, 3, 6 \rangle$
$\langle 5, 9, 10 \rangle$	331	450	$\langle 2, 4, 5 \rangle + 2 \langle 2, 5, 5 \rangle + 2 \langle 3, 4, 5 \rangle + 2 \langle 3, 5, 5 \rangle$
$\langle 5, 9, 11 \rangle$	360	495	$\langle 2, 3, 5 \rangle + \langle 2, 5, 6 \rangle + \langle 2, 6, 6 \rangle + \langle 3, 5, 6 \rangle + \langle 3, 6, 6 \rangle + 2 \langle 3, 3, 6 \rangle$
$\langle 5, 9, 12 \rangle$	384	540	$\langle 2, 3, 6 \rangle + 2 \langle 2, 6, 6 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 6, 6 \rangle$
$\langle 5, 10, 10 \rangle$	364	500	$3 \langle 2, 5, 5 \rangle + 4 \langle 3, 5, 5 \rangle$
$\langle 5, 10, 11 \rangle$	398	550	$\langle 2, 5, 6 \rangle + \langle 3, 5, 5 \rangle + 2 \langle 2, 5, 5 \rangle + 3 \langle 3, 5, 6 \rangle$

$\langle 5, 10, 12 \rangle$	424	600	$3 \langle 2, 5, 6 \rangle + 4 \langle 3, 5, 6 \rangle$
$\langle 5, 11, 11 \rangle$	434	605	$\langle 2, 6, 6 \rangle + \langle 3, 5, 5 \rangle + \langle 3, 6, 6 \rangle + 2 \langle 2, 5, 6 \rangle + 2 \langle 3, 5, 6 \rangle$
$\langle 5, 11, 12 \rangle$	462	660	$\langle 2, 5, 6 \rangle + 2 \langle 2, 6, 6 \rangle + 2 \langle 3, 5, 6 \rangle + 2 \langle 3, 6, 6 \rangle$
$\langle 5, 12, 12 \rangle$	491	720	$3 \langle 2, 6, 6 \rangle + 4 \langle 3, 6, 6 \rangle$
$\langle 6, 6, 6 \rangle$	160	216	$\langle 3, 3, 6 \rangle \cdot \langle 2, 2, 1 \rangle$
$\langle 6, 6, 7 \rangle$	185	252	$\langle 6, 6, 4 \rangle + \langle 6, 6, 3 \rangle$
$\langle 6, 6, 8 \rangle$	203	288	$\langle 2, 2, 2 \rangle \cdot \langle 3, 3, 4 \rangle$
$\langle 6, 6, 9 \rangle$	225	324	$\langle 2, 3, 3 \rangle \cdot \langle 3, 2, 3 \rangle$
$\langle 6, 6, 10 \rangle$	247	360	$3 \langle 3, 3, 4 \rangle + 4 \langle 3, 3, 6 \rangle$
$\langle 6, 6, 11 \rangle$	268	396	$3 \langle 3, 3, 5 \rangle + 4 \langle 3, 3, 6 \rangle$
$\langle 6, 6, 12 \rangle$	280	432	$\langle 2, 2, 2 \rangle \cdot \langle 3, 3, 6 \rangle$
$\langle 6, 7, 7 \rangle$	215	294	$\langle 3, 3, 3 \rangle + 2 \langle 3, 4, 4 \rangle + 4 \langle 3, 3, 4 \rangle$
$\langle 6, 7, 8 \rangle$	239	336	$3 \langle 3, 3, 4 \rangle + 4 \langle 3, 4, 4 \rangle$
$\langle 6, 7, 9 \rangle$	273	378	$\langle 3, 3, 4 \rangle + 2 \langle 3, 3, 5 \rangle + 2 \langle 3, 4, 4 \rangle + 2 \langle 3, 4, 5 \rangle$
$\langle 6, 7, 10 \rangle$	300	420	$3 \langle 3, 3, 5 \rangle + 4 \langle 3, 4, 5 \rangle$
$\langle 6, 7, 11 \rangle$	328	462	$\langle 3, 3, 5 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 4, 5 \rangle + 2 \langle 3, 4, 6 \rangle$
$\langle 6, 7, 12 \rangle$	352	504	$\langle 6, 6, 12 \rangle + \langle 6, 1, 12 \rangle$
$\langle 6, 8, 8 \rangle$	266	384	$\langle 2, 2, 2 \rangle \cdot \langle 3, 4, 4 \rangle$
$\langle 6, 8, 9 \rangle$	300	432	$\langle 2, 4, 3 \rangle \cdot \langle 3, 2, 3 \rangle$
$\langle 6, 8, 10 \rangle$	336	480	$\langle 2, 2, 2 \rangle \cdot \langle 3, 4, 5 \rangle$
$\langle 6, 8, 11 \rangle$	373	528	$\langle 6, 2, 11 \rangle + \langle 6, 6, 11 \rangle$
$\langle 6, 8, 12 \rangle$	390	576	$\langle 2, 4, 4 \rangle \cdot \langle 3, 2, 3 \rangle$
$\langle 6, 9, 9 \rangle$	343	486	$\langle 3, 3, 3 \rangle + 2 \langle 3, 6, 6 \rangle + 4 \langle 3, 3, 6 \rangle$
$\langle 6, 9, 10 \rangle$	375	540	$\langle 2, 3, 5 \rangle \cdot \langle 3, 3, 2 \rangle$
$\langle 6, 9, 11 \rangle$	416	594	$\langle 3, 3, 5 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 5, 6 \rangle + 2 \langle 3, 6, 6 \rangle$
$\langle 6, 9, 12 \rangle$	435	648	$\langle 2, 3, 3 \rangle \cdot \langle 3, 3, 4 \rangle$
$\langle 6, 10, 10 \rangle$	422	600	$\langle 6, 10, 4 \rangle + \langle 6, 10, 6 \rangle$
$\langle 6, 10, 11 \rangle$	463	660	$\langle 6, 6, 11 \rangle + \langle 6, 4, 11 \rangle$
$\langle 6, 10, 12 \rangle$	490	720	$\langle 2, 2, 2 \rangle \cdot \langle 3, 5, 6 \rangle$
$\langle 6, 11, 11 \rangle$	501	726	$\langle 3, 5, 5 \rangle + 2 \langle 3, 6, 6 \rangle + 4 \langle 3, 5, 6 \rangle$
$\langle 6, 11, 12 \rangle$	530	792	$\langle 6, 5, 12 \rangle + \langle 6, 6, 12 \rangle$
$\langle 6, 12, 12 \rangle$	560	864	$\langle 2, 2, 2 \rangle \cdot \langle 3, 6, 6 \rangle$
$\langle 7, 7, 7 \rangle$	250	343	$\langle 4, 4, 4 \rangle + 3 \langle 3, 3, 4 \rangle + 3 \langle 3, 4, 4 \rangle$
$\langle 7, 7, 8 \rangle$	279	392	$\langle 3, 3, 4 \rangle + 2 \langle 4, 4, 4 \rangle + 4 \langle 3, 4, 4 \rangle$
$\langle 7, 7, 9 \rangle$	321	441	$\langle 3, 3, 5 \rangle + \langle 4, 4, 4 \rangle + \langle 4, 4, 5 \rangle + 2 \langle 3, 4, 4 \rangle + 2 \langle 3, 4, 5 \rangle$
$\langle 7, 7, 10 \rangle$	353	490	$\langle 4, 4, 4 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 4, 4 \rangle + 2 \langle 4, 4, 6 \rangle$
$\langle 7, 7, 11 \rangle$	388	539	$\langle 4, 4, 5 \rangle + 2 \langle 3, 3, 6 \rangle + 2 \langle 3, 4, 5 \rangle + 2 \langle 4, 4, 6 \rangle$
$\langle 7, 7, 12 \rangle$	419	588	$2 \langle 3, 3, 6 \rangle + 2 \langle 3, 4, 6 \rangle + 3 \langle 4, 4, 6 \rangle$
$\langle 7, 8, 8 \rangle$	310	448	$3 \langle 3, 4, 4 \rangle + 4 \langle 4, 4, 4 \rangle$
$\langle 7, 8, 9 \rangle$	360	504	$\langle 3, 4, 4 \rangle + 2 \langle 3, 4, 5 \rangle + 2 \langle 4, 4, 4 \rangle + 2 \langle 4, 4, 5 \rangle$
$\langle 7, 8, 10 \rangle$	398	560	$\langle 7, 8, 2 \rangle + \langle 7, 8, 8 \rangle$
$\langle 7, 8, 11 \rangle$	441	616	$\langle 7, 8, 3 \rangle + \langle 7, 8, 8 \rangle$
$\langle 7, 8, 12 \rangle$	474	672	$\langle 7, 8, 4 \rangle + \langle 7, 8, 8 \rangle$
$\langle 7, 9, 9 \rangle$	408	567	$\langle 3, 9, 9 \rangle + \langle 4, 9, 9 \rangle$
$\langle 7, 9, 10 \rangle$	454	630	$\langle 3, 3, 6 \rangle + \langle 3, 4, 4 \rangle + \langle 3, 6, 6 \rangle + \langle 4, 4, 6 \rangle + \langle 4, 6, 6 \rangle + 2 \langle 3, 4, 6 \rangle$
$\langle 7, 9, 11 \rangle$	494	693	$\langle 3, 3, 6 \rangle + \langle 3, 4, 5 \rangle + \langle 3, 4, 6 \rangle + \langle 3, 5, 6 \rangle + \langle 3, 6, 6 \rangle + \langle 4, 5, 6 \rangle + \langle 4, 6, 6 \rangle$
$\langle 7, 9, 12 \rangle$	526	756	$\langle 3, 3, 6 \rangle + 2 \langle 3, 4, 6 \rangle + 2 \langle 3, 6, 6 \rangle + 2 \langle 4, 6, 6 \rangle$
$\langle 7, 10, 10 \rangle$	500	700	$\langle 3, 6, 6 \rangle + \langle 4, 4, 4 \rangle + \langle 4, 6, 6 \rangle + 2 \langle 3, 4, 6 \rangle + 2 \langle 4, 4, 6 \rangle$
$\langle 7, 10, 11 \rangle$	545	770	$\langle 3, 4, 6 \rangle + \langle 3, 5, 6 \rangle + \langle 3, 6, 6 \rangle + \langle 4, 4, 5 \rangle + \langle 4, 4, 6 \rangle + \langle 4, 5, 6 \rangle + \langle 4, 6, 6 \rangle$
$\langle 7, 10, 12 \rangle$	578	840	$\langle 3, 4, 6 \rangle + 2 \langle 3, 6, 6 \rangle + 2 \langle 4, 4, 6 \rangle + 2 \langle 4, 6, 6 \rangle$

$\langle 7, 11, 11 \rangle$	590	847	$\langle 3, 6, 6 \rangle + 2 \langle 3, 5, 6 \rangle + 2 \langle 4, 5, 5 \rangle + 2 \langle 4, 6, 6 \rangle$
$\langle 7, 11, 12 \rangle$	626	924	$\langle 3, 5, 6 \rangle + 2 \langle 3, 6, 6 \rangle + 2 \langle 4, 5, 6 \rangle + 2 \langle 4, 6, 6 \rangle$
$\langle 7, 12, 12 \rangle$	660	1008	$3 \langle 3, 6, 6 \rangle + 4 \langle 4, 6, 6 \rangle$
$\langle 8, 8, 8 \rangle$	343	512	$\langle 2, 2, 2 \rangle \cdot \langle 4, 4, 4 \rangle$
$\langle 8, 8, 9 \rangle$	400	576	$\langle 2, 4, 3 \rangle \cdot \langle 4, 2, 3 \rangle$
$\langle 8, 8, 10 \rangle$	443	640	$\langle 8, 8, 2 \rangle + \langle 8, 8, 8 \rangle$
$\langle 8, 8, 11 \rangle$	492	704	$\langle 8, 8, 4 \rangle + \langle 8, 8, 7 \rangle$
$\langle 8, 8, 12 \rangle$	520	768	$\langle 2, 4, 3 \rangle \cdot \langle 4, 2, 4 \rangle$
$\langle 8, 9, 9 \rangle$	435	648	$\langle 2, 3, 3 \rangle \cdot \langle 4, 3, 3 \rangle$
$\langle 8, 9, 10 \rangle$	500	720	$\langle 2, 3, 5 \rangle \cdot \langle 4, 3, 2 \rangle$
$\langle 8, 9, 11 \rangle$	548	792	$\langle 8, 9, 2 \rangle + \langle 8, 9, 9 \rangle$
$\langle 8, 9, 12 \rangle$	570	864	$\langle 2, 3, 3 \rangle \cdot \langle 4, 3, 4 \rangle$
$\langle 8, 10, 10 \rangle$	559	800	$\langle 4, 4, 4 \rangle + 2 \langle 4, 6, 6 \rangle + 4 \langle 4, 4, 6 \rangle$
$\langle 8, 10, 11 \rangle$	610	880	$\langle 4, 4, 5 \rangle + 2 \langle 4, 4, 6 \rangle + 2 \langle 4, 5, 6 \rangle + 2 \langle 4, 6, 6 \rangle$
$\langle 8, 10, 12 \rangle$	645	960	$3 \langle 4, 4, 6 \rangle + 4 \langle 4, 6, 6 \rangle$
$\langle 8, 11, 11 \rangle$	661	968	$2 \langle 4, 5, 5 \rangle + 2 \langle 4, 5, 6 \rangle + 3 \langle 4, 6, 6 \rangle$
$\langle 8, 11, 12 \rangle$	699	1056	$3 \langle 4, 5, 6 \rangle + 4 \langle 4, 6, 6 \rangle$
$\langle 8, 12, 12 \rangle$	735	1152	$\langle 2, 2, 2 \rangle \cdot \langle 4, 6, 6 \rangle$
$\langle 9, 9, 9 \rangle$	514	729	$\text{Proj}(\llbracket [0, 0], [2] \rrbracket, \langle 9, 9, 10 \rangle)$
$\langle 9, 9, 10 \rangle$	540	810	$\langle 3, 3, 2 \rangle \cdot \langle 3, 3, 5 \rangle$
$\langle 9, 9, 11 \rangle$	600	891	$\text{Proj}(\llbracket [0, 0], [12] \rrbracket, \langle 9, 9, 12 \rangle)$
$\langle 9, 9, 12 \rangle$	600	972	$\langle 3, 3, 2 \rangle \cdot \langle 3, 3, 6 \rangle$
$\langle 9, 10, 10 \rangle$	625	900	$\langle 3, 2, 5 \rangle \cdot \langle 3, 5, 2 \rangle$
$\langle 9, 10, 11 \rangle$	680	990	$\langle 9, 10, 2 \rangle + \langle 9, 10, 9 \rangle$
$\langle 9, 10, 12 \rangle$	708	1080	$\langle 9, 1, 12 \rangle + \langle 9, 9, 12 \rangle$
$\langle 9, 11, 11 \rangle$	754	1089	$\langle 9, 11, 2 \rangle + \langle 9, 11, 9 \rangle$
$\langle 9, 11, 12 \rangle$	768	1188	$\langle 9, 2, 12 \rangle + \langle 9, 9, 12 \rangle$
$\langle 9, 12, 12 \rangle$	800	1296	$\langle 3, 2, 4 \rangle \cdot \langle 3, 6, 3 \rangle$
$\langle 10, 10, 10 \rangle$	693	1000	$\langle 2, 2, 2 \rangle \cdot \langle 5, 5, 5 \rangle$
$\langle 10, 10, 11 \rangle$	765	1100	$3 \langle 5, 5, 5 \rangle + 4 \langle 5, 5, 6 \rangle$
$\langle 10, 10, 12 \rangle$	815	1200	$\langle 4, 4, 6 \rangle + 2 \langle 6, 6, 6 \rangle + 4 \langle 4, 6, 6 \rangle$
$\langle 10, 11, 11 \rangle$	841	1210	$\langle 5, 5, 5 \rangle + 2 \langle 5, 6, 6 \rangle + 4 \langle 5, 5, 6 \rangle$
$\langle 10, 11, 12 \rangle$	894	1320	$\langle 10, 2, 12 \rangle + \langle 10, 9, 12 \rangle$
$\langle 10, 12, 12 \rangle$	936	1440	$\langle 2, 4, 4 \rangle \cdot \langle 5, 3, 3 \rangle$
$\langle 11, 11, 11 \rangle$	922	1331	$\langle 6, 6, 6 \rangle + 3 \langle 5, 5, 6 \rangle + 3 \langle 5, 6, 6 \rangle$
$\langle 11, 11, 12 \rangle$	972	1452	$\langle 2, 11, 12 \rangle + \langle 9, 11, 12 \rangle$
$\langle 11, 12, 12 \rangle$	1022	1584	$\langle 2, 12, 12 \rangle + \langle 9, 12, 12 \rangle$
$\langle 12, 12, 12 \rangle$	1040	1728	$\langle 2, 4, 4 \rangle \cdot \langle 6, 3, 3 \rangle$
$\langle 15, 15, 15 \rangle$	2101	3375	$\langle 7, 8, 8 \rangle + 2 \langle 8, 8, 8 \rangle + \text{TA}(\langle 7, 7, 8 \rangle, \langle 7, 8, 7 \rangle) + \text{TA}(\langle 7, 7, 8 \rangle, \langle 8, 7, 7 \rangle)$
$\langle 18, 18, 18 \rangle$	3200	5832	$\langle 3, 3, 6 \rangle \cdot \langle 6, 6, 3 \rangle$
$\langle 21, 21, 21 \rangle$	5240	9261	$\langle 12, 12, 12 \rangle + 3 \langle 9, 9, 12 \rangle + 3 \langle 9, 12, 12 \rangle$