Towards a qualitative multicriteria decision theory

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Abstract

This paper proposes a principled approach to multicriteria decision making where the worth of decisions along attributes is not supposed to be quantified, as in multiattribute utility theory. In the latter, alternatives are ranked after merging utility functions defined along each attribute. In constrast, our approach presupposes an aggregation of local preference relations, which includes information about the relative importance of coalitions of criteria. It is shown that the requirement for ordinal representation and fusion of preference leads to a very specific family of decision rules, called concordance rules that generalize well-known voting methods.

1 Introduction

It is now acknowledged that decision-making is a worth studying topic in the framework of Artificial Intelligence. Yet, it is again a topic where scientific traditions exist, and have been around for a long time independently of AI. There are at least three decision making problems that have been studied rather independently in the past: individual decision making under uncertainty (DMU), multi-agent decision making (MADM) and multicriteria decision making (MCDM).

DMU research has culminated with the works of Savage (26) and advocates a numerical approach to decision making, whereby uncertainty is represented by a single probability function, preference is encoded by utility functions and acts are ranked according to expected utility. MADM is for a large part based on the works of Arrow (1) who formulated it in a qualitative framework, as a voting problem in which complete preordering relations must be aggregated, thus leading to an impossibility theorem.

However DMU and MADM are similar problems if the set of states of the world is viewed as the set of voters in MADM. It may be surprizing that formally similar problems lead to different settings, one being quantitative and the other qualitative. It is all the more intriguing to compare the two approaches, as AI rather belongs to a non numerical modeling tradition and has started to consider DMU from a qualitative point of view [9][3][7][13]. The difficulties induced by Arrow's impossibility theorem for a proper handling of ordering relations in AI problems have been pointed out in [9]. The axiomatic framework of Savage has been reconsidered in the scope of qualitative decision theory [10][14] and it has been shown that keeping the essential features of Savage's framework while sticking to an ordinal framework leads either to an impossibility theorem or to decision rules that generalize Condorcet's pairwise relative majority rule. It has also been established that the representation of uncertainty stemming from ordinal DMU is precisely the one advocated by Kraus, Lehman and Magidor (19) for non monotonic reasoning.

In this paper, we focus on the third problem MCDM, that is also of the same structure MADM and DMU, the set of criteria playing the the role of voters (resp. of states). Interestingly, there are here two distinct schools of thought in this area, one deriving from the DMU tradition, the other one from voting theory. The first school is exemplified by the works of Keeney and Raiffa (17). This kind of approach is essentially numerical and puts forwards a weigthed sum for the aggregation of scaled utility functions. The second school stems from the works of Roy (24): preferences along each criteria (whether numerical or not) are represented by an outranking relation. Additive concordance rules are used to perform criteria aggregation: they are based on counting the number of criteria which favor an alternative over another.

The emphasis given to ordinal models in this school is more in agreement with qualitative AI methods. Nevertheless, there are very few foundational works in ordinal MCDM, in comparison to multicriteria numerical utility theory. Moreover, in spite of the interest of AI for qualitative preference modelling especially in DMU (e.g. [4][8]) the problem of deriving aggregation procedures compatible with a qualitative approach has not yet be considered in AI, if we except some works in information fusion [21] [20] [2]. The aim of this paper is to fill this gap to some extent, by proposing a generalized, purely ordinal concordance decision rule in MCDM, rigorously justified in an axiomatic way via representation theorems. This work parallels the one started in [10] taking advantage of the similarities between MCDM and DMU. In the next Section we present in more details the two approaches to MCDM and discusses the strong assumptions underlying the numerical aggregation scheme. Section 3 then proposes a generalization of the usual additive concordance rule. Section 4 finally proposes a rigorous axiomatisation of such rules.

2 Preliminary definitions and demarks

A multicriteria decision problem can be characterised by a set $\mathcal{A}$ of *alternatives* (possible actions, objects, candidates) and a set $N = \{1, \ldots, n\}$ of *attributes* or *criteria* used to describe the alternatives. Let X_j denote the set of possible values for component $j \in N$ and $X = X_1 \times \ldots \times X_n$ be the *multiattribute space*. Within X, each alternative $x \in \mathcal{A}$ is represented by the vector $(x_1, \ldots, x_n)$ of attributes values. $\mathcal{A}$ can be identified to its image in X and thus considered as a subset of X. $\mathcal{A}$ will actually be identified with X itself since we need a comparison model allowing to decide whether x is at least as good as y (denoted $x \succsim y$) or not, *whatever* (x, y) in $\mathcal{A}^2$, i.e. a comparison model valid on the entire set X.

As a first consequence, for any pair (x, y) of alternative and for any subset of attributes $A \subset N$, we can soundly construct a mixed alternative xAy whose components are those of x on the elements of A, and those of y on the other attributes:

$$(xAy)_j = \begin{cases} x_j & \text{if} \quad j \in A \\ y_j & \text{if} \quad j \notin A. \end{cases} \qquad j = 1, \ldots, n$$

Each attribute j usually defines a marginal utility $u_j(x_j)$ measuring the attractiveness of the attribute value x_j. We assume that $u_j(x_j) \in [0, 1]$ but any linear scale could be considered as well. In some models, the scale is ordinal in nature and the marginal utility only encodes a ranking of the x_j.

In order to construct the preference relation $\succsim$ on X^2, one uses a *decision rule* defining the preference $x \succsim y$ as a function of vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. As said previously, one can distinguish two approaches to derive $\succsim$.

2.1 The Aggregate then Compare Approach (AC)

This approach is exemplified by the works of Keeney and Raiffa (17) and consists in summarizing each vector x by a quantity $u(x)$. This utility is obtained by aggregation of marginal utilities $u_j(x_j)$, very often a weighted sum. Denoting ψ the aggregation operator, the preference relation $\succsim$ is defined by:

$$x \succsim y \Leftrightarrow \phi(\psi((u_j(x_j)_{j \in N})), \psi((u_j(y_j)_{j \in N})) \geq 0$$

where $(u_j(x_j)_{j \in N})$ is the vector $(u_1(x_1), \ldots, u_n(x_n))$ and ϕ is a comparison function.

A classical choice for ϕ is $\phi(x, y) = x - y$. ψ is very often a weighted sum, but one can also choose another particular instance of the Choquet integral [16] or, in a more qualitative framework, a Sugeno integral [28]. All these general aggregation techniques attach importance weights to groups of criteria: $\mu(A)$ represents the importance of the coalition A, for any $A \subseteq N$. This importance measure must be a *capacity* on N, *i.e.* a mapping defined from 2^N to $[0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(N) = 1$ and $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. An important subclass of capacities is formed by *additive capacities* ($\mu(A) = \sum_{j \in A} \mu(\{j\})$). Other examples are possibility and necessity measures [12].

The Choquet integral is a powerful aggregation operator allowing positive and negative synergies between criteria [16]. When used with an additive capacity, it boils down to the weighted sum. The Sugeno integral is a qualitative counterpart of the Choquet integral. When used with a possibility (resp. a necessity) measure, it boils down to a weighted max (resp. a weighted min) [13].

2.2 The Compare then Aggregate Approach (CA)

This approach consists in comparing, for any pair (x, y) and each attribute j, the attribute values x_j and y_j so as to decide whether x is at least as good as y according to the j^{th} component. This yields n preference indices $\phi_j(x, y)$. These indices are then aggregated before performing the following preference test:

$$x \succsim y \Leftrightarrow \psi(\phi_1(x, y), \ldots, \phi_n(x, y)) \geq 0$$

Example 1 (Additive concordance rule) *Let us define ϕ as: $\phi_j(x, y) = 1$ if $u_j(x_j) > u_j(y_j)$, $\phi_j(x, y) = 0$ if $u_j(x_j) = u_j(y_j)$, $\phi_j(x, y) = -1$ otherwise. Then choosing $\psi(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} w_j \alpha_j$ leads to the* relative majority rule. *When w_j are all equal, we obtain a well known MADM system where $x \succsim y$ iff a majority of voters/attributes is concordant with this preference. In MCDM interpreting the w_j as the weights of criteria leads to a concordance rule used in Electre methods [23; 25]*

Numerous decision rules can actually be obtained as particular instances of this general model. One can find examples in [27; 22; 5; 29; 25]. The ordinal nature of the CA approach is worthwhile noticing. It indeed amounts to constructing n preference *relations* (characterised by functions ϕ_j) which are aggregated to form the overall preference *relation* $\succsim$. We come to this last point in the next section.

2.3 On the commensurability between local preference scales and importance scales

Both AC and CA use n preference scales $(L_j, \geq)$, each being characterized by a set of levels $L_j = \{u_j(x_j), x_j \in X_j\}$ ordered by $\geq$. The two approaches also require an aggregation operation ψ. When ψ explicitly uses an importance measure μ on N (this is the case Sugeno and Choquet integrals) the aggregation operator can be denoted ψ_μ. In the AC approach, ψ_μ is used to aggregate marginal utility indices $u_j(x_j)$ whereas in CA ψ_μ is used to aggregate marginal preference indices $\phi_j(x, y)$ resulting from pairwise comparisons. Anyway there is a specific scale L_μ for criteria importance levels (the range of μ, *i.e.* $L_\mu = \{\mu(A), A \subseteq N\}$). Hence there are two distinct commensurability problems: 1) can the same totally ordered preference scale L_u be attached to all criteria? 2) is there a mapping relating levels of importance in L_μ and levels of satisfaction in L_u.

In the AC approach an affirmative answer is given to both questions. One indeed commonly assumes that the same utility scale $(L_u, \geq)$ is valid for each attribute. A possible choice for this scale is $L_u = \bigcup_{j \in N} L_j$. This choice requires the comparability of utility levels coming from different scales L_j: a strong assumption. On top of this single utility scale, we also need the importance scale L_μ. The conjoint use of scales L_u and L_μ in the definition of preferences implicitly makes the two scales commensurate. In order to explain how this commensurateness between importance and utility is achieved, let us assume a AC model where ψ_μ

satisfies the two following properties:

Idempotence $\forall \alpha \in [0,\ 1],\ \ \psi_\mu(\alpha, \ldots, \alpha) = \alpha$

Coincidence $\forall\ A \subseteq N,\ \psi_\mu(\chi_A(1), \ldots,\ \chi_A(n)) = \mu(A)$ where χ_A is the characteristic function of A in N defined by $\chi_A(j) = 1$ if $j \in A$ and 0 otherwise.

Most aggregation functions satisfy these two properties [1]. Following AC, we can define the preference order $\succsim_\mu$ corresponding to ψ_μ by:

$$x \succsim_\mu y \Leftrightarrow \psi_\mu(u_j(x_j)_{j\in N}) \geq \psi_\mu(u_j(y_j)_{j\in N}).$$

Consider now four alternatives in X defined by:
• the ideal alternative x^* such that $u_j(x_j^*) = 1, \forall j \in N$,
• the anti-ideal alternative x_* such that $u_j(x_*) = 0, \forall j \in N$,
• an alternative a having a constant utility vector $(\alpha, \ldots, \alpha)$, for $\alpha \in L_u$.
• the alternative $x^* A x_*$ for a given proper subset A in N ($A \neq \emptyset$ and $N \setminus A \neq \emptyset$).

It can be shown that:

Proposition 1 *If ψ_μ is idempotent and concident, then for any proper subset A in N , we have:*

$$(x^* A x_*) \succsim_\mu a \Leftrightarrow \mu(A) \geq \alpha$$

This result reveals an implicit comparison between the level $\mu(A)$ of the scale L_μ and the level α of the scale L_u. This shows that, in the CA approach, the intertwining of the scales L_μ and L_u is instrumental in the comparison of alternatives. i.e. that L_μ and L_u need to be commensurate.

Recall that the similarity of the DMU, MADM and MCDM decision frameworks lies in the fact that the set of attributes plays the same role as the set of states and the set of voters. The AC approach, accepting the two commensurability assumptions appears natural in DMU: the set of consequences of acts is indeed often independent of the considered state, and since there is a single decision-maker, there is a single preference scale for the consequences. The second commensurability problem is a matter of comparing degrees of uncertainty of events (the counterpart of degrees of importance) and degrees of preference of consequences. Although clearly distinct notions, uncertainty and preference are equated in DMU provided that the decision maker is able to compare a sure gain and a binary lottery (which characterizes an event).

In the MADM context, the CA approach is much more natural since such commensurability assumptions are difficult to accept. Indeed local preference scales L_j are attached to distinct voters j, hence are hard to reconcile. Moreover the importance of individual voters or groups thereof is generally determined by an external agent, not the voters themselves, hence the commensurability between individual preference scales and the importance scale is not warranted.

In MCDM, the presence of a single decision maker makes the AC approach more natural than in the MADM setting, but it raises an important operational question. Indeed, in order to capture the preferences of decision-makers in the AC model, it would be necessary to ask a huge number of questions aiming at defining exactly how elements of the various utility and importance scales should be intertwined on a common scale. We will show in the next section that concordance methods which are particular instances of the CA approach do not face this problem. From this point of view the MCDM problem appears closer to the MADM setting than to DMU, and it is natural to try and tackle the MCDM problem making as few commensurability assumptions as possible.

[1] In DMU, idempotency holds when a constant act is equated to its unique consequence (a sure gain), and coincidence means that the confidence of event A is the utility of a binary act having extremely good consequences if A occurs, and extremely bad ones if not.

3 Generalized Concordance Rules

The additive concordance rules introduced in 2.2 can be cast in a more general setting. First a preference relation $\succsim_j$ is supposed to exist on each attribute range X_j. It can be derived from the marginal utility functions if any (then $x_j \succsim_j y_j \Leftrightarrow u_j(x_j) \geq u_j(y_j)$) or introduced as such from scratch by the decision maker. Let $\succ_j$ and $\sim_j$ denote the strict preference and the indifference relations derived from $\succ_j$. The following coalition of attributes derives from the marginal preferences:

$$C_{\succsim}(x, y) = \{j \in N, x_j \succsim_j y_j\}$$

$C_{\succsim}(x, y)$ is the set of criteria where x is as least as good as y. Finally, assume an importance relation $\succsim_I$ exists on 2^N, whereby $A \succsim_I B$ means that the group of attributes A is as least as important as the group B. It can be derived from the importance function if any (then $A \succsim_I B \Leftrightarrow \mu(A) \geq \mu(B)$) or introduced as such from scratch by the decision maker. Such a relation is supposed to be reflexive, and monotonic:

$$A \succsim_I B \Rightarrow A \cup C \succsim_I B \quad \text{and} \quad A \succsim_I B \cup C \Rightarrow A \succsim_I B$$

This property is satisfied if $\succsim_I$ derives from a capacity as introduced in Section 2.1. Importance relation derived from additive capacities also obeys the following property of *preadditivity*: $\forall A, B, C \subseteq N$

$$A \cap (B \cup C) = \emptyset \Rightarrow (B \succsim_I C \Leftrightarrow A \cup B \succsim_I A \cup C)$$

However, it is well known that preadditivity of $\succsim_I$ does not imply that it is representable by an additive capacity (see [18]). Let us now define generalized concordance rules (**GC** rules):

Definition 1 *A* generalized concordance rule *defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, $\forall j = 1, \ldots, n$ and the relation $\succsim_I$ on 2^N as follows:*

$$x \succsim y \Leftrightarrow C_{\succsim}(x, y) \succsim_I C_{\succsim}(y, x)$$

This definition is a MCDM counterpart to (and a generalization of) the "lifting rule" proposed by [10] for DMU. When $\succsim_I$ (resp. $\succsim_j$) derive from a capacity function μ (resp. a utility function u_j), or equivalently they are weak orders (and thus always representable by capacity and utility functions), the previous rule becomes (**GC**$_\mu$):

$$x \succsim y \Leftrightarrow \mu(\{j, u_j(x) \geq u_j(y)\}) \geq \mu(\{j, u_j(y) \geq u_j(x)\})$$

The additive concordance rule is recovered when μ is an additive capacity. Remark that:

• Variants of the GC rule can be obtained using $\succ_j$ (resp. $\succ_I$) instead of $\succsim_j$ ($\succsim_I$) in Definition 1.
• Applied to x^*Ax_* and a introduced in §2.3, it is clear that $x^*Ax_* \succsim a \Leftrightarrow A \succsim_I N \setminus A$ because $C_{\succsim}(x^*Ax_*, a) = A$ and $C_{\succsim}(a, x^*Ax_*) = N \setminus A$. One can observe that the result only depends on the inequality $A \succsim_I N \setminus A$ which pertains to two levels of the same scale. More generally, the GC rule does not require any commensurability assumption. Only comparisons within X_j and comparisons between sets of attributes are requested. Hence, when weak orders are considered (rule GC_μ), changing the intertwining of quantities of type $\mu(A)$ with quantities $u_j(x)$ does not affect the preference $\succsim$ induced. Thus, assessing utility functions and the capacity μ should be easier with this model than with the AC approach.
• It is a priori natural to assume that $\succsim_I$ and the $\succsim_j$ are complete and transitive. However, the generalized concordance rule makes sense even if these properties do not hold. The transitivity of the $\succsim_j$ may be questioned: for example, consider an numerical attribute which is naturally ordered (*e.g.* a price). It may happen that any "small variation" of a value on this attribute does not modify the subjective value of the alternative considered. One can indeed imagine that the decision maker remains indifferent between two values x_j and y_j as long as the difference $|x_j - y_j|$ does not exceed a certain threshold. Such preferences are perfectly natural but fail to be transitive. A similar rationale could be developed concerning the transitivity of relation $\succsim_I$.
• Generalized concordance rules fit the the CA approach of Section 2.2 when $\succsim_I$ (resp. $\succsim_j$) derive from a capacity function μ (resp. a utility function u_j), or equivalently when they are weak orders (rule GC_μ). The general CA scheme is recovered when ϕ_j is defined as in 2.2 and ψ is such that :
$\forall \alpha \in [0,1]^n, \psi(\alpha) = \mu(\sigma(\alpha)) - \mu(\sigma(-\alpha))$
where $\sigma(\alpha)$ is a vector with component $\sigma_j(\alpha) = 1$ if $\alpha_j \geq 0$ and 0 otherwise. $\sigma(\alpha)$ is the characteristic vector of $C_{\succsim}(x,y)$ and $\sigma(-\alpha)$ of $C_{\succsim}(x,y)$.

A natural question is whether it is worthwhile considering a non-additive importance function μ to define GC rules. The claim that additive capacities are not expressive enough is based on counterexamples like the following:

Example 2 *We evaluate and compare four candidates applying to a commercial-engineering position. Candidates receive grades according to four points of view: technical skill ($X_1 = \{A, B, C, D, E\}$), commercial skill ($X_2 = \{A, B, C, D, E\}$), age ($X_3 = \{20, \ldots, 60\}$), salary ($X_4 = \{20, \ldots, 100\}$). Within X_1 (resp. X_2), A is the best grade and E is the worse. Numerical values in X_3 and X_4 are to be minimised. The ratings of the 4 candidates $\{x, y, z, w\}$ are:*

candid./attrib.	1	2	3	4
x	*B*	*B*	*31*	*60*
y	*C*	*A*	*31*	*60*
z	*B*	*B*	*49*	*50*
w	*A*	*C*	*26*	*80*

We get:
$C_{\succsim}(x,y) = \{1,3,4\}$ $C_{\succsim}(y,x) = \{2,3,4\}$
$C_{\succsim}(x,z) = \{1,2,3\}$ $C_{\succsim}(z,x) = \{1,2,4\}$
$C_{\succsim}(x,w) = \{2,4\}$ $C_{\succsim}(w,x) = \{1,3\}$

Now assume that the decision maker's choice is x (a reasonable choice since this candidate realizes a good trade-off between the various objectives). An attempt to reconstruct such preferences with the rule (GC_μ) and an additive *μ, leads to the following inequalities:*

$$\begin{array}{rcl} \mu(\{1\}) & > & \mu(\{2\}) \\ \mu(\{3\}) & > & \mu(\{4\}) \\ \mu(\{1\}) + \mu(\{3\}) & < & \mu(\{2\}) + \mu(\{4\}) \end{array}$$

These inequalities being contradictory, the additive rule is unable to describe decision maker's choice.

Another well known argument against additive concordance rules is that they lead to Condorcet effects, *i.e.* that even the strict part of global preference $\succsim$ ($\succ$) may fail to be transitive. However, there are some non additive concordance rules that alleviate these difficulties, *e.g.* the one obtained when μ is a necessity measure (see [10]). For instance the reader could easily check that the previous example can be adressed using a necessity measure for μ.

4 A Characterization of the Generalised Concordance Rule

In order to better understand the descriptive potential of generalized concordance rules, we now characterize preference structures which are compatible with this rule (the proofs are omitted for the sake of brevity). In this paper, both X and N are supposed to be finite (consistently, each X_j admits finite set of values). We also investigate the practical construction of the adequate instance of the rule from a given preference relation $\succsim$ on the entire multiattribute space X. This relation represents the decision maker's preferences. It is assumed to be partially observable on a sufficiently rich part of X. From this initial relation $\succsim$, one can define, for any attribute, a marginal relation $\succsim_j$ restricted to the j^{th} attribute:

$$x_j \succsim_j y_j \Leftrightarrow \forall z \in X,\ (x\{j\}z) \succsim (y\{j\}z) \qquad (1)$$

Then we introduce a first axiom, strongly enforcing the qualitative nature of the model.

AXIOM **NIM** (Neutrality-Independence-Monotony):.
$\forall\, x, y, z, w \in X,\ [C_{\succsim}(x,y) \subseteq C_{\succsim}(z,w)$ and $C_{\succsim}(y,x) \supseteq C_{\succsim}(w,z)] \Rightarrow (x \succsim y \Rightarrow z \succsim w)$

The NIM axiom is a translation to the multiattribute case of an axiom used in Social Choice Theory (see [27]). It can also be seen as a reinforcement of the non-compensation condition used in [15] and [6] adapted to the case of weak preference relations. Under this condition, the preference of x over y only depends on one-dimensional preferences $x \succsim_j y$ defined by (1). NIM implies that an improvement of x in relations $\succsim_j$ cannot downgrade the position of x in $\succsim$.

We introduce a second axiom which gives to any attribute the ability of discriminating at least two elements (x, y) of X.

AXIOM **DI** (Discrimination). $\forall j \in N,\ \exists x, y \in X,\ (x\{j\}y) \succsim y$ and not$(y \succsim (x\{j\}y))$

Finally, we consider an axiom preserving a minimal comparability between the alternatives. The relation $\succsim$

is not necessarily transitive nor complete. However, the incomparability is justified for a pair (x, y) only when at least two attributes j and k are conflicting, *i.e.* $x_j \succ_j y_j$ and $y_k \succ_k x_k$. Such a conflict does not exist when x and y differ on a single attribute. This is the meaning of the following:

AXIOM **MC** (Minimal Comparability). $\forall x, y, z, \in X$, $\forall j \in N$,$(x\{j\}z) \succsim (y\{j\}z)$ or $(y\{j\}z) \succsim (x\{j\}z)$

The five following properties derive from these three fundamental axioms:

- **NI** (Neutrality-Independence): $\forall x, y, z, w \in X$, $[C_{\succsim}(x, y) = C_{\succsim}(z, w)$ and $C_{\succsim}(y, x) = C_{\succsim}(w, z)] \Rightarrow (x \succsim y \Leftrightarrow z \succsim w)$
- **RE** (Reflexivity). $\succsim$ is reflexive on X
- **UN** (Unanimity): $\forall x, y \in X,\ C_{\succsim}(x, y) = N \Rightarrow x \succsim y$
- **IND** (Independence): $\forall A \subseteq N,\ \forall x, y, z, w \in X$, $(xAz) \succsim (yAz) \Leftrightarrow (xAw) \succsim (yAw)$
- **CO** (Consistency): $\forall A, B \in N,\ \forall x, y, z, w \in X$ such that $\forall j \in N, (x_j \succ_j y_j$ and $z_j \succ_j w_j)$, $(xAy) \succsim (xBy) \Leftrightarrow (zAw) \succsim (zBw)$.

The unanimity condition ensures that $\succsim$ refines the Pareto ordering of vectors. IND is the classical condition of preferential independence in multiattribute utility theory and the counterpart of the sure thing principle of DMU. CO means that, when x is uniformly better than y the preference between (xAy) and (xBy) only depends on the choice of A and B. This is a strong counterpart of the weaker $P4$ principle of Savage which projects the preference between acts into a likelihood relation between events.

Proposition 2

$i)$	$NIM \Rightarrow NI$	$iv)$	$NIM \Rightarrow IND$
$ii)$	$MC \Rightarrow RE$	$v)$	$NI \Rightarrow CO$
$iii)$	$NIM + RE \Rightarrow UN$		

Item $iv)$ shows, among other things, that the comparison of $x\{j\}z$ and $y\{j\}z$ is independent from z. It makes the construction of relations $\succsim_j$ from $\succsim$ easier, using equation (1). Indeed, thanks to NI, $\succsim_j$ can be simply defined as follows:

$$x_j \succsim_j y_j \Leftrightarrow \exists z \in X,\ (x\{j\}z) \succsim (y\{j\}z) \qquad (2)$$

When DI and NIM (and thus CO) hold, it possible to extract from the decision maker preference $\succsim$ an importance relation on 2^N:

$$A \succsim_I B \Leftrightarrow (\exists\, x, y \in X : \forall j \in N,\ x_j \succ_j y_j \text{ and } (xAy) \succsim (xBy)) \qquad (3)$$

This definition is very natural. Indeed, when x is uniformly better than y, preferring xAy to xBy is clearly justified by the fact that the coalition of attributes A is considered as more important than the coalition B.

We are now in a position to establish the main result. First of all, let us observe that conditions NIM, DI and MC are compatible. The additive concordance rule indeed satisfies these conditions. The following representation theorem shows that any preference relation $\succsim$ verifying NIM, DI and MC can be represented by a generalised concordance rule.

Theorem 1 *If the decision maker's preference $\succsim$ satisfies NT, NIM and MC then there exists n complete preference relations $\succsim_1, \ldots, \succsim_n$, defined on $X_1, \ldots, X_n$ respectively, and a* monotonic and pre-additive *relation $\succsim_I$ on 2^N, such that:*

$$\forall x, y \in X,\ x \succsim y \Leftrightarrow C_{\succsim}(x, y) \succsim_I C_{\succsim}(y, x)$$

For any attribute j, $\succsim_j$ reveals the decision maker's preferences concerning the consequences of X_j. The relation $\succsim_j$ can be constructed step by step, by observing the decision maker's preferences over pairs of alternatives of type $(x\{j\}z), y\{j\}z)$ for an arbitrary z. This observation is even simpler if relations $\succsim_j$ are supposed to be transitive or quasi transitive ($\succ_j$ transitive). Formula (3) also provides a constructive method to derive the importance relation $\succsim_I$ and thus completes the construction of model. Notice that the entire construction is based on pairwise comparisons. Such comparisons do not require a prohibitive cognitive effort because they only concern alternatives having simple profiles. Moreover, such comparisons do not require any explicit questioning but can be inferred by observing real choices performed by the decision maker.

Notice that the use of a preadditive relation of importance is not necessary in a GC rule, in the sense that Theorem 1 only proves the *existence* of a preadditive $\succsim_I$ involved in a GC rule representing $\succsim$. If we start from a more general monotonic $\succsim_M$ (possibly not preadditive) and construct $\succsim$ via the GC rule, we still get a preference satisfying NIM, NT and MC [2] which thus induces a preadditive $\succsim_I$ by (3).

The reason of this apparent paradox is the following. First note that $C_{\succsim}(x, y) \cup C_{\succsim}(y, x) = N$. So only the pairs (A, B) such that $A \cup B = N$ are compared in the GC rule. It can be proved that:

$$A \succsim_I B \Leftrightarrow A \cup \bar{B} \succsim_M B \cup \bar{A} \qquad (4)$$

$$A \succsim_I B \Leftrightarrow A \succsim_M B \text{ whenever } A \cup B = N \qquad (5)$$

so $\succsim_I$ and $\succsim_M$ coincide on the useful part of $2^N \times 2^N$ and the non preadditivity of $\succsim_M$ cannot be revealed by observing a decision maker which would use a GC rule. Example 2 is a typical case where a decision maker may use a non preadditive relation $\succsim_M$ induced by a necessity measure to describe criteria importance. In this example, criteria importance could also be represented by any preadditive relation $\succsim'_M$ such that $A \succsim_M B$ and $A \succsim'_M B$ coincide for $A \cup B = N$. The impossibility to use an additive capacity in this case stresses the gap existing between preadditivity and additivity.

5 Conclusions

From lack of space, it is not possible to describe the precise structure of the GC rules. It is enough to say that they generally exploit a hierarchy of oligarchies of attributes. Namely, there is a group of attributes that unanimously decides on the

[2] It can be shown that any preference relation $\succsim$ constructed from an importance relation $\succsim_M$ using a GC rule satisfies NIM, NT and MC as soon as the 4 following properties hold:
$\forall j \in N,\ \succsim_j$ is complete, $\forall j \in N, \exists (x, y) \in X^2\ x_j \succ_j y_j$
$\succsim_M$ is reflexive, monotonic, $\forall j \in N,\ N \succ_M N \setminus \{j\}$

preference or incomparability between two alternatives. In case of indifference, the decision is left to another less powerful oligarchy and so on. Future research should also dig into the representation of importance relations induced by theorem 1. Partial results exist in [11] for preferences relations $\succsim$ whose strict part is transitive. Then $\succsim_I$ can be represented by means of a family of weak orders on N itself. Besides, it is clear that GC rules are not the ultimate answer to qualitative MCDM problems. One can easily imagine situations where the preference of a decision-maker can not be expressed this way ; for instance when the preference between x and y depends on a third alternative z, as in MCDM counterparts of stochastic dominance in DMU. Lastly, as suggested by example 2, the approach presented here could benefit to multiagent fusion problems, once adapted to a logical setting.

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