Nonlinear Backward Stochastic Evolutionary Equations Driven by a Space-Time White Noise

Ying Hu, Shanjian Tang

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Abstract

We study the well solvability of nonlinear backward stochastic evolutionary equations driven by a space-time white noise. We first establish a novel a priori estimate for solution of linear backward stochastic evolutionary equations, and then give an existence and uniqueness result for nonlinear backward stochastic evolutionary equations. A dual argument plays a crucial role in the proof of these results. Finally, an example is given to illustrate the existence and uniqueness result.

1 Introduction

Let H be a Hilbert space with $\{e_i\}$ being its orthonormal basis, A an infinitesimal generator which generates a strongly continuous semigroup $\{e^{At}, t \geq 0\}$, and $\mathcal{S}_2(H)$ the Hilbert space of Hilbert-Schmidt operators in H. Denote by W a cylindrical Wiener process in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ being the augmented natural filtration and $\mathcal{P}$ the predictable σ-algebra.

By $L^0_{\mathcal{P}}([0,T]\times\Omega, H)$ we denote the totality of H-valued progressively measurable processes X. For $p \in [1,\infty)$, by $L^p_{\mathcal{P}}([0,T]\times\Omega, H)$ we denote the Banach space of H-valued progressively measurable processes X with $\mathbb{E}\int_0^T \|X_s\|^p ds < \infty$ and by $L^p_{\mathcal{P}}(\Omega, C([0,T], H))$ the subspace of H-valued progressively measurable processes X with strongly continuous trajectories satisfying $\mathbb{E}\max_{s\in[0,T]} \|X_s\|^p < \infty$. Here and below we use the symbol $\|\cdot\|$ to denote a norm when the corresponding space is clear from the context, otherwise we use a subscript. Consider the map

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$f : \Omega \times [0,T] \times H \times \mathcal{S}_2(H) \times H \to H$ where $f(\cdot,0,0,0) \in L^0_{\mathcal{P}}(\Omega \times [0,T], H)$ and there is a positive constant L such that

$$\|f(t,p_1,q_1,s_1) - f(t,p_2,q_2,s_2)\|_H \leq L(\|p_1-p_2\|_H + \|q_1-q_2\|_{\mathcal{S}_2(H)} + \|s_1-s_2\|_H).$$

Linear backward stochastic evolutionary equations arise in the formulation of stochastic maximum principle for optimal control of stochastic partial differential equations, and see among others [1, 8, 4, 5, 7, 10, 11, 12]. The study can be dated back to the work of Bensoussan [1], and to Hu and Peng [8] for a general context. The nonlinear case is given by Hu and Peng [9]. In these works, the underlying Wiener process is assumed to have a trace-class covariance operator—in particular, to be finite-dimensional. Recently, Fuhrman, Hu, and Tessitore [6] discusses a linear backward stochastic evolutionary equation driven by a space-time white noise. The objective of the paper is to study the nonlinear backward stochastic evolutionary equation driven by a space-time white noise.

Consider the following form of nonlinear backward stochastic evolutionary equations (BSEEs)

$$\begin{cases} -dP_t &= \left[A^*P_t + \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i + f\left(t, P_t, Q_t, \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i\right)\right] dt \\ & \quad - \sum_{i=1}^{\infty} Q_t e_i \, d\beta_t^i, \quad t \in (0,T]; \\ P_T &= \eta, \end{cases} \tag{1.1}$$

where $\beta_t^i = \langle e_i, W_t \rangle$, $i = 1, 2...$ is a family of independent Brownian motions, $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$. The unknown process is the pair denoted (P,Q) and takes values in $H \times \mathcal{S}_2(H)$. We will work under the following assumptions, which are assumed to hold throughout the paper.

Hypothesis 1.1

1. *e^{tA}, $t \geq 0$, is a strongly continuous semigroup of bounded linear operators in H. Moreover, $e^{tA} \in \mathcal{S}_2(H)$ for all $t > 0$ and there exist constants $c > 0$ and $\alpha \in [0, 1/2)$ such that $\|e^{tA}\|_{\mathcal{S}_2(H)} \leq ct^{-\alpha}$ for all $t \in (0,T]$.*
2. *The processes C_i are strongly progressively measurable with values in $\mathcal{L}(H)$. Moreover we have $\|C_i(t)\|_{\mathcal{L}(H)} \leq c$, $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $i \in \mathbb{N}$.*
3. *$\sum_{i=1}^{\infty} \|e^{tA}C_i(s)h\|^2 \leq ct^{-2\alpha}\|h\|_H^2$ for all $t \in (0,T]$, $s \geq 0$, and $h \in H$.*

We give the following notion of (mild) solution.

Definition 1.1 *We say that a pair of processes*

$$(P,Q) \in L^2_{\mathcal{P}}(\Omega \times [0,T], H) \times L^2_{\mathcal{P}}(\Omega \times [0,T], \mathcal{S}_2(H))$$

is a mild solution to equation (1.1) *if the following holds:*

1. *The sequence*

$$S^M(s) := \sum_{i=1}^{M} (T-s)^{\alpha} C_i^*(s) Q_s e_i, \quad s \in [0,T]; \quad M = 1, 2, \ldots,$$

converges weakly in $L^2_{\mathcal{P}}(\Omega \times [0,T], H)$.

2. *For any* $t \in [0,T]$,

$$\begin{aligned} P_t &= e^{(T-t)A^*}\eta + \sum_{i=1}^{\infty} \int_t^T e^{(s-t)A^*} C_i^*(s) Q_s e_i ds \\ &\quad + \int_t^T e^{(s-t)A^*} f\left(s, P_s, Q_s, \sum_{i=1}^{\infty} C_i^*(s) Q_s e_i\right) ds \\ &\quad - \sum_{i=1}^{\infty} \int_t^T e^{(s-t)A^*} Q_s e_i d\beta_s^i, \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.2}$$

Note that the integral involving f is well-defined due to the following

$$\begin{aligned} &\int_t^T \left\| e^{(s-t)A^*} f\left(s, P_s, Q_s, \sum_{i=1}^{\infty} C_i^*(s) Q_s e_i\right) \right\|_H ds \\ \leq\ & C \int_t^T \left(\|f(s,0,0,0)\|_H + \|P_s\|_H + \|Q_s\|_{\mathcal{S}_2(H)} + (T-s)^{-\alpha} \left\| (T-s)^{\alpha} \sum_{i=1}^{\infty} C_i^*(s) Q_s e_i \right\|_H \right) ds \\ <\ & \infty. \end{aligned}$$

Note that the term $\sum_{i=1}^{\infty} C_i^* Q e_i$ is not bounded in Q with respect to the Hilbert-Schmidt norm. Its appearance in the drift gives rise to new difficulty in the resolution of the underlying BSEEs, and has to be carefully estimated. In particular, we prove via a dual method novel a priori estimate (see Proposition 2.4 in Section 2 below for details) for solution of linear BSEEs driven by a space-time white noise. The new a priori estimate and the dual arguments are crucial in the subsequent Picard iteration for our nonlinear BSEEs.

The rest of the paper is organized as follows. In Section 2, we prove a new a priori estimate for solution of linear BSEEs driven by a space-time white noise. The nonlinear BSEEs are studied in Section 3. Finally in Section 4, we give an example.

2 Linear BSEE Revisited

(Forward) stochastic evolutionary equations (FSEE) driven by a cylindrical Wiener process have been extensively studied. See, e.g. Da Prato and Zabczyk [2, 3] for excellent expositions and the references therein. Here we give some precise a priori estimate for mild solutions of linear FSEEs, which will play a crucial role in our subsequent analysis.

Lemma 2.1 *For $\gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H)$, the linear stochastic equation*

$$\begin{cases} d\mathcal{Y}_t &= A\mathcal{Y}_t\,dt + \sum_{i=1}^{\infty} C_i(t)\mathcal{Y}_t\,d\beta_t^i + \sum_{i=1}^{\infty} C_i(t)(T-t)^\alpha \gamma_t\,d\beta_t^i, \quad t \in (0,T]; \\ \mathcal{Y}_0 &= 0, \end{cases} \tag{2.1}$$

has a unique mild solution $\mathcal{Y}^\gamma$ in $L^2_{\mathcal{P}}(\Omega, C([0,T],H))$. Furthermore, we have

$$\mathbb{E}\|\mathcal{Y}_t^\gamma\|_H^2 \le C\mathbb{E}\int_0^t (t-s)^{-2\alpha}(T-s)^{2\alpha}\|\gamma_s\|_H^2\,ds. \tag{2.2}$$

For $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $f_0 \in L^0_{\mathcal{P}}(\Omega \times [0,T], H)$ such that for some $\beta \in (0, \frac{1}{2})$

$$\mathbb{E}\int_0^T (T-s)^{2\beta}\|f_0(s)\|_H^2\,ds < \infty, \tag{2.3}$$

the following linear functional G defined by

$$G(\gamma) := E\langle \eta, \mathcal{Y}_T^\gamma\rangle + E\int_0^T \langle f_0(t), \mathcal{Y}_t^\gamma\rangle\,dt, \quad \gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H), \tag{2.4}$$

has a unique linear and continuous extension to $L^2_{\mathcal{P}}(\Omega \times [0,T], H)$, denoted by $\overline{G}$.

Proof. The first two assertions can be found in [6, Theorem 4.3 and Proposition 4.5]. We now prove the last assertion, that is

$$|G(\gamma)|^2 \le C\|\gamma\|^2_{L^2_{\mathcal{P}}(\Omega\times[0,T],H)}, \quad \gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H). \tag{2.5}$$

First from (2.2), we immediately have

$$\mathbb{E}\|\mathcal{Y}_T^\gamma\|_H^2 \le C\|\gamma\|^2_{L^2_{\mathcal{P}}(\Omega\times[0,T],H)}, \quad \gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H).$$

It suffices to prove the following

$$\mathbb{E}\int_0^T (T-s)^{-2\beta}\|\mathcal{Y}_s^\gamma\|_H^2\,ds \le C\|\gamma\|^2_{L^2_{\mathcal{P}}(\Omega\times[0,T],H)}, \quad \gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H).$$

We have from (2.2)

$$\begin{aligned}
& \mathbb{E}\int_0^T (T-s)^{-2\beta}\|\mathcal{Y}_s^\gamma\|_H^2\,ds \\
\leq\ & C\mathbb{E}\int_0^T (T-t)^{-2\beta}\int_0^t (t-s)^{-2\alpha}(T-s)^{2\alpha}\|\gamma_s\|_H^2\,ds\,dt \\
=\ & C\mathbb{E}\int_0^T \int_s^T (T-t)^{-2\beta}(t-s)^{-2\alpha}\,dt\,(T-s)^{2\alpha}\|\gamma_s\|_H^2\,ds \\
=\ & C\mathbb{E}\int_0^T \int_0^1 (1-\theta)^{-2\beta}\theta^{-2\alpha}\,d\theta\,(T-s)^{1-2\beta}\|\gamma_s\|_H^2\,ds \\
\leq\ & CT^{1-2\beta}\int_0^1 (1-\theta)^{-2\beta}\theta^{-2\alpha}\,d\theta\,\|\gamma\|^2_{L^2_{\mathcal{P}}(\Omega\times[0,T],H)}. \qquad (2.6)
\end{aligned}$$

Here we have used in the last equality the transformation of variables: $t = s + (T-s)\theta$. □

Remark 2.2 *Let $N \geq 1$ be an integer. For $\gamma \in L^\infty_{\mathcal{P}}(\Omega\times[0,T],H)$, the linear stochastic equation*

$$\begin{cases} d\mathcal{Y}_t &= A\mathcal{Y}_t\,dt + \sum_{i=1}^{\infty} C_i(t)\mathcal{Y}_t\,d\beta_t^i + \sum_{i=1}^{N} C_i(t)(T-t)^\alpha\gamma_t\,d\beta_t^i, \quad t\in(0,T]; \\ \mathcal{Y}_0 &= 0, \end{cases} \qquad (2.7)$$

has a unique mild solution $\mathcal{Y}^{\gamma,N}$ in $L^2_{\mathcal{P}}(\Omega, C([0,T],H))$. Furthermore, we have

$$\mathbb{E}\|\mathcal{Y}_t^{\gamma,N}\|_H^2 \leq C\mathbb{E}\int_0^t (t-s)^{-2\alpha}(T-s)^{2\alpha}\|\gamma_s\|_H^2\,ds \qquad (2.8)$$

for a positive constant C, which does not depend on N. For $\eta \in L^2(\Omega,\mathcal{F}_T,\mathbb{P},H)$ and $f_0 \in L^0_{\mathcal{P}}(\Omega\times[0,T],H)$ such that for some $\beta\in(0,\frac{1}{2})$

$$\mathbb{E}\int_0^T (T-s)^{2\beta}\|f_0(s)\|_H^2\,ds < \infty, \qquad (2.9)$$

the following linear functional G^N defined by

$$G^N(\gamma) := E\langle\eta,\mathcal{Y}_T^{\gamma,N}\rangle + E\int_0^T \langle f_0(t),\mathcal{Y}_t^{\gamma,N}\rangle\,dt, \quad \gamma\in L^\infty_{\mathcal{P}}(\Omega\times[0,T],H) \qquad (2.10)$$

has a unique linear and continuous extension to $L^2_{\mathcal{P}}(\Omega\times[0,T],H)$, denoted by $\overline{G}^N$. Furthermore, there is a positive constant C such that C does not depend on N and

$$|\overline{G}^N(\gamma)|^2 \leq C\|\gamma\|^2_{L^2_{\mathcal{P}}(\Omega\times[0,T],H)}, \quad \gamma\in L^2_{\mathcal{P}}(\Omega\times[0,T],H). \qquad (2.11)$$

We now recall the following result from [6], concerning the following linear BSEE driven by a white noise:

$$\begin{cases} -dP_t &= [A^*P_t + \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i + f_0(t)]\,dt - \sum_{i=1}^{\infty} Q_t e_i\, d\beta_t^i, \\ P_T &= \eta. \end{cases} \tag{2.12}$$

Lemma 2.3 *Let $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $f_0 \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$. There exists a unique mild solution $(P,Q) \in L^2_{\mathcal{P}}(\Omega \times [0,T], H) \times L^2_{\mathcal{P}}(\Omega \times [0,T], \mathcal{S}_2(H))$ to BSEE (2.12).*

In the subsequent study of the nonlinear case, we need the following a priori estimate for BSEE (2.12).

Proposition 2.4 *(a priori estimate) Let $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and $f_0 \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$. For $\beta \in (\alpha, \frac{1}{2})$ and a mild solution $(P,Q) \in L^2_{\mathcal{P}}(\Omega \times [0,T], H \times \mathcal{S}_2(H))$ to BSEE (2.12), we have*

$$\begin{aligned} & \mathbb{E}\int_t^T ||P_s||_H^2 ds + \mathbb{E}\int_t^T ||Q_s||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_t^T (T-s)^{2\alpha} || \sum_{i=1}^{\infty} C_i(s) Q_s e_i||_H^2 ds \\ \leq\ & C\left(\mathbb{E}||\eta||_H^2 + \mathbb{E}\int_t^T (T-s)^{2\beta} ||f_0||_H^2 ds\right). \end{aligned}$$

Proof. Let us prove the first two terms by duality argument. We have,

$$\mathbb{E}\int_t^T \langle P_s, \rho_s\rangle_H ds + \mathbb{E}\int_t^T \langle Q_s, \Gamma_s\rangle_{\mathcal{S}_2(H)}\, ds \ = \ \mathbb{E}\langle \eta, X_T\rangle_H + \mathbb{E}\int_t^T \langle f_s, X_s\rangle_H ds,$$

where

$$dX_s = (AX_s + \rho_s)ds + \sum_{i=1}^{\infty} C_i(s) X_s d\beta_s^i + \sum_{i=1}^{\infty} \Gamma_s e_i d\beta_s^i, \quad s \in (t,T]; \quad X_t = 0.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\int_t^T \langle P_s, \rho_s\rangle_H ds + \mathbb{E}\int_t^T \langle Q_s, \Gamma_s\rangle_{\mathcal{S}_2(H)} ds \\
= \;& \mathbb{E}\langle \eta, X_T\rangle_H + \mathbb{E}\int_t^T \langle f_s, X_s\rangle_H ds \\
\leq \;& \mathbb{E}\left[||\eta||_H^2\right]^{\frac{1}{2}} \mathbb{E}[||X_T||_H^2]^{\frac{1}{2}} + \int_t^T \mathbb{E}[||X_s||_H^2]^{\frac{1}{2}} \mathbb{E}[||f_s||_H^2]^{\frac{1}{2}}\, ds \\
\leq \;& \mathbb{E}\left[||\eta||_H^2\right]^{\frac{1}{2}} \mathbb{E}[||X_T||_H^2]^{\frac{1}{2}} + \sup_{t\leq s\leq T} \mathbb{E}[||X_s||_H^2]^{\frac{1}{2}} \int_t^T \mathbb{E}[||f_s||_H^2]^{\frac{1}{2}}\, ds \\
\leq \;& \mathbb{E}\left[||\eta||_H^2\right]^{\frac{1}{2}} \mathbb{E}[||X_T||_H^2]^{\frac{1}{2}} + \sup_{t\leq s\leq T} \mathbb{E}[||X_s||_H^2]^{\frac{1}{2}} \int_t^T (T-s)^{\beta} \mathbb{E}[||f_s||_H^2]^{\frac{1}{2}} (T-s)^{-\beta}\, ds \\
\leq \;& \mathbb{E}\left[||\eta||_H^2\right]^{\frac{1}{2}} \mathbb{E}[||X_T||_H^2]^{\frac{1}{2}} + \frac{T^{1-2\beta}}{1-2\beta} \sup_{t\leq s\leq T} \mathbb{E}[||X_s||_H^2]^{\frac{1}{2}} \mathbb{E}\left[\int_t^T (T-s)^{2\beta} ||f_s||_H^2\, ds\right]^{\frac{1}{2}}.
\end{aligned}$$

Using the inequality from [6, Proposition 4.4]

$$\sup_{s\in[t,T]} \mathbb{E}\|X_s\|_H^2 \leq C\|(\rho,\Gamma)\|^2_{L^2_{\mathcal{P}}(\Omega\times[t,T],H\times\mathcal{S}_2(H))}$$

with

$$\|(\rho,\Gamma)\|^2_{L^2_{\mathcal{P}}(\Omega\times[t,T],H\times\mathcal{S}_2(H))} := \mathbb{E}\int_t^T \|\rho_s\|_H^2 ds + \mathbb{E}\int_t^T \|\Gamma_s\|^2_{\mathcal{S}_2(H)} ds,$$

we have

$$\begin{aligned}
&\mathbb{E}\int_t^T \langle P_s, \rho_s\rangle_H ds + \mathbb{E}\int_t^T \langle Q_s, \Gamma_s\rangle_{\mathcal{S}_2(H)} ds \\
\leq \;& C\left(\mathbb{E}\|\eta\|_H^2 + \mathbb{E}\int_t^T (T-s)^{2\beta} ||f_s||_H^2\, ds\right)^{\frac{1}{2}} \|(\rho,\Gamma)\|_{L^2_{\mathcal{P}}(\Omega\times[t,T],H\times\mathcal{S}_2(H))}.
\end{aligned}$$

This implies the desired a priori estimate for the norm of (P, Q).

To complete the proof, we consider once again the duality:

$$\mathbb{E}\int_t^T \langle (T-s)^{\alpha} \sum_{i=1}^{\infty} C_i Q_s e_i, \gamma_s\rangle ds \;=\; \mathbb{E}\langle \eta, X_T\rangle + \mathbb{E}\int_t^T \langle f_s, X_s\rangle ds,$$

where $X_t = 0$ and

$$dX_s = AX_s ds + \sum_{i=1}^{\infty} C_i(s) X_s d\beta_s^i + \sum_{i=1}^{\infty} C_i(s)(T-s)^{\alpha}\gamma_s d\beta_s^i, \quad s\in[t,T].$$

We have

$$\begin{aligned}
&\mathbb{E}\int_t^T \langle (T-s)^\alpha \sum_{i=1}^\infty C_i Q_s e_i, \gamma_s\rangle ds \\
= \;& \mathbb{E}\langle \eta, X_T\rangle + \mathbb{E}\int_t^T \langle (T-s)^\beta f_s, (T-s)^{-\beta} X_s\rangle ds \\
\leq \;& \mathbb{E}\left[||\eta||_H^2\right]^{\frac{1}{2}} \mathbb{E}\left[||X_T||_H^2\right]^{\frac{1}{2}} \\
& + \mathbb{E}\left[\int_t^T (T-s)^{2\beta}||f_s||_H^2 ds\right]^{\frac{1}{2}} \mathbb{E}\left[\int_t^T (T-s)^{-2\beta}||X_s||_H^2 ds\right]^{\frac{1}{2}}.
\end{aligned}$$

On the one hand, we have from [6, Proposition 4.5],

$$\mathbb{E}\|X_T\|_H^2 \leq C\int_t^T (T-l)^{-2\alpha}(T-l)^\alpha \mathbb{E}\|\gamma_l\|_H^2 dl = C\int_t^T \mathbb{E}\|\gamma_l\|_H^2 dl$$

and

$$\begin{aligned}
&\mathbb{E}\int_t^T (T-s)^{-2\beta}||X_s||_H^2 ds \\
\leq \;& C\int_t^T (T-s)^{-2\beta}\int_t^s (s-l)^{-2\alpha}(T-l)^{2\alpha}\mathbb{E}\|\gamma_l\|_H^2 dl ds \\
= \;& C\int_t^T \int_l^T (T-s)^{-2\beta}(s-l)^{-2\alpha} ds (T-l)^{2\alpha}\mathbb{E}\|\gamma_l\|_H^2 dl \\
= \;& C\left(\int_0^1 (1-\theta)^{-2\beta}\theta^{-2\alpha} d\theta\right)\int_t^T (T-l)^{1-2\beta}\mathbb{E}\|\gamma_l\|_H^2 dl \\
\leq \;& CT^{1-2\beta}\left(\int_0^1 (1-\theta)^{-2\beta}\theta^{-2\alpha} d\theta\right)\int_t^T \mathbb{E}\|\gamma_l\|_H^2 dl.
\end{aligned}$$

Note that in the last equality, we have used the transformation of variables: $s = l + (T-l)\theta$. Concluding the above, we have

$$\begin{aligned}
&\mathbb{E}\int_t^T \langle (T-s)^\alpha \sum_{i=1}^\infty C_i Q_s e_i, \gamma_s\rangle ds \\
\leq \;& C\left(\mathbb{E}||\eta||_H^2 + \mathbb{E}\int_t^T (T-s)^{2\beta}||f_s||_H^2 ds\right)^{\frac{1}{2}} \|\gamma\|_{L^2_{\mathcal{P}}(\Omega\times[t,T],H)}.
\end{aligned}$$

Then we have the last desired a priori estimate. □

Proposition 2.5 *Suppose that*

$$\mathbb{E}||\eta||_H^2 + \mathbb{E}\int_0^T (T-s)^{2\beta}||f_0||_H^2 ds < \infty.$$

There exists a unique solution to linear BSEE (2.12) *such that*

$$\mathbb{E}\int_0^T ||P_s||_H^2 ds + \mathbb{E}\int_0^T ||Q_s||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_0^T (T-s)^{2\alpha}||\sum_{i=1}^{\infty} C_i(s)Q_s e_i||_H^2 ds < \infty.$$

Proof. Uniqueness is an immediate consequence of Proposition 2.4. It remains to consider the existence assertion.

Define for any integer $k > T$,

$$f_0^k(t) = f_0(t)\chi_{[0,T-1/k]}(t), \quad t \in [0,T].$$

We have for any $k > T$,

$$\begin{aligned} & k^{-2\beta}E\int_0^T \|f_0^k(s)\|_H^2 \, ds = k^{-2\beta}E\int_0^{T-1/k} \|f_0(s)\|_H^2 \, ds \\ \leq \quad & E\int_0^{T-1/k} (T-s)^{2\beta}\|f_0(s)\|_H^2 \, ds \leq E\int_0^T (T-s)^{2\beta}\|f_0(s)\|_H^2 \, ds < \infty. \end{aligned} \tag{2.13}$$

Therefore, $f_0^k \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$, and in view of Lemma 2.3, BSEE (2.12) for $f_0 = f_0^k$ has a unique mild solution (P^k, Q^k) for any integer $k > T$.

Moreover, for $k > l > T$, we have

$$E\int_0^T (T-s)^{2\beta}\|f_0^k(s) - f_0^l(s)\|_H^2 \, ds \leq E\int_{T-1/l}^{T-1/k} (T-s)^{2\beta}\|f_0(s)\|_H^2 \, ds \to 0 \tag{2.14}$$

as $k, l \to \infty$. Applying Proposition 2.4, we see that $\{(P^k, Q^k)\}$ is a Cauchy sequence in the space $L^2_{\mathcal{P}}(\Omega \times [0,T], H \times \mathcal{S}_2(H))$, and the sequence of processes $\{(T-s)^{\alpha}\sum_{i=1}^{\infty} C_i(s)Q_s^k e_i, s \in [0,T]; k > T\}$ is a Cauchy sequence in the space $L^2_{\mathcal{P}}(\Omega \times [0,T], H)$. Thus, they have limits $(P, Q, (T-t)^{\alpha}S) \in L^2_{\mathcal{P}}(\Omega \times [0,T], H \times \mathcal{S}_2(H) \times H)$, which satisfies the following equation:

$$\begin{cases} -dP_t &= [A^*P_t + S(t) + f_0(t)]\, dt - \displaystyle\sum_{i=1}^{\infty} Q_t e_i \, d\beta_t^i, \\ P_T &= \eta. \end{cases} \tag{2.15}$$

To show that (P, Q) is a mild solution to BSEE (2.12), it is sufficient for us to prove that

$$S(t) = \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i, \tag{2.16}$$

with the limit being defined in the following weak sense:

$$\{(T-t)^\alpha \sum_{i=1}^{N} C_i^*(t) Q_t e_i, t \in [0,T]\} \stackrel{N\to\infty}{\longrightarrow} \{(T-t)^\alpha S(t), t \in [0,T]\}$$

weakly in the Hilbert space $L^2_{\mathcal{P}}(\Omega \times [0,T], H)$.

Note that $\mathcal{Y}^\gamma$ is the solution to the stochastic equation (2.1) for $\gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H)$. We have the following duality:

$$\begin{aligned} & E\int_0^T \langle (T-t)^\alpha \sum_{i=1}^{\infty} C_i^*(t) Q^k e_i, \gamma_t\rangle \, dt = E\int_0^T \sum_{i=1}^{\infty} \langle Q^k e_i, (T-t)^\alpha C_i(t)\gamma_t\rangle \, dt \\ = \;& E\langle \eta, \mathcal{Y}_T^\gamma\rangle + E\int_0^T \langle f_0^k(t), \mathcal{Y}_t^\gamma\rangle \, dt. \end{aligned} \tag{2.17}$$

Passing to the limit $k \to \infty$, we have for $\gamma \in L^\infty_{\mathcal{P}}(\Omega \times [0,T], H)$,

$$E\int_0^T \langle (T-t)^\alpha S(t), \gamma_t\rangle \, dt \;=\; E\langle \eta, \mathcal{Y}_T^\gamma\rangle + E\int_0^T \langle f_0(t), \mathcal{Y}_t^\gamma\rangle \, dt = G(\gamma). \tag{2.18}$$

Since the process $(T-t)^\alpha S(t), t \in [0,T]$ lies in $L^2_{\mathcal{P}}(\Omega \times [0,T], H)$, in view of Lemma 2.1, we have for $\gamma \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$

$$E\int_0^T \langle (T-t)^\alpha S(t), \gamma_t\rangle \, dt \;=\; \overline{G}(\gamma). \tag{2.19}$$

On the other hand, we have the duality:

$$\begin{aligned} & E\int_0^T \left\langle (T-t)^\alpha \sum_{i=1}^{N} C_i^*(t) Q^k e_i, \gamma_t \right\rangle dt = E\int_0^T \sum_{i=1}^{N} \langle Q^k e_i, (T-t)^\alpha C_i(t)\gamma_t\rangle \, dt \\ = \;& E\left\langle \eta, \mathcal{Y}_T^N \right\rangle + E\int_0^T \left\langle (T-t)^\beta f_0^k(t), (T-t)^{-\beta}\mathcal{Y}_t^N \right\rangle dt. \end{aligned} \tag{2.20}$$

Setting $k \to \infty$, in view of (2.8), we have

$$E\int_0^T \left\langle (T-t)^\alpha S^N(t), \gamma_t \right\rangle dt \;=\; E\left\langle \eta, \mathcal{Y}_T^N \right\rangle + E\int_0^T \left\langle f_0(t), \mathcal{Y}_t^N \right\rangle dt = G^N(\gamma) \tag{2.21}$$

with

$$S^N(t) := \sum_{i=1}^{N} C_i^*(t) Q e_i, \quad t \in [0,T]. \tag{2.22}$$

In view of Remark 2.2, we have

$$\mathbb{E}\int_0^T \|(T-t)^\alpha S^N(t)\|_H^2\, dt = \|\overline{G}^N(\cdot)\|^2 \le C \tag{2.23}$$

with C being independent of N. Then, the set $\{(T-\cdot)^\alpha S^N(\cdot), N=1,2,\ldots\}$ is weakly compact, and thus has a weakly convergent subsequence. Let $(T-\cdot)\overline{S}(\cdot)$ be one weak limit. Then in view of equality (2.21), we have for $\gamma \in L^2_{\mathcal{P}}(\Omega\times[0,T],H)$,

$$E\int_0^T \left\langle (T-t)^\alpha \overline{S}(t), \gamma_t \right\rangle\, dt \;\; = \;\; \overline{G}(\gamma) = E\int_0^T \langle (T-t)^\alpha S(t), \gamma_t\rangle\, dt. \tag{2.24}$$

Therefore, $\overline{S} = S$, and the desired equality (2.16) is true. □

3 Main Result

In this section, we state and prove the following result.

Theorem 3.1 *For $\eta \in L^2(\Omega,\mathcal{F}_T,\mathbb{P},H)$ and the map $f:\Omega\times[0,T]\times H\times\mathcal{S}_2(H)\times H\to H$ where $f(\cdot,0,0,0)\in L^0_{\mathcal{P}}(\Omega\times[0,T],H)$ such that*

$$\mathbb{E}\int_0^T (T-s)^{2\beta}\|f(s,0,0,0)\|_H^2\, ds < \infty \tag{3.1}$$

for some $\beta\in(\alpha,\frac{1}{2})$, and

$$\|f(t,p_1,q_1,s_1)-f(t,p_2,q_2,s_2)\|_H \le L(\|p_1-p_2\|_H + \|q_1-q_2\|_{\mathcal{S}_2(H)} + \|s_1-s_2\|_H) \tag{3.2}$$

for a positive constant L. There exists a unique solution (P,Q) for (1.1) such that

$$\mathbb{E}\int_0^T ||P_s||_H^2 ds + \mathbb{E}\int_0^T ||Q_s||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_0^T (T-s)^{2\alpha}||\sum_{i=1}^\infty C_i(s)Q_s e_i||_H^2 ds < \infty.$$

Proof. (i) Uniqueness. Let (P^i,Q^i) be a mild solution to BSEE (1.1) for $i=1,2$. Define

$$\widehat{P} := P^1 - P^2, \quad \widehat{Q} := Q^1 - Q^2;$$

and

$$\widehat{f}(t) := f(t,P_t^1,Q_t^1,\sum_{i=1}^\infty C_i^*(t)Q_t^1 e_i) - f(t,P_t^2,Q_t^2,\sum_{i=1}^\infty C_i^*(t)Q_t^2 e_i), \quad t\in[0,T].$$

We have

$$\begin{cases} -d\widehat{P}_t &= [A^*\widehat{P}_t + \sum_{i=1}^{\infty} C_i^*(t)\widehat{Q}_t e_i + \widehat{f}(t)]\,dt - \sum_{i=1}^{\infty} \widehat{Q}_t e_i\, d\beta_t^i, \quad t \in [0,T]; \\ \widehat{P}_T &= 0. \end{cases} \tag{3.3}$$

It suffices to show that $\widehat{P} = 0$ and $\widehat{Q} = 0$ on the interval $[T-\varepsilon_0, T]$ for a sufficiently small $\varepsilon_0 > 0$. From Proposition 2.4, we have

$$\begin{aligned} & \mathbb{E}\int_t^T ||\widehat{P}_s||_H^2 ds + \mathbb{E}\int_t^T ||\widehat{Q}_s||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_t^T (T-s)^{2\alpha} || \sum_{i=1}^{\infty} C_i(s)\widehat{Q}_s e_i||_H^2 ds \\ \leq\ & C\left(\mathbb{E}\int_t^T (T-s)^{2\beta} ||\widehat{f}(s)||_H^2 ds\right), \end{aligned}$$

and further in view of the Lipschitz continuity of f,

$$\begin{aligned} & \mathbb{E}\int_t^T ||\widehat{P}_s||_H^2 ds + \mathbb{E}\int_t^T ||\widehat{Q}_s||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_t^T (T-s)^{2\alpha} || \sum_{i=1}^{\infty} C_i(s)\widehat{Q}_s e_i||_H^2 ds \\ \leq\ & C\mathbb{E}\int_t^T (T-s)^{2(\beta-\alpha)} \left(\|\widehat{P}_s\|_H^2 + ||\widehat{Q}_s||_{\mathcal{S}_2(H)}^2 + ||(T-s)^{2\alpha} \sum_{i=1}^{\infty} C_i(s)\widehat{Q}_t e_i||_H^2 \right) ds \\ \leq\ & C\varepsilon^{2(\beta-\alpha)}\mathbb{E}\int_t^T \left(\|\widehat{P}_s\|_H^2 + ||\widehat{Q}_s||_{\mathcal{S}_2(H)}^2 + ||(T-s)^{2\alpha} \sum_{i=1}^{\infty} C_i(s)\widehat{Q}_s e_i||_H^2 \right) ds. \end{aligned}$$

Thus we have the desired uniqueness on the interval $[T-\varepsilon_0, T]$ for a sufficiently small $\varepsilon_0 > 0$. Iteratively in a backward way, we can show the uniqueness on the whole interval $[0,T]$.

We use the Picard iteration to construct a sequence of solutions to linear BSEEs, and show that its limit is a solution to the nonlinear BSEE (1.1). Noting that $f(\cdot,0,0,0)$ verifies the integrability (3.1), in view of Proposition 2.5, the following BSEE

$$\begin{cases} -dP_t^1 &= [A^*P_t^1 + \sum_{i=1}^{\infty} C_i^*(t)Q_t^1 e_i]\,dt + f(t,0,0,0)\,dt - \sum_{i=1}^{\infty} Q_t^1 e_i\, d\beta_t^i, \quad t \in [0,T]; \\ P_T^1 &= \eta \end{cases} \tag{3.4}$$

has a unique mild solution (P^1, Q^1), and BSEE

$$\begin{cases} -dP_t^{k+1} &= [A^*P_t^{k+1} + \sum_{i=1}^{\infty} C_i^*(t)Q_t^{k+1} e_i]\,dt + f(t, P_t^k, Q_t^k, \sum_{i=1}^{\infty} C_i^*(t)Q_t^k e_i)\ dt \\ & \quad - \sum_{i=1}^{\infty} Q_t^{k+1} e_i\, d\beta_t^i, \quad t \in [0,T]; \\ P_T^{k+1} &= \eta, \end{cases} \tag{3.5}$$

has a unique mild solution (P^{k+1}, Q^{k+1}) with $k = 1, 2, \ldots$, such that for $k = 0, 1, 2, \ldots$,

$$\mathbb{E}\int_0^T ||P_s^{k+1}||_H^2 ds + \mathbb{E}\int_0^T ||Q_s^{k+1}||_{\mathcal{S}_2(H)}^2 ds + \mathbb{E}\int_0^T (T-s)^{2\alpha} ||\sum_{i=1}^{\infty} C_i(s) Q_s^{k+1} e_i||_H^2 ds < \infty.$$

From Proposition 2.4 and the Lipschitz continuity of f, we can show the following for $t \in [T-\varepsilon, T]$

$$\begin{aligned}
&\mathbb{E}\int_t^T ||P_s^{k+1} - P_s^k||_H^2 ds + \mathbb{E}\int_t^T ||Q_s^{k+1} - Q_s^k||_{\mathcal{S}_2(H)}^2 ds \\
&\quad + \mathbb{E}\int_t^T (T-s)^{2\alpha} ||\sum_{i=1}^{\infty} C_i(s)[Q_s^{k+1} - Q_s^k] e_i||_H^2 ds \\
\leq\ & C\varepsilon^{2(\beta-\alpha)} \left(\mathbb{E}\int_t^T ||P_s^{k-1} - P_s^k||_H^2 ds + \mathbb{E}\int_t^T ||Q_s^{k-1} - Q_s^k||_{\mathcal{S}_2(H)}^2 ds \right) \\
&\quad + C\varepsilon^{2(\beta-\alpha)} \mathbb{E}\int_t^T (T-s)^{2\alpha} ||\sum_{i=1}^{\infty} C_i(s)[Q_s^{k-1} - Q_s^k] e_i||_H^2 \, ds.
\end{aligned}$$

Choose $\varepsilon_1 > 0$ such that $C\varepsilon_1^{2(\beta-\alpha)} = \frac{1}{2}$. Then the sequence

$$(P_s^k, Q_s^k, (T-s)^{2\alpha} ||\sum_{i=1}^{\infty} C_i(s) Q_s^k e_i||_H^2), \quad s \in [T-\varepsilon_1, T]$$

converges strongly in $L^2_{\mathcal{P}}(\Omega \times [T-\varepsilon_1, T], H \times \mathcal{S}_2(H) \times H)$ to a triplet

$$\{(P_t, Q_t, (T-t)^{\alpha} S(t)), t \in [T-\varepsilon_1, T]\}.$$

Moreover, we have

$$\begin{cases} -dP_t &= [A^* P_t + S(t) + f(t, P_t, Q_t, S(t))]\, dt - \sum_{i=1}^{\infty} Q_t e_i \, d\beta_t^i, \quad t \in [0, T]; \\ P_T &= \eta. \end{cases} \tag{3.6}$$

It remains to prove the following weak convergence:

$$\{(T-t)^{\alpha} \sum_{i=1}^{N} C_i^*(t) Q_t e_i, t \in [T-\varepsilon_1, T]\} \overset{N\to\infty}{\longrightarrow} \{(T-t)^{\alpha} S(t), t \in [T-\varepsilon_1, T]\} \tag{3.7}$$

weakly in the Hilbert space $L^2_{\mathcal{P}}(\Omega \times [T-\varepsilon_1, T], H)$.

For $\gamma \in L^{\infty}_{\mathcal{P}}(\Omega \times [T-\varepsilon_1, T], H)$, let $\mathcal{X}^{\gamma}$ be the unique mild solution of the following stochastic equation:

$$\begin{cases} d\mathcal{X}_t &= A\mathcal{X}_t\, dt + \sum_{i=1}^{\infty} C_i(t)\mathcal{X}_t\, d\beta_t^i + \sum_{i=1}^{\infty} C_i(t)(T-t)^{\alpha}\gamma_t\, d\beta_t^i, \quad t \in (T-\varepsilon_1, T]; \\ \mathcal{X}_{T-\varepsilon_1} &= 0. \end{cases} \tag{3.8}$$

Using (3.5), we have the following duality:

$$\begin{aligned} & E\int_{T-\varepsilon_1}^{T} \langle (T-t)^{\alpha} \sum_{i=1}^{\infty} C_i^*(t) Q_t^{k+1} e_i, \gamma_t\rangle\, dt \\ = \;& E\langle \eta, \mathcal{X}_T^{\gamma}\rangle + E\int_{T-\varepsilon_1}^{T} \langle f(t, P_t^k, Q_t^k, \sum_{i=1}^{\infty} C_i^*(t) Q_t^k e_i), \mathcal{X}_t^{\gamma}\rangle\, dt. \end{aligned} \tag{3.9}$$

By setting $k \to \infty$, we have

$$E\int_{T-\varepsilon_1}^{T} \langle (T-t)^{\alpha} S(t), \gamma_t\rangle\, dt \;=\; E\langle \eta, \mathcal{X}_T^{\gamma}\rangle + E\int_{T-\varepsilon_1}^{T} \langle f(t, P_t, Q_t, S(t)), \mathcal{X}_t^{\gamma}\rangle\, dt. \tag{3.10}$$

On the other hand, in view of BSEE (3.5), we have the duality:

$$\begin{aligned} & E\int_{T-\varepsilon_1}^{T} \left\langle (T-t)^{\alpha} \sum_{i=1}^{N} C_i^*(t) Q^{k+1} e_i, \gamma_t \right\rangle dt = E\int_{T-\varepsilon_1}^{T} \sum_{i=1}^{N} \langle Q^{k+1} e_i, (T-t)^{\alpha} C_i(t)\gamma_t\rangle\, dt \\ = \;& E\left\langle \eta, \mathcal{X}_T^{\gamma,N}\right\rangle + E\int_{T-\varepsilon_1}^{T} \left\langle f(t, P_t^k, Q_t^k, \sum_{i=1}^{\infty} C_i^*(t) Q^k e_i), \mathcal{X}_t^{\gamma,N}\right\rangle dt. \end{aligned} \tag{3.11}$$

Here, $\mathcal{X}^{\gamma,N}$ is the unique mild solution of the following stochastic equation:

$$\begin{cases} d\mathcal{X}_t &= A\mathcal{X}_t\, dt + \sum_{i=1}^{\infty} C_i(t)\mathcal{X}_t\, d\beta_t^i + \sum_{i=1}^{N} C_i(t)(T-t)^{\alpha}\gamma_t\, d\beta_t^i, \quad t \in (T-\varepsilon_1, T]; \\ \mathcal{X}_{T-\varepsilon_1} &= 0. \end{cases} \tag{3.12}$$

Setting $k \to \infty$, we have

$$E\int_{T-\varepsilon_1}^{T} \left\langle (T-t)^{\alpha} \sum_{i=1}^{N} C_i^*(t) Q e_i, \gamma_t \right\rangle dt \;=\; E\left\langle \eta, \mathcal{X}_T^{\gamma,N}\right\rangle + E\int_{T-\varepsilon_1}^{T} \left\langle f(t, P_t, Q_t, S(t)), \mathcal{X}_t^{\gamma,N}\right\rangle dt.$$

Note that we only have the following weaker integrability on $f(\cdot, P, Q, S)$:

$$\mathbb{E}\int_{T-\varepsilon_1}^{T} (T-t)^{2\alpha} \|f(t, P_t, Q_t, S(t))\|_H^2\, dt < \infty.$$

Subsequently, in view of [6, Theorem 4.3], we have

$$\begin{aligned}
&\lim_{N\to\infty} E\int_{T-\varepsilon_1}^{T} \langle (T-t)^{\alpha} \sum_{i=1}^{N} C_i^*(t)Qe_i, \gamma_t\rangle \, dt \\
= \;& E\langle \eta, \mathcal{X}_T^{\gamma}\rangle + E\int_{T-\varepsilon_1}^{T} \langle f(t, P_t, Q_t, S(t)), \mathcal{X}_t^{\gamma}\rangle \, dt. \qquad (3.13)
\end{aligned}$$

Proceeding identically as in the proof of the previous equality (2.16), we have the weak convergence (3.7). In this way, we get the existence of BSEE (1.1) on the interval $[T-\varepsilon_1, T]$. In a backward way, we can show its existence iteratively on the intervals $[T-2\varepsilon_1, T-\varepsilon_1], \ldots, [0, T-n_0\varepsilon_1]$ for the greatest integer n_0 such that $l\varepsilon_1 < T$, and thus the existence on the whole interval $[0,T]$. □

4 Example

Set $H := L^2(0,1)$ and consider an H-valued cylindrical Wiener process $\{W_t, t \geq 0\}$. A is the realization of the second derivative operator in H with Dirichlet boundary conditions. So $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$ and $A\phi = \phi''$ for all $\phi \in \mathcal{D}(A)$. Choose an orthonormal basis in $L^2(0,1)$ with $\sup_i \sup_{x\in(0,1)} |e_i(x)| < \infty$, for instance a trigonometrical basis. Let $\sigma \in L_{\mathcal{P}}^{\infty}(\Omega \times (0,T), L^{\infty}(0,1))$. Define $C_i(t) : H \to H$ by

$$(C_i(t)\phi)(x) := \sigma(t,x)e_i(x)\phi(x), \quad (t,x) \in [0,T] \times [0,1]$$

for $\phi \in H$. We have $A^* = A$ and $C_i^* = C_i$ with $i = 1, 2, \ldots$. From Da Prato and Zabczyk [3], we see that (A, C) satisfies Hypothesis 1.1.

Then for suitable conditions on (η, f), our Theorem 3.1 can be applied to give the existence and uniqueness of a mild solution to the following backward stochastic partial differential equation driven by a space-time white noise:

$$\left\{\begin{array}{lcl}
-dP_t(x) &=& \left[\dfrac{d^2}{dx^2}P_t(x) + \sigma(t,x)Q_t(x)\right] dt + f\left(t, x, P_t, Q_t, \displaystyle\sum_{i=1}^{\infty} \sigma(t)Q_t\right) dt \\
& & -Q_t(x)\, dW_t(x), \quad (t,x) \in [0,T) \times (0,1); \\
P_t(0) &=& P_t(1) = 0, \quad t \in [0,T]; \\
P_T(x) &=& \eta(x), \quad x \in [0,1].
\end{array}\right. \qquad (4.1)$$

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