



HAL
open science

Sweeping processes with prescribed behaviour on jumps

Vincenzo Recupero, Filippo Santambrogio

► **To cite this version:**

Vincenzo Recupero, Filippo Santambrogio. Sweeping processes with prescribed behaviour on jumps. 2017. hal-01570514

HAL Id: hal-01570514

<https://hal.science/hal-01570514>

Preprint submitted on 30 Jul 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SWEEPING PROCESSES WITH PRESCRIBED BEHAVIOUR ON JUMPS

VINCENZO RECUPERO AND FILIPPO SANTAMBROGIO

ABSTRACT. We present a generalized formulation of sweeping process where the behaviour of the solution is prescribed at the jump points of the driving moving set. An existence and uniqueness theorem for such formulation is proved. As a consequence we derive a formulation and an existence/uniqueness theorem for sweeping processes driven by an arbitrary BV moving set, whose evolution is not necessarily right continuous. Applications to the play operator of elastoplasticity are also shown.

1. INTRODUCTION

Sweeping processes are a class of evolution differential inclusions introduced by J.J. Moreau in a series of articles [54, 55, 56, 57] which culminated in the celebrated paper [60], originating a research that is still active. The original formulation introduced in [54, 57] reads as follows. Let \mathcal{H} be a real Hilbert space, $a, b \in \mathbb{R}$, $a < b$, and, for every time $t \in [a, b]$, let $\mathcal{C}(t)$ be a given nonempty, closed and convex subset of \mathcal{H} such that the mapping $t \mapsto \mathcal{C}(t)$ is Lipschitz continuous when the family of closed subsets of \mathcal{H} is endowed with the Hausdorff metric. One has to find a Lipschitz continuous function $y : [a, b] \rightarrow \mathcal{H}$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \quad (1.1)$$

$$-y'(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b], \quad (1.2)$$

$$y(0) = y_0, \quad (1.3)$$

y_0 being a prescribed point in $\mathcal{C}(a)$. Here \mathcal{L}^1 is the Lebesgue measure, and $N_{\mathcal{C}(t)}$ is the exterior normal cone to $\mathcal{C}(t)$ at $y(t)$ (all the precise definitions will be given in Section 2). When the interior of $\mathcal{C}(t)$ is nonempty, the process defined by (1.1)-(1.3) has a nice and useful geometrical-mechanical interpretation which we recall from [60]: “the moving point $u(t)$ remains at rest as long as it happens to lie in the interior of $\mathcal{C}(t)$; when caught up with the boundary of the moving set, it can only proceed in an inward normal direction, as if pushed by this boundary, so as to go on belonging to $\mathcal{C}(t)$ ”. Moreau was originally motivated by plasticity and friction dynamics (cf. [57, 59, 61]), but now sweeping processes have found applications to nonsmooth mechanics (see, e.g., [53, 43, 62]), to economics (cf., e.g., [33, 25, 31]), to electrical circuits (see, e.g., [2, 10, 1, 3]), to crowd motion modeling (cf., e.g., [48, 45, 49, 47, 46, 26]), and to other fields (see, e.g., the references in the recent paper [72]).

In [60] the formulation (1.1)-(1.3) is extended to the case when the mapping $t \mapsto \mathcal{C}(t)$ is of bounded variation and right continuous in the following natural way: it is proved that there is a unique $y \in BV^r([a, b]; \mathcal{H})$, the space of right continuous \mathcal{H} -valued functions of bounded variation, such that there exist a positive measure μ and a μ -integrable function $v : [a, b] \rightarrow \mathcal{H}$

2010 *Mathematics Subject Classification.* 34A60, 49J52, 34G25, 47J20.

Key words and phrases. Sweeping processes, Evolution variational inequalities, Play operator, Convex sets, Functions of bounded variation.

Supported by the INdAM - GNAMPA Project 2016 “Sweeping processes: teoria, controllo e applicazioni a problemi rate independent”.

satisfying

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \quad (1.4)$$

$$Dy = v\mu, \quad (1.5)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b], \quad (1.6)$$

$$y(a) = y_0, \quad (1.7)$$

where Dy denotes distributional derivative of y , which is a measure since $y \in BV$. A relevant particular case is provided by the case when $\mathcal{C}(t) = u(t) - \mathcal{Z}$, with $u \in BV^r([0, T]; \mathcal{H})$, and $\mathcal{Z} \subseteq \mathcal{H}$ closed, convex and nonempty, namely the sweeping process driven by a set with constant shape. The resulting solution operator $\mathbf{P} : BV^r([a, b]; \mathcal{H}) \rightarrow BV^r([a, b]; \mathcal{H})$ associating with u the unique function y satisfying (1.4)-(1.7) with $\mathcal{C}(t) = u(t) - \mathcal{Z}$ is called *vector play operator* (see, e.g., [37, 38, 39, 63, 65, 67]) and has an important role in elasto-plasticity and hysteresis (cf., e.g., [36, 76, 12, 38, 50, 51]).

The theoretical analysis of problem (1.4)-(1.7) has been expanded in various directions: the case of \mathcal{C} continuous was first dealt in [53], the nonconvex case has been studied in several papers, e.g. [73, 74, 75, 15, 17, 20, 5, 23, 70, 8, 29, 71, 32, 6, 69]; for stochastic versions see, e.g., [13, 14, 7, 18], while periodic solutions can be found in [16]. The continuous dependence properties of various sweeping problems are investigated, e.g., in [60, 11, 64, 41, 65, 66, 34, 35], and the control problems are studied, e.g., in [21, 22, 24].

When the moving set \mathcal{C} jumps, the geometrical interpretation of the sweeping process (1.4)-(1.7) has to be revisited by analyzing the behaviour of the solution y at jump points t of \mathcal{C} : at such points it can be showed (cf. [60]) that $y(t) = y(t+) = \text{Proj}_{\mathcal{C}(t)}(u(t-))$, where Proj is the classical projection operator, thus y instantaneously moves from $\mathcal{C}(t-)$ to $\mathcal{C}(t) = \mathcal{C}(t+)$ along the shortest path which allows to satisfy the constraint (1.4). Although this is a very natural requirement of formulation (1.4)-(1.7), this is not the only one: let us see, for instance, two cases where a different behaviour can be prescribed at the jump points of \mathcal{C} .

First let us consider the problem of extending the formulation (1.4)-(1.7) to the case of arbitrary moving sets \mathcal{C} of bounded variation, not necessarily right continuous. Of course one could replace $\mathcal{C}(t)$ by $\mathcal{C}(t+)$ in the differential inclusions (1.6), but this would not take into account of the position of $\mathcal{C}(t)$ at a jump point t : it would be very natural, instead, to expect that for such t one should find $y(t+) = \text{Proj}_{\mathcal{C}(t+)}(\text{Proj}_{\mathcal{C}(t)}(y(t-)))$.

Another situation is given by the play operator \mathbf{P} : when a jump point t of the ‘‘input function’’ $u \in BV^r([a, b]; \mathcal{H})$ occurs, it would be very natural to expect that $y(t+)$ would be the final point of the play operator driven by the segment $(1 - \sigma)u(t-) + \sigma u(+)$, $\sigma \in [0, 1]$, in other words we would expect that y behaves as if u traversed the segment joining $u(t-)$ and $u(t+)$ with ‘‘infinite velocity’’ at every jump points t . Nevertheless it can be proved (cf. [65, Section 5.3] and [40]) that in general the solution $y = \mathbf{P}(u)$ provided by (1.4)-(1.7) does not satisfy this property, which would be also natural if we recall that the play operator is *rate independent*, i.e. $\mathbf{P}(u \circ \psi) = \mathbf{P}(u) \circ \psi$ for every increasing surjective reparametrization of time $\psi : [a, b] \rightarrow [a, b]$.

The aim of this paper is to provide a general formulation which take into account of all these possible different behaviors at jumps point of the driving moving set \mathcal{C} . To be more precise if $t \mapsto \mathcal{C}(t)$ is right continuous of bounded variation, we prove that there exists a unique function $y \in BV^r([a, b]; \mathcal{H})$ such that $y(t) \in \mathcal{C}(t) = \mathcal{C}(t+)$ for every $t \in [a, b]$ and

$$Dy = v\mu, \quad (1.8)$$

$$v(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [a, b], \quad (1.9)$$

$$y(t+) = g_t(y(t-)) \quad \text{for every discontinuity point } t \text{ of } \mathcal{C}, \quad (1.10)$$

$$y(0) = y_0, \quad (1.11)$$

where $g_t : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ is a family of functions prescribing the behaviour of y at every jump point t of \mathcal{C} (a sort of family of “initial conditions at the jump points of the datum”, but we actually consider a more general situation). The case of the arbitrary moving set \mathcal{C} , not necessarily continuous, can be immediately deduced by taking $g_t = \text{Proj}_{\mathcal{C}(t+)} \circ \text{Proj}_{\mathcal{C}(t)}$, the “double projection”.

The paper is organized as follows. In the next section we present some technical preliminaries and in Section 3 we state our main result. This result will be proved in Section 4 and finally in the last Section 5 we present some applications and consequences of our main results.

2. PRELIMINARIES

In this section we recall the main definitions and tools needed in the paper. The set of integers greater than or equal to 1 will be denoted by \mathbb{N} . Given an interval I of the real line \mathbb{R} , if $\mathcal{B}(I)$ indicates the family of Borel sets in I , $\mu : \mathcal{B}(I) \rightarrow [0, \infty]$ is a measure, $p \in [1, \infty]$, and E is a Banach space, then the space of E -valued functions which are p -integrable with respect to μ will be denoted by $L^p(I, \mu; E)$ or simply by $L^p(\mu; E)$. We do not identify two functions which are equal μ -almost everywhere. The one dimensional Lebesgue measure is denoted by \mathcal{L}^1 and the Dirac delta at a point $t \in \mathbb{R}$ is denoted by δ_t . For the theory of integration of vector valued functions we refer, e.g., to [44, Chapter VI].

2.1. Functions with values in a metric space. In this subsection we assume that

$$(X, d) \text{ is a complete metric space,} \tag{2.1}$$

where we admit that d is an extended metric, i.e. X is a set and $d : X \times X \rightarrow [0, \infty]$ satisfies the usual axioms of a distance, but may take on the value ∞ . The notion of completeness remains unchanged. The general topological notions of interior, closure and boundary of a subset $A \subseteq X$ will be respectively denoted by $\text{int}(A)$, $\text{cl}(A)$ and ∂A . We also set $d(x, A) := \inf_{a \in A} d(x, a)$. If (Y, d_Y) is a metric space then the continuity set of a function $f : Y \rightarrow X$ is denoted by $\text{Cont}(f)$, while $\text{Discont}(f) := Y \setminus \text{Cont}(f)$. The set of continuous X -valued functions defined on Y is denoted by $C(Y; X)$. For $S \subseteq Y$ we write $\text{Lip}(f, S) := \sup\{d(f(s), f(t))/d_Y(t, s) : s, t \in S, s \neq t\}$, $\text{Lip}(f) := \text{Lip}(f, Y)$, the Lipschitz constant of f , and $\text{Lip}(Y; X) := \{f : Y \rightarrow X : \text{Lip}(f) < \infty\}$, the set of X -valued Lipschitz continuous functions on Y . If E is a Banach space with norm $\|\cdot\|_E$ and $S \subseteq Y$, we set $\|f\|_{\infty, S} := \sup_{t \in S} \|f(t)\|_E$ for every function $f : Y \rightarrow E$. We recall now the notion of BV function with values in a metric space (see, e.g., [4, 77]).

Definition 2.1. *Given an interval $I \subseteq \mathbb{R}$, a function $f : I \rightarrow X$, and a subinterval $J \subseteq I$, the (pointwise) variation of f on J is defined by*

$$V(f, J) := \sup \left\{ \sum_{j=1}^m d(f(t_{j-1}), f(t_j)) : m \in \mathbb{N}, t_j \in J \forall j, t_0 < \dots < t_m \right\}.$$

If $V(f, I) < \infty$ we say that f is of bounded variation on I and we set $BV(I; X) := \{f : I \rightarrow X : V(f, I) < \infty\}$.

It is well known that the completeness of X implies that every $f \in BV(I; X)$ admits one-sided limits $f(t-), f(t+)$ at every point $t \in I$, with the convention that $f(\inf I-) := f(\inf I)$ if $\inf I \in I$, and $f(\sup I+) := f(\sup I)$ if $\sup I \in I$, and that $\text{Discont}(f)$ is at most countable. We set $BV^r(I; X) := \{f \in BV(I; X) : f(t) = f(t+) \forall t \in I\}$ and if I is bounded we have $\text{Lip}(I; X) \subseteq BV(I; X)$.

2.2. Convex sets in Hilbert spaces. Throughout the remainder of the paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x, y) \mapsto \langle x, y \rangle \\ \|x\| := \langle x, x \rangle^{1/2} \end{cases}, \quad (2.2)$$

and we endow \mathcal{H} with the natural metric defined by $d(x, y) := \|x - y\|$, $x, y \in \mathcal{H}$. We set

$$\mathcal{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} : \mathcal{K} \text{ nonempty, closed and convex}\}.$$

If $\mathcal{K} \in \mathcal{C}_{\mathcal{H}}$ and $x \in \mathcal{H}$, then $\text{Proj}_{\mathcal{K}}(x)$ is the projection on \mathcal{K} , i.e. $y = \text{Proj}_{\mathcal{K}}(x)$ is the unique point such that $d(x, \mathcal{K}) = \|x - y\|$, and it is also characterized by the two conditions

$$y \in \mathcal{K}, \quad \langle x - y, v - y \rangle \leq 0 \quad \forall v \in \mathcal{K}.$$

If $\mathcal{K} \in \mathcal{C}_{\mathcal{H}}$ and $x \in \mathcal{K}$, then $N_{\mathcal{K}}(x)$ denotes the (*exterior*) *normal cone of \mathcal{K} at x* :

$$N_{\mathcal{K}}(x) := \{u \in \mathcal{H} : \langle u, v - x \rangle \leq 0 \ \forall v \in \mathcal{K}\} = \text{Proj}_{\mathcal{K}^c}^1(x) - x. \quad (2.3)$$

It is well known that the multivalued mapping $x \mapsto N_{\mathcal{K}}(x)$ is *monotone*, i.e. $\langle u_1 - u_2, x_1 - x_2 \rangle \geq 0$ whenever $x_j \in \mathcal{K}$, $u_j \in N_{\mathcal{K}}(x_j)$, $j = 1, 2$ (see, e.g., [9, Exemple 2.8.2, p.46]). We endow the set $\mathcal{C}_{\mathcal{H}}$ with the Hausdorff distance. Here we recall the definition.

Definition 2.2. The Hausdorff distance $d_{\mathcal{H}} : \mathcal{C}_{\mathcal{H}} \times \mathcal{C}_{\mathcal{H}} \rightarrow [0, \infty]$ is defined by

$$d_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) := \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\}, \quad \mathcal{A}, \mathcal{B} \in \mathcal{C}_{\mathcal{H}}.$$

The metric space $(\mathcal{C}_{\mathcal{H}}, d_{\mathcal{H}})$ is complete (cf. [19, Theorem II-14, Section II.3.14, p. 47]).

2.3. Differential measures. We recall that a \mathcal{H} -valued measure on I is a map $\mu : \mathcal{B}(I) \rightarrow \mathcal{H}$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint sets in $\mathcal{B}(I)$. The *total variation of μ* is the positive measure $|\mu| : \mathcal{B}(I) \rightarrow [0, \infty]$ defined by

$$|\mu|(B) := \sup \left\{ \sum_{n=1}^{\infty} \|\mu(B_n)\| : B = \bigcup_{n=1}^{\infty} B_n, B_n \in \mathcal{B}(I), B_h \cap B_k = \emptyset \text{ if } h \neq k \right\}.$$

The vector measure μ is said to be *with bounded variation* if $|\mu|(I) < \infty$. In this case the equality $\|\mu\| := |\mu|(I)$ defines a norm on the space of measures with bounded variation (see, e.g. [28, Chapter I, Section 3]).

If $\nu : \mathcal{B}(I) \rightarrow [0, \infty]$ is a positive bounded Borel measure and if $g \in L^1(I, \nu; \mathcal{H})$, then $g\nu$ will denote the vector measure defined by $g\nu(B) := \int_B g d\nu$ for every $B \in \mathcal{B}(I)$. In this case $|g\nu|(B) = \int_B \|g(t)\| d\nu$ for every $B \in \mathcal{B}(I)$ (see [28, Proposition 10, p. 174]). Moreover a vector measure μ is called ν -*absolutely continuous* if $\mu(B) = 0$ whenever $B \in \mathcal{B}(I)$ and $\nu(B) = 0$.

Assume that $\mu : \mathcal{B}(I) \rightarrow \mathcal{H}$ is a vector measure with bounded variation and let $f : I \rightarrow \mathcal{H}$ and $\phi : I \rightarrow \mathbb{R}$ be two *step maps with respect to μ* , i.e. there exist $f_1, \dots, f_m \in \mathcal{H}$, $\phi_1, \dots, \phi_m \in \mathcal{H}$ and $A_1, \dots, A_m \in \mathcal{B}(I)$ mutually disjoint such that $|\mu|(A_j) < \infty$ for every j and $f = \sum_{j=1}^m \mathbb{1}_{A_j} f_j$, $\phi = \sum_{j=1}^m \mathbb{1}_{A_j} \phi_j$, where $\mathbb{1}_S$ is the characteristic function of a set S , i.e. $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such step functions we define $\int_I \langle f, \mu \rangle := \sum_{j=1}^m \langle f_j, \mu(A_j) \rangle \in \mathbb{R}$ and $\int_I \phi d\mu := \sum_{j=1}^m \phi_j \mu(A_j) \in \mathcal{H}$. If $St(|\mu|; \mathcal{H})$ (resp. $St(|\mu|)$) is the set of \mathcal{H} -valued (resp. real valued) step maps with respect to μ , then the maps $St(|\mu|; \mathcal{H}) \rightarrow \mathcal{H} : f \mapsto \int_I \langle f, \mu \rangle$ and $St(|\mu|) \rightarrow \mathcal{H} : \phi \mapsto \int_I \phi d\mu$ are linear and continuous when $St(|\mu|; \mathcal{H})$ and $St(|\mu|)$ are endowed with the L^1 -seminorms $\|f\|_{L^1(|\mu|; \mathcal{H})} := \int_I \|f\| d|\mu|$ and $\|\phi\|_{L^1(|\mu|)} := \int_I |\phi| d|\mu|$. Therefore they admit unique continuous extensions $l_{\mu} : L^1(|\mu|; \mathcal{H}) \rightarrow \mathbb{R}$ and $J_{\mu} : L^1(|\mu|) \rightarrow \mathcal{H}$, and we set

$$\int_I \langle f, d\mu \rangle := l_{\mu}(f), \quad \int_I \phi d\mu := J_{\mu}(\phi), \quad f \in L^1(|\mu|; \mathcal{H}), \quad \phi \in L^1(|\mu|).$$

If ν is bounded positive measure and $g \in L^1(\nu; \mathcal{H})$, arguing first on step functions, and then taking limits, it is easy to check that $\int_I \langle f, d(g\nu) \rangle = \int_I \langle f, g \rangle d\nu$ for every $f \in L^\infty(\mu; \mathcal{H})$. The following results (cf., e.g., [28, Section III.17.2-3, pp. 358-362]) provide a connection between functions with bounded variation and vector measures which will be implicitly used in the paper.

Theorem 2.1. *For every $f \in BV(I; \mathcal{H})$ there exists a unique vector measure of bounded variation $\mu_f : \mathcal{B}(I) \rightarrow \mathcal{H}$ such that*

$$\begin{aligned} \mu_f(]c, d[) &= f(d-) - f(c+), & \mu_f([c, d]) &= f(d+) - f(c-), \\ \mu_f([c, d[) &= f(d-) - f(c-), & \mu_f(]c, d]) &= f(d+) - f(c+). \end{aligned}$$

whenever $c < d$ and the left hand side of each equality makes sense. Conversely, if $\mu : \mathcal{B}(I) \rightarrow \mathcal{H}$ is a vector measure with bounded variation, and if $f_\mu : I \rightarrow \mathcal{H}$ is defined by $f_\mu(t) := \mu(] \inf I, t[\cap I)$, then $f_\mu \in BV(I; \mathcal{H})$ and $\mu_{f_\mu} = \mu$.

Proposition 2.1. *Let $f \in BV(I; \mathcal{H})$, let $g : I \rightarrow \mathcal{H}$ be defined by $g(t) := f(t-)$, for $t \in \text{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \rightarrow \mathbb{R}$ be defined by $V_g(t) := V(g,] \inf I, t[\cap I)$. Then $\mu_g = \mu_f$ and $|\mu_f| = \mu_{V_g} = V(g, I)$.*

The measure μ_f is called *Lebesgue-Stieltjes measure* or *differential measure* of f . Let us see the connection with the distributional derivative. If $f \in BV(I; \mathcal{H})$ and if $\bar{f} : \mathbb{R} \rightarrow \mathcal{H}$ is defined by

$$\bar{f}(t) := \begin{cases} f(t) & \text{if } t \in I \\ f(\inf I) & \text{if } \inf I \in \mathbb{R}, t \notin I, t \leq \inf I \\ f(\sup I) & \text{if } \sup I \in \mathbb{R}, t \notin I, t \geq \sup I \end{cases}, \quad (2.4)$$

then, as in the scalar case, it turns out (cf. [65, Section 2]) that $\mu_f(B) = D\bar{f}(B)$ for every $B \in \mathcal{B}(\mathbb{R})$, where $D\bar{f}$ is the distributional derivative of \bar{f} , i.e.

$$- \int_{\mathbb{R}} \varphi'(t) \bar{f}(t) dt = \int_{\mathbb{R}} \varphi dD\bar{f} \quad \forall \varphi \in C_c^1(\mathbb{R}; \mathbb{R}),$$

$C_c^1(\mathbb{R}; \mathbb{R})$ being the space of real continuously differentiable functions on \mathbb{R} with compact support. Observe that $D\bar{f}$ is concentrated on I : $D\bar{f}(B) = \mu_f(B \cap I)$ for every $B \in \mathcal{B}(I)$, hence in the remainder of the paper, if $f \in BV(I; \mathcal{H})$ then we will simply write

$$Df := D\bar{f} = \mu_f, \quad f \in BV(I; \mathcal{H}), \quad (2.5)$$

and from the previous discussion it follows that

$$\|Df\| = |Df|(I) = \|\mu_f\| = V(f, I) \quad \forall f \in BV^r(I; \mathcal{H}). \quad (2.6)$$

3. MAIN RESULT

We are now in position to state the main theorem of the paper.

Theorem 3.1. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b]; \mathcal{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq]a, b]$, and that for every $t \in S$ we are given a function $g_t : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ such that $\text{Lip}(g_t) \leq 1$ and*

$$\sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} < \infty. \quad (3.1)$$

Then there exists a unique $y \in BV^r([a, b]; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \quad (3.2)$$

$$Dy = v\mu, \quad (3.3)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b] \setminus S, \quad (3.4)$$

$$y(t) = g_t(y(t-)) \quad \forall t \in S, \quad (3.5)$$

$$y(a) = y_0. \quad (3.6)$$

Moreover if $a \leq s < t \leq b$ we have

$$V(y, [s, t]) \leq V(\mathcal{C}, [s, t]) + \sum_{r \in S \cap [s, t]} (\|g_r - Id\|_{\infty, \mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-), \mathcal{C}(r))). \quad (3.7)$$

Finally if $y_{0,j} \in \mathcal{C}(a)$, $j = 1, 2$ and y_j is the only function such that there is a measure $\mu_j : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v_j \in L^1(\mu_j; \mathcal{H})$ for which (3.2)-(3.6) hold with y, v, μ, y_0 replaced respectively by $y_j, v_j, \mu_j, y_{0,j}$, then

$$t \mapsto \|y_1(t) - y_2(t)\|^2 \text{ is nonincreasing.}$$

In Section 5 we will show a series of results that can be deduced from Theorem 3.1.

4. PROOFS

We start by recalling the existence and uniqueness result of classical sweeping processes due to J.J. Moreau (cf. [60]).

Theorem 4.1. *If $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b[; \mathcal{C}_{\mathcal{H}})$, and $y_0 \in \mathcal{C}(a)$, then there exists a unique $y \in BV^r([a, b[; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which*

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[, \quad (4.1)$$

$$Dy = v\mu, \quad (4.2)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in]a, b[, \quad (4.3)$$

$$y(a) = y_0. \quad (4.4)$$

Moreover

$$V(y, [s, t]) \leq V(\mathcal{C}, [s, t]) \quad \forall s, t \in [a, b[, s < t. \quad (4.5)$$

Finally if $y_{0,j} \in \mathcal{C}(a)$, $j = 1, 2$ and y_j is the only function such that there is a measure $\mu_j : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v_j \in L^1(\mu_j; \mathcal{H})$ for which (4.1)-(4.4) hold with y, v, μ, y_0 replaced respectively by $y_j, v_j, \mu_j, y_{0,j}$, then

$$t \mapsto \|y_1(t) - y_2(t)\|^2 \text{ is nonincreasing.} \quad (4.6)$$

Concerning Theorem 4.1 let us observe that the existence and uniqueness of the solution y is proved in [60, Proposition 3b], while formula (4.5) is proved in [60, Proposition 2c]. Finally the last statement is proved in [60, Proposition 2b].

It is important to compare Theorem 4.1 to our main result, Theorem 3.1: it is possible to see that Theorem 3.1 includes the statement of Theorem 4.1 when $g_t : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ is given by the projection onto $\mathcal{C}(t)$, since in this case $\|g_t - Id\|_{\infty, \mathcal{C}(t-)} \leq d_{\mathcal{H}}(\mathcal{C}(t-), \mathcal{C}(t))$. The goal of our main theorem is to allow for different prescribed behaviors at points $t \in S$, and the proof will be based on a suitable combination of Theorem 4.1 with some explicit applications of the maps g_t .

In the following Lemma we provide an integral formulation of the sweeping process.

Lemma 4.1. *Assume that $-\infty < a < b < \infty$, $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ is a measure, $B \in \mathcal{B}([a, b[)$, and $\mu(B) \neq 0$. If $\mathcal{C} \in BV^r([a, b[; \mathcal{C}_{\mathcal{H}})$, $v \in L^1(\mu; \mathcal{H})$, $y \in BV^r([a, b[; \mathcal{H})$, and $y(t) \in \mathcal{C}(t)$ for every $t \in [a, b[$, then the following two conditions are equivalent.*

- (i) $-v(t) \in N_{\mathcal{C}(t)}(y(t))$ for μ -a.e. $t \in B$.
- (ii) $\int_B \langle y(t) - z(t), v(t) \rangle d\mu(t) \leq 0$ for every μ -measurable $z : [0, T] \rightarrow \mathcal{H}$ such that $z(t) \in \mathcal{C}(t)$ for every $t \in B$.

Proof. Let us start by assuming that (i) holds and let $z : [a, b[\rightarrow \mathcal{H}$ be a μ -measurable function such that $z(t) \in \mathcal{C}(t)$ for every $t \in B$. Then it follows that

$$\langle y(t) - z(t), v(t) \rangle \leq 0 \quad \forall t \in B,$$

and integrating over B we infer condition (ii). Now assume that (ii) is satisfied and observe that $\mathfrak{V} = \{(t, [t - h, t + h] \cap B) : h > 0, t \in B\}$ is a μ -Vitali relation covering B according to the definition given in [30, Section 2.8.16, p. 151]. Recall that if $f \in L^1(\mu, B; \mathcal{H})$ then there exists a μ -zero measure set Z such that $f(B \setminus Z)$ is separable (see, e.g., [44, Property M11, p. 124]), therefore from (the proof) of [30, Corollary 2.9.9., p. 156] it follows that

$$\lim_{h \searrow 0} \frac{1}{\mu([t - h, t + h] \cap B)} \int_{[t - h, t + h] \cap B} \|f(\tau) - f(t)\|_E d\mu(\tau) = 0 \quad \text{for } \mu\text{-a.e. } t \in B. \quad (4.7)$$

In [30] the points s satisfying (4.7) are called μ -Lebesgue points of f on B with respect to the μ -Vitali relation \mathfrak{V} covering B . Let L be the set of μ -Lebesgue points for both $\tau \mapsto v(\tau)$ and $\tau \mapsto \langle y(\tau), v(\tau) \rangle$ on B with respect to \mathfrak{V} , fix $t \in B$, and choose $\zeta_t \in \mathcal{C}(t)$ arbitrarily. Thanks to [19, Theorem III.9, p. 67] there exists a μ -measurable function $\zeta : [a, b[\rightarrow \mathcal{H}$ such that $\zeta(t) = \zeta_t$ and $\zeta(\tau) \in \mathcal{C}(\tau)$ for every $\tau \in [a, b[$, therefore a straightforward computation shows that

$$\lim_{h \searrow 0} \frac{1}{\mu([t - h, t + h] \cap B)} \int_{[t - h, t + h] \cap B} \langle \zeta(\tau), v(\tau) \rangle d\mu(\tau) = \langle \zeta_t, v(t) \rangle.$$

The function $z(\tau) := \mathbb{1}_{[a, b[\cap [t - h, t + h]}(\tau) \zeta(\tau) + \mathbb{1}_{[a, b[\setminus [t - h, t + h]}(\tau) y(\tau)$, $\tau \in [a, b[$, is well defined for every sufficiently small $h > 0$, therefore $z(\tau) \in \mathcal{C}(\tau)$ for every $\tau \in [a, b[$, thus taking z in condition (iii) we get

$$\int_{[t - h, t + h] \cap B} \langle y(\tau), v(\tau) \rangle d\mu(\tau) \leq \int_{[t - h, t + h] \cap B} \langle \zeta_t, v(\tau) \rangle d\mu(\tau).$$

Dividing this inequality by $\mu([t - h, t + h] \cap B)$ and taking the limit as $h \searrow 0$ we get $\langle y(t) - \zeta_t, v(t) \rangle \leq 0$. Therefore we have proved that

$$\langle y(t) - z, v(t) \rangle \leq 0 \quad \forall z \in \mathcal{C}(t), \quad \text{for } \mu\text{-a.e. } t \in B,$$

i.e. condition (i) holds. □

Now it is convenient to recall the notion of normalized arc-length parametrization for a metric-space-valued curve, provided by the following proposition (cf., e.g., [30, 65])

Proposition 4.1. *Assume that (2.1) is satisfied and that $f \in BV([a, b]; X)$. We define $\ell_f : [a, b] \rightarrow [a, b]$ by*

$$\ell_f(t) := \begin{cases} a + \frac{b - a}{V(f, [a, b])} V(f, [a, t]) & \text{if } V(f, [a, b]) \neq 0 \\ a & \text{if } V(f, [a, b]) = 0 \end{cases} \quad t \in [a, b].$$

If $X = \mathcal{H}$, a Hilbert space, and $f \in BV^r([a, b]; \mathcal{H})$, then there is a unique $\tilde{f} \in \text{Lip}([a, b]; \mathcal{H})$ such that $\text{Lip}(\tilde{f}) \leq V(f, [a, b])/(b - a)$ and

$$f = \tilde{u} \circ \ell_f, \quad (4.8)$$

$$\tilde{u}(\ell_f(t-)(1 - \lambda) + \ell_f(t)\lambda) = (1 - \lambda)f(t-) + \lambda f(t) \quad \forall t \in [a, b], \quad \forall \lambda \in [0, 1]. \quad (4.9)$$

The normalization factor in Definition 4.1 is not necessary in the proofs of our theorems but it is consistent and simplifies the statement of Theorem 5.4.

In the following lemma we show that we can take $\mu = \text{D}\ell_{\mathcal{C}}$, and in this case we have an explicit bound for the density of $\text{D}y$ with respect to $\text{D}\ell_{\mathcal{C}}$.

Lemma 4.2. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b[; \mathcal{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(c)$, and y is the only function such that there is a measure $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which (4.1)–(4.4) hold. Then there is a unique (up to $\text{D}\ell_{\mathcal{C}}$ -equivalence) $w \in L^1(\text{D}\ell_{\mathcal{C}}; \mathcal{H})$ such that*

$$\text{D}y = w \text{D}\ell_{\mathcal{C}}, \quad (4.10)$$

and we have

$$-w(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \text{D}\ell_{\mathcal{C}}\text{-a.e. } t \in]a, b[, \quad (4.11)$$

$$\|w(t)\| \leq V(\mathcal{C}, [a, b])/(b - a) \quad \text{for } \text{D}\ell_{\mathcal{C}}\text{-a.e. } t \in [a, b[. \quad (4.12)$$

Proof. The existence of a measure $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ such that (4.1)–(4.4) hold is guaranteed by Theorem 4.1, which, by estimate (4.5), also implies that $\text{D}y$ is absolutely continuous with respect to $\text{D}\ell_{\mathcal{C}}$ and the existence and uniqueness of w is a consequence of the vectorial Radon-Nikodym theorem [44, Corollary 4.2, Section VII.4, p. 204]. From Lemma 4.1 we infer that for every μ -measurable function $z : [a, b[\rightarrow \mathcal{H}$ such that $z(t) \in \mathcal{C}(t)$ for every $t \in [a, b[$ we have that

$$\begin{aligned} \int_{[a, b[} \langle y(t) - z(t), w(t) \rangle \text{dD}\ell_{\mathcal{C}}(t) &= \int_{[a, b[} \langle y(t) - z(t), \text{d}(w \text{D}\ell_{\mathcal{C}})(t) \rangle \\ &= \int_{[a, b[} \langle y(t) - z(t), \text{dD}y(t) \rangle = \int_{[a, b[} \langle y(t) - z(t), \text{d}(v\mu)(t) \rangle \\ &= \int_{[a, b[} \langle y(t) - z(t), v(t) \rangle \text{d}\mu(t) \leq 0, \end{aligned}$$

therefore using again Lemma 4.1 we get (4.11). Finally using [30, Corollary 2.9.9., p. 156] as in formula (4.7) in the proof of Lemma 4.1, and exploiting estimate (4.5), we get that for $\text{D}\ell_{\mathcal{C}}$ -a.e. $t \in]a, b[$ we have

$$\begin{aligned} \|w(t)\| &= \lim_{h \searrow 0} \frac{1}{\text{D}\ell_{\mathcal{C}}([t - h, t + h])} \int_{[t - h, t + h]} \|w(\tau)\| \text{dD}\ell_{\mathcal{C}}(\tau) \\ &= \lim_{h \searrow 0} \frac{|\text{D}y|([t - h, t + h])}{\text{D}\ell_{\mathcal{C}}([t - h, t + h])} = \lim_{h \searrow 0} \frac{V(\mathcal{C}, [a, b]) V(y, [t - h, t + h])}{(b - a) V(\mathcal{C}, [t - h, t + h])} \leq \frac{V(\mathcal{C}, [a, b])}{b - a}, \end{aligned}$$

and the lemma is completely proved, since $\text{D}\ell_{\mathcal{C}}(\{0\}) = 0$. \square

Remark 4.1. The previous Lemma can also be proved by using the representation formula [68, Formula (69)].

Now we can prove our main result in the particular case when the behaviour of the solution is prescribed on a finite number of points.

Lemma 4.3. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b[; \mathcal{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq]a, b[$, and that for every $t \in S$ we are given $g_t : \mathcal{H} \rightarrow \mathcal{C}(t)$ such that*

$$\sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} < \infty. \quad (4.13)$$

If $F \subseteq S$ is a finite set, then there exists a unique $y \in BV^r([a, b[; \mathcal{H})$ such that there are a measure $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ satisfying

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[, \quad (4.14)$$

$$Dy = v\mu, \quad (4.15)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\setminus F, \quad (4.16)$$

$$y(t) = g_t(y(t-)) \quad \forall t \in F, \quad (4.17)$$

$$y(a) = y_0. \quad (4.18)$$

Moreover, if $a \leq s < t \leq b$, we have

$$V(y, [s, t]) \leq V(\mathcal{C}, [s, t]) + \sum_{r \in F \cap [s, t]} (\|g_r - Id\|_{\infty, \mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-), \mathcal{C}(r))), \quad (4.19)$$

one can take

$$\mu = D\ell_{\mathcal{C}} + \sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} \delta_t, \quad (4.20)$$

and for such μ the function v satisfying (4.15)-(4.16) is unique up to μ -equivalence and we have

$$\|v(t)\| \leq \max\{1, V(\mathcal{C}, [a, b]) / (b - a)\} \quad \text{for } \mu\text{-a.e. } t \in [a, b[. \quad (4.21)$$

Proof. It is not restrictive to assume that $\|g_t - Id\|_{\infty, \mathcal{C}(t-)} \neq 0$ for every $t \in S$. Let us set $t_0 := a$, $t_n := b$ and suppose that $F = \{t_1, \dots, t_{n-1}\}$ for some $n \in \mathbb{N}$ with $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$. Let us call \mathcal{C}_j the restriction of \mathcal{C} to the interval $[t_{j-1}, t_j]$. Observe that we have

$$\begin{aligned} \ell_{\mathcal{C}_j}(t) &= t_{j-1} + \frac{t_j - t_{j-1}}{V(\mathcal{C}, [t_{j-1}, t_j])} V(\mathcal{C}, [t_{j-1}, t]) \\ &= t_{j-1} + \frac{t_j - t_{j-1}}{V(\mathcal{C}, [t_{j-1}, t_j])} \frac{V(\mathcal{C}, [a, b])}{b - a} (\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t_{j-1})) \end{aligned}$$

therefore

$$D\ell_{\mathcal{C}_j} = \frac{t_j - t_{j-1}}{V(\mathcal{C}, [t_{j-1}, t_j])} \frac{V(\mathcal{C}, [a, b])}{b - a} D\ell_{\mathcal{C}} \quad (4.22)$$

and by applying Theorem 4.1 and Lemma 4.2 we get that for every $j \in \{1, \dots, n\}$ there is a unique $y^j \in BV^r([t_{j-1}, t_j[; \mathcal{H})$ and a unique $v^j \in L^1(D\ell_{\mathcal{C}}, [t_{j-1}, t_j[; \mathcal{H})$ such that

$$y^j(t) \in \mathcal{C}(t) \quad \forall t \in [t_{j-1}, t_j[, \quad (4.23)$$

$$Dy^j = v^j D\ell_{\mathcal{C}} \quad \text{on } \mathcal{B}([t_{j-1}, t_j[), \quad (4.24)$$

$$-v^j(t) \in N_{\mathcal{C}(t)}(y^j(t)) \quad \text{for } D\ell_{\mathcal{C}}\text{-a.e. } t \in [t_{j-1}, t_j[, \quad (4.25)$$

$$y^j(t_{j-1}) = \begin{cases} y_0 & \text{if } j = 1 \\ g_{t_{j-1}}(y^{j-1}(t_{j-1}-)) & \text{if } j \in \{2, \dots, n\} \end{cases}, \quad (4.26)$$

and, using (4.12) and (4.22), we have

$$\|v^j(t)\| \leq \frac{t_j - t_{j-1}}{V(\mathcal{C}, [t_{j-1}, t_j])} \frac{V(f, [a, b])}{b - a} \frac{V(\mathcal{C}, [t_{j-1}, t_j])}{t_j - t_{j-1}} = \frac{V(f, [a, b])}{b - a} \quad \text{for } D\ell_{\mathcal{C}}\text{-a.e. } t \in]t_{j-1}, t_j[. \quad (4.27)$$

Now we define $y : [a, b[\rightarrow \mathcal{H}$ by setting

$$y(t) := \sum_{j=1}^n \mathbb{1}_{[t_{j-1}, t_j[}(t) y^j(t) \quad (4.28)$$

and $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ by (4.20). Observe that y is right continuous and satisfies (4.14), (4.17), and (4.18). Moreover μ is a positive measure, $\mu(\{0\}) = 0$, and thanks to right continuity of y we have that

$$\begin{aligned} \mathbf{V}(y, [s, t]) &= \sum_{j=1}^n |\mathbf{D}y|([t_{j-1}, t_j[\cap [s, t]) \\ &= \sum_{j=1}^n \mathbf{V}(y,]t_{j-1}, t_j[\cap [s, t]) + \sum_{t_j \in F \cap [s, t]} \|y(t_j) - y(t_j-)\| \\ &= \sum_{j=1}^n \mathbf{V}(y^j,]t_{j-1}, t_j[\cap [s, t]) + \sum_{F \cap [s, t]} \|g_{t_j}(y^j(t_j-)) - y^j(t_j-)\| \\ &\leq \mathbf{V}(\mathcal{C}, [s, t]) + \sum_{t \in F \cap [s, t]} \|g_t - Id\|_{\infty, \mathcal{C}(t-)}, \end{aligned}$$

hence y has bounded variation and (4.19) holds. Now we consider a set $B \in \mathcal{B}([a, b[)$ such that $\mu(B) = 0$. For every $j \in \{1, \dots, n\}$ we have $0 = \mu(B \cap]t_{j-1}, t_j[) = \mathbf{D}l_{\mathcal{C}}(B \cap]t_{j-1}, t_j[)$, thus from (4.24) we infer that $\mathbf{D}y^j(B \cap]t_{j-1}, t_j[) = 0$, and, since $y = y^j$ on $]t_{j-1}, t_j[$, it follows that $\mathbf{D}y = \mathbf{D}y^j$ on $\mathcal{B}(]t_{j-1}, t_j[)$, so that $\mathbf{D}y(B \cap]t_{j-1}, t_j[) = 0$. We also have that $0 = \mu(B \cap \{t_j\}) = \|g_{t_j} - Id\|_{\infty, \mathcal{C}(t_j-)} \delta_{t_j}(B)$ for every $j \in \{1, \dots, n-1\}$, thus $\mathbf{D}y(\{t_j\}) = y(t_j) - y(t_j-) = g_{t_j}(y(t_j-)) - y(t_j-) = 0$. Therefore we infer that $\mathbf{D}y(B) = 0$ whenever $B \in \mathcal{B}([a, b[)$ and $\mu(B) = 0$, so that $\mathbf{D}y$ is μ -absolutely continuous and by the vectorial Radon-Nikodym theorem [44, Corollary 4.2, Section VII.4, p. 204] there exists a unique (up to μ -equivalence) function $v \in L^1(\mu; \mathcal{H})$ such that $\mathbf{D}y = v\mu$. It follows that on $\mathcal{B}(]t_{j-1}, t_j[)$ we have $\mathbf{D}y^j = \mathbf{D}y = v\mu = v \mathbf{D}l_{\mathcal{C}}$, thus $v = v^j$ for $\mathbf{D}l_{\mathcal{C}}$ -a.e. $t \in]t_{j-1}, t_j[$ and v satisfies (4.16) thanks to (4.25). Let us also observe that for every $t \in F$ we have

$$y(t) - y(t-) = \mathbf{D}y(\{t\}) = \int_{\{t\}} v \, d\mu = (\mathbf{D}l_{\mathcal{C}}(\{t\}) + \|g_t - Id\|_{\infty, \mathcal{C}(t-)} v(t)) \quad \forall t \in F$$

therefore

$$\|v(t)\| = \frac{\|y(t) - y(t-)\|}{\mathbf{D}l_{\mathcal{C}}(\{t\}) + \|g_t - Id\|_{\infty, \mathcal{C}(t-)}} \leq \frac{\|g_t(y(t-)) - y(t-)\|}{\|g_t - Id\|_{\infty, \mathcal{C}(t-)}} \leq 1 \quad \forall t \in F, \quad (4.29)$$

thus formula (4.21) follows from (4.27). The uniqueness of y is a consequence of its construction. \square

We will need the following weak compactness theorem for measures [27, Theorem 5, p. 105], which we state in a form which is suitable to our purposes.

Theorem 4.2. *Let $I \subseteq \mathbb{R}$ be an interval and let M be a subset of the vector space of measures $\nu : \mathcal{B}(I) \rightarrow \mathcal{H}$ with bounded variation endowed with the norm $\|\nu\| := |\nu|(I)$. Assume that M is bounded. Then M is weakly sequentially precompact if and only if there is a bounded positive measure $\mu : \mathcal{B}(I) \rightarrow [0, \infty[$ such that*

$$\forall \varepsilon > 0 \exists \delta > 0 \quad : \quad \left(B \in \mathcal{B}(I), \mu(B) < \delta \implies \sup_{\nu \in M} |\nu|(B) < \varepsilon \right). \quad (4.30)$$

It is worthwhile to mention that the above theorem is concerned with the notion of weak convergence of measures (in duality with the space of linear continuous functionals on the space of measures) and not with weak-* convergence of measures (in duality with continuous functions). Anyway, if one does not want an equivalence but only an implication, it is clearly true that the same condition above also provides weakly-* sequential precompactness of M .

Theorem 4.2 is stated in [27, Theorem 5, p. 105] as a topological precompactness result. An inspection in the proof easily shows that this is actually a sequential precompactness theorem, since an isometric isomorphism reduces it to the well-known Dunford-Pettis weak sequential precompactness theorem in $L^1(\mu; \mathcal{H})$ (see, e.g., [27, Theorem 1, p. 101]).

We are now in position to prove the following theorem which immediately implies our main result Theorem 3.1.

Theorem 4.3. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV^r([a, b[; \mathcal{C}_{\mathcal{H}})$, $y_0 \in \mathcal{C}(a)$, $S \subseteq]a, b[$, and that for every $t \in S$ we are given $g_t : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ such that $\text{Lip}(g_t) \leq 1$ and*

$$\sum_{t \in S} \|g_t - \text{Id}\|_{\infty, \mathcal{C}(t-)} < \infty. \quad (4.31)$$

Then there exists a unique $y \in BV^r([a, b[; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[, \quad (4.32)$$

$$Dy = v\mu, \quad (4.33)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\setminus S, \quad (4.34)$$

$$y(t) = g_t(y(t-)) \quad \forall t \in S, \quad (4.35)$$

$$y(a) = y_0. \quad (4.36)$$

Moreover (3.7) holds whenever $a \leq s < t \leq b$. Finally the function $t \mapsto \|y_1(t) - y_2(t)\|^2$ is nonincreasing whenever $y_{0,j} \in \mathcal{C}(a)$, $j = 1, 2$, and y_j is the only function such that there is a measure $\mu_j : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ and a function $v_j \in L^1(\mu_j; \mathcal{H})$ for which (4.32)-(4.36) hold with y, v, μ, y_0 replaced respectively by $y_j, v_j, \mu_j, y_{0,j}$.

Proof. We may assume again that $\|g_t - \text{Id}\|_{\infty, \mathcal{C}(t-)} \neq 0$ for every $t \in S$, thus from (4.31) it follows that S is at most countable, and we may assume it contains infinitely many elements, since the finite case is considered in Lemma 4.3. Let $\mu : \mathcal{B}([a, b[) \rightarrow [0, \infty[$ be defined by

$$\mu = D\ell_{\mathcal{C}} + \sum_{t \in S} \|g_t - \text{Id}\|_{\infty, \mathcal{C}(t-)} \delta_t. \quad (4.37)$$

If $S = \{s_n : n \in \mathbb{N}\}$, we set $S_n := \{s_1, \dots, s_n\}$ for every $n \in \mathbb{N}$ and from Lemma 4.3 it follows that there is a unique $y_n \in BV^r([a, b[; \mathcal{H})$ and a unique $v_n \in L^1(\mu; \mathcal{H})$ satisfying

$$y_n(t) \in \mathcal{C}(t) \quad \forall t \in [a, b[, \quad (4.38)$$

$$Dy_n = v_n\mu, \quad (4.39)$$

$$-v_n(t) \in N_{\mathcal{C}(t)}(y_n(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b[\setminus S_n, \quad (4.40)$$

$$y_n(t) = g_t(y_n(t-)) \quad \forall t \in S_n, \quad (4.41)$$

$$y_n(a) = y_0. \quad (4.42)$$

Moreover if $a \leq s < t \leq b$ we have

$$V(y_n, [s, t]) \leq V(\mathcal{C}, [s, t]) + \sum_{r \in S \cap [s, t]} (\|g_r - \text{Id}\|_{\infty, \mathcal{C}(r-)} - d_{\mathcal{H}}(\mathcal{C}(r-), \mathcal{C}(r))) < \infty \quad (4.43)$$

and

$$\|v_n(t)\| \leq V(\mathcal{C}, [a, b])/(b-a) \quad \text{for } \mu\text{-a.e. } t \in [a, b[. \quad (4.44)$$

Recall that $S_{n+1} \setminus S_n = \{s_{n+1}\}$ and assume that $S_n = \{t_1, \dots, t_n\}$ with $t_{j-1} < t_j$ for every j , and that $t_{h-1} < s_{n+1} < t_h$ for some $h \in \{2, \dots, n\}$ (the cases $s_{n+1} < t_1$ and $t_n < s_{n+1}$ are dealt with similarly). Then $y_{n+1}(t) = y_n(t)$ for every $t \in]a, s_{n+1}[$, while at $t \in [s_{n+1}, t_h[$ the distance between y_n and y_{n+1} can be estimated by using (4.6), Theorem 4.1, and (4.38)–(4.41) as follows:

$$\begin{aligned} \|y_{n+1}(t) - y_n(t)\| &\leq \|y_{n+1}(s_{n+1}) - y_n(s_{n+1})\| \\ &= \|g_{s_{n+1}}(y_{n+1}(s_{n+1}-)) - \text{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-))\| \\ &= \|g_{s_{n+1}}(y_n(s_{n+1}-)) - \text{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-))\| \\ &\leq \|g_{s_{n+1}}(y_n(s_{n+1}-)) - y_n(s_{n+1}-)\| \\ &\quad + \|y_n(s_{n+1}-) - \text{Proj}_{\mathcal{C}(s_{n+1})}(y_n(s_{n+1}-))\| \\ &\leq \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})) \quad \forall t \in [s_{n+1}, t_h[. \end{aligned} \quad (4.45)$$

If $h < n$ and $t \in [t_h, t_{h+1}[$, from (4.6), (4.38)–(4.41), and (4.45), we infer that

$$\begin{aligned} \|y_{n+1}(t) - y_n(t)\| &\leq \|g_{t_h}(y_{n+1}(t_h-)) - g_{t_h}(y_n(t_h-))\| \\ &\leq \|y_{n+1}(t_h-) - y_n(t_h-)\| \\ &= \lim_{t \rightarrow t_h-} \|y_{n+1}(t) - y_n(t)\| \\ &\leq \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})), \end{aligned}$$

and iterating this procedure we get the same estimate for $t \in [t_k, t_{k+1}[$ and $h \leq k < n$, thus

$$\|y_{n+1} - y_n\|_{\infty} \leq \|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})),$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \|y_n - y_{n+1}\|_{\infty} &\leq \sum_{n=1}^{\infty} \left(\|g_{s_{n+1}} - Id\|_{\infty, \mathcal{C}(s_{n+1}-)} + d_{\mathcal{H}}(\mathcal{C}(s_{n+1}-), \mathcal{C}(s_{n+1})) \right) \\ &\leq \sum_{t \in S} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} + V(\mathcal{C}, [a, b]) < \infty \end{aligned}$$

i.e. y_n is uniformly Cauchy and there exists $y : [a, b[\rightarrow \mathcal{H}$ such that

$$y_n \rightarrow y \quad \text{uniformly on } [a, b[. \quad (4.46)$$

Moreover $y \in BV([a, b[; \mathcal{H})$ and (3.7) holds by (4.43) and the semicontinuity of the variation w.r.t. to the pointwise convergence, and y satisfies (4.32) by virtue of the closedness of $\mathcal{C}(t)$. Conditions (4.35)–(4.36) are trivially satisfied because for every $t \in S$ the sequence $y_n(t)$ is definitively constant, equal to $g_t(y(t-))$. Let us observe that thanks to (4.44) we have that

$$|Dy_n|(B) = \int_B \|v_n(t)\| \, d\mu(t) \leq \int_B \frac{V(\mathcal{C}, [a, b])}{b-a} \, d\mu(t) = \frac{V(\mathcal{C}, [a, b])}{b-a} \mu(B) \quad \forall B \in \mathcal{B}([a, b[)$$

hence by the Dunford-Pettis Theorem 4.2 for measures and from [35, Lemma 7.1] we infer that

$$Dy_n \rightharpoonup Dy \quad (4.47)$$

Since $\|v_n\|_{L^2(\mu; \mathcal{H})} \leq V(\mathcal{C}, [a, b])\sqrt{\mu([a, b])}/(b-a) < \infty$, there exists $v \in L^2(\mu; \mathcal{H})$ such that, up to subsequences,

$$v_n \rightharpoonup v \quad \text{in } L^2(\mu; \mathcal{H}) \text{ as } n \rightarrow \infty, \quad (4.48)$$

and thanks to (4.40) and Lemma 4.1 we have that

$$\int_{[a,b[\setminus S} \langle y_n(t) - z(t), v_n(t) \rangle d\mu(t) \leq 0 \quad \forall n \in \mathbb{N}, \quad (4.49)$$

therefore taking the limit as $n \rightarrow \infty$ from (4.46) and (4.48) we infer that

$$\int_{[a,b[\setminus S} \langle y(t) - z(t), v(t) \rangle d\mu \leq 0 \quad (4.50)$$

which, by Lemma 4.1, is equivalent to (4.34). Estimate (3.7) follows from (4.43) and from the lower semicontinuity of the variation. If $\phi : [a, b[\rightarrow \mathcal{H}$ is an arbitrary bounded Borel function then $\mu \mapsto \int_{[a,b[} \langle \phi(t), d\mu(t) \rangle$ is a continuous linear functional on the space of measures with bounded variation and we have

$$\lim_{n \rightarrow \infty} \int_{[a,b[} \langle \phi(t), dDy_n(t) \rangle = \int_{[a,b[} \langle \phi(t), dDy(t) \rangle.$$

On the other hand we have

$$\int_{[a,b[} \langle \phi(t), dDy_n(t) \rangle = \int_{[a,b[} \langle \phi(t), v_n(t) \rangle d\mu(t) \quad \forall n \in \mathbb{N},$$

thus taking the limit as $n \rightarrow \infty$ thanks to (4.47) and (4.48) we get

$$\int_{[a,b[} \langle \phi(t), dDy(t) \rangle = \int_{[a,b[} \langle \phi(t), v(t) \rangle d\mu(t)$$

therefore the arbitrariness of ϕ yields $Dy = v\mu$ (cf., e.g., [28, Proposition 35, p. 326]). Then (4.33) is proved and the existence part of the theorem is done. If $B \in \mathcal{B}([a, b[)$ then, by [58, Proposition 2] and by the monotonicity of the normal cone we have

$$\begin{aligned} \int_{B \setminus S} dD(\|y_1(\cdot) - y_2(\cdot)\|^2) &\leq 2 \int_{B \setminus S} \langle y_1 - y_2, dD(y_1 - y_2) \rangle \\ &= 2 \int_{B \setminus S} \langle y_1(t) - y_2(t), v_1(t) - v_2(t) \rangle d\mu(t) \leq 0, \end{aligned} \quad (4.51)$$

while if $t \in S$ then we have

$$\begin{aligned} D(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) &= \|y_1(t) - y_2(t)\|^2 - \|y_1(t-) - y_2(t-)\|^2 \\ &= \|g_t(y_1(t-)) - g_t(y_2(t-))\| - \|y_1(t-) - y_2(t-)\|^2 \\ &\leq \|y_1(t-) - y_2(t-)\| - \|y_1(t-) - y_2(t-)\|^2 = 0, \end{aligned} \quad (4.52)$$

therefore for every $B \in \mathcal{B}([a, b[)$ we find

$$\begin{aligned} D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B) &= D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \setminus S) + D(\|y_1(\cdot) - y_2(\cdot)\|^2)(B \cap S) \\ &= \int_{B \setminus S} dD(\|y_1(\cdot) - y_2(\cdot)\|^2) + \sum_{t \in B \cap S} D(\|y_1(\cdot) - y_2(\cdot)\|^2)(\{t\}) \leq 0 \end{aligned}$$

which implies that $t \mapsto \|y_1(t) - y_2(t)\|^2$ is nonincreasing and leads to the uniqueness of the solution. As a consequence the whole sequence y_n converges uniformly to y . \square

5. APPLICATIONS

In this section we discuss some consequences and particular cases of Theorem 3.1.

5.1. Sweeping processes with arbitrary BV driving set and prescribed behaviour on jumps. We first consider the case of a sweeping processes with prescribed behaviour on jumps, where the driving moving set is not assumed to be right continuous.

Theorem 5.1. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in BV([a, b]; \mathcal{C}_{\mathcal{H}})$, $y_0 \in \mathcal{H}$, and that for every $t \in \text{Discont}(\mathcal{C}) \cup \{a\}$ we are given $g_t^l : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ and $g_t^r : \mathcal{C}(t) \rightarrow \mathcal{C}(t+)$ such that*

$$\text{Lip}(g_t^l) \leq 1, \quad \text{Lip}(g_t^r) \leq 1 \quad \forall t \in \text{Discont}(\mathcal{C}) \setminus \{a\} \quad (5.1)$$

and

$$\sum_{t \in \text{Discont}(\mathcal{C})} \|g_t^l - \text{Id}\|_{\infty, \mathcal{C}(t-)} < \infty, \quad \sum_{t \in \text{Discont}(\mathcal{C})} \|g_t^r - \text{Id}\|_{\infty, \mathcal{C}(t)} < \infty. \quad (5.2)$$

Then there exists a unique $y \in BV([a, b]; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [a, b], \quad (5.3)$$

$$Dy = v\mu, \quad (5.4)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in \text{Cont}(\mathcal{C}), \quad (5.5)$$

$$y(t) = g_t^l((y(t-))), \quad y(t+) = g_t^r(g_t^l((y(t-)))) \quad \forall t \in \text{Discont}(\mathcal{C}) \setminus \{a\}, \quad (5.6)$$

$$y(a) = g_a^l(y_0), \quad y(a+) = g_a^r(a)(y(a)). \quad (5.7)$$

Moreover if $a \leq s < t \leq b$ then

$$V(y, [s, t]) \leq V(\mathcal{C}, [s, t])$$

$$+ \sum_{\sigma \in \text{Discont}(\mathcal{C}) \cap [s, t]} \left(\|g_{\sigma}^l - \text{Id}\|_{\infty, \mathcal{C}(\sigma-)} + \|g_{\sigma}^r - \text{Id}\|_{\infty, \mathcal{C}(\sigma)} - d_{\mathcal{H}}(\mathcal{C}(\sigma-), \mathcal{C}(\sigma)) + d_{\mathcal{H}}(\mathcal{C}(\sigma), \mathcal{C}(\sigma+)) \right).$$

Proof. We can apply Theorem 3.1 with $g_t := g_t^r \circ g_t^l$ and find a unique function $\hat{y} \in BV^r([a, b]; \mathcal{H})$ such that there is a measure $\hat{\mu} : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $\hat{v} \in L^1(\hat{\mu}; \mathcal{H})$ for which

$$\hat{y}(t) \in \mathcal{C}(t+) \quad \forall t \in [a, b] \quad (5.8)$$

$$D\hat{y} = \hat{v}\hat{\mu}, \quad (5.9)$$

$$-\hat{v}(t) \in N_{\mathcal{C}(t+)}(\hat{y}(t)) = N_{\mathcal{C}(t)}(\hat{y}(t)) \quad \text{for } \hat{\mu}\text{-a.e. } t \in \text{Cont}(\mathcal{C}), \quad (5.10)$$

$$\hat{y}(t) = g_t^r(g_t^l(\hat{y}(t-))) \quad \forall t \in \text{Discont}(\mathcal{C}) \setminus \{a\}, \quad (5.11)$$

$$\hat{y}(a) = g_a^r(y_0). \quad (5.12)$$

Then the theorem is satisfied if we take $y : [a, b] \rightarrow \mathcal{H}$, $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$, and $v : [a, b] \rightarrow \mathcal{H}$ defined by

$$y(t) := \begin{cases} \hat{y}(t) & \text{if } t \in \text{Cont}(\mathcal{C}) \\ g_t^l(\hat{y}(t-)) & \text{if } t \in \text{Discont}(\mathcal{C}) \end{cases},$$

$\mu := \hat{\mu}$, and

$$v(t) := \begin{cases} \hat{v}(t) & \text{if } t \in \text{Cont}(\mathcal{C}) \\ 0 & \text{if } t \in \text{Discont}(\mathcal{C}) \text{ and } \hat{\mu}(\{t\}) = 0 \\ \frac{y(t+) - y(t-)}{\hat{\mu}(\{t\})} & \text{if } t \in \text{Discont}(\mathcal{C}) \text{ and } \hat{\mu}(\{t\}) \neq 0 \end{cases}.$$

(observe that the first condition in (5.2) ensures that $y \in BV([a, b]; \mathcal{H})$). \square

Remark 5.1. Let us observe that we can actually prove a result which is more general than Theorem 5.1: indeed we can prescribe the behavior of the solution $y(t)$ also on a countable set of points t where \mathcal{C} is continuous. In order to do that we need to assume that the families $g_t^l : \mathcal{C}(t-) \rightarrow \mathcal{C}(t)$ and $g_t^r : \mathcal{C}(t) \rightarrow \mathcal{C}(t+)$ are indexed by $t \in S$, where $S \subseteq [a, b]$, $\text{Lip}(g_t^l) \leq 1$

and $\text{Lip}(g_t^r) \leq 1$ for every $t \in S$, and $\sum_{t \in S} \|g_t^l - \text{Id}\|_{\infty, \mathcal{C}(t-)} < \infty$ and $\sum_{t \in S} \|g_t^r - \text{Id}\|_{\infty, \mathcal{C}(t)} < \infty$. Therefore it follows that if $\mathcal{C} \in \text{BV}([a, b]; \mathcal{C}_{\mathcal{H}})$, then there exists a unique $y \in \text{BV}([a, b]; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which (5.3), (5.4), and (5.7) hold together with

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [a, b] \setminus S, \quad (5.13)$$

$$y(t) = g_t^l((y(t-))), \quad y(t+) = g_t^r(g_t^l((y(t-)))) \quad \forall t \in S \setminus \{a\}. \quad (5.14)$$

Observe that in this case S is a fortiori at most countable and y may jump even when \mathcal{C} does not jump.

5.2. Sweeping processes with arbitrary BV driving moving set. Another consequence of Theorem 5.1 is the existence and uniqueness theorem for sweeping processes with arbitrary BV driving moving set.

Theorem 5.2. *Assume that $-\infty < a < b < \infty$, $\mathcal{C} \in \text{BV}([a, b]; \mathcal{C}_{\mathcal{H}})$ and $y_0 \in \mathcal{H}$. Then there exists a unique $y \in \text{BV}([a, b]; \mathcal{H})$ such that there is a measure $\mu : \mathcal{B}([a, b]) \rightarrow [0, \infty[$ and a function $v \in L^1(\mu; \mathcal{H})$ for which*

$$y(t) \in \mathcal{C}(t), \quad (5.15)$$

$$\text{D}y = v\mu, \quad (5.16)$$

$$-v(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in \text{Cont}(\mathcal{C}), \quad (5.17)$$

$$y(t) = \text{Proj}_{\mathcal{C}(t)}(y(t-)), \quad y(t+) = \text{Proj}_{\mathcal{C}(t+)}(y(t)) \quad \forall t \in \text{Discont}(\mathcal{C}) \setminus \{a\} \quad (5.18)$$

$$y(a) = \text{Proj}_{\mathcal{C}(a)}(y_0), \quad y(a+) = \text{Proj}_{\mathcal{C}(a+)}(y(a)). \quad (5.19)$$

Moreover $V(y, [s, t]) \leq V(\mathcal{C}, [s, t])$ whenever $a \leq s < t \leq b$.

Proof. It is enough to apply Theorem 5.1 with $g_t^l := \text{Proj}_{\mathcal{C}(t+)}$ and $g_t^r := \text{Proj}_{\mathcal{C}(t)}$. \square

5.3. The play operator. The play operator is the solution operator of the sweeping process driven by a moving set $\mathcal{C}(t)$ with constant shape, i.e. $\mathcal{C}(t) = u(t) - \mathcal{Z}$, where $u \in \text{BV}^r([a, b]; \mathcal{H})$ and $\mathcal{Z} \in \mathcal{C}_{\mathcal{H}}$. In the following result we restate here the existence Theorem 4.1 in this particular case by using the integral formulation of Lemma 4.1 and we collect some other well-known results (see also [39], where the Young integral is used, and [35, Section 5] containing a slightly different integral formulation).

Theorem 5.3. *Assume that $-\infty < a < b < \infty$, $\mathcal{Z} \in \mathcal{C}_{\mathcal{H}}$, $u \in \text{BV}^r([a, b]; \mathcal{H})$ and $z_0 \in \mathcal{Z}$. Then there exists a unique $y \in \text{BV}^r([a, b]; \mathcal{H})$ such that*

$$y(t) \in u(t) - \mathcal{Z} \quad \forall t \in [a, b], \quad (5.20)$$

$$\int_{[a, b]} \langle z(t) - u(t) + y(t), \text{dD}y(t) \rangle \leq 0 \quad \text{for every } \mu\text{-measurable } z : [a, b] \rightarrow \mathcal{Z}, \quad (5.21)$$

$$u(0) - y(0) = z_0. \quad (5.22)$$

The solution operator $\text{P} : \text{BV}^r([a, b]; \mathcal{H}) \times \mathcal{Z} \rightarrow \text{BV}^r([a, b]; \mathcal{H})$ associating with $(u, z_0) \in \text{BV}^r([a, b]; \mathcal{H}) \times \mathcal{Z}$ the unique function $y = \text{P}(u, z_0)$ satisfying (5.20)-(5.22) is called play operator and it is rate independent, i.e.

$$\text{P}(u \circ \psi, z_0) = \text{P}(u, z_0) \circ \psi \quad \forall u \in \text{BV}^r([a, b]; \mathcal{H}) \quad (5.23)$$

whenever $\psi \in C([a, b]; [a, b])$ is nondecreasing and surjective. We have that $\text{P}(\text{Lip}([a, b]; \mathcal{H}) \times \mathcal{Z}) \subseteq \text{Lip}([a, b]; \mathcal{H})$ and if $u \in \text{Lip}([a, b]; \mathcal{H})$ then $\text{P}(z_0, u) = y$ is the unique function satisfying (5.20), (5.22), and

$$\langle z(t) - u(t) - y(t), y'(t) \rangle \leq 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b], \quad \forall z \in \mathcal{Z}. \quad (5.24)$$

The restriction $\mathsf{P} : \text{Lip}([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow \text{Lip}([a, b]; \mathcal{H})$ is continuous if $\text{Lip}([a, b]; \mathcal{H})$ is endowed with both the BV-norm metric

$$d_{BV}(v, w) := \|v - w\|_{\infty, [a, b]} + \mathsf{V}(v - w; [a, b]),$$

and the strict metric

$$d_s(v, w) := \|v - w\|_{L^1(\mathcal{L}^1; \mathcal{H})} + |\mathsf{V}(v; [a, b]) - \mathsf{V}(w; [a, b])|,$$

and the operator $\mathsf{P} : BV^r([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a, b]; \mathcal{H})$ is its unique continuous extension w.r.t. the topology induced by d_{BV} .

Let us observe that the integral formulation (5.20)-(5.22) is a direct consequence of Theorem 4.1 and Lemma 4.1. Formulas (5.23) and (5.24) are well-known (see, e.g., [60, Section 3i, Section 3c] and [38, Proposition 3.9, p. 33]). The continuity of the restriction of P w.r.t. d_{BV} is proved in [38, Theorem 3.12, p. 34], while its continuity w.r.t. $d_s(v, w)$ is proved in [65, Theorem 5.5]. The BV-norm continuity of P on $BV^r([a, b]; \mathcal{H}) \times \mathcal{Z}$ is proved in [35, Theorem 3.3].

The operator $\mathsf{P} : BV^r([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a, b]; \mathcal{H})$ in general is not continuous w.r.t. to the strict metric d_s (cf. [65, Theorem 3.7]) but its restriction $\mathsf{P} : \text{Lip}([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow \text{Lip}([a, b]; \mathcal{H})$ can be continuously extended to another operator if $BV^r([a, b]; \mathcal{H})$ is endowed with the strict topology in the domain, and with the L^1 -topology in the codomain. Let us recall the precise result proved in [65, Theorem 3.7].

Theorem 5.4. *The operator $\mathsf{P} : \text{Lip}([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow \text{Lip}([a, b]; \mathcal{H})$ admits a unique continuous extension $\overline{\mathsf{P}} : BV^r([a, b]; \mathcal{H}) \times \mathcal{Z} \longrightarrow BV^r([a, b]; \mathcal{H})$ if $BV^r([a, b]; \mathcal{H})$ is endowed with the strict topology d_s in the domain, and with the $L^1(\mathcal{L}^1; \mathcal{H})$ -topology in the codomain. We have*

$$\overline{\mathsf{P}}(u, z_0) = \mathsf{P}(\tilde{u}, z_0) \circ \ell_u \quad \forall u \in BV^r([a, b]; \mathcal{H}), \quad (5.25)$$

where $\tilde{u} \in \text{Lip}([a, b]; \mathcal{H})$ is the arc length reparametrization introduced in Proposition 4.1. In general $\overline{\mathsf{P}} \neq \mathsf{P}$.

The meaning of the extension $\overline{\mathsf{P}}$ and of formula (5.25) is clear: we reparametrize by the arc length the function $u = \tilde{u} \circ \ell_u$ and we apply the play operator to the Lipschitz reparametrization \tilde{u} , which is a segment on the jump sets $[\ell_u(t-), \ell_u(t+)]$. Then we “throw away” the jump sets from $\mathsf{P}(\tilde{u}, z_0)$ by reinserting ℓ_u and we obtain $\overline{\mathsf{P}}(u, z_0) = \mathsf{P}(\tilde{u}, z_0) \circ \ell_u$. A further motivation to this procedure is the rate independence of P (but observe that ℓ_u is not continuous) and we could also say that we are filling in the jumps of u with a segment traversed with “infinite velocity”. The continuity property of $\overline{\mathsf{P}}$ in Theorem 5.4 confirms this interpretation. Now we are going to show that $\overline{\mathsf{P}}(u, z_0)$ can also be obtained as the solution of a sweeping processes with a suitable prescribed behaviour on jumps.

Theorem 5.5. *Assume that $\mathcal{Z} \in \mathcal{C}_{\mathcal{H}}$, $u \in BV^r([a, b]; \mathcal{H})$ and $z_0 \in \mathcal{Z}$. For every $x, y \in \mathcal{H}$ let $\text{seg}_{x, y} : [0, 1] \longrightarrow \mathcal{H}$ be defined by $\text{seg}_{x, y}(t) := (1 - t)x + ty$, $t \in [0, 1]$. Then there exists a unique $y \in BV^r([a, b]; \mathcal{H})$ such that*

$$y(t) \in u(t) - \mathcal{Z} \quad \forall t \in [a, b] \quad (5.26)$$

$$\int_{\text{Cont}(u)} \langle z(t) - u(t) + y(t), dDy(t) \rangle \leq 0 \quad \text{for every } \mu\text{-measurable } z : [a, b] \longrightarrow \mathcal{Z}, \quad (5.27)$$

$$y(t) = \mathsf{P}(\text{seg}_{u(t-), u(t)}, u(t-) - y(t-))(1) \quad \forall t \in \text{Discont}(\mathcal{C}), \quad (5.28)$$

$$y(0) = y_0. \quad (5.29)$$

Moreover if $u \in BV^r([a, b]; \mathcal{H})$, $z_0 \in \mathcal{Z}$, $s, t \in [a, b]$, $s < t$, then we have

$$y = \mathsf{P}(\tilde{u}, z_0) \circ \ell_u = \overline{\mathsf{P}}(u, z_0), \quad (5.30)$$

and

$$V(\bar{P}(u), [s, t]) \leq V(u, [s, t]). \quad (5.31)$$

Proof. We set $\mathcal{C}(t) := u(t) - \mathcal{Z}$, $y_0 := u(0) - z_0$, $S := \text{Discont}(u)$, and

$$g_t(x) := P(\text{seg}_{u(t-), u(t)}, u(t-) - x)(1), \quad t \in S.$$

If $t \in \text{Discont}(u)$ and $x \in \mathcal{C}(t-)$ then $g_t(x)$ is the solution of the sweeping process driven by $\mathcal{K}_u(\tau) := (1 - \tau)u(t-) - \tau u(t) - \mathcal{Z}$, $\tau \in [0, 1]$, and having x as initial condition, thus from (4.5) we infer that $V(P(\text{seg}_{u(t-), u(t)}, u(t-) - x), [0, 1]) \leq V(\mathcal{K}_u, [0, 1]) = \|u(t) - u(t-)\|$ and we have

$$\begin{aligned} \sum_{t \in \text{Discont}(u)} \|g_t - Id\|_{\infty, \mathcal{C}(t-)} &= \sum_{t \in \text{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} \|g_t(x) - x\| \\ &= \sum_{t \in \text{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} \|P(\text{seg}_{u(t-), u(t)}, u(t-) - x)(1) - x\| \\ &\leq \sum_{t \in \text{Discont}(u)} \sup_{x \in \mathcal{C}(t-)} V(P(\text{seg}_{u(t-), u(t)}, u(t-) - x), [0, 1]) \\ &\leq \sum_{t \in \text{Discont}(u)} \|u(t) - u(t-)\| \leq V(u, [0, T]) \end{aligned}$$

which together with (3.7) implies (5.31). Moreover thanks to (4.6) we have that $\|g_t(x_1) - g_t(x_2)\| \leq \|x_1 - x_2\|$ therefore we can apply Theorem 3.1 and infer the existence of a unique y satisfying (5.26)–(5.29). Now we show that (5.30) holds. Thanks to the chain rule [65, Theorem A.7] we have that $D\bar{P}(u, z_0) = w D\ell_u$ with

$$w(t) = \begin{cases} (P(\tilde{u}), z_0)'(t) & \text{if } t \in \text{Cont}(u) \\ \frac{P(\tilde{u}, z_0)(\ell_u(t)) - P(\tilde{u}, z_0)(\ell_u(t-))}{\ell_u(t) - \ell_u(t-)} & \text{if } t \in \text{Discont}(u) \end{cases}$$

and if we set

$$N := \{\sigma \in [a, b] : \langle z - \tilde{u}(\sigma) + P(\tilde{u}, z_0)(\sigma), (P(\tilde{u}, z_0))'(\sigma) \rangle > 0 \text{ for some } z \in \mathcal{Z}\}$$

from (5.24) we deduce that $\mathcal{L}^1(N) = 0$, hence, thanks to [65, Lemma A.5], we have that

$$\begin{aligned} D\ell_{\mathcal{C}}(\{t \in \text{Cont}(\ell_{\mathcal{C}}) : \langle z - \tilde{u}(\ell_u(t)) + P(\tilde{u}, z_0)(\ell_u(t)), (P(\tilde{u}), z_0)'(\ell_u(t)) \rangle > 0 \text{ for some } z \in \mathcal{Z}\}) \\ = D\ell_{\mathcal{C}}(\{t \in \text{Cont}(\ell_{\mathcal{C}}) : \ell_{\mathcal{C}}(t) \in N\}) = \mathcal{L}^1(N) = 0. \end{aligned}$$

This implies that if $z : [a, b] \rightarrow \mathcal{H}$ is μ -measurable and $z(t) \in \mathcal{Z}$ for every $t \in [a, b]$, then

$$\begin{aligned} \int_{\text{Cont}(u)} \langle z(t) - u(t) + P(\tilde{u}, z_0)(\ell_u(t)), dD(P(\tilde{u}, z_0) \circ \ell_u) \rangle \\ = \int_{\text{Cont}(u)} \langle z(t) - u(t) + P(\tilde{u}, z_0)(\ell_u(t)), (P(\tilde{u}, z_0))'(\ell_u(t)) \rangle dD\ell_{\mathcal{C}}(t) \leq 0 \end{aligned}$$

and (5.27) is satisfied. Finally (5.28) is trivially satisfied by $\bar{P}(u, z_0)$. \square

REFERENCES

- [1] V. Acary, O. Bonnefon, B. Brogliato, “Nonsmooth Modeling and Simulation for Switched Circuits. Lecture notes in electrical engineering.” Springer (2011).
- [2] K. Addy, S. Adly, B. Brogliato, D. Goeleven, *A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics*, Nonlinear Anal. Hybrid Syst. **1** (2007), 30-43.

- [3] S. Adly, T. Haddad, L. Thibault, *Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities*, Math. Program. Ser. B **148** (2014), 5-47.
- [4] L. Ambrosio, *Metric space valued functions of bounded variation*, Ann. Sc. Norm. Sup. Pisa **17** (1990), 439-478.
- [5] H. Benabdellah, *Existence of solutions to the nonconvex sweeping process*, J. Differential Equations **164** (2000), 286-295.
- [6] F. Bernicot, J. Venel, *Differential inclusions with proximal normal cones in Banach spaces*, J. Convex Anal. **17** (2010), 451-484.
- [7] F. Bernicot, J. Venel, *Stochastic perturbations of sweeping process*, J. Differential Equations **251** (2011), 1195-1224.
- [8] M. Bounkhel, L. Thibault, *Nonconvex sweeping process and prox-regularity in Hilbert space*, J. Nonlinear Convex Anal. **6** (2005), 359-374.
- [9] H. Brezis, "Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", North-Holland Mathematical Studies, Vol. 5, North-Holland Publishing Company, Amsterdam, 1973.
- [10] B. Brogliato, L. Thibault, *Existence and uniqueness of solutions for non-autonomous complementarity dynamical systems*, J. Convex Anal. **17** (2010), 961-990.
- [11] M. Brokate, P. Krejčí, H. Schnabel, *On uniqueness in evolution quasivariational inequalities*, J. Convex Anal. **11** (2004), 111-130.
- [12] M. Brokate, J. Sprekels, "Hysteresis and Phase Transitions", Applied Mathematical Sciences, **121**, Springer-Verlag, New York, 1996.
- [13] C. Castaing, *Version aléatoire de problème de raflee par un convexe variable*, C.R. Acad. Sci. Paris, Sér. A **277** (1973), 1057-1059.
- [14] C. Castaing, *Equations différentielles. Rafle par un convexe aléatoire à variation continue forte*, C. R. Acad. Sci. Paris, Sér. A **282** (1976), 515-518.
- [15] C. Castaing, T.X. Duc Ha, M. Valadier, *Evolution equations governed by the sweeping process*, Set-Valued Anal. **1** (1993), 109-139.
- [16] C. Castaing, M.D.P. Monteiro Marques, *BV periodic solutions of an evolution problem associated with continuous moving convex sets*, Set-Valued Anal. **3** (1995), 381-399.
- [17] C. Castaing, M.D.P. Monteiro Marques, *Evolution problems associated with non-convex closed moving sets with bounded variation*, Portugaliae Mathematica **53** (1996), 73-87.
- [18] C. Castaing, M.D.P. Monteiro Marques, P. Raynaud de Fitte, *A Skorokhod problem governed by a closed convex moving set*, J. Convex Anal. **23** (2016), 387-423
- [19] C. Castaing, M. Valadier, "Convex Analysis and Measurable Multifunctions", Springer, Berlin - Heidelberg - New York, 1977.
- [20] G. Colombo, V.V. Goncharov, *The sweeping process without convexity*, Set-Valued Anal. **7** (1999), 357-374.
- [21] G. Colombo, R. Henrion, N.D. Hoang, B.S. Mordukhovich, *Discrete approximations of a controlled sweeping process*, Set-Valued Var. Anal. **23** (2015), 69-86.
- [22] G. Colombo, R. Henrion, Nguyen D. Hoang, B. S. Mordukhovich, *Optimal control of the sweeping process: the polyhedral case*, J. Differential Equations **260** (2016), 3397-3447.
- [23] G. Colombo, M.D.P. Monteiro Marques, *Sweeping by a continuous prox-regular set*, J. Differential Equations **187** (2003), 46-62.
- [24] G. Colombo, M. Palladino, *The minimum time function for the controlled Moreau's sweeping process*, SIAM J. Control **54** (2016), 2036-2062.
- [25] B. Cornet, *Existence of slow solutions for a class of differential inclusions*, J. Math. Anal. Appl. **96** (1983), 130-147.
- [26] S. Di Marino, B. Maury, F. Santambrogio, *Measure sweeping processes*, J. Convex Anal. **23** (2016), 567-601.
- [27] J. Diestel, J.J. Uhl, "Vector Measures", American Mathematical Society, Providence, 1977.
- [28] N. Dinculeanu, "Vector Measures", International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Berlin, 1967.
- [29] J.F. Edmond and L. Thibault, *BV solutions of nonconvex sweeping process differential inclusions with perturbation*, J. Differential Equations **226** (2006), 135-179.
- [30] H. Federer, "Geometric Measure Theory", Springer-Verlag, Berlin-Heidelberg, 1969.
- [31] S. Flam, J.-B. Hiriart-Urruty, A. Jourani, *Feasibility in finite time*, J. Dynamical Control Systems **15** (2009), 537-555.
- [32] T. Haddad, A. Jourani, L. Thibault, *Reduction of sweeping process to unconstrained differential inclusion*, Pac. J. Optim. **4** (2008), 493-512.
- [33] C. Henry, *An existence theorem for a class of differential equations with multivalued right-hand side*, J. Math. Anal. Appl. **41** (1973), 179-186.

- [34] O. Klein, V. Recupero *Hausdorff metric BV discontinuity of sweeping processes*, J. Phys.: Conf. Ser., **727** (2016), 012006.
- [35] J. Kopfová, V. Recupero, *BV-norm continuity of sweeping processes driven by a set with constant shape*, J. Differential Equations **261** (2016), 5875-5899.
- [36] M. A. Krasnosel'skii, A. V. Pokrovskii, "Systems with Hysteresis", Springer-Verlag, Berlin Heidelberg, 1989.
- [37] P. Krejčí, *Vector hysteresis models*, European J. Appl. Math. **2** (1991), 281-292.
- [38] P. Krejčí, "Hysteresis, Convexity and Dissipation in Hyperbolic Equations", Gakuto International Series Mathematical Sciences and Applications, Vol. 8, Gakkōtoshō, Tokyo, 1997.
- [39] P. Krejčí, P. Laurençot, *Generalized variational inequalities*, J. Convex Anal. **9** (2002), 159-183.
- [40] P. Krejčí, V. Recupero, *Comparing BV solutions of rate independent processes*, J. Convex Anal. **21** (2014), 121-146.
- [41] P. Krejčí, T. Roche, *Lipschitz continuous data dependence of sweeping processes in BV spaces*, Discrete Contin. Dyn. Syst. Ser. B **15** (2011), 637-650.
- [42] P. Krejčí, A. Vladimirov, *Polyhedral sweeping processes with oblique reflection in the space of regulated functions*, Set-Valued Anal. **11** (2003), 91-110.
- [43] M. Kunze, M. D. P. Monteiro Marques, *An introduction to Moreau's sweeping processes*, Impact in Mechanical Systems - Analysis and Modelling, B. Brogliato (Ed.) Lecture Notes in Physics **551**, Springer (2000), 1-60.
- [44] S. Lang, "Real and Functional Analysis - Third Edition", Springer Verlag, New York, 1993.
- [45] B. Maury, A. Roudneff-Chupin, F. Santambrogio, *A macroscopic crowd motion model of gradient flow type*, Math. Models Methods Appl., **20**, 1787-1821, 2010.
- [46] B. Maury, A. Roudneff-Chupin and F. Santambrogio, *Congestion-driven dendritic growth*, Discrete Contin. Dyn. Syst., **34** (4), 1575-1604, 2014.
- [47] B. Maury, A. Roudneff-Chupin, F. Santambrogio, J. Venel, *Handling Congestion in Crowd Motion Modeling*, Netw. Heterog. Media, **6** (3), 485-519, 2011.
- [48] B. Maury, J. Venel, *A mathematical framework for a crowd motion model*, C. R. Math. Acad. Sci. Paris **346** (2008), 1245-1250.
- [49] B. Maury, J. Venel, *A discrete contact model for crowd motion*, ESAIM: M2AN **45** (2011), 145-168.
- [50] A. Mielke, *Evolution in rate-independent systems*, In "Handbook of Differential Equations, Evolutionary Equations, vol. 2", C. Dafermos, E. Feireisl editors, Elsevier, 2005, 461-559.
- [51] A. Mielke, T. Roubíček, *Rate Independent Systems, Theory and Applications*, Springer, 2015.
- [52] M.D.P. Monteiro Marques, *Perturbations convexes semi-continues supérieurement dans les espaces de Hilbert*, Sem. Anal. Convexe Montpellier (1984), exposé 2.
- [53] M.D.P. Monteiro Marques, "Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction", Birkhauser Verlag, Basel, 1993.
- [54] J. J. Moreau, *Rafle par un convexe variable, I*, Sem. d'Anal. Convexe, Montpellier, Vol. 1 (1971), Exposé No. 15.
- [55] J. J. Moreau, *Rafle par un convexe variable, II*, Sem. d'Anal. Convexe, Montpellier, Vol. 2 (1972), Exposé No. 3.
- [56] J. J. Moreau, *Problème d'évolution associé à un convexe mobile dun espace hilbertien*, C.R. Acad. Sci. Paris Sér. A-B **276** (1973), A791-A794.
- [57] J. J. Moreau, *On unilateral constraints, friction and plasticity*, in "New Variational Techniques in Mathematical Physics" (G. Capriz and G. Stampacchia, Ed.) pp. 173-322, C.I.M.E. II Ciclo 1973, Ediz. Cremonese, Roma, 1974.
- [58] J.J. Moreau, *Sur les mesures différentielles de fonctions vectorielles et certains problèmes d'évolution*, C. R. Math. Acad. Sci. Paris Sér. A **282** (1976), 837-840.
- [59] J. J. Moreau, *Application of convex analysis to the treatment of elastoplastic systems*, in "Application of Methods of Functional Analysis to Problems in Mechanics" (P. Germain, B. Nayroles, Eds.) Lecture Notes in Mathematics Vol. 503, Springer, Berlin-Heidelberg-New York (1976), 56-89.
- [60] J.J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations **26** (1977), 347-374.
- [61] J. J. Moreau, *An introduction to unilateral dynamics*, in "Novel Approaches in Civil Engineering (M. Frémond, F. Maceri Eds.)", Springer, Berlin (2002).
- [62] L. Paoli, *Multibody Dynamics with Unilateral Constraints and Dry Friction: How the Contact Dynamics Approach May Handle Coulomb's Law Indeterminacies?* J. Convex Anal. **23** (2016), 849-876.
- [63] V. Recupero, *The play operator on the rectifiable curves in a Hilbert space*, Math. Methods Appl. Sci. **31** (2008), 1283-1295.
- [64] V. Recupero, *Sobolev and strict continuity of general hysteresis operators*, Math. Methods Appl. Sci. **32** (2009), 2003-2018.

- [65] V. Recupero, *BV solutions of rate independent variational inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sc. (5) **10** (2011), 269-315.
- [66] V. Recupero, *A continuity method for sweeping processes*, J. Differential Equations **251** (2011) 2125-2142.
- [67] V. Recupero, *BV continuous sweeping processes*, J. Differential Equations **259** (2015), 4253-4272.
- [68] V. Recupero, *Sweeping processes and rate independence*, J. Convex Anal. **23** (2016), 921-946.
- [69] M. Sene, L. Thibault, *Regularization of dynamical systems associated with prox-regular moving sets*, J. Nonlin. Convex Anal. **15** (2014), 647-663.
- [70] L. Thibault, *Sweeping process with regular and nonregular sets*, J. Differential Equations **193** (2003), 1-26.
- [71] L. Thibault, *Regularization of nonconvex sweeping process in Hilbert space*, Set-Valued Anal. **16** (2008), 319-333.
- [72] L. Thibault *Moreau Sweeping Process with Bounded Truncated Retraction*, J. Convex Anal. **23** (2016), 1051-1098.
- [73] M. Valadier, *Quelques problèmes d'entraînement unilatéral en dimension finie*, Sem. Anal. Convexe Montpellier (1988), Exposé 8.
- [74] M. Valadier, *Lipschitz approximation of the sweeping (or Moreau) process*, J. Differential Equations **88** (1990), 248-264.
- [75] M. Valadier, *Rafle et viabilité*, Sem. Anal. Convexe Montpellier (1992), Exposé 17.
- [76] A. Visintin, "Differential Models of Hysteresis", Applied Mathematical Sciences, Vol. 111, Springer-Verlag, Berlin Heidelberg, 1994.
- [77] W. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York (1989).

Vincenzo Recupero, DIPARTIMENTO DI SCIENZE MATEMATICHE, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, I-10129 TORINO, ITALY.

E-mail address: vincenzo.recupero@polito.it

Filippo Santambrogio, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UNIV. PARIS-SUD, CNRS, UNIVERSITÉ PARIS-SACLAY, F-91405 ORSAY CEDEX, FRANCE.

E-mail address: filippo.santambrogio@math.u-psud.fr