



Method of Homogenization for the Study of the Propagation of Electromagnetic Waves in a Composite

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Part 1: Modeling, Scaling, Existence and Uniqueness Results



Motivations

- Propagation of the electromagnetic field resulting from a lightning strike in and near a composite material. In particular, we focus on composite material of an aircraft.
- The latest aircraft Airbus A350 and Boeing B757 are made of advanced materials such that composites, titanium and advanced aluminium.
- A revolutionary material. The advantages of composite material : the weight. It is lighter than the metal and it is a strong and durable material, insensitive to corrosion.

Motivations

- Traditionally, the old aircraft's fuselage was made of aluminum. Aluminium conducts electricity one thousand more than composite. Composite consists in carbon fibers enclosed in epoxy resin, and therefore it is the resin that causes the composite to be less conductive than aluminium.
- A carbon fuselage is not as conductive as one made of metal. Modern aircrafts have seen also the increasing reliance on electronic avionics systems instead of mechanical controls and electromechanical instrumentation. For these reasons, aircraft manufacturers are very sensitive to lightning protection and pay special attention to aircraft certification through testing and analysis.



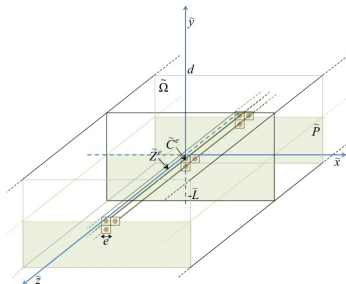
6 / 45



7 / 45

Modeling

- It is composed by air above composite material. Composite consists of electrical conducting carbon fibers distributed in periodic inclusion in epoxy resin. We study the behavior of the EM field in this domain.



- \tilde{P} is the domain containing the material. And the periodic cell

$$\tilde{Z}^e = \left[-\frac{e}{2}, \frac{e}{2}\right] \times [-e, 0] \times \mathbb{R}, \quad (1)$$

- e is the lateral size of the basic cell of the periodic microstructure of the material.

Modeling

- The propagation of this field in the composite will be modeled by Time-Harmonic Maxwell's equations.

$$\nabla \times \tilde{H} - i\tilde{\omega}\epsilon_0\epsilon^*\tilde{E} = \sigma\tilde{E}, \quad \text{Maxwell - Ampere equation} \quad (2)$$

$$\nabla \times \tilde{E} + i\tilde{\omega}\mu_0\tilde{H} = 0, \quad \text{Maxwell - Faraday equation} \quad (3)$$

$$\nabla \cdot (\epsilon_0\epsilon^*\tilde{E}) = \tilde{\rho}, \quad (4)$$

$$\nabla \cdot (\mu_0\tilde{H}) = 0, \quad (5)$$

- where $\tilde{E}(t, \tilde{x}, \tilde{y}, \tilde{z}) = \Re(\tilde{E}(\tilde{x}, \tilde{y}, \tilde{z}) \exp^{i\tilde{\omega}t})$
- μ_0 the permeability of the vacuum,
- $\epsilon = \epsilon_0\epsilon^*$ the permittivity and ϵ^* the relative permittivity with $\epsilon^* = \epsilon_a$ in the air, ϵ_r in the resin, ϵ_c in the carbon.

3D model

- We consider that carbon fibers are conductive but not perfect conductor, $\sigma_c = 40000 \text{ S.m}^{-1}$.
- Epoxy resin is not conductive. In our model, we doped the resin that is to say we added black carbon or graphene. Then the electrical conductivity of the resin increases from 10^{-12} to 10^{-3} S.m^{-1} .
- During the electrical solicitation of a lightning strike, the air becomes suddenly conductive. The ionized channel of a lightning strike is very conductive, we take $\sigma_c = 4200 \text{ S.m}^{-1}$.

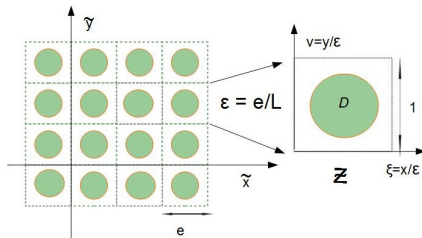
Homogenized problem

- Homogenization has been developed for periodic structures. The aim of Homogenization theory is to describe the average properties of composite material at the macroscopic level, taking into account their microscopic arrangement. ([A.Bensoussan, J. L.Lions, G.Papanicolaou], [D.Cioranescu et P.Donato], E.Sanchez-Palencia,...)



Periodic Homogenization

- The macroscopic scale, which corresponds to the scale of observation is represented by the variable x . The microscopic scale is the characteristic scale of the environment, it corresponds to the size of heterogeneities. The ratio between these two typical scales is denoted ε , where ε is a small parameter. The variable describing phenomena on a microscopic scale is $\frac{x}{\varepsilon}$.
- In our problem the carbon fibers are distributed periodically in the resin. The size of the period, composed by carbon fiber and resin, is small compared to the size of the plate, we denote ε their ratio.



Periodic Homogenization

- For models describing phenomena at very small length scales, the discretization to perform numerical simulation should be of smaller length in comparison to the length scale of the model. The computational cost of such simulation is very high. Sometimes, even impossible. Homogenization can help approximate the solutions to the Microscopic models.

The problem

$$\left\{ \begin{array}{ll} \nabla \times \nabla \times \tilde{E} + (-\tilde{\omega}^2 \mu_0 \epsilon_0 \epsilon^* + i\tilde{\omega} \mu_0 \sigma) \tilde{E} = 0 & \text{in } \tilde{\Omega}. \\ \nabla \times \tilde{E} \times e_2 = -i\tilde{\omega} \mu_0 \tilde{H}_d(\tilde{x}, \tilde{z}) \times e_2 & \text{on } \tilde{\Gamma}_d \\ \nabla \times \tilde{E} \times e_2 = 0 & \text{on } \tilde{\Gamma}_L, \\ \nabla \cdot [(-\tilde{\omega}^2 \mu_0 \epsilon_0 \epsilon^* + i\tilde{\omega} \mu_0 \sigma) \tilde{E}] = 0, \end{array} \right. \quad (6)$$

- with \tilde{H}_d , the magnetic field induced by the peak of the current of the first return stroke.
- e_2 , outward normal.

We non-dimensionalize the problem

- We propose a rescaling of the system, we consider a set of characteristic sizes related to our problem: $\bar{\omega}$ the characteristic pulsation, $\bar{\sigma}$ the characteristic electric conductivity, \bar{E} the characteristic electric magnitude and the characteristic thickness of the plate \bar{L} . Physical factors are then rewritten using those values. We obtain a new set of dimensionless and unitless variables and fields in which the system is rewritten. With the dimensionless variables: $\mathbf{x} = (x, y, z)$ with $x = \frac{\tilde{x}}{\bar{L}}$, $y = \frac{\tilde{y}}{\bar{L}}$ and $z = \frac{\tilde{z}}{\bar{L}}$

$$\tilde{E}(\bar{\omega}\omega, \bar{L}x, \bar{L}y, \bar{L}z) = E(\omega, \mathbf{x}) * \bar{E} \quad (7)$$

- E : the unitless value and \bar{E} : the characteristic value.
- Taking the partial derivative with respect to \tilde{x}

$$\frac{\partial E}{\partial \mathbf{x}}(\omega, \mathbf{x}) = \frac{\bar{L}}{\bar{E}} \frac{\partial \tilde{E}}{\partial \tilde{\mathbf{x}}}(\bar{\omega}\omega, \bar{L}_x, \bar{L}_y, \bar{L}_z), \quad (8)$$

System into dimensionless variables

- Substituting those dimensionless variables and fields into equation (6), we obtain:

$$\nabla \times \nabla \times E(\omega, \mathbf{x}) + \left(-\frac{4\pi^2 \bar{L}^2}{\bar{\lambda}^2} \omega^2 + i \frac{\bar{L}^2}{\bar{\delta}^2} \frac{\sigma_a}{\bar{\sigma}} \omega\right) E(\omega, \mathbf{x}) = 0 \text{ for } 0 \leq \bar{L}y \leq d,$$

$$\nabla \times \nabla \times E(\omega, \mathbf{x}) + \left(-\frac{4\pi^2 \bar{L}^2}{\bar{\lambda}^2} \epsilon_r \omega^2 + i \frac{\bar{L}^2}{\bar{\delta}^2} \frac{\sigma_r}{\bar{\sigma}} \omega\right) E(\omega, \mathbf{x}) = 0 \text{ for } (\bar{L}x, \bar{L}y, \bar{L}z)$$

$$\nabla \times \nabla \times E(\omega, \mathbf{x}) + \left(-\frac{4\pi^2 \bar{L}^2}{\bar{\lambda}^2} \epsilon_c \omega^2 + i \frac{\bar{L}^2}{\bar{\delta}^2} \frac{\sigma_c}{\bar{\sigma}} \omega\right) E(\omega, \mathbf{x}) = 0 \text{ for } (\bar{L}x, \bar{L}y, \bar{L}z)$$

- with

$$\bar{\lambda} = \frac{2\pi c}{\bar{\omega}}, \quad (9)$$

which is the characteristic wave length and

$$\bar{\delta} = \frac{1}{\sqrt{\bar{\omega} \bar{\sigma} \mu_0}}, \quad (10)$$

which is the characteristic skin thickness. $\bar{\delta}$ corresponds to the order of magnitude of the penetration length of the electric field in the

System into dimensionless variables

- with the boundary conditions

$$\nabla \times E(\omega, \mathbf{x}) \times \mathbf{e}_2 = -i\omega \bar{\omega} \mu_0 \frac{\bar{L}}{\bar{E}} \overline{H_d} \overline{H_d}(\bar{L}x, \bar{L}z) \times \mathbf{e}_2 \quad \text{when } (\bar{L}x, \bar{L}y, \bar{L}z) \in \widetilde{\Gamma}_\alpha$$

$$\nabla \times E(\omega, \mathbf{x}) \times \mathbf{e}_2 = 0 \quad \text{when } (\bar{L}x, \bar{L}y, \bar{L}z) \in \widetilde{\Gamma}_L.$$

- where $\bar{\omega} \mu_0 \frac{\bar{L}}{\bar{E}} \overline{H_d}$ being order 1 with the characteristic magnetic field $\overline{H_d} = \frac{I}{2\pi r}$, $I = 200$ kA is the current density and r the radius of the lightning, and the characteristic electric field $\bar{E} = 20$ kV/m.
- The characteristic thickness of the plate \bar{L} is about 10^{-3} m and the size of the basic cell e is about 10^{-5} m. We define the ratio $\frac{e}{\bar{L}}$ equals a small parameter ε :

$$\frac{e}{\bar{L}} \sim 10^{-2} = \varepsilon. \quad (11)$$

System into dimensionless variables

- Concerning the characteristic electric conductivity it seems to be reasonable to take for $\bar{\sigma}$ the value of the effective electric conductivity of the composite material.
- The effective longitudinal electric conductivity, (the arithmetic average), is expressed by the equation:

$$\bar{\sigma} = \sigma_{\text{long}} = f_c \sigma_c + (1 - f_c) \sigma_r, \quad (12)$$

where f_c is the volume fraction of the carbon fiber.

- The effective transverse electric conductivity, (the harmonic average), is expressed by

$$\bar{\sigma} = \sigma_{\text{trans}} = \frac{1}{\frac{f_c}{\sigma_c} + \frac{(1-f_c)}{\sigma_r}}. \quad (13)$$

- In our study we consider the case for $\bar{\omega} = 10^6 \text{ rad.s}^{-1}$, which corresponds to the air ionized, a resin doped and the effective longitudinal electric conductivity of the carbon fibers.

Problem in ε

- Rewriting the problem in terms of ε

$$\begin{cases} \nabla \times \nabla \times E^\varepsilon + (-\omega^2 \varepsilon^\eta \epsilon^\star + i \omega \sigma^\varepsilon(x, y, z)) E^\varepsilon = 0 & \text{in } \Omega \\ \nabla \times E^\varepsilon \times e_2 = -i \omega H_d(x, z) \times e_2 & \text{on } \Gamma_d, \\ \nabla \times E^\varepsilon \times e_2 = 0 & \text{on } \Gamma_L. \\ \nabla \cdot [(-\omega^2 \varepsilon^\eta \epsilon^\star + i \omega \sigma^\varepsilon) E^\varepsilon] = 0 & \text{in } \Omega, \end{cases} \quad (14)$$

- with $\varepsilon^\eta = \frac{4\pi^2 \bar{L}^2}{\lambda^2}$,

$$\sigma^\varepsilon(x, y, z) = \Sigma^\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) \begin{cases} \Sigma_a^\varepsilon & \text{in } \Omega_a, \\ \Sigma_r^\varepsilon & \text{in } \Omega_r, \\ \Sigma_c^\varepsilon & \text{in } \Omega_c, \end{cases} \quad (15)$$

- where $\Sigma_a^\varepsilon = \frac{\sigma_a}{\sigma} \frac{\bar{L}^2}{\delta^2}$, $\Sigma_r^\varepsilon = \frac{\sigma_r}{\sigma} \frac{\bar{L}^2}{\delta^2}$ and $\Sigma_c^\varepsilon = \frac{\sigma_c}{\sigma} \frac{\bar{L}^2}{\delta^2}$

Weak formulation

- Integrating (14) over Ω and using the Green's formula we obtain the weak formulation

$$\left\{ \begin{array}{l} \text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \text{ such that for any } V \in \mathbf{X}^\varepsilon(\Omega) \\ a^{\varepsilon,\eta}(E^\varepsilon, V) = \int_{\Omega} \nabla \times E^\varepsilon \cdot \nabla \times \overline{V} \, d\mathbf{x} \\ \quad + \int_{\Omega_a^\varepsilon} (-\omega^2 \varepsilon^\eta + i\omega \Sigma_a^\varepsilon) E^\varepsilon \cdot \overline{V} \, d\mathbf{x} \\ \quad + \int_{\Omega_c^\varepsilon} (-\omega^2 \varepsilon^\eta \epsilon_c + i\omega \Sigma_c^\varepsilon) E^\varepsilon \cdot \overline{V} \, d\mathbf{x} + \int_{\Omega_r^\varepsilon} (-\omega^2 \varepsilon^\eta \epsilon_r + i\omega \Sigma_r^\varepsilon) E^\varepsilon \cdot \overline{V} \, d\mathbf{x} \\ \quad = \int_{\Gamma_d} (\nabla \times E^\varepsilon \times \mathbf{e}_2) \cdot \overline{V}_T \, d\sigma \\ \quad = \int_{\Gamma_d} -i\omega H_d \times \mathbf{e}_2 \cdot \overline{V}_T \, d\sigma \end{array} \right.$$

Existence and unicity

- With the variational space :

$$\begin{aligned} \mathbf{X}^\varepsilon(\Omega) = \{u \in \mathbf{X}(\Omega) \mid & (-\omega^2 \varepsilon^\eta + i\omega \Sigma_a^\varepsilon)u|_{\Omega_a^\varepsilon} \cdot e_2 = (-\omega^2 \varepsilon^\eta \varepsilon_r + i\omega \Sigma_r^\varepsilon)u|_{\Omega_r^\varepsilon} \cdot \\ & (-\omega^2 \varepsilon^\eta \varepsilon_r + i\omega \Sigma_r^\varepsilon)u|_{\Omega_r^\varepsilon} \cdot n|_{\Omega_r^\varepsilon}^\varepsilon = (-\omega^2 \varepsilon^\eta \varepsilon_c + i\omega \Sigma_c^\varepsilon)u|_{\Omega_c^\varepsilon} \cdot n|_{\Omega_c^\varepsilon}^\varepsilon\}. \end{aligned} \quad (17)$$

- with the next space:

$$\begin{aligned} \mathbf{X}(\Omega) = \{u \in \mathbf{H}(\text{curl}, \Omega) \mid & \text{Div}u|_{\Omega_a^\varepsilon} \in L^2(\Omega_a^\varepsilon), \\ & \text{Div}u|_{\Omega_r^\varepsilon} \in L^2(\Omega_r^\varepsilon), \text{Div}u|_{\Omega_c^\varepsilon} \in L^2(\Omega_c^\varepsilon)\}. \end{aligned} \quad (18)$$

- equipped with the norm

$$\begin{aligned} \|u\|_{\mathbf{X}^\varepsilon(\Omega)}^2 = & \|u\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Div}u|_{\Omega_a^\varepsilon}\|_{L^2(\Omega_a^\varepsilon)}^2 + \|\text{Div}u|_{\Omega_r^\varepsilon}\|_{L^2(\Omega_r^\varepsilon)}^2 \\ & + \|\text{Div}u|_{\Omega_c^\varepsilon}\|_{L^2(\Omega_c^\varepsilon)}^2 + \|\text{curl}u\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \quad (19)$$

Existence and unity of the solution

- $a^{\varepsilon, \eta}$ is not coercive, it is necessary to regularize it by adding terms involving the divergence of E^ε in Ω_a^ε , Ω_r^ε and Ω_c^ε , since the continuity of the divergence is broken through the interfaces $\partial\Omega_a^\varepsilon$, $\partial\Omega_r^\varepsilon$ and $\partial\Omega_c^\varepsilon$.
- we define the regularized formulation of problem :

$$\left\{ \begin{array}{l} \text{Find } E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega) \text{ such that for any } V \in \mathbf{X}^\varepsilon(\Omega) \\ a_R^{\varepsilon, \eta}(E^\varepsilon, V) = a^{\varepsilon, \eta}(E^\varepsilon, V) + s \int_{\Omega_a^\varepsilon} \nabla \cdot E^\varepsilon \nabla \cdot \bar{V} \, d\mathbf{x} \\ + s \int_{\Omega_r^\varepsilon} \nabla \cdot E^\varepsilon \nabla \cdot \bar{V} \, d\mathbf{x} + s \int_{\Omega_c^\varepsilon} \nabla \cdot E^\varepsilon \nabla \cdot \bar{V} \, d\mathbf{x} \\ = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot \bar{V}_T \, d\sigma. \end{array} \right. \quad (20)$$

Existence and unity of the solution

- For any $\varepsilon > 0$ and any $\eta \geq 0$, sesquilinear form $a_R^{\varepsilon, \eta}(\cdot, \cdot)$ is continuous over $\mathbf{X}^\varepsilon(\Omega)$ thanks to the continuity conditions. It is also coercive thanks to this proposition :

Proposition

For any $\varepsilon > 0$, for any $\eta \geq 0$ and for any $s > 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, there exists a positive constant C_0 depending on $\varepsilon_r, \varepsilon_c, s, \omega$ but not on ε such that:

$$\forall E^\varepsilon \in \mathbf{X}^\varepsilon(\Omega), \quad \Re[\exp(-i\frac{\pi}{4}) a_R^{\varepsilon, \eta}(E^\varepsilon, E^\varepsilon)] \geq C_0 \|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)} \quad (21)$$

Existence and unity of the solution

- The sesquilinear form $a_R^{\varepsilon,\eta}$ is continuous, bounded, coercive thanks to the above proposition and the right hand side is continuous on $X^\varepsilon(\Omega)$, then regularized problem has a unique solution in $X^\varepsilon(\Omega)$ thanks to the Lax-Milgram Lemma.
- The problems are equivalents for an appropriate choice of s , then the terms $s.\text{div}$ disappear.

Part II : Homogenization

Homogenized problem

- We use two methods of homogenization : an asymptotic analysis and the two-scale convergence, as ε goes to 0, which gives a rigorous justification of the homogenization result.
- We study the following equation for $\Sigma_a^\varepsilon = \varepsilon$, $\Sigma_r^\varepsilon = \varepsilon^4$ and $\Sigma_c^\varepsilon = 1$ and $\eta = 5$

$$\begin{aligned} \nabla \times \nabla \times E^\varepsilon - \omega^2 \varepsilon^5 E^\varepsilon + i\omega \left[\left(\mathbf{1}_C^\varepsilon \left(\frac{\mathbf{x}}{\varepsilon} \right) + \varepsilon^4 \mathbf{1}_R^\varepsilon \left(\frac{\mathbf{x}}{\varepsilon} \right) \right) \mathbf{1}_{\{y < 0\}} \right. \\ \left. + \varepsilon \mathbf{1}_{\{y > 0\}} \right] E^\varepsilon = 0, \end{aligned} \quad (22)$$

- with the microscopic cell $\mathcal{Z} = [-\frac{1}{2}, \frac{1}{2}] \times [-1, 0]^2$.
- By these methods we can obtain cell problem and the homogenized problem.

Asymptotic expansion

- If \mathbf{x} and $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ denote, respectively, the macroscopic and microscopic variables, the asymptotic expansion method consists in approaching the unknown E^ε by the series, and resolve the new problem in $\Omega \times \mathcal{Z}$..

$$E^\varepsilon(\mathbf{x}) = E_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon E_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \varepsilon^2 E_2(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \dots, \quad (23)$$

- Plugging the asymptotic expansion in the equation, gathering the coefficients with the same power of ε , we get:

$$\left\{ \begin{array}{l} \frac{1}{\varepsilon^2} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \\ + \frac{1}{\varepsilon} [\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_1(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \\ + \nabla_{\mathbf{y}} \times \nabla_{\mathbf{x}} \times E_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})] \\ + \dots \end{array} \right\} = 0. \quad (24)$$

- ε is small, ε^i can be neglected, then we extract a cascade of equations

$$\left\{ \begin{array}{l} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) = 0 \\ \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_1(\mathbf{x}, \mathbf{y}) \\ + \nabla_{\mathbf{y}} \times \nabla_{\mathbf{x}} \times E_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) = 0 \end{array} \right. \quad (25)$$

Asymptotic expansion

- with the divergence equation

$$\nabla_{\mathbf{y}} \cdot (i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y})) = 0, \quad (26)$$

- and the boundary conditions

$$\begin{cases} \left(\frac{1}{\varepsilon} \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{x}} \times E_0(\mathbf{x}, \mathbf{y}) \right) \\ = -i\omega H_d \times n, \quad \mathbf{x} \in R^3, \quad \mathbf{y} \in \mathcal{Z}. \end{cases} \quad (27)$$

- Taking the first equation of the system and the divergence equation to obtain :

$$\begin{cases} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) = 0, \\ \nabla_{\mathbf{y}} \cdot \{i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y})\} = 0. \end{cases} \quad (28)$$

Asymptotic expansion

- Multiplying the first equation by E_0 and integrating by parts over \mathcal{Z} leads to

$$\begin{aligned} & \int_{\mathcal{Z}} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) E_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ &= \int_{\mathcal{Z}} |\nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y})|^2 \, d\mathbf{y} \\ &= 0. \end{aligned} \tag{29}$$

- We deduce that the equation is equivalent to

$$\nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) = 0, \tag{30}$$

for any $\mathbf{y} \in \mathcal{Z}$.

- $E_0(\mathbf{x}, \mathbf{y})$ can be decomposed as

$$E_0(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_0(\mathbf{x}, \mathbf{y}), \tag{31}$$

where $\Phi_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega; \mathbf{H}_{\#}^1(\mathcal{Z}))$ and where $E(\mathbf{x}) \in \mathbf{L}^2(\Omega)$ is the average of E_0 over the \mathbf{y} variable.

Two-scale convergence

- We show rigorously with two-scale convergence that the solution of problem converge to the solution of the homogenized problem when ε goes to 0.
- Two-scale convergence was developed by G Allaire, G.Nguetseng

- The main theorem

Théorème

(Nguetseng). Let $u^\varepsilon(\mathbf{x}) \in \mathbf{L}^2(\Omega)$. Suppose there exists a constant $c > 0$ such that for all ε

$$\|u^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq c. \quad (32)$$

- Then there exists a subsequence of ε (still denoted u^ε) and $u_0(\mathbf{x}, \mathbf{y}) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$ such that:

$$u^\varepsilon(\mathbf{x}) \text{ two-scale converges to } u_0(\mathbf{x}, \mathbf{y}). \quad (33)$$

- we have for all $v(\mathbf{x}) \in C_0(\overline{\Omega})$ and all $w(\mathbf{y}) \in \mathbf{L}^2_\#(\mathcal{Z})$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u^\varepsilon(\mathbf{x}) \cdot v(\mathbf{x}) w\left(\frac{\mathbf{x}}{\varepsilon}\right) d\mathbf{x} \\ = \int_{\Omega} \int_{\mathcal{Z}} u_0(\mathbf{x}, \mathbf{y}) \cdot v(\mathbf{x}) w(\mathbf{y}) d\mathbf{x} d\mathbf{y}. \end{aligned} \quad (34)$$

- We have

Théorème

(Estimate) For any $\varepsilon > 0$, for any $\eta \geq 0$, there exists a positive constant ω_0 which does not depend on ε and such that for all $\omega \in (0, \omega_0)$, $E^\varepsilon \in X^\varepsilon(\Omega)$ solution of the regularized problem, satisfies

$$\|E^\varepsilon\|_{\mathbf{X}^\varepsilon(\Omega)} \leq C \quad (35)$$

with $C = \frac{c_{\gamma t} c_{\gamma T}}{c_0} \|H_d\|_{H(\text{curl}, \Omega)}$.

Théorème

Under assumptions of Theorem Estimate, sequence E^ε solution of the initial problem, converges to $E(\mathbf{x}) \in \mathbf{L}^2(\Omega)$ which is the unique solution of the homogenized problem:

$$\begin{cases} \theta_1 \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) + i\omega\theta_2 E(\mathbf{x}) = 0 & \text{in } \Omega, \\ \theta_1 \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times \mathbf{e}_2 = -i\omega H_d \times \mathbf{e}_2 & \text{on } \Gamma_d, \\ \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times \mathbf{e}_2 = 0 & \text{on } \Gamma_L. \end{cases} \quad (36)$$

with $\theta_1 = \int_{\mathcal{Z}} \text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y}) \, d\mathbf{y}$ and $\theta_2 = \int_{\mathcal{Z}} \mathbf{1}_C(\mathbf{y})(\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \, d\mathbf{y}$. And where the scalar function χ is the unique solution, up to an additive constant in the Hilbert space of \mathcal{Z} periodic functions $H_{\#}^1(\mathcal{Z})$, of

$$\begin{cases} \Delta_{\mathbf{y}}(\chi(\mathbf{y})) = 0 & \text{in } \mathcal{Z} \setminus \partial\Omega_C, \\ \left[\frac{\partial \chi}{\partial n} \right] = -n_j & \text{on } \partial\Omega_C \text{ and } [\chi] = 0 & \text{on } \partial\Omega_C. \end{cases} \quad (37)$$

where $[f]$ is the jump across the surface of $\partial\Omega_C$, n_j , $j = \{1, 2, 3\}$ is the projection on the axis \mathbf{e}_j of the normal of $\partial\Omega_C$.

- Step 1: Two-scale convergence.** Due to the estimate, E^ε is bounded in $\mathbf{L}^2(\Omega)$. Hence, up to a subsequence, E^ε two-scale converges to $E_0(\mathbf{x}, \mathbf{y})$ belonging to $\mathbf{L}^2(\Omega, \mathbf{L}^2_\#(\mathcal{Z}))$. That means for any $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}_0^1(\Omega, \mathbf{C}_\#^1(\mathcal{Z}))$, we have:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} E^{\varepsilon}(\mathbf{x}) \cdot V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) d\mathbf{x} = \int_{\Omega} \int_{\mathcal{Z}} E_0(\mathbf{x}, \mathbf{y}) \cdot V(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x}. \quad (38)$$

- Step 2: Deduction of the constraint equation.** We multiply Equation by oscillating test function $V^\varepsilon(\mathbf{x}) = V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$ where $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}_0^1(\Omega, \mathbf{C}_\#^1(\mathcal{Z}))$:

$$\begin{aligned} & \int_{\Omega} \nabla \times E^{\varepsilon}(\mathbf{x}) \cdot (\nabla_{\mathbf{x}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) + \frac{1}{\varepsilon} \nabla_{\mathbf{y}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \\ & + [-\omega^2 \varepsilon^5 k(\epsilon) + i\omega(\mathbf{1}_C^{\varepsilon}(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4 \mathbf{1}_R^{\varepsilon}(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y < 0\}} \\ & + \varepsilon \mathbf{1}_{\{y > 0\}}] E^{\varepsilon} \cdot V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) d\mathbf{x} \\ & = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times V(x, 1, z, \xi, \frac{1}{\varepsilon}, \zeta)) \times \mathbf{e}_2 d\sigma. \end{aligned} \quad (39)$$

- Integrating by parts, we get:

$$\begin{aligned}
 & \int_{\Omega} E^{\varepsilon}(\mathbf{x}) \cdot (\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{x}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \\
 & + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon^2} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) + [-\omega^2 \varepsilon^5 k(\varepsilon) \\
 & + i\omega(\mathbf{1}_C^{\varepsilon}(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4 \mathbf{1}_R^{\varepsilon}(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y < 0\}} + \varepsilon \mathbf{1}_{\{y > 0\}}] E^{\varepsilon}(\mathbf{x}) \cdot V^{\varepsilon}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) d\mathbf{x} \\
 & = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times V(x, 1, z, \xi, \frac{1}{\varepsilon}, \zeta)) \times \mathbf{e}_2 d\sigma.
 \end{aligned} \tag{40}$$

- Now we multiply (40) by ε^2 and we pass to the two-scale limit, applying Theorem (Nguetseng) we obtain:

$$\int_{\Omega} \int_{\mathcal{Z}} E_0(\mathbf{x}, \mathbf{y}) (\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \mathbf{y})) d\mathbf{y} d\mathbf{x} = 0. \tag{41}$$

- We deduce the constraint equation for the profile E_0 :

$$\nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times E_0(\mathbf{x}, \mathbf{y}) = 0. \tag{42}$$

- **Step 3. Looking for the solutions to the constraint equation.**
- It is the same demonstration than asymptotic expansion then we conclude that $E_0(\mathbf{x}, \mathbf{y})$ can be decomposed as

$$E_0(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_0(\mathbf{x}, \mathbf{y}). \quad (43)$$

- **Step 4. Equations for $E(\mathbf{x})$ and $\Phi_0(\mathbf{x}, \mathbf{y})$.**
- The divergence equation is multiplied with $V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = \varepsilon v(\mathbf{x}) \psi(\frac{\mathbf{x}}{\varepsilon})$, where $v \in \mathbf{C}_0^1(\Omega)$ and $\psi \in \mathbf{H}_{\#}^1(\mathcal{Z})$. Theorem Nguetseng and integration by parts yields for all $\psi \in \mathbf{H}_{\#}^1(\mathcal{Z})$ and $v \in \mathbf{C}_0^1(\Omega)$

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \cdot \{ -\omega^2 \varepsilon^5 k(\varepsilon) E^{\varepsilon}(\mathbf{x}) + i\omega [(\mathbf{1}_C(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4 \mathbf{1}_R(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y < 0\}} + \varepsilon \mathbf{1}_{\{y > 0\}}] E^{\varepsilon} \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \{ -\omega^2 \varepsilon^5 k(\varepsilon) E^{\varepsilon}(\mathbf{x}) + i\omega [\mathbf{1}_C(\frac{\mathbf{x}}{\varepsilon}) + \varepsilon^4 \mathbf{1}_R(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y < 0\}} \\ &+ \varepsilon \mathbf{1}_{\{y > 0\}}] E^{\varepsilon} \} \cdot (\varepsilon \nabla v(\mathbf{x}) \psi(\frac{\mathbf{x}}{\varepsilon}) + v(\mathbf{x}) \nabla_{\mathbf{y}} \psi(\frac{\mathbf{x}}{\varepsilon})) d\mathbf{x} \\ &= - \int_{\Omega} \int_{\mathcal{Z}} v(\mathbf{x}) \nabla_{\mathbf{y}} \psi(\mathbf{y}) \cdot [i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y})] d\mathbf{y} d\mathbf{x} = 0. \end{aligned} \quad (44)$$

- from which it follows that

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y})] = 0. \quad (45)$$

- with E_0 given by the decomposition (43). So we obtain the local equation

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) \{E(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_0(\mathbf{x}, \mathbf{y})\}] d\mathbf{y} = 0. \quad (46)$$

- The potential Φ_0 may be written on the form

$$\Phi_0(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^3 \chi_j(\mathbf{y}) e_j \cdot E(\mathbf{x}) = \chi(\mathbf{y}) \cdot E(\mathbf{x}), \quad (47)$$

- we get:

$$E_0(\mathbf{x}, \mathbf{y}) = (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x}). \quad (48)$$

- Inserting E_0 in the divergence equation, we obtain

$$\nabla_{\mathbf{y}} \cdot [i\omega \mathbf{1}_C(\mathbf{y}) (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y}))] = 0. \quad (49)$$

- We build oscillating test functions satisfying the constraint and use them in weak formulation. We define test function $V(\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x}, \mathbf{y})$, $V(\mathbf{x}, \mathbf{y}) \in \mathbf{C}_0^1(\Omega, \mathbf{C}_{\#}^1(\mathcal{Z}))$ and we inject in (40) test function $V^\varepsilon = V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$, which gives:

$$\begin{aligned} & \int_{\Omega} E^\varepsilon(\mathbf{x}) \cdot \left(\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) + \frac{2}{\varepsilon} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right. \\ & \quad \left. + \frac{1}{\varepsilon^2} \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right) + [-\omega^2 \varepsilon^5 k(\varepsilon) + i\omega((\mathbf{1}_C^\varepsilon(\frac{\mathbf{x}}{\varepsilon}) \\ & \quad + \varepsilon^4 \mathbf{1}_R^\varepsilon(\frac{\mathbf{x}}{\varepsilon})) \mathbf{1}_{\{y < 0\}} + \varepsilon \mathbf{1}_{\{y > 0\}})] E^\varepsilon(\mathbf{x}) \cdot V(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) d\mathbf{x} \\ & = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times V^\dagger(\mathbf{x}, 1, z, \xi, \zeta)) \times \mathbf{e}_2 d\sigma, \end{aligned} \quad (50)$$

- with $V(\mathbf{x}, 1, z, \xi, \nu, \zeta) = V^\dagger(\mathbf{x}, 1, z, \xi, \zeta)$ the restriction on V which does not depend on ν . The term containing the constraint, the third one, disappears. Passing to the limit $\varepsilon \rightarrow 0$ and replacing the expression of V by the term $\alpha(\mathbf{x}) + \nabla_{\mathbf{y}}\beta(\mathbf{x}, \mathbf{y})$, we have

$$\begin{aligned} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times V(\mathbf{x}, \mathbf{y}) &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\alpha(\mathbf{x})) + \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}\beta(\mathbf{x}, \mathbf{y})) \\ &= \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}\beta(\mathbf{x}, \mathbf{y})) \end{aligned}$$

- Since $\nabla_{\mathbf{y}} \times (\nabla_{\mathbf{y}}) = 0$, the term $\frac{2}{\varepsilon} \nabla_{\mathbf{x}} \times \nabla_{\mathbf{y}} \times \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})$ vanishes. Therefore, (50) becomes:

$$\begin{aligned}
 & \int_{\Omega} \int_{\mathcal{Z}} E_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \\
 & \quad + i\omega \mathbf{1}_C(\mathbf{y}) E_0(\mathbf{x}, \mathbf{y}) \cdot (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \, dy d\mathbf{x} \\
 & = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times (\alpha(x, 1, z) + \nabla_{\mathbf{y}} \beta(x, 1, z, \xi, \zeta))) \times \mathbf{e}_2 \, d\sigma.
 \end{aligned} \tag{52}$$

- Now in (52) we replace expression E_0 giving by (48). We obtain

$$\begin{aligned}
 & \int_{\Omega} \int_{\mathcal{Z}} (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x}) \cdot (\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \\
 & \quad + i\omega \mathbf{1}_C(\mathbf{y}) (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x})) \cdot (\alpha(\mathbf{x}) + \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y})) \, dy d\mathbf{x} \\
 & = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times (\alpha(x, 1, z) + \nabla_{\mathbf{y}} \beta(x, 1, z, \xi, \zeta))) \times \mathbf{e}_2 \, d\sigma.
 \end{aligned} \tag{53}$$

- Taking $\alpha(\mathbf{x}) = 0$ in weak formulation, we obtain

$$\begin{aligned} \int_{\Omega} \int_{\mathcal{Z}} (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y}) \\ + i\omega \mathbf{1}_C(\mathbf{y}) (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x}) \cdot \nabla_{\mathbf{y}} \beta(\mathbf{x}, \mathbf{y}) d\mathbf{y} d\mathbf{x} = 0. \end{aligned} \quad (54)$$

- Integrating by parts

$$\begin{aligned} \int_{\Omega} \int_{\mathcal{Z}} & -\nabla_{\mathbf{y}} \cdot \{(\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x})\} \beta(\mathbf{x}, \mathbf{y}) \\ & - i\omega \nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})(\text{Id} - \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x})\} \beta(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} d\mathbf{x} = 0. \end{aligned} \quad (55)$$

- And since $\nabla_{\mathbf{y}} \cdot \{\mathbf{1}_C(\mathbf{y})(\text{Id} + \nabla_{\mathbf{y}}\chi(\mathbf{y}))E(\mathbf{x})\} = 0$ we obtain

$$\int_{\Omega} \int_{\mathcal{Z}} -\nabla_{\mathbf{y}} \cdot \{(\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x})\} \beta(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} d\mathbf{x} = 0. \quad (56)$$

- which gives the cell problem

$$\nabla_{\mathbf{y}} \cdot [\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})] = 0. \quad (57)$$

- From (49) and (57), the scalar function χ is the unique solution, thanks to Lax-Milgram Lemma, up to an additive constant in the Hilbert space of \mathcal{Z} periodic function $H_{\#}^1(\mathcal{Z})$ of the following boundary value problem

$$\begin{cases} \Delta_{\mathbf{y}}(\chi(\mathbf{y})) = 0 & \text{in } \mathcal{Z} \setminus \partial\Omega_C, \\ [\frac{\partial\chi}{\partial n}] = -n_j & \text{on } \partial\Omega_C, \\ [\chi] = 0 & \text{on } \partial\Omega_C. \end{cases} \quad (58)$$

- where $[f]$ is the jump across the surface of $\partial\Omega_C$, n_j , $j = \{1, 2, 3\}$ is the projection on the axis e_j of the normal of $\partial\Omega_C$. (58) can be seen as an electrostatic problem. Solving (49) and (57) reduces to look for a potential induced by surface density of charges. Then χ is this potential induced by the charges on the interface of carbon fiber.

- Setting $\beta(\mathbf{x}, \mathbf{y}) = 0$ in (53) and integrating by parts, we get

$$\begin{aligned} \int_{\Omega} \int_{\mathcal{Z}} (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) \cdot \alpha(\mathbf{x}) \\ + i\omega \mathbf{1}_C(\mathbf{y}) (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) E(\mathbf{x}) \alpha(\mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\ = -i\omega \int_{\Gamma_d} H_d \times \mathbf{e}_2 \cdot (\mathbf{e}_2 \times \alpha(x, 1, z)) \times \mathbf{e}_2 \, d\sigma. \end{aligned} \quad (59)$$

- which gives the following well posed problem for $E(\mathbf{x})$

$$\begin{cases} \theta_1 \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times E(\mathbf{x}) + i\omega \theta_2 E(\mathbf{x}) = 0 & \text{in } \Omega, \\ \theta_1 \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times \mathbf{e}_2 = -i\omega H_d \times \mathbf{e}_2 & \text{on } \Gamma_d, \\ \nabla_{\mathbf{x}} \times E(\mathbf{x}) \times \mathbf{e}_2 = 0 & \text{on } \Gamma_L. \end{cases} \quad (60)$$

- with $\theta_1 = \int_{\mathcal{Z}} \text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y}) \, d\mathbf{y}$ and $\theta_2 = \int_{\mathcal{Z}} \mathbf{1}_C(\mathbf{y}) (\text{Id} + \nabla_{\mathbf{y}} \chi(\mathbf{y})) \, d\mathbf{y}$.

Numerical simulations

- Numerical simulations were done using FreeFem++, it is a software to solve numerically partial differential equations (PDE) with finite elements methods.
- We resolve cell problem using Lagrange P2 finite elements.

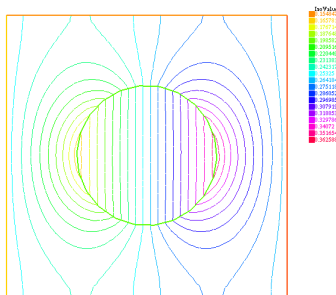


Figure: Plot of χ_1

Numerical simulations

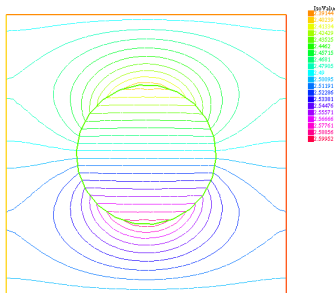


Figure: Plot of χ^2

Thank you