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# Simplest examples of inverse scattering on the plane at fixed energy\*

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We consider the inverse scattering problem for the two-dimensional Schrödinger equation at fixed positive energy. Our results include inverse scattering reconstructions from the simplest scattering amplitudes. In particular, we give a complete analytic solution of the phased and phaseless inverse scattering problems for the single-point potentials (of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type). Then we study numerical inverse scattering reconstructions from the simplest scattering amplitudes using the Riemann-Hilbert-Manakov problem of the soliton theory. Finally, we apply the later numerical inverse scattering results for constructing related numerical solutions for equations of the Novikov-Veselov hierarchy at fixed positive energy.

**Keywords:** inverse scattering, Schrödinger equation, numerical analysis, Novikov-Veselov equation

**Subjects:** 35R30 (inverse problems for PDEs), 65N21 (numerical analysis of inverse problems for PDEs), 35P25 (scattering theory), 35J10 (Schrödinger operator), 35Q53 (KdV-like equations);

## 1 Introduction

We consider the two-dimensional Schrödinger equation at fixed positive energy  $E$ :

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E > 0, \quad (1)$$

where  $v$  is a real-valued sufficiently regular potential on  $\mathbb{R}^2$  with sufficient decay at infinity. For this equation we consider the classical scattering solutions  $\psi^+ = \psi^+(x, k)$ , specified by the following asymptotics:

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + C(|k|) \frac{e^{i|k||x|}}{|x|^{1/2}} f(k, |k| \frac{x}{|x|}) + o(|x|^{-\frac{1}{2}}), \quad |x| \rightarrow +\infty, \\ x \in \mathbb{R}^2, \quad k \in \mathbb{R}^2, \quad k^2 = E, \quad C(|k|) &= -\pi i \sqrt{2\pi} e^{-i\pi/4} |k|^{\frac{1}{2}}, \end{aligned} \quad (2)$$

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\*Dedicated to S. P. Novikov on the occasion of his 80th birthday

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with a priori unknown coefficient  $f$ . The function  $f$  of (2) is known as the scattering amplitude for equation (1) and is defined on

$$\mathcal{M}_E = \mathbb{S}^1_{\sqrt{E}} \times \mathbb{S}^1_{\sqrt{E}}, \quad (3)$$

$$\mathbb{S}^1_{\sqrt{E}} = \{m \in \mathbb{R}^2 \mid m^2 = E\}. \quad (4)$$

It is known that  $f$  possesses the following properties:

$$f(k, l) = f(-l, -k) \quad (\text{reciprocity}), \quad (5)$$

$$f(k, l) - \overline{f(l, k)} = \frac{\pi}{i\sqrt{E}} \int_{\mathbb{S}^1_{\sqrt{E}}} f(k, m) \overline{f(l, m)} dm \quad (\text{unitarity}), \quad (6)$$

where  $k, l \in \mathbb{S}^1_{\sqrt{E}}$ . For possible assumptions on  $v$  assuring existence and uniqueness of  $\psi^+$  at fixed  $k$  and properties (5), (6) for  $f$  see, e.g., [14, 6, 10].

Note also that

$$f_y(k, l) = e^{i(k-l)y} f(k, l), \quad k, l \in \mathbb{S}^1_{\sqrt{E}}, \quad (7)$$

where  $f$  is the scattering amplitude for  $v$  and  $f_y$  is the scattering amplitude for the translated potential  $v_y = v(\cdot - y)$ ,  $y \in \mathbb{R}^2$ .

For equation (1), the problem of finding  $\psi^+$ ,  $f$  from  $v$  is known as the direct scattering problem; the problem of finding  $v$  from  $f$  is known as the inverse scattering problem; and the problem of finding  $v$  from  $|f|^2$  is known as the phaseless inverse scattering problem.

In addition to equation (1) we consider its isospectral deformations at fixed  $E$  given by the Novikov-Veselov equation and its higher order analogs, see [16, 17, 6]. These equations admit a representation in the form of L-A-B Manakov triple (introduced in [11]), where  $L = -\Delta + v - E$ . We recall that the first non-trivial equation of the Novikov-Veselov hierarchy can be written as:

$$\begin{aligned} \partial_t v &= 4 \operatorname{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ \partial_{\bar{z}} w &= -3\partial_z v, \quad w = w(x, t), \quad w(x, t) \rightarrow 0, \quad |x| \rightarrow +\infty, \\ E &> 0, \quad t \in \mathbb{R}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \quad (8)$$

where  $v = v(x, t)$ ,  $w = w(x, t)$ ,  $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ ,  $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ .

Analog of the Gardner-Green-Kruskal-Miura relations for the equations of the Novikov-Veselov hierarchy and for the scattering amplitude  $f = f(k, l, t)$  are as follows:

$$\begin{aligned} f(k, l, t) &= \exp(2itE^{\frac{2n+1}{2}} (\cos((2n+1)\varphi_k) - \cos((2n+1)\varphi_l))) f(k, l, 0), \\ k &= \sqrt{E}(\cos \varphi_k, \sin \varphi_k), \quad l = \sqrt{E}(\cos \varphi_l, \sin \varphi_l), \end{aligned} \quad (9)$$

where  $n$  is the number of equation in the hierarchy; see [18] for the classical Gardner-Green-Kruskal-Miura relations and [13, 8] for (9).

**Remark 1.** (a) Properties (5), (6) are invariant with respect to transformations of  $f$  given by (7) and (9). (b) The differential scattering cross-section  $|f|^2$  is invariant with respect to transformations of  $f$  given by (7) and (9).

In [13, 14] it was shown that if  $f$  is smooth, satisfies (5), (6) and

$$\|f\|_{L^2(\mathcal{M}_E)} < \frac{\sqrt{E}}{3\pi}, \quad (10)$$

then there is a smooth, real-valued, decaying at infinity potential  $v$  such that  $f$  is the scattering amplitude for  $v$  at fixed  $E$ . In addition, this  $v$  is reconstructed from  $f$  via the algorithm suggested in [13, 14] and simplified in [15]. This final algorithm of [15] is recalled in Section 2.

Note that these results of [13, 14, 15] are obtained using, in particular, the Riemann-Hilbert-Manakov problem of the soliton theory (see [12]) and results of [5] and [7]. In turn, the algorithm of [15] is implemented numerically in [3].

The results of the present work include inverse scattering reconstructions from some simplest functions  $f$  satisfying (5), (6) at fixed  $E$ . In particular, in this framework we give a complete analytic solution of the phased and phaseless inverse scattering problems for the single-point potentials (of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type)  $v_{\alpha,y}(x)$ ,  $\alpha \in \mathbb{R} \cup \{\infty\}$ ,  $y \in \mathbb{R}^2$ , see Subsection 4.1.

Then we give numerical inverse scattering reconstructions from some simplest scattering amplitudes  $f$  satisfying (5), (6) at fixed  $E > 0$  using the numerical implementation (in MATLAB) of [3] of the algorithm recalled in Section 2.

First of all, in this connection, we study reconstructions from constant  $f$  satisfying (5), (6) at fixed  $E$ . In particular, such  $f$  arise as scattering amplitudes of the single-point potentials  $v_{\alpha,y}(x)$  for  $y = 0$  (i.e. supported at zero). Note that already for this simplest case there are no explicit analytic reconstruction formulas for regular potentials. Our numerical results for this case develop studies of [4, 2]. These results are presented in details in Subsection 4.2.

Then, using the numerical inverse scattering implementation of [3], we study reconstructions from functions  $f$  arising as scattering amplitudes of multi-point potentials (scatterers)

$$v(x) = \sum_{j=1}^N v_{\alpha_j, y_j}(x), \quad x \in \mathbb{R}^2, \alpha_j \in \mathbb{R}, y_j \in \mathbb{R}^2, \quad (11)$$

consisting of  $N$  single-point scatterers  $v_{\alpha_j, y_j}(x)$ , where each point scatterer is described by its internal parameter  $\alpha_j$  and position  $y_j$  (and  $y_i \neq y_j$  for  $i \neq j$ ). This can be also considered as the first use of the multi-point potentials of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type for testing inverse scattering algorithms. Possibility of such tests was mentioned in [10]. These results are presented in details in Subsection 5.1.

We emphasize that our aforementioned numerical reconstructions are obtained using the results of [15, 3] and can be considered as regular approximations to the initial multi-point potentials which are quite singular.

Finally, using the scattering amplitudes for the multi-point potentials, relations (9) and the inverse scattering implementation of [3] we obtain the related numerical solutions for equations of the Novikov-Veselov hierarchy. In particular, these results also illustrate non-uniqueness in the inverse scattering problem without phase information at fixed energy. See Subsection 5.2 for details.

## 2 Inverse scattering algorithm

It is convenient to use the following notations:

$$\begin{aligned} z &= x_1 + ix_2, & \bar{z} &= x_1 - ix_2, \\ \lambda &= E^{-1/2}(k_1 + ik_2), & \lambda' &= E^{-1/2}(l_1 + il_2), \end{aligned} \quad (12)$$

where  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $k = (k_1, k_2) \in \mathbb{S}_{\sqrt{E}}^1$ ,  $l = (l_1, l_2) \in \mathbb{S}_{\sqrt{E}}^1$ . In these notations

$$\begin{aligned} k_1 &= \frac{1}{2}E^{1/2}(\lambda + \lambda^{-1}), & k_2 &= \frac{i}{2}E^{1/2}(\lambda^{-1} - \lambda), \\ l_1 &= \frac{1}{2}E^{1/2}(\lambda' + \lambda'^{-1}), & l_2 &= \frac{i}{2}E^{1/2}(\lambda'^{-1} - \lambda'), \end{aligned} \quad (13)$$

where  $\lambda, \lambda' \in T$ ,

$$T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}. \quad (14)$$

Using formulas (3), (4), (12), (13), (14) one can see that

$$\mathbb{S}_{\sqrt{E}}^1 \cong T, \quad \mathcal{M}_E \cong T \times T. \quad (15)$$

In addition, in these notations functions  $\psi^+$ ,  $f$  of (2) can be written as

$$\psi^+ = \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \quad (16)$$

where  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

The algorithm of [15] for finding  $v$  on  $\mathbb{R}^2$  from  $f$  on  $\mathcal{M}_E$  has the following scheme:

$$f \longrightarrow h_{\pm} \longrightarrow \mu^+ \longrightarrow \mu_- \longrightarrow v, \quad (17)$$

and consists of the following steps:

Step 1. Find functions  $h_{\pm}(\lambda, \lambda', E)$ ,  $\lambda, \lambda' \in T$ , from the following linear integral equations:

$$\begin{aligned} h_{\pm}(\lambda, \lambda', E) - \pi \int_T h_{\pm}(\lambda, \lambda'', E) \chi\left(\pm i \left[\frac{\lambda}{\lambda''} - \frac{\lambda'}{\lambda}\right]\right) \times \\ \times f(\lambda'', \lambda', E) |d\lambda''| = f(\lambda, \lambda', E), \end{aligned} \quad (18)$$

where

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases} \quad (19)$$

Step 2. Solve the following linear integral equation for  $\mu^+(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ ,  $E > 0$ :

$$\mu^+(z, \lambda, E) + \int_T B(\lambda, \lambda', z, E) \mu^+(z, \lambda', E) |d\lambda'| = 1, \quad (20)$$

where

$$\begin{aligned} B(\lambda, \lambda', z, E) &= \frac{1}{2} \int_T h_-(\zeta, \lambda', z, E) \chi \left( -i \left[ \frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1-0)} \\ &\quad - \frac{1}{2} \int_T h_+(\zeta, \lambda', z, E) \chi \left( i \left[ \frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1+0)}, \end{aligned} \quad (21)$$

$$\begin{aligned} h_{\pm}(\lambda, \lambda', z, E) &= h_{\pm}(\lambda, \lambda', E) \times \\ &\quad \times \exp \left( -i \frac{\sqrt{E}}{2} ((\lambda - \lambda') \bar{z} - (\lambda^{-1} - \lambda'^{-1}) z) \right), \end{aligned} \quad (22)$$

and  $\lambda, \lambda' \in T$ ,  $z \in \mathbb{C}$ ,  $E > 0$ .

Step 3. Define function  $\mu_-(z, \lambda, E)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ ,  $E > 0$ , by the formula

$$\begin{aligned} \mu_-(z, \lambda, E) &= \mu^+(z, \lambda, E) + \pi i \int_T h_-(\lambda, \lambda', z, E) \times \\ &\quad \times \chi \left( -i \left[ \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) \mu^+(z, \lambda', E) |d\lambda'|, \end{aligned} \quad (23)$$

where function  $h_-(\lambda, \lambda', z, E)$  is given by (22) and  $\chi$  is defined by (19).

Step 4. Potential  $v = v(x, E)$ ,  $x \in \mathbb{R}^2$ ,  $E > 0$ , is given by the formula

$$v(x, E) = \frac{\sqrt{E}}{\pi} \int_T \partial_z \mu_-(z, \zeta, E) d\zeta, \quad (24)$$

where  $z = x_1 + ix_2$ ,  $x = (x_1, x_2)$ ,  $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ .

**Remark 2.** As it was mentioned in the introduction, if  $f$  satisfies (5), (6), (10), then there exists a smooth real-valued decaying at infinity potential  $v$  such that  $f$  is the scattering amplitude for  $v$  at fixed  $E > 0$ . In this result condition (10) can be replaced by a much weaker condition that all integral equations in (18), (20) are uniquely solvable. In addition, in notations of the present section condition (10) can be written as

$$\|f\|_{L^2(T \times T)} < \frac{1}{3\pi}. \quad (25)$$

Note also that

$$v(x, \tau^2 E) = \tau^2 v(\tau x, E), \quad E > 0, \tau > 0, \quad (26)$$

for  $v(x, E)$  reconstructed via (17) from  $f$  which is independent of  $E$ , i.e.  $f = f(\lambda, \lambda')$ .

### 3 Scattering functions for multi-point potentials

The scattering theory for multi-point potentials  $v$  mentioned in formula (11) of the introduction is presented, in particular, in [1, 9, 10]. In addition, all single-point potentials  $v_{\alpha,y}(x)$ ,  $\alpha \in \mathbb{R} \setminus \{0\}$ ,  $x, y \in \mathbb{R}^2$ , can be considered as renormalizations of delta functions  $\varepsilon\delta(x-y)$  with negative coefficients  $\varepsilon$ .

We recall that for the multi-point potentials  $v$  of formula (11) the classical scattering functions  $\psi^+$  and  $f$  are given by explicit formulas as follows.

For the classical scattering eigenfunctions  $\psi^+$  the following formulas hold:

$$\psi^+(x, k) = e^{ikx} + \sum_{j=1}^N q_j^+(k) G^+(x - y_j, k), \quad (27)$$

$$x \in \mathbb{R}^2, k \in \mathbb{S}_{\sqrt{E}}^1, y_j \in \mathbb{R}^2, y_j \neq y_m \text{ for } j \neq m,$$

$$G^+(x, k) = -\frac{i}{4} H_0^{(1)}(|x||k|), \quad x \in \mathbb{R}^2, k \in \mathbb{S}_{\sqrt{E}}^1, \quad (28)$$

where  $H_0^{(1)}$  is the Hankel function of the first kind of order zero and  $q^+(k) = (q_1^+(k), \dots, q_N^+(k))$  is the solution of the following linear system:

$$A^+(k)q^+(k) = b^+(k), \quad (29)$$

where  $A^+(k) \in M_N(\mathbb{C})$ ,  $b^+(k) \in \mathbb{C}^N$  are given by

$$A_{m,j}^+(k) = \begin{cases} 1 + \frac{\alpha_m}{4\pi}(\pi i - 2 \ln |k|), & m = j, \\ -\alpha_m G^+(y_m - y_j, k), & m \neq j, \end{cases} \quad (30)$$

$$b^+(k) = (\alpha_1 e^{iky_1}, \dots, \alpha_N e^{iky_N}), \quad \alpha_1, \dots, \alpha_N \in \mathbb{R}. \quad (31)$$

For the classical scattering amplitude  $f$  the following formula holds:

$$f(k, l) = \frac{1}{(2\pi)^2} \sum_{j=1}^N q_j^+(k) e^{-ily_j}, \quad k, l \in \mathbb{S}_{\sqrt{E}}^1, \quad (32)$$

where  $q_j^+(k)$  are the same as in (27), (29).

## 4 Reconstructions from constant $f$

### 4.1 Analytic inverse scattering for the single-point potentials

The simplest functions on  $\mathcal{M}_E$  are constants. Therefore, the results given below in this section are of particular interest.

**Lemma 1.** *Let  $f \equiv f_0$  on  $\mathcal{M}_E$  for fixed  $E > 0$ , where  $f_0$  is a complex constant. Then  $f$  satisfies (5), (6) if and only if  $f_0 \in S$ , where*

$$\begin{aligned} S &= \{ \zeta \in \mathbb{C} \mid \zeta - \bar{\zeta} = -2i\pi^2 \zeta \bar{\zeta} \} \\ &= \{ \zeta \in \mathbb{C} \mid |\zeta + \frac{i}{2\pi^2}| = \frac{1}{2\pi^2} \}. \end{aligned} \quad (33)$$

Lemma 1 follows from direct substitution of  $f_0$  into (5), (6).

One can see that  $S$  is the circle centered at  $-\frac{i}{2\pi^2}$  of radius  $\frac{1}{2\pi^2}$ .

Using (29), (32) (for  $N = 1, y_1 = 0$ ) one can see that the scattering amplitude for the single point potential  $v_{\alpha,y}, y = 0$ , at fixed energy  $E$  is given by the following formula:

$$f(k, l) \equiv f_\alpha(E),$$

$$f_\alpha(E) = \frac{1}{(2\pi)^2} \frac{\alpha}{1 + \frac{\alpha}{4\pi}(\pi i - \ln E)}, \quad E > 0, \alpha \in \mathbb{R}. \quad (34)$$

Assuming that  $f_\alpha(E)$  is defined as in (34), we have the following result.

**Theorem 1.** *Let  $\zeta = f_\alpha(E)$  for fixed  $E > 0$ . Then  $\zeta \in S$  for any  $\alpha \in \mathbb{R} \cup \{\infty\}$ . Conversely, for any  $\zeta \in S$  there exists the unique  $\alpha \in \mathbb{R} \cup \{\infty\}$  such that  $\zeta = f_\alpha(E)$  and this  $\alpha$  is given by the following formula:*

$$\alpha = \frac{(2\pi)^2 \zeta}{1 - \pi \zeta (\pi i - \ln E)}. \quad (35)$$

*Proof of Theorem 1.* The fact that the scattering amplitudes  $f$  (of Section 3) for the multi-point potentials satisfy (5), (6), the corollary of (34) that  $f = f_\alpha(E)$  is constant at fixed  $E$  and  $\alpha$ , and Lemma 1 imply that  $\zeta \in S$  if  $\zeta = f_\alpha(E)$ . The property that  $\zeta \in S$  if  $\zeta = f_\alpha(E)$  can also be verified by the direct calculation using the precise formula for  $f_\alpha(E)$  in (34).

Conversely, consider the equation

$$f_\alpha(E) = \zeta \quad \text{with respect to } \alpha \in \mathbb{C} \cup \{\infty\}, \quad (36)$$

for fixed  $\zeta \in S$  and  $E > 0$ . One can see that this equation is uniquely solvable and that the solution is given by formula (35). Direct calculations also show that  $\alpha = \bar{\alpha}$ .

Theorem 1 is proved.  $\square$

**Remark 3.** Theorem 1 gives a complete solution of the inverse scattering problem for the single point potentials  $v_{\alpha,y}, \alpha \in \mathbb{R} \cup \{\infty\}, y = 0$  (uniqueness, reconstruction, characterization). In view of formula (7), this solution admits a straightforward generalization to the case of  $v_{\alpha,y}$  with  $\alpha \in \mathbb{R} \cup \{\infty\}, y \in \mathbb{R}^2$ .

**Remark 4.** The property that  $\zeta \in S$  if  $\zeta = f_\alpha(E)$  can be considered as a relationship between the amplitude and phase of a single point scatterer  $v_{\alpha,y}, \alpha \in \mathbb{R}, y = 0$ . For a single point potential centered at zero, such a relationship was obtained in [4, 2] in the form:

$$\sin \phi = -|\beta|/4, \quad \text{where}$$

$$\beta = |\beta| \exp(i\phi), \quad \beta = (2\pi)^2 f(k, l). \quad (37)$$

However, in [4, 2] this relation is not yet related to the unitarity property (6) of the scattering amplitude  $f$ .



In addition, Theorem 1 implies the following corollary for inverse scattering without phase information.

**Corollary 1.** *Let  $E > 0$  be fixed. Then for any  $\sigma \in [0, \frac{1}{\pi^4}]$  the values of parameter  $\alpha \in \mathbb{R} \cup \{\infty\}$  such that  $\sigma = |f_\alpha(E)|^2$  are given by (35) with*

$$\zeta = \pm \sqrt{\sigma(1 - \sigma\pi^4)} - i\sigma\pi^2, \quad (38)$$

where the expression in (38) is single-valued for  $\sigma = 0$  and  $\sigma = \frac{1}{\pi^4}$  and is double-valued for  $\sigma \in (0, \frac{1}{\pi^4})$ . In addition, for any  $\sigma \in (\frac{1}{\pi^4}, \infty)$  there exists no  $\alpha \in \mathbb{R} \cup \{\infty\}$  such that  $\sigma = |f_\alpha(E)|^2$ .

**Remark 5.** Let  $\zeta = f_{\alpha_1}(E_1)$  for some fixed  $\alpha_1 \in \mathbb{R} \cup \{\infty\}$ ,  $E_1 > 0$ . Then for any  $E_2 > 0$  there exists the unique  $\alpha_2 \in \mathbb{R} \cup \{\infty\}$  such that  $\zeta = f_{\alpha_2}(E_2)$ , and this  $\alpha_2$  is given by the following formula:

$$\alpha_2 = \frac{\alpha_1}{1 + \frac{\alpha_1}{4\pi} \ln \frac{E_2}{E_1}}. \quad (39)$$

## 4.2 Numerical reconstructions from $f \in S$

In contrast to Theorem 1, we have no explicit formula for finding a regular potential  $v(x, E)$  with constant scattering amplitude  $f \in S \setminus \{0\}$  at fixed energy  $E > 0$ . However, the related numerical reconstructions using the numerical implementation of [3] of the algorithm recalled in Section 2 are presented below in this section.

It is convenient to use the following parametrization of the circle  $S$  of (33):

$$S = \{\zeta = \zeta(\varphi) \mid \zeta(\varphi) = \frac{1}{2\pi^2}(-i + e^{i\varphi}), \varphi \in [-\pi, \pi)\}. \quad (40)$$

Let

$$\begin{aligned} \mathcal{A}_\pm &= \{\zeta \in \mathbb{C} \mid \pm(\zeta + \bar{\zeta} + 2\pi\zeta\bar{\zeta}\ln E) > 0\} \\ &= \{\zeta \in \mathbb{C} \mid \pm(|\zeta + \frac{1}{2\pi\ln E}| - \frac{1}{2\pi\ln E}) > 0\}. \end{aligned} \quad (41)$$

Note that

$$\begin{aligned} \alpha < 0 \text{ for } \zeta \in \mathcal{A}_- \cap S, \quad \alpha > 0 \text{ for } \zeta \in \mathcal{A}_+ \cap S, \\ \alpha = 0 \text{ for } \zeta = 0, \quad \alpha = \infty \text{ for } \zeta = \frac{1}{\pi(\pi i - \ln E)}, \end{aligned} \quad (42)$$

where  $\alpha$  is given by (35) for fixed  $E > 0$ .

Figure 1 illustrates reconstructions  $v(x, E)$  from scattering amplitudes  $f(k, l) \equiv \zeta(\varphi)$  for  $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$  as well as for  $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$ , where  $\zeta(\varphi)$  is given in (40) and  $E = 100$ . We show the real parts of the reconstructed potentials  $v(x, E)$  only. The reason is that in our cases the imaginary parts are very small in comparison with the real parts. In addition, the domain of negative  $\alpha$  for  $E = 100$  corresponds to  $\varphi \in (90^\circ, 158.6^\circ)$ , and  $\alpha = \infty$  corresponds to  $158.6^\circ$ .

Note that numerical examples illustrating reconstructions from  $f(k, l) \equiv \zeta(\varphi)$  with  $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$  at fixed  $E > 0$  were already given in [2] in the framework of inverse scattering in acoustics.

However, in addition to remarks of [2], it is interesting to note that for the potential  $v(x, E)$  shown at Figure 1 for  $\varphi = -89^\circ$  equation (1) can not be interpreted already as the acoustic Helmholtz equation with variable sound speed  $c(x) > 0$ . More precisely, in this case equation (1) can not be rewritten as

$$-\Delta\psi - \left(\frac{\omega^2}{c(x)^2} - \frac{\omega^2}{c_0^2}\right)\psi = \frac{\omega^2}{c_0^2}\psi, \quad (43)$$

where

$$\begin{aligned} v(x, E) &= -\left(\frac{\omega^2}{c(x)^2} - \frac{\omega^2}{c_0^2}\right), \quad c(x) > 0, \quad c_0 > 0, \\ \omega &> 0, \quad \frac{\omega^2}{c_0^2} = E, \quad E = 100. \end{aligned} \quad (44)$$

The reason is that in this example

$$\max_x v(x, E) > E. \quad (45)$$

In connection with the results of Subsection 4.1, it is important to note that the reconstruction shown in Figure 1 (right) is positive at zero, whereas all single-point potentials  $v_{\alpha, y}(x)$ ,  $\alpha \in \mathbb{R} \cup \{\infty\}$ ,  $\alpha \neq 0$ ,  $y \in \mathbb{R}^2$ , can be considered as renormalized  $\delta$ -functions  $\varepsilon\delta(x)$  with negative  $\varepsilon$ . On the other hand, the reconstruction shown in Figure 1 (left) looks indeed as a regularized  $\varepsilon\delta(x)$  with negative  $\varepsilon$ . To our knowledge, the reconstructions of Figure 1 (left) were not yet given in the literature.

It is important to note that reconstructions shown in Figure 1 for  $\varphi = 201^\circ$  and  $\varphi = -21^\circ$  are obtained from scattering amplitudes  $f$  which differ only by their phases. However, these reconstructions differ by their signs as well as by the order of their amplitudes, and illustrate non-uniqueness in the phaseless inverse scattering problem (mentioned in the introduction) in the simplest case!

Finally, note that condition (10) for  $f(k, l) \equiv \zeta(\varphi)$  is fulfilled for  $\varphi \in (70.8^\circ, 109.2^\circ)$  only, where  $\zeta$  is defined as in (40) and  $E = 100$ . However, as it was already pointed out in [3, 2], the algorithm recalled in Section 2 works well much beyond limitation (10).

## 5 Further reconstruction examples

Let  $f_{\alpha_1, \dots, \alpha_N, y_1, \dots, y_N}$  denote the scattering amplitude of (32) for fixed  $\alpha_1, \dots, \alpha_N \in \mathbb{R} \cup \{\infty\}$  and  $y_1, \dots, y_N \in \mathbb{R}^2$ .

### 5.1 Reconstructions from scattering amplitudes for multi-point potentials

Developing results of Subsection 4.2 we also obtain numerical reconstructions  $v(x, E)$  from some scattering amplitudes  $f_{\alpha_1, \dots, \alpha_N, y_1, \dots, y_N}$  for multi-point potentials via the numerical implementation of [3] of the algorithm recalled above in Section 2. These reconstructions  $v(x, E)$  are illustrated by Figure 2 for the case of 3-point potentials and  $E = 100$ . In particular, in these examples we have

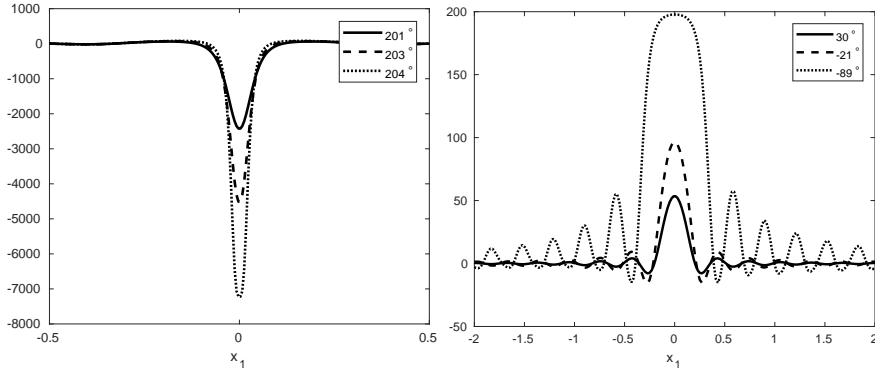


Figure 1: Cross-sections of potentials  $v(x, E)$  (real part) numerically reconstructed from the scattering amplitudes  $f(k, l) \equiv \zeta(\varphi)$ ,  $\varphi = 201^\circ$  ( $\alpha \approx 5.15$ ),  $203^\circ$  ( $\alpha \approx 4.98$ ),  $204^\circ$  ( $\alpha \approx 4.9$ ) (left) and  $\varphi = 30^\circ$  ( $\alpha \approx 1.25$ ),  $-21^\circ$  ( $\alpha \approx 1.86$ ),  $-89^\circ$  ( $\alpha \approx 2.71$ ) (right). Here  $\zeta(\varphi)$  is defined as in (40) and  $E = 100$ .

that the positions of points of the initial multi-point potentials are reconstructed very properly. In addition, we have

$$v(x, E) \approx \sum_{j=1}^N v_{\alpha_j, y_j}(x, E), \quad (46)$$

where  $v_{\alpha_j, y_j}(x, E)$  denote the reconstructions from the scattering amplitudes of the single-point potentials  $v_{\alpha_j, y_j}(x)$ .

## 5.2 Evolutions according to equations of the Novikov-Veselov hierarchy

We recall the following scheme, established in [7, 13, 14, 15], for constructing solutions  $v(x, t, E)$ ,  $x \in \mathbb{R}^2$ ,  $t \in \mathbb{R}$ , of the Novikov-Veselov equation (8) and its higher order analogs at fixed  $E > 0$ :

$$f(k, l) \longrightarrow f(k, l, t) \longrightarrow v(x, t, E), \quad (47)$$

consisting of the following steps:

- Step 1. Given a smooth function  $f(k, l)$  on  $\mathcal{M}_E$  satisfying (5), (6) and (10) at fixed  $E > 0$ , define  $f(k, l, t)$ ,  $t \in \mathbb{R} \setminus \{0\}$ , using (9) with  $f(k, l, 0) = f(k, l)$ , where  $n$  is the number of equation in the Novikov-Veselov hierarchy.
- Step 2. Construct  $v = v(x, t, E)$  using scheme (17) of Section 2 with  $f = f(k, l, t)$  for each fixed  $t \in \mathbb{R}$ .

Then  $v(x, t, E)$  satisfies the  $n$ -th equation of the Novikov-Veselov hierarchy at fixed  $E > 0$ .

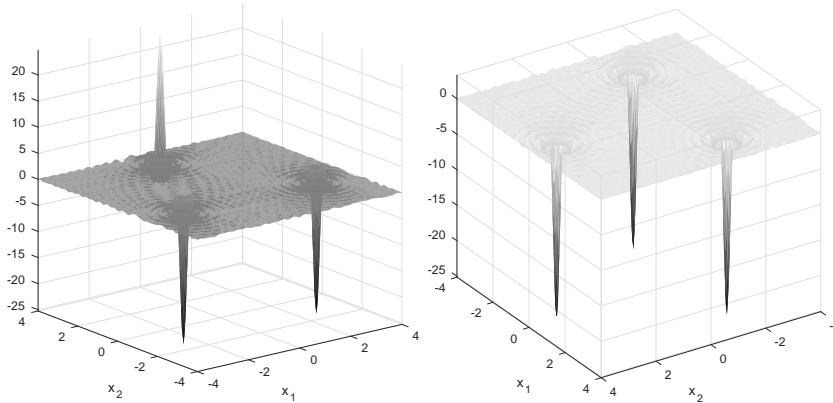


Figure 2: Numerical reconstructions from the scattering amplitudes of 3-point potentials  $f_{\alpha_1, \alpha_2, \alpha_3, y_1, y_2, y_3}$  with  $y_k = (3 \cos(\frac{2\pi k}{3}), 3 \sin(\frac{2\pi k}{3}))$ . Left:  $\alpha_1 = 0.7$ ,  $\alpha_2 = \alpha_3 = -1$ ; right:  $\alpha_1 = \alpha_2 = \alpha_3 = -1$ . Here  $E = 100$ .

Note that in a similar way with Remark 2, condition (10) can be weakened to the condition of the unique solvability of all involved integral equations.

In this section, using the numerical implementation of [3] of the algorithm recalled in Section 2, we present numerical solutions of the Novikov-Veselov equation (8) and its higher order analogs using scheme (47). In our examples we use some scattering amplitudes  $f$  of Section 3 for single- and multi-point potentials as the initial data of this scheme.

Our numerical results are illustrated by Figures 2, 3, 4. In particular, Figure 3 shows  $v(x, t, E)$  for the case of single-point potential. In this case  $v(x, 0, E)$  looks as the reconstructions shown in Figure 1 (left) with

$$\min_x v(x, 0, E) \approx -26.31. \quad (48)$$

Besides, Figures 2 (right) and 4 show  $v(x, 0, E)$  and  $v(x, t, E)$  for the case of 3-point potentials.

Note that these numerical solutions  $v(x, t, E)$  of the Novikov-Veselov equation and its higher order analogs at fixed  $E > 0$  illustrate the significant impact of the phase of the scattering amplitude  $f$  on the form of the reconstructed potential. These solutions (for fixed  $t$ ) can be considered as non-trivial examples of non-uniqueness in the phaseless inverse scattering problem mentioned in the introduction.

## 6 Acknowledgements

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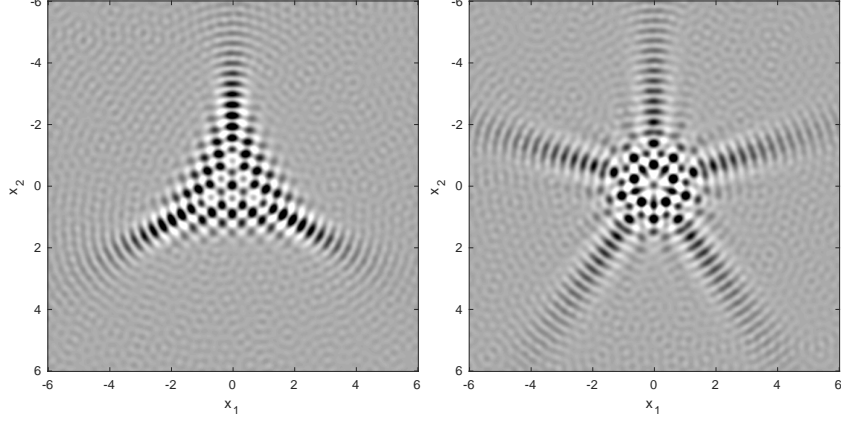


Figure 3: Solution  $v(x, t, E)$  of the  $n$ -th equation of the Novikov-Veselov hierarchy constructed from scattering amplitude  $f(k, l) \equiv f_\alpha(E)$ . Left:  $n = 1$ ,  $\alpha = -1$ ,  $t = 2E^{-3/2}$ ; right:  $n = 2$ ,  $\alpha = -1$ ,  $t = E^{-5/2}$ . Here  $f_\alpha(E)$  is defined as in (34) and  $E = 100$ . The color indicates the value varying from  $-2$  (black) to  $1$  (white).

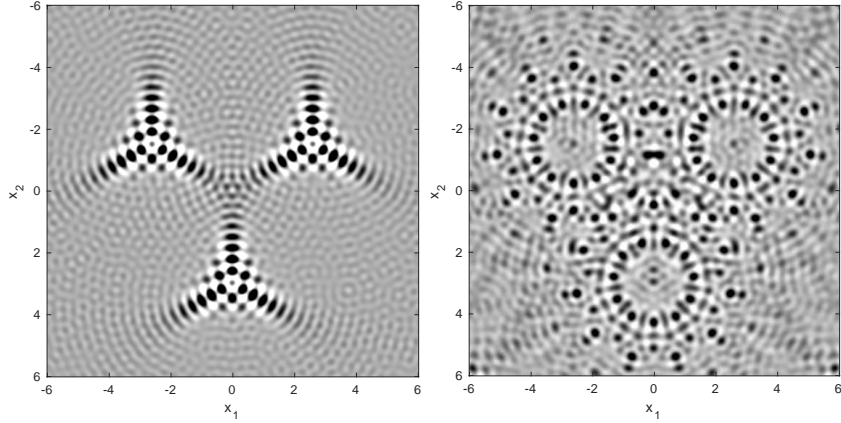


Figure 4: Solution  $v(x, t, E)$  of the  $n$ -th equation of the Novikov-Veselov hierarchy constructed from scattering amplitude  $f(k, l) = f_{\alpha_1, \alpha_2, \alpha_3, y_1, y_2, y_3}(k, l)$ ,  $\alpha_1 = \alpha_2 = \alpha_3 = -1$ ,  $y_k = (3 \cos(\frac{2\pi k}{3}), 3 \sin(\frac{2\pi k}{3}))$ . Left:  $n = 1$ ,  $t = 2.5E^{-3/2}$ ; right:  $n = 5$ ,  $t = E^{-11/2}$ . Here  $E = 100$  and the color indicates the value varying from  $-2$  (black) to  $1$  (white).

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