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Simplest examples of inverse scattering on the plane at fixed energy*

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We consider the inverse scattering problem for the two-dimensional Schrödinger equation at fixed positive energy. Our results include inverse scattering reconstructions from the simplest scattering amplitudes. In particular, we give a complete analytic solution of the phased and phaseless inverse scattering problems for the single-point potentials (of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type). Then we study numerical inverse scattering reconstructions from the simplest scattering amplitudes using the Riemann-Hilbert-Manakov problem of the soliton theory. Finally, we apply the later numerical inverse scattering results for constructing related numerical solutions for equations of the Novikov-Veselov hierarchy at fixed positive energy.

Keywords: inverse scattering, Schrödinger equation, numerical analysis, Novikov-Veselov equation

Subjects: 35R30 (inverse problems for PDEs), 65N21 (numerical analysis of inverse problems for PDEs), 35P25 (scattering theory), 35J10 (Schrödinger operator), 35Q53 (KdV-like equations);

1 Introduction

We consider the two-dimensional Schrödinger equation at fixed positive energy E :

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E > 0, \quad (1)$$

where v is a real-valued sufficiently regular potential on \mathbb{R}^2 with sufficient decay at infinity. For this equation we consider the classical scattering solutions $\psi^+ = \psi^+(x, k)$, specified by the following asymptotics:

$$\begin{aligned} \psi^+(x, k) &= e^{ikx} + C(|k|) \frac{e^{i|k||x|}}{|x|^{1/2}} f(k, |k| \frac{x}{|x|}) + o(|x|^{-\frac{1}{2}}), \quad |x| \rightarrow +\infty, \\ x \in \mathbb{R}^2, \quad k \in \mathbb{R}^2, \quad k^2 = E, \quad C(|k|) &= -\pi i \sqrt{2\pi} e^{-i\pi/4} |k|^{\frac{1}{2}}, \end{aligned} \quad (2)$$

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with a priori unknown coefficient f . The function f of (2) is known as the scattering amplitude for equation (1) and is defined on

$$\mathcal{M}_E = \mathbb{S}^1_{\sqrt{E}} \times \mathbb{S}^1_{\sqrt{E}}, \quad (3)$$

$$\mathbb{S}^1_{\sqrt{E}} = \{m \in \mathbb{R}^2 \mid m^2 = E\}. \quad (4)$$

It is known that f possesses the following properties:

$$f(k, l) = f(-l, -k) \quad (\text{reciprocity}), \quad (5)$$

$$f(k, l) - \overline{f(l, k)} = \frac{\pi}{i\sqrt{E}} \int_{\mathbb{S}^1_{\sqrt{E}}} f(k, m) \overline{f(l, m)} dm \quad (\text{unitarity}), \quad (6)$$

where $k, l \in \mathbb{S}^1_{\sqrt{E}}$. For possible assumptions on v assuring existence and uniqueness of ψ^+ at fixed k and properties (5), (6) for f see, e.g., [14, 6, 10].

Note also that

$$f_y(k, l) = e^{i(k-l)y} f(k, l), \quad k, l \in \mathbb{S}^1_{\sqrt{E}}, \quad (7)$$

where f is the scattering amplitude for v and f_y is the scattering amplitude for the translated potential $v_y = v(\cdot - y)$, $y \in \mathbb{R}^2$.

For equation (1), the problem of finding ψ^+ , f from v is known as the direct scattering problem; the problem of finding v from f is known as the inverse scattering problem; and the problem of finding v from $|f|^2$ is known as the phaseless inverse scattering problem.

In addition to equation (1) we consider its isospectral deformations at fixed E given by the Novikov-Veselov equation and its higher order analogs, see [16, 17, 6]. These equations admit a representation in the form of L-A-B Manakov triple (introduced in [11]), where $L = -\Delta + v - E$. We recall that the first non-trivial equation of the Novikov-Veselov hierarchy can be written as:

$$\begin{aligned} \partial_t v &= 4 \operatorname{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ \partial_{\bar{z}} w &= -3\partial_z v, \quad w = w(x, t), \quad w(x, t) \rightarrow 0, \quad |x| \rightarrow +\infty, \\ E &> 0, \quad t \in \mathbb{R}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \quad (8)$$

where $v = v(x, t)$, $w = w(x, t)$, $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$.

Analog of the Gardner-Green-Kruskal-Miura relations for the equations of the Novikov-Veselov hierarchy and for the scattering amplitude $f = f(k, l, t)$ are as follows:

$$\begin{aligned} f(k, l, t) &= \exp(2itE^{\frac{2n+1}{2}} (\cos((2n+1)\varphi_k) - \cos((2n+1)\varphi_l))) f(k, l, 0), \\ k &= \sqrt{E}(\cos \varphi_k, \sin \varphi_k), \quad l = \sqrt{E}(\cos \varphi_l, \sin \varphi_l), \end{aligned} \quad (9)$$

where n is the number of equation in the hierarchy; see [18] for the classical Gardner-Green-Kruskal-Miura relations and [13, 8] for (9).

Remark 1. (a) Properties (5), (6) are invariant with respect to transformations of f given by (7) and (9). (b) The differential scattering cross-section $|f|^2$ is invariant with respect to transformations of f given by (7) and (9).

In [13, 14] it was shown that if f is smooth, satisfies (5), (6) and

$$\|f\|_{L^2(\mathcal{M}_E)} < \frac{\sqrt{E}}{3\pi}, \quad (10)$$

then there is a smooth, real-valued, decaying at infinity potential v such that f is the scattering amplitude for v at fixed E . In addition, this v is reconstructed from f via the algorithm suggested in [13, 14] and simplified in [15]. This final algorithm of [15] is recalled in Section 2.

Note that these results of [13, 14, 15] are obtained using, in particular, the Riemann-Hilbert-Manakov problem of the soliton theory (see [12]) and results of [5] and [7]. In turn, the algorithm of [15] is implemented numerically in [3].

The results of the present work include inverse scattering reconstructions from some simplest functions f satisfying (5), (6) at fixed E . In particular, in this framework we give a complete analytic solution of the phased and phaseless inverse scattering problems for the single-point potentials (of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type) $v_{\alpha,y}(x)$, $\alpha \in \mathbb{R} \cup \{\infty\}$, $y \in \mathbb{R}^2$, see Subsection 4.1.

Then we give numerical inverse scattering reconstructions from some simplest scattering amplitudes f satisfying (5), (6) at fixed $E > 0$ using the numerical implementation (in MATLAB) of [3] of the algorithm recalled in Section 2.

First of all, in this connection, we study reconstructions from constant f satisfying (5), (6) at fixed E . In particular, such f arise as scattering amplitudes of the single-point potentials $v_{\alpha,y}(x)$ for $y = 0$ (i.e. supported at zero). Note that already for this simplest case there are no explicit analytic reconstruction formulas for regular potentials. Our numerical results for this case develop studies of [4, 2]. These results are presented in details in Subsection 4.2.

Then, using the numerical inverse scattering implementation of [3], we study reconstructions from functions f arising as scattering amplitudes of multi-point potentials (scatterers)

$$v(x) = \sum_{j=1}^N v_{\alpha_j, y_j}(x), \quad x \in \mathbb{R}^2, \alpha_j \in \mathbb{R}, y_j \in \mathbb{R}^2, \quad (11)$$

consisting of N single-point scatterers $v_{\alpha_j, y_j}(x)$, where each point scatterer is described by its internal parameter α_j and position y_j (and $y_i \neq y_j$ for $i \neq j$). This can be also considered as the first use of the multi-point potentials of the Bethe-Peierls-Fermi-Zeldovich-Berezin-Faddeev type for testing inverse scattering algorithms. Possibility of such tests was mentioned in [10]. These results are presented in details in Subsection 5.1.

We emphasize that our aforementioned numerical reconstructions are obtained using the results of [15, 3] and can be considered as regular approximations to the initial multi-point potentials which are quite singular.

Finally, using the scattering amplitudes for the multi-point potentials, relations (9) and the inverse scattering implementation of [3] we obtain the related numerical solutions for equations of the Novikov-Veselov hierarchy. In particular, these results also illustrate non-uniqueness in the inverse scattering problem without phase information at fixed energy. See Subsection 5.2 for details.

2 Inverse scattering algorithm

It is convenient to use the following notations:

$$\begin{aligned} z &= x_1 + ix_2, & \bar{z} &= x_1 - ix_2, \\ \lambda &= E^{-1/2}(k_1 + ik_2), & \lambda' &= E^{-1/2}(l_1 + il_2), \end{aligned} \quad (12)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathbb{S}_{\sqrt{E}}^1$, $l = (l_1, l_2) \in \mathbb{S}_{\sqrt{E}}^1$. In these notations

$$\begin{aligned} k_1 &= \frac{1}{2}E^{1/2}(\lambda + \lambda^{-1}), & k_2 &= \frac{i}{2}E^{1/2}(\lambda^{-1} - \lambda), \\ l_1 &= \frac{1}{2}E^{1/2}(\lambda' + \lambda'^{-1}), & l_2 &= \frac{i}{2}E^{1/2}(\lambda'^{-1} - \lambda'), \end{aligned} \quad (13)$$

where $\lambda, \lambda' \in T$,

$$T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}. \quad (14)$$

Using formulas (3), (4), (12), (13), (14) one can see that

$$\mathbb{S}_{\sqrt{E}}^1 \cong T, \quad \mathcal{M}_E \cong T \times T. \quad (15)$$

In addition, in these notations functions ψ^+ , f of (2) can be written as

$$\psi^+ = \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \quad (16)$$

where $\lambda, \lambda' \in T$, $z \in \mathbb{C}$, $E > 0$.

The algorithm of [15] for finding v on \mathbb{R}^2 from f on \mathcal{M}_E has the following scheme:

$$f \longrightarrow h_{\pm} \longrightarrow \mu^+ \longrightarrow \mu_- \longrightarrow v, \quad (17)$$

and consists of the following steps:

Step 1. Find functions $h_{\pm}(\lambda, \lambda', E)$, $\lambda, \lambda' \in T$, from the following linear integral equations:

$$\begin{aligned} h_{\pm}(\lambda, \lambda', E) - \pi \int_T h_{\pm}(\lambda, \lambda'', E) \chi\left(\pm i \left[\frac{\lambda}{\lambda''} - \frac{\lambda'}{\lambda}\right]\right) \times \\ \times f(\lambda'', \lambda', E) |d\lambda''| = f(\lambda, \lambda', E), \end{aligned} \quad (18)$$

where

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases} \quad (19)$$

Step 2. Solve the following linear integral equation for $\mu^+(z, \lambda, E)$, $z \in \mathbb{C}$, $\lambda \in T$, $E > 0$:

$$\mu^+(z, \lambda, E) + \int_T B(\lambda, \lambda', z, E) \mu^+(z, \lambda', E) |d\lambda'| = 1, \quad (20)$$

where

$$\begin{aligned} B(\lambda, \lambda', z, E) = & \frac{1}{2} \int_T h_-(\zeta, \lambda', z, E) \chi \left(-i \left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1-0)} \\ & - \frac{1}{2} \int_T h_+(\zeta, \lambda', z, E) \chi \left(i \left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1+0)}, \end{aligned} \quad (21)$$

$$\begin{aligned} h_{\pm}(\lambda, \lambda', z, E) = & h_{\pm}(\lambda, \lambda', E) \times \\ & \times \exp \left(-i \frac{\sqrt{E}}{2} ((\lambda - \lambda') \bar{z} - (\lambda^{-1} - \lambda'^{-1}) z) \right), \end{aligned} \quad (22)$$

and $\lambda, \lambda' \in T$, $z \in \mathbb{C}$, $E > 0$.

Step 3. Define function $\mu_-(z, \lambda, E)$, $z \in \mathbb{C}$, $\lambda \in T$, $E > 0$, by the formula

$$\begin{aligned} \mu_-(z, \lambda, E) = & \mu^+(z, \lambda, E) + \pi i \int_T h_-(\lambda, \lambda', z, E) \times \\ & \times \chi \left(-i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) \mu^+(z, \lambda', E) |d\lambda'|, \end{aligned} \quad (23)$$

where function $h_-(\lambda, \lambda', z, E)$ is given by (22) and χ is defined by (19).

Step 4. Potential $v = v(x, E)$, $x \in \mathbb{R}^2$, $E > 0$, is given by the formula

$$v(x, E) = \frac{\sqrt{E}}{\pi} \int_T \partial_z \mu_-(z, \zeta, E) d\zeta, \quad (24)$$

where $z = x_1 + ix_2$, $x = (x_1, x_2)$, $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$.

Remark 2. As it was mentioned in the introduction, if f satisfies (5), (6), (10), then there exists a smooth real-valued decaying at infinity potential v such that f is the scattering amplitude for v at fixed $E > 0$. In this result condition (10) can be replaced by a much weaker condition that all integral equations in (18), (20) are uniquely solvable. In addition, in notations of the present section condition (10) can be written as

$$\|f\|_{L^2(T \times T)} < \frac{1}{3\pi}. \quad (25)$$

Note also that

$$v(x, \tau^2 E) = \tau^2 v(\tau x, E), \quad E > 0, \tau > 0, \quad (26)$$

for $v(x, E)$ reconstructed via (17) from f which is independent of E , i.e. $f = f(\lambda, \lambda')$.

3 Scattering functions for multi-point potentials

The scattering theory for multi-point potentials v mentioned in formula (11) of the introduction is presented, in particular, in [1, 9, 10]. In addition, all single-point potentials $v_{\alpha,y}(x)$, $\alpha \in \mathbb{R} \setminus \{0\}$, $x, y \in \mathbb{R}^2$, can be considered as renormalizations of delta functions $\varepsilon\delta(x-y)$ with negative coefficients ε .

We recall that for the multi-point potentials v of formula (11) the classical scattering functions ψ^+ and f are given by explicit formulas as follows.

For the classical scattering eigenfunctions ψ^+ the following formulas hold:

$$\psi^+(x, k) = e^{ikx} + \sum_{j=1}^N q_j^+(k) G^+(x - y_j, k), \quad (27)$$

$$x \in \mathbb{R}^2, k \in \mathbb{S}_{\sqrt{E}}^1, y_j \in \mathbb{R}^2, y_j \neq y_m \text{ for } j \neq m,$$

$$G^+(x, k) = -\frac{i}{4} H_0^{(1)}(|x||k|), \quad x \in \mathbb{R}^2, k \in \mathbb{S}_{\sqrt{E}}^1, \quad (28)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero and $q^+(k) = (q_1^+(k), \dots, q_N^+(k))$ is the solution of the following linear system:

$$A^+(k)q^+(k) = b^+(k), \quad (29)$$

where $A^+(k) \in M_N(\mathbb{C})$, $b^+(k) \in \mathbb{C}^N$ are given by

$$A_{m,j}^+(k) = \begin{cases} 1 + \frac{\alpha_m}{4\pi}(\pi i - 2 \ln |k|), & m = j, \\ -\alpha_m G^+(y_m - y_j, k), & m \neq j, \end{cases} \quad (30)$$

$$b^+(k) = (\alpha_1 e^{iky_1}, \dots, \alpha_N e^{iky_N}), \quad \alpha_1, \dots, \alpha_N \in \mathbb{R}. \quad (31)$$

For the classical scattering amplitude f the following formula holds:

$$f(k, l) = \frac{1}{(2\pi)^2} \sum_{j=1}^N q_j^+(k) e^{-ily_j}, \quad k, l \in \mathbb{S}_{\sqrt{E}}^1, \quad (32)$$

where $q_j^+(k)$ are the same as in (27), (29).

4 Reconstructions from constant f

4.1 Analytic inverse scattering for the single-point potentials

The simplest functions on \mathcal{M}_E are constants. Therefore, the results given below in this section are of particular interest.

Lemma 1. *Let $f \equiv f_0$ on \mathcal{M}_E for fixed $E > 0$, where f_0 is a complex constant. Then f satisfies (5), (6) if and only if $f_0 \in S$, where*

$$\begin{aligned} S &= \{ \zeta \in \mathbb{C} \mid \zeta - \bar{\zeta} = -2i\pi^2 \zeta \bar{\zeta} \} \\ &= \{ \zeta \in \mathbb{C} \mid |\zeta + \frac{i}{2\pi^2}| = \frac{1}{2\pi^2} \}. \end{aligned} \quad (33)$$

Lemma 1 follows from direct substitution of f_0 into (5), (6).

One can see that S is the circle centered at $-\frac{i}{2\pi^2}$ of radius $\frac{1}{2\pi^2}$.

Using (29), (32) (for $N = 1, y_1 = 0$) one can see that the scattering amplitude for the single point potential $v_{\alpha,y}, y = 0$, at fixed energy E is given by the following formula:

$$f(k, l) \equiv f_\alpha(E),$$

$$f_\alpha(E) = \frac{1}{(2\pi)^2} \frac{\alpha}{1 + \frac{\alpha}{4\pi}(\pi i - \ln E)}, \quad E > 0, \alpha \in \mathbb{R}. \quad (34)$$

Assuming that $f_\alpha(E)$ is defined as in (34), we have the following result.

Theorem 1. *Let $\zeta = f_\alpha(E)$ for fixed $E > 0$. Then $\zeta \in S$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$. Conversely, for any $\zeta \in S$ there exists the unique $\alpha \in \mathbb{R} \cup \{\infty\}$ such that $\zeta = f_\alpha(E)$ and this α is given by the following formula:*

$$\alpha = \frac{(2\pi)^2 \zeta}{1 - \pi \zeta (\pi i - \ln E)}. \quad (35)$$

Proof of Theorem 1. The fact that the scattering amplitudes f (of Section 3) for the multi-point potentials satisfy (5), (6), the corollary of (34) that $f = f_\alpha(E)$ is constant at fixed E and α , and Lemma 1 imply that $\zeta \in S$ if $\zeta = f_\alpha(E)$. The property that $\zeta \in S$ if $\zeta = f_\alpha(E)$ can also be verified by the direct calculation using the precise formula for $f_\alpha(E)$ in (34).

Conversely, consider the equation

$$f_\alpha(E) = \zeta \quad \text{with respect to } \alpha \in \mathbb{C} \cup \{\infty\}, \quad (36)$$

for fixed $\zeta \in S$ and $E > 0$. One can see that this equation is uniquely solvable and that the solution is given by formula (35). Direct calculations also show that $\alpha = \bar{\alpha}$.

Theorem 1 is proved. \square

Remark 3. Theorem 1 gives a complete solution of the inverse scattering problem for the single point potentials $v_{\alpha,y}, \alpha \in \mathbb{R} \cup \{\infty\}, y = 0$ (uniqueness, reconstruction, characterization). In view of formula (7), this solution admits a straightforward generalization to the case of $v_{\alpha,y}$ with $\alpha \in \mathbb{R} \cup \{\infty\}, y \in \mathbb{R}^2$.

Remark 4. The property that $\zeta \in S$ if $\zeta = f_\alpha(E)$ can be considered as a relationship between the amplitude and phase of a single point scatterer $v_{\alpha,y}, \alpha \in \mathbb{R}, y = 0$. For a single point potential centered at zero, such a relationship was obtained in [4, 2] in the form:

$$\sin \phi = -|\beta|/4, \quad \text{where}$$

$$\beta = |\beta| \exp(i\phi), \quad \beta = (2\pi)^2 f(k, l). \quad (37)$$

However, in [4, 2] this relation is not yet related to the unitarity property (6) of the scattering amplitude f .

In addition, Theorem 1 implies the following corollary for inverse scattering without phase information.

Corollary 1. *Let $E > 0$ be fixed. Then for any $\sigma \in [0, \frac{1}{\pi^4}]$ the values of parameter $\alpha \in \mathbb{R} \cup \{\infty\}$ such that $\sigma = |f_\alpha(E)|^2$ are given by (35) with*

$$\zeta = \pm \sqrt{\sigma(1 - \sigma\pi^4)} - i\sigma\pi^2, \quad (38)$$

where the expression in (38) is single-valued for $\sigma = 0$ and $\sigma = \frac{1}{\pi^4}$ and is double-valued for $\sigma \in (0, \frac{1}{\pi^4})$. In addition, for any $\sigma \in (\frac{1}{\pi^4}, \infty)$ there exists no $\alpha \in \mathbb{R} \cup \{\infty\}$ such that $\sigma = |f_\alpha(E)|^2$.

Remark 5. Let $\zeta = f_{\alpha_1}(E_1)$ for some fixed $\alpha_1 \in \mathbb{R} \cup \{\infty\}$, $E_1 > 0$. Then for any $E_2 > 0$ there exists the unique $\alpha_2 \in \mathbb{R} \cup \{\infty\}$ such that $\zeta = f_{\alpha_2}(E_2)$, and this α_2 is given by the following formula:

$$\alpha_2 = \frac{\alpha_1}{1 + \frac{\alpha_1}{4\pi} \ln \frac{E_2}{E_1}}. \quad (39)$$

4.2 Numerical reconstructions from $f \in S$

In contrast to Theorem 1, we have no explicit formula for finding a regular potential $v(x, E)$ with constant scattering amplitude $f \in S \setminus \{0\}$ at fixed energy $E > 0$. However, the related numerical reconstructions using the numerical implementation of [3] of the algorithm recalled in Section 2 are presented below in this section.

It is convenient to use the following parametrization of the circle S of (33):

$$S = \{\zeta = \zeta(\varphi) \mid \zeta(\varphi) = \frac{1}{2\pi^2}(-i + e^{i\varphi}), \varphi \in [-\pi, \pi)\}. \quad (40)$$

Let

$$\begin{aligned} \mathcal{A}_\pm &= \{\zeta \in \mathbb{C} \mid \pm(\zeta + \bar{\zeta} + 2\pi\zeta\bar{\zeta}\ln E) > 0\} \\ &= \{\zeta \in \mathbb{C} \mid \pm(|\zeta + \frac{1}{2\pi\ln E}| - \frac{1}{2\pi\ln E}) > 0\}. \end{aligned} \quad (41)$$

Note that

$$\begin{aligned} \alpha < 0 \text{ for } \zeta \in \mathcal{A}_- \cap S, \quad \alpha > 0 \text{ for } \zeta \in \mathcal{A}_+ \cap S, \\ \alpha = 0 \text{ for } \zeta = 0, \quad \alpha = \infty \text{ for } \zeta = \frac{1}{\pi(\pi i - \ln E)}, \end{aligned} \quad (42)$$

where α is given by (35) for fixed $E > 0$.

Figure 1 illustrates reconstructions $v(x, E)$ from scattering amplitudes $f(k, l) \equiv \zeta(\varphi)$ for $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as well as for $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$, where $\zeta(\varphi)$ is given in (40) and $E = 100$. We show the real parts of the reconstructed potentials $v(x, E)$ only. The reason is that in our cases the imaginary parts are very small in comparison with the real parts. In addition, the domain of negative α for $E = 100$ corresponds to $\varphi \in (90^\circ, 158.6^\circ)$, and $\alpha = \infty$ corresponds to 158.6° .

Note that numerical examples illustrating reconstructions from $f(k, l) \equiv \zeta(\varphi)$ with $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ at fixed $E > 0$ were already given in [2] in the framework of inverse scattering in acoustics.

However, in addition to remarks of [2], it is interesting to note that for the potential $v(x, E)$ shown at Figure 1 for $\varphi = -89^\circ$ equation (1) can not be interpreted already as the acoustic Helmholtz equation with variable sound speed $c(x) > 0$. More precisely, in this case equation (1) can not be rewritten as

$$-\Delta\psi - \left(\frac{\omega^2}{c(x)^2} - \frac{\omega^2}{c_0^2}\right)\psi = \frac{\omega^2}{c_0^2}\psi, \quad (43)$$

where

$$\begin{aligned} v(x, E) &= -\left(\frac{\omega^2}{c(x)^2} - \frac{\omega^2}{c_0^2}\right), \quad c(x) > 0, \quad c_0 > 0, \\ \omega &> 0, \quad \frac{\omega^2}{c_0^2} = E, \quad E = 100. \end{aligned} \quad (44)$$

The reason is that in this example

$$\max_x v(x, E) > E. \quad (45)$$

In connection with the results of Subsection 4.1, it is important to note that the reconstruction shown in Figure 1 (right) is positive at zero, whereas all single-point potentials $v_{\alpha, y}(x)$, $\alpha \in \mathbb{R} \cup \{\infty\}$, $\alpha \neq 0$, $y \in \mathbb{R}^2$, can be considered as renormalized δ -functions $\varepsilon\delta(x)$ with negative ε . On the other hand, the reconstruction shown in Figure 1 (left) looks indeed as a regularized $\varepsilon\delta(x)$ with negative ε . To our knowledge, the reconstructions of Figure 1 (left) were not yet given in the literature.

It is important to note that reconstructions shown in Figure 1 for $\varphi = 201^\circ$ and $\varphi = -21^\circ$ are obtained from scattering amplitudes f which differ only by their phases. However, these reconstructions differ by their signs as well as by the order of their amplitudes, and illustrate non-uniqueness in the phaseless inverse scattering problem (mentioned in the introduction) in the simplest case!

Finally, note that condition (10) for $f(k, l) \equiv \zeta(\varphi)$ is fulfilled for $\varphi \in (70.8^\circ, 109.2^\circ)$ only, where ζ is defined as in (40) and $E = 100$. However, as it was already pointed out in [3, 2], the algorithm recalled in Section 2 works well much beyond limitation (10).

5 Further reconstruction examples

Let $f_{\alpha_1, \dots, \alpha_N, y_1, \dots, y_N}$ denote the scattering amplitude of (32) for fixed $\alpha_1, \dots, \alpha_N \in \mathbb{R} \cup \{\infty\}$ and $y_1, \dots, y_N \in \mathbb{R}^2$.

5.1 Reconstructions from scattering amplitudes for multi-point potentials

Developing results of Subsection 4.2 we also obtain numerical reconstructions $v(x, E)$ from some scattering amplitudes $f_{\alpha_1, \dots, \alpha_N, y_1, \dots, y_N}$ for multi-point potentials via the numerical implementation of [3] of the algorithm recalled above in Section 2. These reconstructions $v(x, E)$ are illustrated by Figure 2 for the case of 3-point potentials and $E = 100$. In particular, in these examples we have

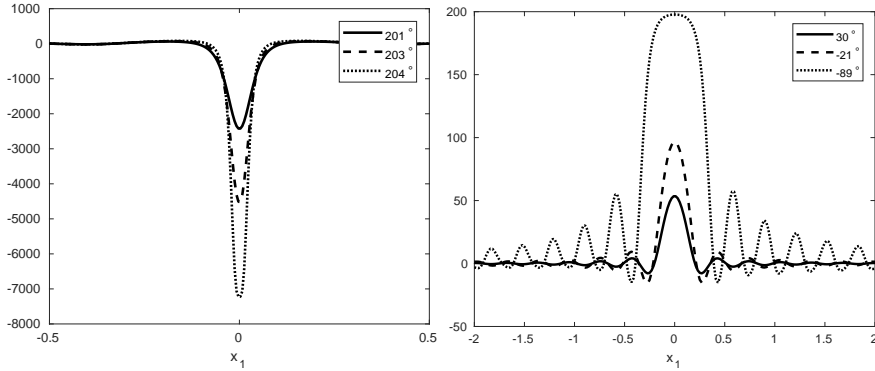


Figure 1: Cross-sections of potentials $v(x, E)$ (real part) numerically reconstructed from the scattering amplitudes $f(k, l) \equiv \zeta(\varphi)$, $\varphi = 201^\circ$ ($\alpha \approx 5.15$), 203° ($\alpha \approx 4.98$), 204° ($\alpha \approx 4.9$) (left) and $\varphi = 30^\circ$ ($\alpha \approx 1.25$), -21° ($\alpha \approx 1.86$), -89° ($\alpha \approx 2.71$) (right). Here $\zeta(\varphi)$ is defined as in (40) and $E = 100$.

that the positions of points of the initial multi-point potentials are reconstructed very properly. In addition, we have

$$v(x, E) \approx \sum_{j=1}^N v_{\alpha_j, y_j}(x, E), \quad (46)$$

where $v_{\alpha_j, y_j}(x, E)$ denote the reconstructions from the scattering amplitudes of the single-point potentials $v_{\alpha_j, y_j}(x)$.

5.2 Evolutions according to equations of the Novikov-Veselov hierarchy

We recall the following scheme, established in [7, 13, 14, 15], for constructing solutions $v(x, t, E)$, $x \in \mathbb{R}^2$, $t \in \mathbb{R}$, of the Novikov-Veselov equation (8) and its higher order analogs at fixed $E > 0$:

$$f(k, l) \longrightarrow f(k, l, t) \longrightarrow v(x, t, E), \quad (47)$$

consisting of the following steps:

- Step 1. Given a smooth function $f(k, l)$ on \mathcal{M}_E satisfying (5), (6) and (10) at fixed $E > 0$, define $f(k, l, t)$, $t \in \mathbb{R} \setminus \{0\}$, using (9) with $f(k, l, 0) = f(k, l)$, where n is the number of equation in the Novikov-Veselov hierarchy.
- Step 2. Construct $v = v(x, t, E)$ using scheme (17) of Section 2 with $f = f(k, l, t)$ for each fixed $t \in \mathbb{R}$.

Then $v(x, t, E)$ satisfies the n -th equation of the Novikov-Veselov hierarchy at fixed $E > 0$.

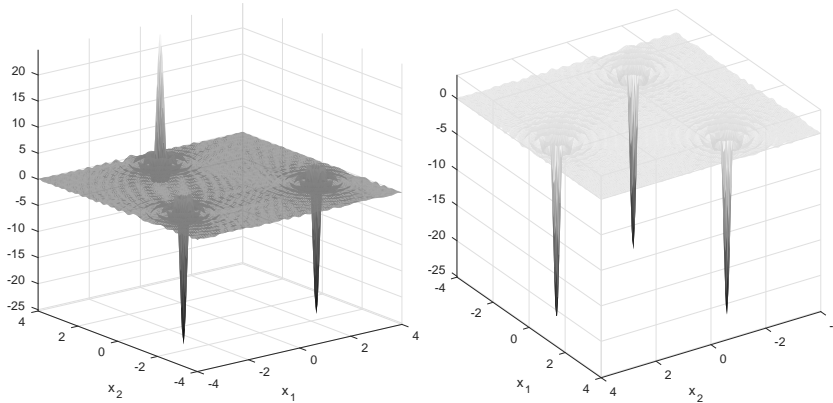


Figure 2: Numerical reconstructions from the scattering amplitudes of 3-point potentials $f_{\alpha_1, \alpha_2, \alpha_3, y_1, y_2, y_3}$ with $y_k = (3 \cos(\frac{2\pi k}{3}), 3 \sin(\frac{2\pi k}{3}))$. Left: $\alpha_1 = 0.7$, $\alpha_2 = \alpha_3 = -1$; right: $\alpha_1 = \alpha_2 = \alpha_3 = -1$. Here $E = 100$.

Note that in a similar way with Remark 2, condition (10) can be weakened to the condition of the unique solvability of all involved integral equations.

In this section, using the numerical implementation of [3] of the algorithm recalled in Section 2, we present numerical solutions of the Novikov-Veselov equation (8) and its higher order analogs using scheme (47). In our examples we use some scattering amplitudes f of Section 3 for single- and multi-point potentials as the initial data of this scheme.

Our numerical results are illustrated by Figures 2, 3, 4. In particular, Figure 3 shows $v(x, t, E)$ for the case of single-point potential. In this case $v(x, 0, E)$ looks as the reconstructions shown in Figure 1 (left) with

$$\min_x v(x, 0, E) \approx -26.31. \quad (48)$$

Besides, Figures 2 (right) and 4 show $v(x, 0, E)$ and $v(x, t, E)$ for the case of 3-point potentials.

Note that these numerical solutions $v(x, t, E)$ of the Novikov-Veselov equation and its higher order analogs at fixed $E > 0$ illustrate the significant impact of the phase of the scattering amplitude f on the form of the reconstructed potential. These solutions (for fixed t) can be considered as non-trivial examples of non-uniqueness in the phaseless inverse scattering problem mentioned in the introduction.

6 Acknowledgements

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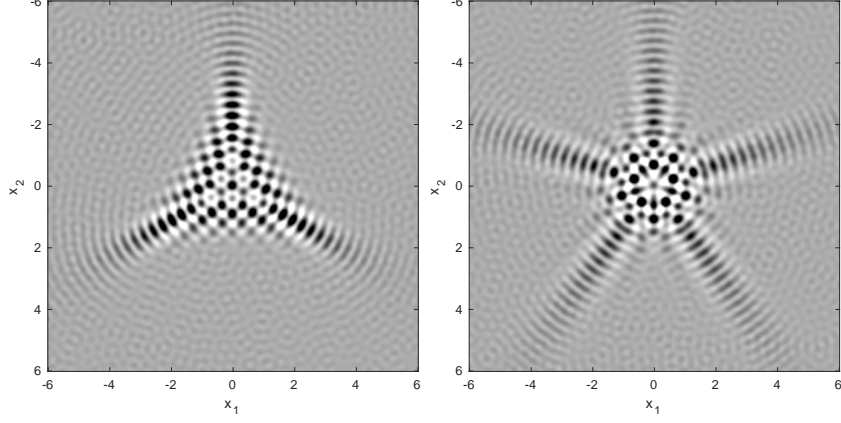


Figure 3: Solution $v(x, t, E)$ of the n -th equation of the Novikov-Veselov hierarchy constructed from scattering amplitude $f(k, l) \equiv f_\alpha(E)$. Left: $n = 1$, $\alpha = -1$, $t = 2E^{-3/2}$; right: $n = 2$, $\alpha = -1$, $t = E^{-5/2}$. Here $f_\alpha(E)$ is defined as in (34) and $E = 100$. The color indicates the value varying from -2 (black) to 1 (white).

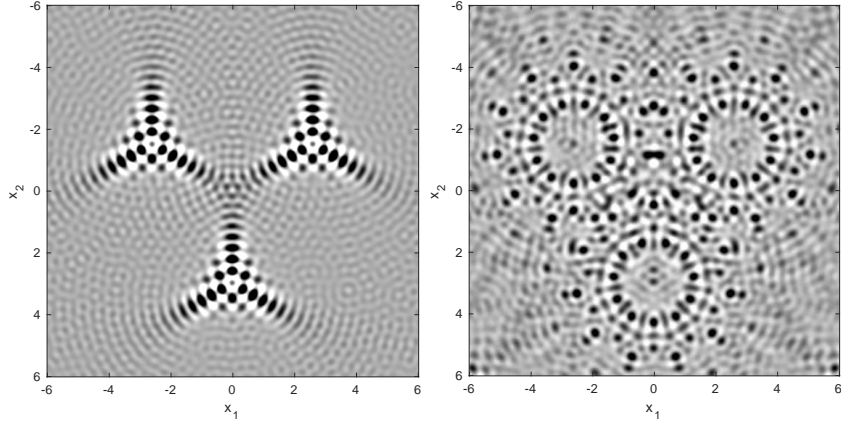


Figure 4: Solution $v(x, t, E)$ of the n -th equation of the Novikov-Veselov hierarchy constructed from scattering amplitude $f(k, l) = f_{\alpha_1, \alpha_2, \alpha_3, y_1, y_2, y_3}(k, l)$, $\alpha_1 = \alpha_2 = \alpha_3 = -1$, $y_k = (3 \cos(\frac{2\pi k}{3}), 3 \sin(\frac{2\pi k}{3}))$. Left: $n = 1$, $t = 2.5E^{-3/2}$; right: $n = 5$, $t = E^{-11/2}$. Here $E = 100$ and the color indicates the value varying from -2 (black) to 1 (white).

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