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A new transmuted family of distributions: Properties and estimation with applications

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Abstract

In this paper, a new transmuted general family of distributions is introduced and studied. In particular, we obtain explicit expressions for the moments, the incomplete moments, the moment generating function, entropies measures and order statistics. Further, we introduce a bivariate extension of the new family. Some sub-models of the family are introduced, involving Burr, Gompertz, Weibull and gamma distributions. We discuss the different methods of estimation of the model parameters and a simulation study is investigated for one of the special models to check the asymptotic behavior of the estimates under all estimation methods. Further, the potentiality of the family is illustrated by fitting two real data sets to the mentioned sub-models.

Keywords— Transmuted distribution, Generated family, T-X transformation, Quantile function, Rényi and Mathai-Hanboldand entropy, Estimation.

2000 Mathematics Subject Classification:— 60E05, 62E15, 62F10.

1 Introduction

Literature of lifetime distributions is rich with various continuous univariate distributions and still growing rapidly. Several extensions of some well-known lifetime distributions have been developed during the last two decades for modeling and analysis many types of real life data that having different random nature. This development is followed by many approaches for generating new families of distributions because that the family contains many distributions and hence increase the chance of modeling a large number of real data. Among these families, there are the Exponential-G by [13], the Marshal-Olkin-G (MO-G) by [14], the transmuted distributions by [20], the beta-G by [11], the Kumaraswamy-G (Kw-G) by [8], the McDonald-G (Mc-G) by [2], the gamma-G (type 1) by [22], the gamma-G (type 2) by [18], the log-gamma-G by [7], the logistic-G by [21], the exponentiated generalized-G by [9], the Transformed-Transformer (T-X) by [5], the exponentiated (T-X) by [6], the Kumaraswamy odd log-logistic-G (KwOLL-G) by [3], the generalized transmuted-G by [16], the beta transmuted-H families by [1] and the cubic rank transmuted distributions by [12].

1.1 Genesis of the family

Here, we introduce a new family of distribution called the transmuted general (T-G) family of distributions. In order to motivate its interest, let us first present a simple case of the well-know T-X family introduced by [5]. For a general baseline cumulative distribution function (cdf) denoted by $G(x)$, a new cdf is given by

$$F(x) = \int_0^{G(x)} p(t)dt, \quad x \in \mathbb{R}, \quad (1.1)$$

where $p(t)$ denotes a probability density function (pdf) with support on $[0, 1]$. As noticed in [4], the famous transmuted distributions family with cdf given by $F(x) = (1+\lambda)G(x) - \lambda G^2(x)$, $\lambda \in [-1, 1]$, [20], corresponds to the cdf given by (1.1) defined with the pdf: $p(t) = 1 + \lambda - 2\lambda t$. As a new remark, one observe that $p(t)$ can be expressed as generalized mixture of two pdfs with support on $[0, 1]$:

$$p(t) = (1 - \lambda)p_1(t) + \lambda p_2(t),$$

where $p_1(t) = 1$ and $p_2(t) = 2(1 - t)$. This fact motivates the consideration of a new family of distributions derived to (1.1) and generalized mixture of two pdfs; for any pdfs $p_1(t)$ and $p_2(t)$ with support on $[0, 1]$ and λ such that $\inf_{t \in [0,1]} [(1 - \lambda)p_1(t) + \lambda p_2(t)] > 0$, the following function $F(x)$ is a cdf:

$$F(x) = \int_0^{G(x)} [(1 - \lambda)p_1(t) + \lambda p_2(t)] dt, \quad x \in \mathbb{R}.$$

A new family of distributions arises by keeping $p_1(t) = 1$ for the sake of simplicity but choosing a new pdf for $p_2(t)$, with different nature. In this paper, we focus our attention on the decreasing convex pdf given by $p_2(t) = \frac{2}{(1+t)^2}$. The corresponding cdf is given by

$$F(x) = \int_0^{G(x)} \left[(1 - \lambda) + \lambda \frac{2}{(1+t)^2} \right] dt = \left[1 - \lambda + \frac{2\lambda}{1 + G(x)} \right] G(x), \quad \lambda \in [-1, 1], x \in \mathbb{R}. \quad (1.2)$$

An equivalent expression of (1.2) is given by

$$F(x) = G(x) + \lambda \frac{G(x) [1 - G(x)]}{1 + G(x)}, \quad x \in \mathbb{R}.$$

To the best of our knowledge, this family of distributions is new in the literature.

1.2 Aim and organization of the paper

The aim of this paper is to explore the mathematical and practical aspects of the T-G family, and to show its importance and its flexibility using applications to real data sets. In particular, we explicit expressions for the moments, incomplete moments, generating function, mean deviation and residual and reversed residual life moments. Stochastic ordering and reliability parameter are studied. General results for order statistics are derived. Entropies measures (Rényi entropy and Mathai-Hanboldand entropy) are determined. Some sub-models of the family are introduced, involving Burr, Gompertz, Weibull and gamma distributions. A bivariate extension of the new family is also introduced. We provide simulation results of different estimation methods (maximum likelihood, least square, minimum spacing, weighted least square and Cramer Von Mises) for one of the special models to check the asymptotic behavior of the estimates under those methods. Further, applications on two different real data sets are given, with fair comparisons with recent flexible distribution.

This paper is unfolded as follows. Section 2 is devoted to the mathematical properties of the T-G family. Special models are introduced in Section 3. An extension to the bivariate case is explored in Section 4. Section 5 presents estimation methods for the parameters of the T-G family. Simulation results and applications to real data sets are given in Section 6.

2 Mathematical properties

We say that a random variable follows the T-G distribution if it has the cdf given by (1.2). Hereafter, we consider a random variable X following the T-G distribution.

2.1 Shape of the pdf

Let $g(x)$ be the baseline pdf, the pdf of X is given by

$$f(x) = \left[1 - \lambda + \frac{2\lambda}{(1 + G(x))^2} \right] g(x), \quad \lambda \in [-1, 1], x \in \mathbb{R}. \quad (2.1)$$

An equivalent expression is given by

$$f(x) = \frac{\{(1 + \lambda) + (1 - \lambda)G(x) [2 + G(x)]\} g(x)}{(1 + G(x))^2}, \quad \lambda \in [-1, 1], x \in \mathbb{R}.$$

The critical points of $f(x)$ are the roots of the equation:

$$\frac{g'(x)}{g(x)} - 2 \frac{g(x)}{1 + G(x)} + \frac{2g(x)(\lambda - 1)[1 + G(x)]}{1 + \lambda - (1 - \lambda)G(x)[2 + G(x)]} = 0.$$

2.2 Expansions for the pdf and the cdf

By using generalized binomial expansion we can show that the pdf (2.1) of X has the expansion

$$f(x) = \sum_{j=0}^{\infty} a_j h_j(x), \quad (2.2)$$

where $a_0 = 1 + \lambda$, $a_j = 2\lambda(-1)^j$ for $j \geq 1$ and $h_j(x) = (j + 1)G^j(x)g(x)$ for $j \geq 0$.

The cdf of X can be expressed as

$$F(x) = \sum_{j=0}^{\infty} a_j H_{j+1}(x), \quad (2.3)$$

where $H_{j+1}(x) = G^{j+1}(x)$ for $j \geq 0$.

Note that $h_j(x)$ is the pdf of the exp-G distribution with power parameter $j + 1$, and $H_{j+1}(x)$ the associated cdf. Therefore (2.2) (resp. (2.3)) reveals that the pdf (resp. cdf) of the T-G distribution is a linear combination of pdfs (resp. cdfs) of the exp-G distributions. Hence some properties of the exp-G distribution can be used to determine some mathematical properties of the T-G distribution. Technical details on the the exp-G distribution can be found in [13].

2.3 Quantile function

For $\lambda \in [-1, 1)$, the quantile function of X is given by

$$F^{-1}(x) = G^{-1} \left[\frac{1 - x + \lambda - \sqrt{x^2 + x(2 - 6\lambda) + (1 + \lambda)^2}}{2(\lambda - 1)} \right], \quad x \in (0, 1). \quad (2.4)$$

For $\lambda = 1$, we have

$$F^{-1}(x) = G^{-1} \left[\frac{x}{2 - x} \right], \quad x \in (0, 1). \quad (2.5)$$

Thus, for any random variable U following the uniform $U(0, 1)$ distribution, the random variable X given by $X = F^{-1}(U)$ follows the T-G distribution.

2.4 Hazard rate function

The hazard rate function (hrf) of X is given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\{(1 + \lambda) + (1 - \lambda)G(x)[2 + G(x)]\}g(x)}{[1 + G(x)][1 - \lambda G(x) - (1 - \lambda)G^2(x)]}, \quad \lambda \in [-1, 1], \quad x \in \mathbb{R}.$$

The critical points of $h(x)$ are obtained from the equation:

$$\frac{g'(x)}{g(x)} - \frac{g(x)}{1 + G(x)} + \frac{[\lambda - 2(\lambda - 1)G(x)]g(x)}{1 - \lambda G(x) + (\lambda - 1)G^2(x)} + \frac{2(1 - \lambda)[1 + G(x)]g(x)}{1 + \lambda - (1 - \lambda)G(x)[2 + G(x)]} = 0.$$

2.5 Moments, moment generating function and mean deviations

Using the expansion (2.2), the r^{th} moment of X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{j=0}^{\infty} a_j \int_{-\infty}^{\infty} x^r h_j(x) dx. \quad (2.6)$$

Similarly, denoting by $\mathbf{1}_A$ the indicator random variable on A , the r^{th} incomplete moment of X can be obtained as

$$\mu^r(x) = E(X^r \mathbf{1}_{\{X \leq x\}}) = \sum_{j=0}^{\infty} a_j T_{j,r}(x), \quad (2.7)$$

where $T_{j,r}(x) = \int_{-\infty}^x t^r h_j(t) dt$.

The moment generating function of X is given by

$$M(t) = E(e^{tX}) = \sum_{j=0}^{\infty} a_j M_j(t),$$

where $M_j(t) = \int_0^{\infty} e^{tx} h_j(x) dx$.

The mean deviations of X about the mean and median, respectively, can be put as

$$D_{\mu} = E(|X - \mu|) = 2\mu F(\mu) - 2\mu^1(\mu) \quad (2.8)$$

and

$$D_M = E(|X - M|) = \mu - 2\mu^1(M), \quad (2.9)$$

where $\mu = E(X)$ is given by (2.6), $F(\mu)$ is easily calculated from (1.2) and $\mu^1(M)$ is obtained from (2.7) with $M = F^{-1}(\frac{1}{2})$, using (2.4) or (2.5).

2.6 Residual and reversed residual life moments

Using the binomial expansion, the r^{th} residual life moments of X can be obtained as

$$\begin{aligned} M_r(t) &= E((X - t)^r | X > t) = \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} t^{r-j} E(X^j | X > t) \\ &= \frac{1}{1 - F(t)} \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} t^{r-j} [E(X^j) - \mu^j(t)], \end{aligned}$$

where $E(X^j)$ is given by (2.6) and $\mu^j(t)$ is obtained from (2.7).

Similarly, the r^{th} reversed residual life moments of X can be obtained as

$$m_r(t) = E((t - X)^r | X \leq t) = \sum_{j=0}^r \binom{r}{j} (-1)^j t^{r-j} E(X^j | X \leq t) = \frac{1}{F(t)} \sum_{j=0}^r \binom{r}{j} (-1)^j t^{r-j} \mu^j(t).$$

2.7 Stochastic ordering

This subsection is devoted to the stochastic ordering, which is a common concept to show the ordering mechanism in life time distributions. A random variable X is said to be stochastically smaller than a random variable Y in the

- stochastic order ($X \leq_{st} Y$) if the associated cdfs satisfy: $F_X(x) \geq F_Y(x)$ for all x .
- hazard rate order ($X \leq_{hr} Y$) if the associated hrfs satisfy: $h_X(x) \geq h_Y(x)$ for all x .
- likelihood ratio order ($X \leq_{lr} Y$) if the ratio of the associated pdfs given by $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

When the supports of X and Y have a common finite left end-point, the following chain of implications holds:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

Further details can be found in [19]. Let X be a random variable following the T-G distribution with parameter λ_1 with pdf $f_X(x)$ and Y be a random variable following the T-G distribution with parameter λ_2 with pdf $f_Y(x)$. Let us prove that $X \leq_{lr} Y$ under some assumptions on λ_1 and λ_2 . We have

$$f_X(x) = \left[1 - \lambda_1 + \frac{2\lambda_1}{(1 + G(x))^2} \right] g(x) = \frac{[1 + \lambda_1 + (1 - \lambda_1)[2 + G(x)]G(x)]G(x)}{(1 + G(x))^2} g(x)$$

and

$$f_Y(x) = \frac{[1 + \lambda_2 + (1 - \lambda_2)[2 + G(x)]G(x)]G(x)}{(1 + G(x))^2} g(x).$$

Hence

$$\frac{f_X(x)}{f_Y(x)} = \frac{1 + \lambda_1 + (1 - \lambda_1)z}{1 + \lambda_2 + (1 - \lambda_2)z},$$

where $z = [2 + G(x)]G(x)$. Taking derivative with respect to x , we obtain

$$\left(\frac{f_X(x)}{f_Y(x)} \right)' = \frac{2(\lambda_2 - \lambda_1)z'}{(1 + \lambda_2 + (1 - \lambda_2)z)^2},$$

where $z' = 2[1 + G(x)]g(x)$. Since $z' \geq 0$ and $(1 + \lambda_2 + (1 - \lambda_2)z)^2 > 0$, if $\lambda_2 < \lambda_1$, we have $\left(\frac{f_X(x)}{f_Y(x)} \right)' < 0$, implying that $\frac{f_X(x)}{f_Y(x)}$ decreases in x , so $X \leq_{lr} Y$.

2.8 Stress-strength reliability

In reliability theory, a common situation is that the life of a component has a random strength subjected to a random stress. The random strength can be modeled by a random variable X and the random stress can be modeled by a random variable Y . The probability that the component functions satisfactorily is given by $R = P(Y < X)$, which is a well-known measure of component reliability with many applications. Let X be a random variable following the T-G distribution with parameter λ_1 with pdf $f_X(x)$ and Y be a random variable following the T-G distribution with parameter λ_2 with cdf $F_Y(x)$. Owing to the expansions (2.2) and (2.3), we have

$$\begin{aligned} R &= P(Y < X) = \int_{-\infty}^{\infty} P(Y < X | \{X = x\}) f_X(x) dx = \int_{-\infty}^{\infty} f_X(x) F_Y(x) dx \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_j^{(\lambda_1)} a_i^{(\lambda_2)} \int_{-\infty}^{\infty} h_j(x) H_{i+1}(x) dx, \end{aligned}$$

where $a_j^{(\lambda)}$ is defined by $a_0^{(\lambda)} = 1 + \lambda$ and $a_j^{(\lambda)} = 2\lambda(-1)^j$ for $j \geq 1$ and $\lambda \in \{\lambda_1, \lambda_2\}$.

2.9 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let X_1, X_2, \dots, X_n is a random sample of size n of X and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the corresponding order statistics obtained from the sample. The pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x) (1-F(x))^{n-i} f(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F^{j+i-1}(x) f(x).$$

Using finite and generalized binomial series expansions, we have

$$\begin{aligned} F^{j+i-1}(x) &= \left[1 - \lambda + \frac{2\lambda}{1+G(x)} \right]^{j+i-1} G^{j+i-1}(x) \\ &= G^{j+i-1}(x) \sum_{k=0}^{j+i-1} \binom{j+i-1}{k} (1-\lambda)^{j+i-1-k} (2\lambda)^k [1+G(x)]^{-k} \\ &= \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} b_{i,j,k,\ell} H_{j+i+\ell-1}(x), \end{aligned}$$

where

$$b_{i,j,k,\ell} = \binom{j+i-1}{k} \binom{-k}{\ell} (1-\lambda)^{j+i-1-k} (2\lambda)^k.$$

Owing to the expansion (2.2), we can write

$$F^{j+i-1}(x) f(x) = \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} a_m b_{i,j,k,\ell} H_{j+i+\ell-1}(x) h_m(x),$$

therefore

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{j,k,\ell,m} H_{j+i+\ell-1}(x) h_m(x),$$

where

$$c_{j,k,\ell,m} = \frac{n!}{(i-1)!(n-i)!} \binom{n-i}{j} (-1)^j a_m b_{i,j,k,\ell}.$$

The r^{th} moment of $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \int_{-\infty}^{\infty} x^r f_{i:n}(x) dx = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} c_{j,k,\ell,m} \int_{-\infty}^{\infty} x^r H_{j+i+\ell-1}(x) h_m(x) dx.$$

Note that the integral term only depends on the $g(x)$ and $G(x)$ the pdf and cdf of base line distribution.

2.10 Rényi entropy

The Rényi entropy of X is given by

$$I_\delta = \frac{1}{1-\delta} E[f^{\delta-1}(X)] = \frac{1}{1-\delta} \int_{-\infty}^{\infty} f^\delta(x) dx \quad (2.10)$$

for $\delta > 0$ and $\delta \neq 1$. Using finite and generalized binomial series expansions (noticing that $\left| \lambda \left(\frac{2}{(1+G(x))^2} - 1 \right) \right| < 1$), we have

$$\begin{aligned}
f^\delta(x) &= \left[1 + \lambda \left(\frac{2}{(1+G(x))^2} - 1 \right) \right]^\delta g^\delta(x) = g^\delta(x) \sum_{k=0}^{\infty} \binom{\delta}{k} \lambda^k \left(\frac{2}{(1+G(x))^2} - 1 \right)^k \\
&= g^\delta(x) \sum_{k=0}^{\infty} \sum_{\ell=0}^k \binom{\delta}{k} \binom{k}{\ell} \lambda^k (-1)^{k-\ell} 2^\ell [1+G(x)]^{-2\ell} \\
&= g^\delta(x) \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{m=0}^{\infty} \binom{\delta}{k} \binom{k}{\ell} \binom{-2\ell}{m} \lambda^k (-1)^{k-\ell} 2^\ell H_m(x). \tag{2.11}
\end{aligned}$$

Hence we have

$$I_\delta = \frac{1}{1-\delta} \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{m=0}^{\infty} \binom{\delta}{k} \binom{k}{\ell} \binom{-2\ell}{m} \lambda^k (-1)^{k-\ell} 2^\ell \int_{-\infty}^{\infty} H_m(x) g^\delta(x) dx.$$

Note that the integral term only depends on the $g(x)$ and $G(x)$ the pdf and cdf of base line distribution. Finally, let us mention that the Shannon entropy of a random variable X is defined by $E(-\log[f(X)])$ is the special case of the Rényi entropy when $\delta \rightarrow 1$.

2.11 Mathai-Haubold entropy

Classical Shannon entropy has been generalized in many directions. One of them is the δ -generalized entropy introduced by [15]. It is defined by

$$f_{MH}(x) = \frac{1}{\delta-1} \left[\int f^{2-\delta}(x) dx - 1 \right].$$

Similar arguments to (2.11) gives

$$f^{2-\delta}(x) = g^{2-\delta}(x) \sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{m=0}^{\infty} \binom{2-\delta}{k} \binom{k}{\ell} \binom{-2\ell}{m} \lambda^k (-1)^{k-\ell} 2^\ell H_m(x).$$

Therefore

$$f_{MH}(x) = \frac{1}{\delta-1} \left[\sum_{k=0}^{\infty} \sum_{\ell=0}^k \sum_{m=0}^{\infty} \binom{2-\delta}{k} \binom{k}{\ell} \binom{-2\ell}{m} \lambda^k (-1)^{k-\ell} 2^\ell \int_{-\infty}^{\infty} H_m(x) g^{2-\delta}(x) dx - 1 \right].$$

The solution of above integral depends on any arbitrary base line distribution.

3 Special models

Let us now introduce some special T-G distributions based on some well-known distributions.

3.1 Transmuted Burr (TB) distribution

If Burr XII distribution has pdf and cdf given, respectively, by $g(x) = ckx^{c-1}(1+x^c)^{-k-1}$, $k, c, x > 0$, and $G(x) = 1 - (1+x^c)^{-k}$, then the cdf and pdf of T-B distribution are given by

$$F(x) = 1 - (1+x^c)^{-k} + \lambda \frac{[1 - (1+x^c)^{-k}][1+x^c]^{-k}}{2 - (1+x^c)^{-k}}, \quad x > 0$$

and

$$f(x) = ckx^{c-1}(1+x^c)^{-k-1} + \lambda \frac{ckx^{c-1}(1+x^c)^{-k-1} \left\{ 1 - [1 - (1+x^c)^{-k}] [3 - [1+x^c]^{-k}] \right\}}{[2 - [1+x^c]^{-k}]^2}, \quad x > 0.$$

Using these functions, we immediatly obtain the expression of the hrf. Figure 1 shows some plots of pdfs and hrfs for TB distribution with arbitrary choices for the parameters (c, k, λ) .

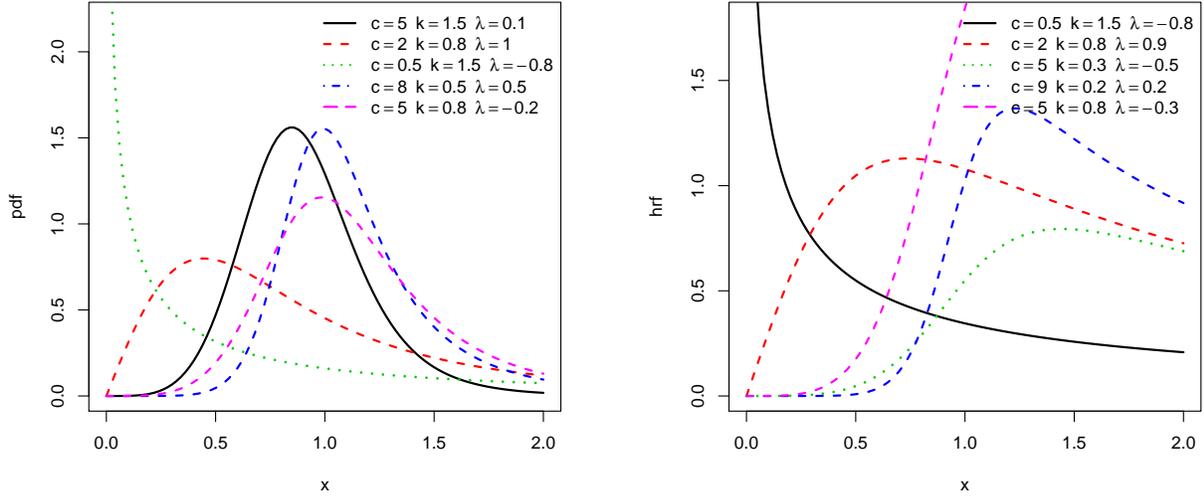


Figure 1: Plots of pdf and hrf for TB distribution with different parameter values.

3.2 Transmuted Gompertz (TGz) distribution

If Gompertz distribution has pdf and cdf given, respectively, by $g(x) = a \exp [bx - \frac{a}{b} (e^x - 1)]$, $a, b, x > 0$, and $G(x) = 1 - \exp [-\frac{a}{b} (e^{bx} - 1)]$, then the cdf and pdf of T-Gz distribution are given by

$$F(x) = \left[1 - \exp \left[-\frac{a}{b} (e^{bx} - 1) \right] \right] \left[1 + \lambda \frac{\exp \left[-\frac{a}{b} (e^{bx} - 1) \right]}{2 - \exp \left[-\frac{a}{b} (e^{bx} - 1) \right]} \right], \quad x > 0$$

and

$$f(x) = a \exp \left[bx - \frac{a}{b} (e^x - 1) \right] \left[1 + \lambda \frac{\left\{ 1 - [1 - \exp \left[-\frac{a}{b} (e^{bx} - 1) \right]] [3 - \exp \left[-\frac{a}{b} (e^{bx} - 1) \right]] \right\}}{[2 - \exp \left[-\frac{a}{b} (e^{bx} - 1) \right]]^2} \right], \quad x > 0.$$

The expression of the hrf follows immediatly. Figure 2 shows some plots of pdfs and hrfs for TGz distribution with arbitrary choices for the parameters (a, b, λ) .

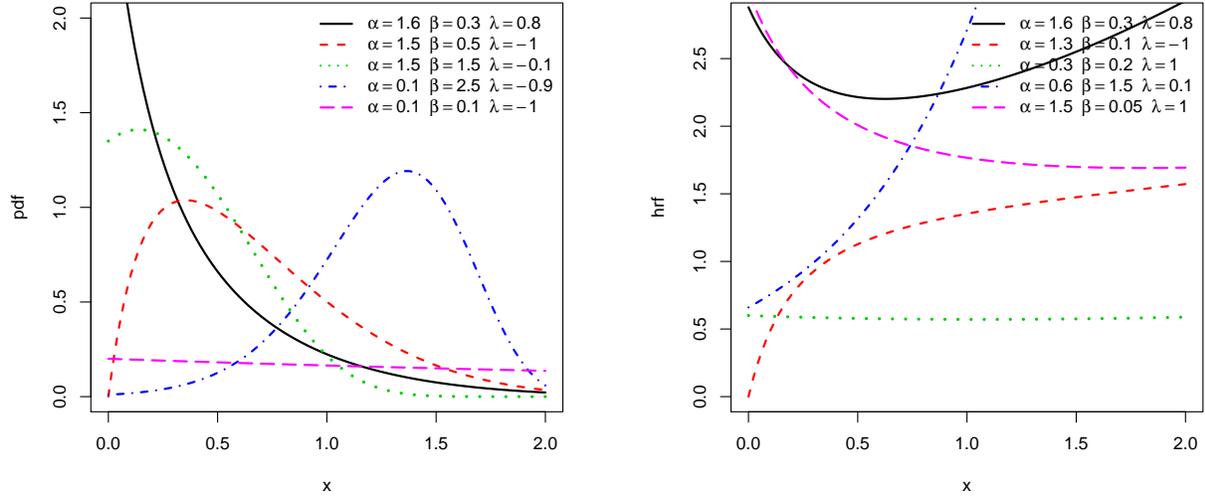


Figure 2: Plots of pdf and hrf for TGz distribution with different parameter values.

3.3 Transmuted Weibull (TW) distribution

If Weibull distribution has pdf and cdf given, respectively, by $g(x) = \alpha \beta x^{\beta-1} e^{-\alpha x^\beta}$, $\alpha, \beta, x > 0$, and $G(x) = 1 - e^{-\alpha x^\beta}$, then the cdf and pdf of T-W distribution are given by

$$F(x) = 1 - e^{-\alpha x^\beta} + \frac{\lambda [1 - e^{-\alpha x^\beta}]}{2 - e^{-\alpha x^\beta}} e^{-\alpha x^\beta}, \quad x > 0$$

and

$$f(x) = \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} + \frac{\lambda \alpha \beta x^{\beta-1} e^{-\alpha x^\beta}}{[2 - e^{-\alpha x^\beta}]^2} \left\{ 1 - [1 - e^{-\alpha x^\beta}] [3 - e^{-\alpha x^\beta}] \right\}, \quad x > 0.$$

The expression of the hrf follows immediately. Figure 3 presents some plots of pdfs and hrfs for TW distribution with arbitrary choices for the parameters (α, β, λ) .

3.4 Transmuted gamma (TGa) distribution

If Gamma distribution has pdf and cdf given, respectively, by $g(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$, $a, b, x > 0$, and $G(x) = \frac{\gamma(a, bx)}{\Gamma(a)}$, where $\gamma(a, bx) = b^a \int_0^x t^{a-1} e^{-bt} dt$, then the cdf and pdf of T-Ga distribution are given by

$$F(x) = \frac{\gamma(a, bx)}{\Gamma(a)} \left[1 + \lambda \frac{1 - \frac{\gamma(a, bx)}{\Gamma(a)}}{1 + \frac{\gamma(a, bx)}{\Gamma(a)}} \right], \quad x > 0$$

and

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \left[1 + \lambda \frac{\left\{ 1 - \left[\frac{\gamma(a, bx)}{\Gamma(a)} \left[2 - \frac{\gamma(a, bx)}{\Gamma(a)} \right] \right\}}{\left[1 + \frac{\gamma(a, bx)}{\Gamma(a)} \right]^2} \right], \quad x > 0.$$

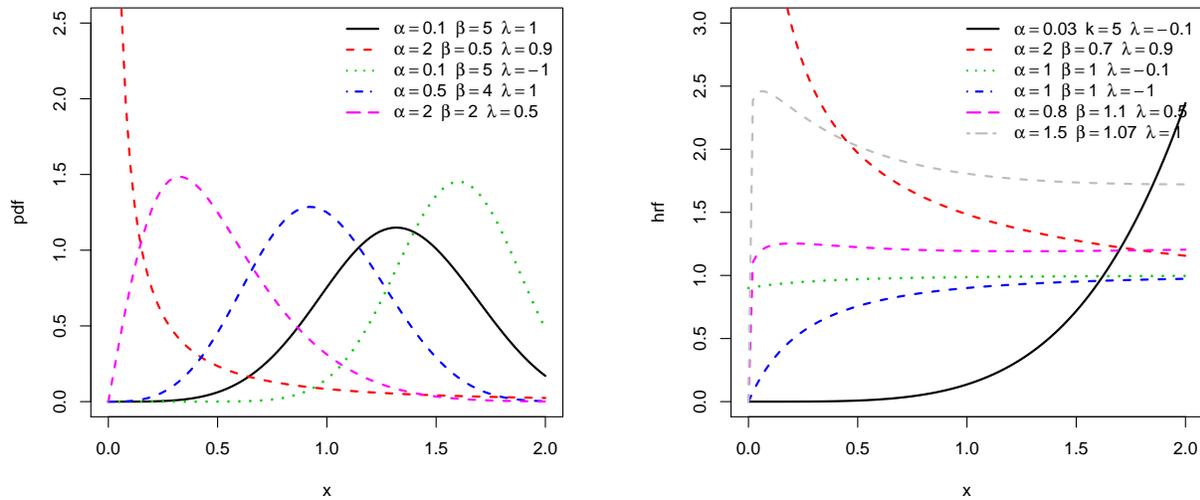


Figure 3: Plots of pdf and hrf for TW distribution with different parameter values.

We easily obtain the expression of the hrf. Figure 4 presents some plots of pdfs and hrfs for TGa distribution with arbitrary choices for the parameters (a, b, λ) .

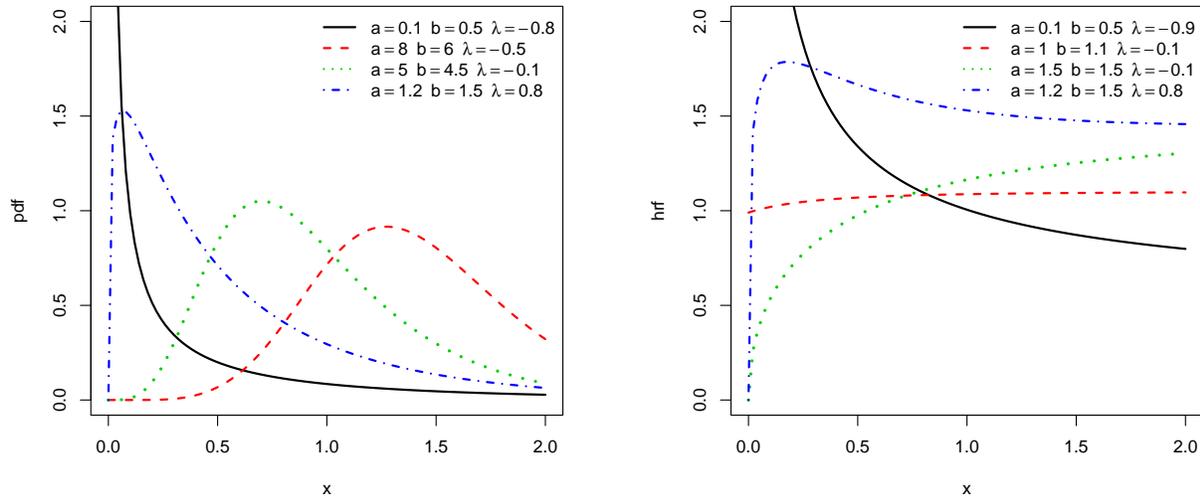


Figure 4: Plots of pdf and hrf for TGa distribution with different parameter values.

4 Bivariate extension

We now propose a bivariate version of the T-G family of distributions. Let (X, Y) be a bivariate random variable with the joint cdf given by

$$F_{X,Y}(x, y) = \left[1 - \lambda + \frac{2\lambda}{1 + G(x, y)} \right] G(x, y), \quad \lambda \in [-1, 1], (x, y) \in \mathbb{R}^2,$$

where $G(x, y)$ denoted a bivariate cdf. Let $G_1(x)$ and $G_2(y)$ be the corresponding marginal cdfs, $g_1(x)$ and $g_2(y)$ be the corresponding marginal pdfs and $g(x, y)$ be the corresponding pdf. Then the marginal cdfs of (X, Y) are given by

$$F_X(x) = \left[1 - \lambda + \frac{2\lambda}{1 + G_1(x)} \right] G_1(x)$$

and

$$F_Y(y) = \left[1 - \lambda + \frac{2\lambda}{1 + G_2(y)} \right] G_2(y).$$

The marginal pdfs of (X, Y) are given by

$$f_X(x) = \left[1 - \lambda + \frac{2\lambda}{(1 + G_1(x))^2} \right] g_1(x)$$

and

$$f_Y(y) = \left[1 - \lambda + \frac{2\lambda}{(1 + G_2(y))^2} \right] g_2(y).$$

The pdf of (X, Y) is given by

$$f(x, y) = \left[1 - \lambda + \frac{2\lambda}{(1 + G(x, y))^2} \right] \theta(x, y),$$

where

$$\theta(x, y) = g(x, y) - \frac{4\lambda}{(1 + G(x, y))[(1 - \lambda)(1 + G(x, y))^2 + 2\lambda]} \frac{\partial G(x, y)}{\partial x} \frac{\partial G(x, y)}{\partial y}.$$

The conditional cdfs are given by

$$F_{X|Y}(x | y) = \frac{(1 - \lambda)(1 + G(x, y)) + 2\lambda}{(1 - \lambda)(1 + G_2(y)) + 2\lambda} \times \frac{G(x, y)(1 + G_2(y))}{(1 + G(x, y))G_2(y)}$$

and

$$F_{Y|X}(y | x) = \frac{(1 - \lambda)(1 + G(x, y)) + 2\lambda}{(1 - \lambda)(1 + G_1(x)) + 2\lambda} \times \frac{G(x, y)(1 + G_1(x))}{(1 + G(x, y))G_1(x)}.$$

The conditional pdfs are given by

$$f_{X|Y}(x | y) = \frac{(1 - \lambda)(1 + G(x, y))^2 + 2\lambda}{(1 - \lambda)(1 + G_2(y))^2 + 2\lambda} \times \frac{\theta(x, y)(1 + G_2(y))^2}{(1 + G(x, y))^2 g_2(y)}$$

and

$$f_{Y|X}(y | x) = \frac{(1 - \lambda)(1 + G(x, y))^2 + 2\lambda}{(1 - \lambda)(1 + G_1(x))^2 + 2\lambda} \times \frac{\theta(x, y)(1 + G_1(x))^2}{(1 + G(x, y))^2 g_1(x)}.$$

5 Different estimation methods

In this section, we will give the estimates of the T-G family by five different methods of estimations, i.e., Maximum Likelihood method, Least Square method, Weighted Least Square method, Maximum Product Spacing method and Cramer Von Mises method.

5.1 Maximum likelihood method (MLE)

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of T-G from complete samples only. Let x_1, x_2, \dots, x_n be observed values from the T-G distribution with parameters λ and ξ . Let $\Theta = (\lambda, \xi)^T$ be the parameter vector. The total log-likelihood function for Θ is given by

$$\ell(\Theta) = \sum_{i=1}^n \log[g(x_i)] - 2 \sum_{i=1}^n \log[1 - G(x_i)] + \sum_{i=1}^n \log[(1 + \lambda) + (1 - \lambda)G(x_i)].$$

Maximizing $\ell(\Theta)$ with respect to λ and ξ , we have following system of non linear equations:

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \sum_{i=1}^n \left[\frac{1 - G(x_i) \{2 + G(x_i)\}}{(1 + \lambda) + (1 - \lambda)G(x_i) \{2 + G(x_i)\}} \right] = 0,$$

$$\frac{\partial \ell(\Theta)}{\partial \xi} = \sum_{i=1}^n \left[\frac{g'(x_i)_\xi}{g(x_i)} \right] - 2 \sum_{i=1}^n \left[\frac{G'(x_i)_\xi}{1 + G(x_i)} \right] + \sum_{i=1}^n \left[\frac{2(1 - \lambda)G'(x_i)_\xi \{1 + G(x_i)\}}{(1 + \lambda) + (1 - \lambda)G(x_i) \{2 + G(x_i)\}} \right] = 0,$$

where $g'(x_i)_\xi = \frac{\partial}{\partial \xi} g(x_i)$ and $G'(x_i)_\xi = \frac{\partial}{\partial \xi} G(x_i)$. This system of non-linear equations can be solved numerically by any software to obtained the estimates $\hat{\lambda}_{MLE}$ and $\hat{\xi}_{MLE}$.

5.2 Least square method (LSE)

Let x_1, x_2, \dots, x_n be the observed values from the T-G distribution with parameters λ and ξ , in increasing order. By considering the associated order statistics $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, note that $E[F(X_{i:n})] = \frac{i}{n+1}$. Least square estimates can be obtained by minimizing the following expression

$$S(\Theta) = \sum_{i=1}^n [F(x_i) - E[F(X_{i:n})]]^2 = \sum_{i=1}^n \left[F(x_i) - \frac{i}{n+1} \right]^2.$$

Minimizing $S(\Theta)$ with respect to λ and ξ , we have following system of non linear equations:

$$\frac{\partial S(\Theta)}{\partial \lambda} = 2 \sum_{i=1}^n \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\lambda = 0,$$

$$\frac{\partial S(\Theta)}{\partial \xi} = 2 \sum_{i=1}^n \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i)_\xi = 0,$$

where $F'(x_i)_\lambda = \frac{\partial}{\partial \lambda} F(x_i)$ and $F'(x_i)_\xi = \frac{\partial}{\partial \xi} F(x_i)$. This system of non-linear equations can be solved numerically by any software to obtained the estimates $\hat{\lambda}_{LSE}$ and $\hat{\xi}_{LSE}$.

5.3 Minimum spacing method (MPS)

This method is based on an idea that the differences (spacings) of the consecutive points should be identically distributed. Let x_1, x_2, \dots, x_n be the observed values from the T-G distribution with parameters λ and ξ , in increasing order. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i},$$

where $D_i = F(x_i) - F(x_{i-1})$ with $F(x_0) = 0$ and $F(x_{n+1}) = 1$. We have

$$\log(GM) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log [F(x_i) - F(x_{i-1})].$$

Using the cdf in (1.2), we have

$$\log(GM) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \left[\left[1 - \lambda + \frac{2\lambda}{1+G(x_i)} \right] G(x_i) - \left[1 - \lambda + \frac{2\lambda}{1+G(x_{i-1})} \right] G(x_{i-1}) \right].$$

Minimizing $\log(GM)$ with respect to λ and ξ , we have following system of non linear equations:

$$\begin{aligned} \frac{\partial \log(GM)}{\partial \lambda} &= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{\left[-1 + \frac{2}{1+G(x_i)} \right] G(x_i) - \left[-1 + \frac{2}{1+G(x_{i-1})} \right] G(x_{i-1})}{\left[\left[1 - \lambda + \frac{2\lambda}{1+G(x_i)} \right] G(x_i) - \left[1 - \lambda + \frac{2\lambda}{1+G(x_{i-1})} \right] G(x_{i-1}) \right]} \right] = 0, \\ \frac{\partial \log(GM)}{\partial \xi} &= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[\frac{\left\{ \left[1 - \lambda + \frac{2\lambda}{1+G(x_i)} \right] G'(x_i) \xi + \left[\frac{2\lambda}{[1+G(x_i)]^2} \right] G(x_i) \right\}}{\left[\left[1 - \lambda + \frac{2\lambda}{1+G(x_i)} \right] G(x_i) - \left[1 - \lambda + \frac{2\lambda}{1+G(x_{i-1})} \right] G(x_{i-1}) \right]} \right. \\ &\quad \left. - \frac{\left\{ \left[1 - \lambda + \frac{2\lambda}{1+G(x_{i-1})} \right] G'(x_{i-1}) \xi + \left[\frac{2\lambda}{[1+G(x_{i-1})]^2} \right] G(x_{i-1}) \right\}}{\left[\left[1 - \lambda + \frac{2\lambda}{1+G(x_i)} \right] G(x_i) - \left[1 - \lambda + \frac{2\lambda}{1+G(x_{i-1})} \right] G(x_{i-1}) \right]} \right] = 0. \end{aligned}$$

This system of non-linear equations can be solved numerically by any software to obtained the estimates $\hat{\lambda}_{MPE}$ and $\hat{\xi}_{MPE}$.

5.4 Weighted least square (WLS)

Let x_1, x_2, \dots, x_n be the observed values from the T-G distribution with parameters λ and ξ , in increasing order. The likelihood function for weighted least square estimates is given by

$$W(\Theta) = \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1} \right]^2.$$

Minimizing $W(\Theta)$ with respect to λ and ξ , we have following system of non linear equations:

$$\begin{aligned} \frac{\partial W(\Theta)}{\partial \lambda} &= 2 \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i) \lambda = 0, \\ \frac{\partial W(\Theta)}{\partial \xi} &= 2 \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1} \right] F'(x_i) \xi = 0. \end{aligned}$$

These equations can be solved numerically to obtained the estimates $\hat{\lambda}_{wls}$ and $\hat{\xi}_{wls}$.

5.5 Cramer Von Mises (CVM)

Cramer von Mises is a type of minimum distance estimators. Let x_1, x_2, \dots, x_n be the observed values from the T-G distribution with parameters λ and ξ , in increasing order. The likelihood function for Cramer Von Mises estimates is given by

$$C(\Theta) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_i) - \frac{2i-1}{2n} \right]^2.$$

Minimizing $C(\Theta)$ with respect to λ and ξ , we have following system of non linear equations:

$$\frac{\partial C(\Theta)}{\partial \lambda} = 2 \sum_{i=1}^n \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\lambda = 0,$$

$$\frac{\partial C(\Theta)}{\partial \xi} = 2 \sum_{i=1}^n \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\xi = 0.$$

This system of non-linear equations can be solved numerically to obtained the estimates $\hat{\lambda}_{CME}$ and $\hat{\xi}_{CME}$.

6 Simulation results and applications to real data

In this section, we present some experimental results to compare the performance of the different estimators proposed in the previous section. Then we use two real data sets set to compare TB, TW, TGa and TGz to TGE(transmuted generalized exponential distribution), TLE (Transmuted linear exponential distribution), GT-W (generalized transmuted Weibull), Burr XII, Weibull, Gamma and Gompertz distributions.

6.1 Simulation results

We perform extensive Monte Carlo simulations to compare the performance of the different estimators (CVM, LSE, WLS, MPS and MLE), mainly with respect to their biases and mean-squared errors (MSEs) for different sample sizes and for different parameter values. For obtaining different estimators, the number of replications is 10000 in all the simulations. We have considered different sample sizes $n = 10, 50, 100$ and 250. For obtaining frequentist estimators, we consider TGz with $a = 10, b = 2.5, \lambda = 0.5$, because it was the superior among the considered sub-models of the T-G family as it will be shown in next subsection. The simulation results are provided in Table 1.

Table 1: Simulation results for TGz.

n	parameters	CVM	LSE	WLS	MPS	MLE
20	Bias(\hat{a})	1.334	4.044	3.495	0.385	0.275
	MSE(\hat{a})	1.643	5.158	4.291	0.949	1.191
	Bias(\hat{b})	0.300	0.240	0.264	0.017	0.111
	MSE(\hat{b})	0.300	0.336	0.400	0.388	0.480
	Bias($\hat{\lambda}$)	0.077	0.092	0.085	0.041	0.050
	MSE($\hat{\lambda}$)	0.125	0.203	0.185	0.070	0.089
50	Bias(\hat{a})	1.121	3.777	3.870	0.580	0.033
	MSE(\hat{a})	1.233	4.153	4.096	1.296	0.799
	Bias(\hat{b})	0.300	0.268	0.368	0.201	0.034
	MSE(\hat{b})	0.300	0.294	0.438	0.347	0.339
	Bias($\hat{\lambda}$)	0.071	0.091	0.092	0.032	0.034
	MSE($\hat{\lambda}$)	0.114	0.201	0.205	0.048	0.058
100	Bias(\hat{a})	1.166	3.459	4.179	0.535	0.010
	MSE(\hat{a})	1.869	3.685	4.313	1.265	0.589
	Bias(\hat{b})	0.307	0.249	0.427	0.179	0.035
	MSE(\hat{b})	0.465	0.285	0.472	0.326	0.269
	Bias($\hat{\lambda}$)	0.021	0.088	0.099	0.021	0.023
	MSE($\hat{\lambda}$)	0.033	0.191	0.219	0.032	0.040
250	Bias(\hat{a})	0.747	3.452	4.263	0.290	0.028
	MSE(\hat{a})	1.510	3.560	4.337	1.049	0.438
	Bias(\hat{b})	0.198	0.261	0.440	0.102	0.024
	MSE(\hat{b})	0.346	0.281	0.462	0.240	0.221
	Bias($\hat{\lambda}$)	0.014	0.089	0.100	0.012	0.017
	MSE($\hat{\lambda}$)	0.022	0.191	0.223	0.020	0.028

The results indicate that the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

6.2 Empirical reliability data examples

In this subsection, we evaluate the performance of the T-G family of distributions by fitting four sub-models of this family, namely TB, TW, TGa and TGz to two reliability data sets. The data sets are described as follows.

The first data set (data set 1) refer to the 50 observations with hole and sheet thickness are 12 mm and 3.15 mm reported by [10]. The data are: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

The second data set (data set 2) relating to the strengths of 1.5 cm glass fibers given by [17]. This set was obtained by workers at the UK National Physical Laboratory. The observations are as follows: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

The descriptive statistics of the two data sets are given in Table 3. From this table, both data sets are under-dispersed. The data set 1 is approximately symmetric and the data set is moderately skewed. Moreover, the first data set is platykurtic while the second one is leptokurtic.

For all the compared distributions, the maximum likelihood method is used to estimate the parameters and also their standard errors are obtained. The model adequacy measures: Anderson-Darling (A^*), Cramer-von Mises (W^*) and Kolmogorov-Smirnov (K-S) statistic with its p-value are used to compare these distributions, where the smaller values of these statistics and larger p-value give the best fit to the data. The obtained results are given in Tables 2 and 7, and we conclude that the considered sub-models of the family are good competitor to the compared distributions. Also, it is found that the smallest values of A^* , W^* , K-S and largest values of p-value are obtained for the TGz distribution, and by this the TGz distribution has a good performance among the compared distributions. Also, this conclusion is asserted again by Figures 5 and 6. Consequently, the variance-covariance matrices of the MLEs of the TGz distribution for both data sets are obtained, noting that the diagonal entries of those matrices are the variances of the MLEs of the TGz parameters for each data set, while the other entries can led to positive and negative correlations for pairs of the estimates. Also, 95% and 99% confidence intervals of the TGz parameters are given in Tables 5 and 6 for both data sets. Moreover, some measures of the TGz distribution for both data sets are obtained by using its theoretical properties and are shown in Table 4. Making use of Tables 3 and 4, we can note the closeness between measures of the TGz distribution and the corresponding descriptive statistics of both data sets.

Table 2: MLEs, their standard errors (in parentheses), A*, W*, KS, and P-value for data set 1.

Distribution	MLE's	A*	W*	KS	P-value		
TB(c,k, λ)	2.148 (0.321)	38.854 (16.111)	-0.0097 (0.555)	0.660	0.107	0.109	0.581
TW(α,β,λ)	35.904 (15.729)	2.108 (0.334)	-0.025 (0.562)	0.642	0.104	0.109	0.587
TGa(a,b, λ)	2.818 (0.678)	18.218 (3.924)	-0.293 (0.442)	1.045	0.173	0.142	0.236
TGz(a,b, λ)	2.285 (1.404)	8.765 (4.719)	-0.421 (0.228)	0.445	0.072	0.097	0.736
TGE(α,θ,λ)	2.693 (0.839)	12.493 (1.666)	-0.547 (0.318)	1.076	0.178	0.147	0.234
TLE(β,θ,λ)	0.612 (1.895)	65.579 (14.184)	-0.378 (0.441)	0.570	0.096	0.107	0.630
GT-W(η,σ,λ)	0.170 (0.023)	1.991 (0.330)	-0.271 (0.430)	0.607	0.098	0.117	0.612
Burr(c,k)	2.154 (0.240)	39.124 (15.013)	-	0.675	0.108	0.120	0.571
Weibull(α,β)	36.145 (14.393)	2.117 (0.245)	-	0.649	0.108	0.110	0.569
Gamma(a,b)	3.029 (0.575)	18.564 (3.836)	-	1.120	0.192	0.145	0.170
Gompertz(a,b)	1.605 (0.512)	10.221 (1.753)	-	0.475	0.0875	0.135	0.568

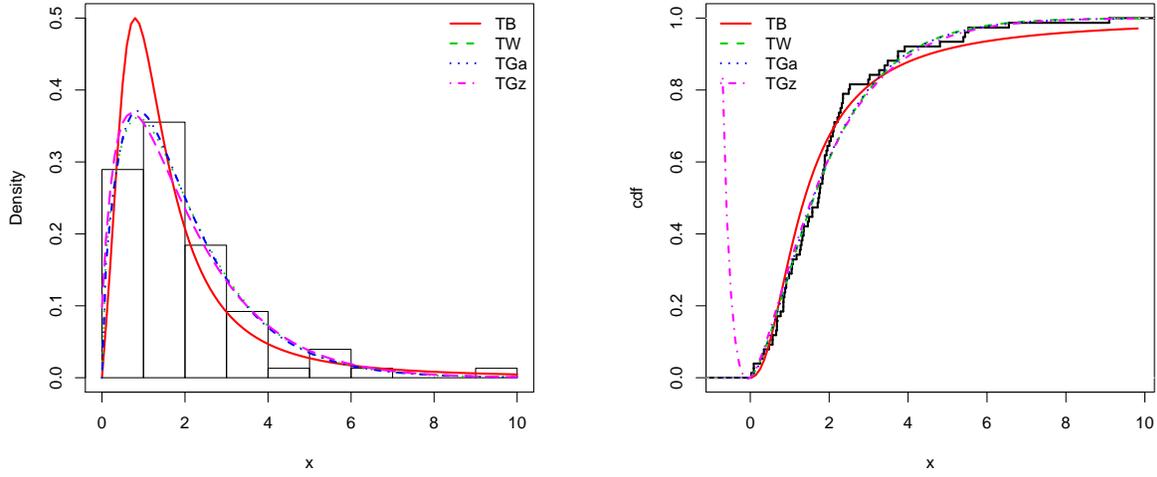


Figure 5: The estimated pdf and cdf for data set 1.

Table 3: Descriptive statistics for the two data sets.

Statistics	Mean	SD	Median	MD Mean	MD Median	Skewness	Kurtosis	Entropy
Data set 1	0.16	0.08	0.16	0.065	0.07	0.07	2.21	2.47
Data set 2	1.51	0.32	1.59	0.24	0.23	-0.88	3.92	3.80

Table 4: Some moments of the TGz distribution for the two data sets.

Statistics	Mean	S.D	Median	Variance	Skewness	Kurtosis	Entropy	MD Mean	MD Median
Data set 1	0.163	0.079	0.177	0.006	0.093	2.606	2.68	0.065	0.069
Data set 2	1.52	0.32	1.60	0.11	-0.90	4.20	3.86	0.25	0.23

Table 5: Confidence intervals of TGz for given data set 1.

CI	a	b	λ
95%	[0, 5.036]	[0, 18.014]	[0, 0.025]
99%	[0, 5.907]	[0, 20.940]	[0, 0.167]

Table 6: Confidence intervals of TGz for given data set 2.

CI	a	b	λ
95%	[0, 0.051]	[2.242, 4.111]	[0, 0.253]
99%	[0, 0.061]	[1.946, 4.407]	[0, 0.532]

Table 7: MLEs, their standard errors (in parentheses), A^* , W^* , KS, and P-value for data set 2.

Distribution	MLE's			A^*	W^*	KS	P-value
TB(c,k,λ)	5.889 (1.208)	0.492 (0.115)	-0.909 (0.173)	5.751	1.055	0.293	0.003
TW(α,β,λ)	0.077 (0.034)	5.467 (0.680)	-0.314 (0.366)	1.242	0.226	0.144	0.442
TGa(a,b,λ)	16.354 (3.177)	11.241 (2.105)	-0.465 (0.266)	2.921	0.532	0.201	0.223
TGz(a,b,λ)	0.020 (0.016)	3.177 (0.477)	-0.629 (0.450)	0.808	0.140	0.124	0.685
Burr(c,k)	7.482 (1.280)	0.320 (0.065)	-	6.128	1.132	0.330	0.002
Weibull(α,β)	0.059 (0.020)	5.77 (0.575)	-	1.303	0.237	0.152	0.307
Gamma(a,b)	17.439 (3.078)	11.573 (2.072)	-	3.117	0.568	0.216	0.105
Gompertz(a,b)	0.009 (0.004)	3.618 (0.294)	-	0.842	0.144	0.130	0.437

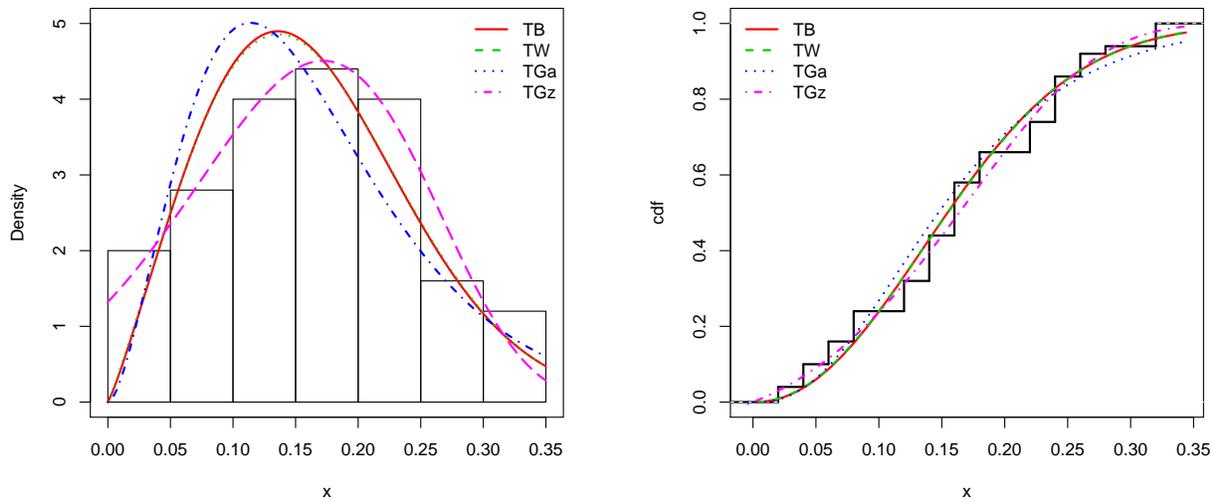


Figure 6: The estimated pdf and cdf for data set 2.

The variance-covariance matrices of the MLEs of the TGz distribution for data set 1 is given as

$$\begin{pmatrix} 5.779 & -11.185 & -2.841 \\ -11.185 & 22.275 & 5.421 \\ -2.841 & 5.421 & 1.509 \end{pmatrix}.$$

The variance-covariance matrices of the MLEs of the TGz distribution for data set 2 is given as

$$\begin{pmatrix} 0.001 & -0.007 & -0.006 \\ -0.007 & 0.226 & 0.163 \\ -0.006 & 0.163 & 0.202 \end{pmatrix}.$$

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