Variance of the volume of random real algebraic submanifolds II

Thomas Letendre, Martin Puchol

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Abstract

Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and let M be its real locus. We study the vanishing locus Z_{s_d} in M of a random real holomorphic section s_d of $\mathcal{E}\otimes\mathcal{L}^d$, where $\mathcal{L}\to\mathcal{X}$ is an ample line bundle and $\mathcal{E}\to\mathcal{X}$ is a rank r Hermitian bundle, $r\in\{1,\ldots,n\}$. We establish the asymptotics of the variance of the linear statistics associated with Z_{s_d}, as d goes to infinity. These asymptotics are of order $d^{r-\frac{n}{2}}$. As a special case, we get the asymptotic variance of the volume of Z_{s_d}.

The present paper extends the results of [22], by the first-named author, in essentially two ways. First, our main theorem covers the case of maximal codimension ($r=n$), which was left out in [22]. Second, we show that the leading constant in our asymptotics is positive. This last result is proved by studying the Wiener–Itō expansion of the linear statistics associated with the common zero set in $\mathbb{RP}^n$ of r independent Kostlan–Shub–Smale polynomials.

Keywords: Random submanifolds, Kac–Rice formula, linear statistics, Kostlan–Shub–Smale polynomials, Bergman kernel, real projective manifold, Wiener–Itō expansion.

Mathematics Subject Classification 2010: 14P99, 32A25, 53C40, 60G15, 60G57.

1 Introduction

In recent years, the study of random submanifolds has been a very active research area [9, 14, 15, 25, 26, 29]. There exist several models of random submanifolds, built on the following principle. Given M a dimension n ambient manifold and $r\in\{1,\ldots,n\}$, we consider the common zero set of r independent random functions on M. Under some technical assumption, this zero set is almost surely a codimension r smooth submanifold.

In this paper, we are interested in a model of random real algebraic submanifolds in a projective manifold. It was introduced in this generality by Gayet and Welschinger in [13] and studied in [14, 15, 21, 22], among others. This model is the real counterpart of the random complex algebraic submanifolds considered by Bleher, Shiffman and Zelditch [6, 30, 31].

Framework. Let us describe more precisely our framework. More details are given in Sect. 2, below. Let $\mathcal{X}$ be a smooth complex projective manifold of dimension $n\geqslant 1$. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ and let $\mathcal{E}$ be a rank $r\in\{1,\ldots,n\}$ holomorphic vector bundle over $\mathcal{X}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures and that the real locus of $\mathcal{X}$ is not empty. We denote by M this real locus which is a smooth closed (i.e. compact without boundary) manifold of real dimension n.

Let $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ denote Hermitian metrics on $\mathcal{E}$ and $\mathcal{L}$ respectively, which are compatible with the real structures. We assume that $h_{\mathcal{L}}$ has positive curvature ω, so that ω is a Kähler form on $\mathcal{X}$. This ω induces a Riemannian metric g on $\mathcal{X}$, hence on M. Let us denote by $|\mathrm{d}V_M|$ the Riemannian volume measure on M induced by g.

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For any $d \in \mathbb{N}$, the measure $|\mathrm{d}V_M|$ and the metrics $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$ induce a Euclidean inner product on the space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of global real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ (see Eq. (2.2)). Given $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by $Z_s = s^{-1}(0) \cap M$ the real zero set of s. For d large enough, for almost every s with respect to the Lebesgue measure, Z_s is a codimension r smooth closed submanifold of M, possibly empty. We denote by $|\mathrm{d}V_s|$ the Riemannian volume measure on Z_s induced by g. In the following, we consider $|\mathrm{d}V_s|$ as a Radon measure on M, that is a continuous linear form on $(\mathcal{C}^0(M), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the sup norm.

Remark 1.1. If $n = r$ then Z_s is a finite subset of M for almost every s. In this case, $|\mathrm{d}V_s|$ is the sum of the unit Dirac masses on the points of Z_s.

Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then $|\mathrm{d}V_{s_d}|$ is a random positive Radon measure on M. We set $Z_d = Z_{s_d}$ and $|\mathrm{d}V_d| = |\mathrm{d}V_{s_d}|$ in order to simplify notations. We are interested in the asymptotic distribution of the linear statistics $\langle |\mathrm{d}V_d|\ , \phi \rangle = \int_{Z_d} \phi\, |\mathrm{d}V_d|$, where $\phi : M \to \mathbb{R}$ is a continuous test-function. In particular, $\langle |\mathrm{d}V_d|\ , \mathbf{1} \rangle$ is the volume of Z_d (its cardinal if $n = r$), where $\mathbf{1}$ is the unit constant function on M.

As usual, we denote by $\mathbb{E}[X]$ the mathematical expectation of the random vector X. The asymptotic expectation of $\langle |\mathrm{d}V_d|\ , \phi \rangle$ was computed in [21, Sect. 5.3].

Theorem 1.2 ([21])**.** *Let $\mathcal{X}$ be a complex projective manifold of positive dimension n defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \ldots, n\}$ Hermitian vector bundle and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as $d \to +\infty$:*

$$\forall \phi \in \mathcal{C}^0(M), \qquad \mathbb{E}[\langle |\mathrm{d}V_d|\ , \phi \rangle] = d^{\frac{r}{2}} \left(\int_M \phi\, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_\infty O\big(d^{\frac{r}{2}-1}\big)\,.$$

Moreover the error term $O\big(d^{\frac{r}{2}-1}\big)$ does not depend on ϕ.

The asymptotic variance of $\langle |\mathrm{d}V_d|\ , \phi \rangle$, as d goes to infinity, was proved to be a $O\big(d^{r-\frac{n}{2}}\big)$ when the codimension of Z_d is $r < n$ (see [22, Thm. 1.6]). Our first main theorem (Thm. 1.6 below) extends this result to the maximal codimension case.

Statement of the main results. Before we state our main result, let us introduce some more notations. We denote by $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])\,(Y - \mathbb{E}[Y])]$ the covariance of the real random variables X and Y. Let $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$ denote the variance of X. Finally, we call *variance* of $|\mathrm{d}V_d|$ and we denote by $\mathrm{Var}(|\mathrm{d}V_d|)$ the symmetric bilinear form on $\mathcal{C}^0(M)$ defined by:

$$\forall \phi_1, \phi_2 \in \mathcal{C}^0(M), \qquad \mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1, \phi_2) = \mathrm{Cov}(\langle |\mathrm{d}V_d|\ , \phi_1 \rangle, \langle |\mathrm{d}V_d|\ , \phi_2 \rangle)\,.$$

Definition 1.3. Let $\phi \in \mathcal{C}^0(M)$, we denote by ϖ_ϕ its *continuity modulus*, which is the function from $(0, +\infty)$ to $[0, +\infty)$ defined by:

$$\varpi_\phi : \varepsilon \longmapsto \sup\left\{ |\phi(x) - \phi(y)| \;\middle|\; (x, y) \in M^2, \rho_g(x, y) \leqslant \varepsilon \right\},$$

where $\rho_g(\cdot, \cdot)$ stands for the geodesic distance on (M, g).

We denote by $\mathcal{M}_{rn}(\mathbb{R})$ the space of matrices of size $r \times n$ with real coefficients.

Definition 1.4. Let $L : V \to V'$ be a linear map between two Euclidean spaces. We denote the *Jacobian* of L by $\left|\det{}^{\perp}(L)\right| = \sqrt{\det\,(LL^*)}$, where $L^* : V' \to V$ is the adjoint operator of L. Similarly, let $A \in \mathcal{M}_{rn}(\mathbb{R})$, we define its *Jacobian* to be $\left|\det{}^{\perp}(A)\right| = \sqrt{\det\,(AA^{\mathrm{t}})}$.

Definition 1.5. For every $t > 0$, we define $(X(t), Y(t))$ to be a centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ such that the following hold:

- the couples $(X_{ij}(t), Y_{ij}(t))$ with $i, j \in \{1, \ldots, n\}$ are independent from one another,
- the variance matrix of $(X_{ij}(t), Y_{ij}(t))$ is:

$$\begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & e^{-\frac{t}{2}}\left(1 - \frac{t}{1-e^{-t}}\right) \\ e^{-\frac{t}{2}}\left(1 - \frac{t}{1-e^{-t}}\right) & 1 - \frac{te^{-t}}{1-e^{-t}} \end{pmatrix} \quad \text{if } j = 1, \text{ and} \quad \begin{pmatrix} 1 & e^{-\frac{t}{2}} \\ e^{-\frac{t}{2}} & 1 \end{pmatrix} \quad \text{otherwise.}$$

We can now state our main result.

Theorem 1.6. *Let $\mathcal{X}$ be a complex projective manifold of dimension $n \geqslant 1$ defined over the reals, we assume that its real locus M is non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ Hermitian vector bundle and let $\mathcal{L} \to \mathcal{X}$ be a positive Hermitian line bundle, both equipped with compatible real structures. For every $d \in \mathbb{N}$, let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.*

Let $\beta \in (0, \frac{1}{2})$, then there exists $C_\beta > 0$ such that, for all $\alpha \in (0,1)$, for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, the following holds as $d \to +\infty$:

$$\mathrm{Var}(|\mathrm{d}V_d|)(\phi_1, \phi_2) = d^{r-\frac{n}{2}} \left(\int_M \phi_1 \phi_2 \, |\mathrm{d}V_M| \right) \left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} + \delta_{rn} \frac{2}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \right) + \|\phi_1\|_\infty \|\phi_2\|_\infty O\big(d^{r-\frac{n}{2}-\alpha}\big) + \|\phi_1\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\big(d^{r-\frac{n}{2}}\big), \quad (1.1)$$

where δ_{rn} is the Kronecker symbol, equal to 1 if $r = n$ and 0 otherwise, and

$$\mathcal{I}_{n,r} = \frac{1}{2} \int_0^{+\infty} \left(\frac{\mathbb{E}\left[\left|\det{}^\perp(X(t))\right| \left|\det{}^\perp(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r \left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \right)^2 \right) t^{\frac{n-2}{2}} \, \mathrm{d}t < +\infty. \quad (1.2)$$

Moreover the error terms $O\big(d^{r-\frac{n}{2}-\alpha}\big)$ and $O\big(d^{r-\frac{n}{2}}\big)$ in (1.1) do not depend on (ϕ_1, ϕ_2).

Remark 1.7. Applying Thm. 1.6 with $\phi_1 = \mathbf{1} = \phi_2$ gives the asymptotic variance of the Riemannian volume of Z_d.

Theorem 1.8. *For any $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, the universal constant:*

$$\frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{(2\pi)^r} \mathcal{I}_{n,r} + \delta_{rn} \frac{2}{\mathrm{Vol}\left(\mathbb{S}^n\right)}$$

appearing in Thm. 1.6 is positive.

Remark 1.9. Thm. 1.8 was proved for $r = n = 1$ in [11], and for $r = n \geqslant 2$ in [2]. Note that Thm. 1.8 states that $\mathcal{I}_{n,r} > 0$ if $r < n$, but this is not necessarily the case when $r = n$. Indeed, $\mathcal{I}_{1,1} < 0$ by [11, Prop. 3.1 and Rem. 1].

Let us state some corollaries of Thm. 1.6. Cor. 1.10, 1.11 and 1.12 below are extensions to the case $r \leqslant n$ of Cor. 1.9, 1.10 and 1.11 of [22], respectively. The proofs that Thm. 1.6 implies Cor. 1.10, 1.11 and 1.12 were given in [22, Sect. 5] in the case $r < n$. They are still valid for $r \leqslant n$. We do not reproduce these proofs in the present paper.

Corollary 1.10 (Concentration in probability)**.** *In the same setting as Thm. 1.6, let $\alpha > -\frac{n}{4}$ and let $\phi \in \mathcal{C}^0(M)$. Then, for every $\varepsilon > 0$, we have:*

$$\mathbb{P}\left(d^{-\frac{r}{2}} \left| \langle |\mathrm{d}V_d| \, , \phi \rangle - \mathbb{E}[\langle |\mathrm{d}V_d| \, , \phi \rangle] \right| > d^\alpha \varepsilon \right) = \frac{1}{\varepsilon^2} O\Big(d^{-(\frac{n}{2}+2\alpha)} \Big),$$

where the error term is independent of ε, but depends on ϕ.

Corollary 1.11. *In the same setting as Thm. 1.6, let $U \subset M$ be an open subset, then as $d \to +\infty$ we have $\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\big(d^{-\frac{n}{2}}\big)$.*

Let us denote by $\mathrm{d}\nu_d$ the standard Gaussian measure on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ (see (2.1)). Let $\mathrm{d}\nu$ denote the product measure $\bigotimes_{d \in \mathbb{N}} \mathrm{d}\nu_d$ on $\prod_{d \in \mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then we have the following.

Corollary 1.12 (Strong law of large numbers)**.** *In the setting of Thm. 1.6, let us assume $n \geqslant 3$. Let $(s_d)_{d \in \mathbb{N}} \in \prod_{d \in \mathbb{N}} \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ be a random sequence of sections. Then, $\mathrm{d}\nu$-almost surely,*

$$d^{-\frac{r}{2}} |\mathrm{d}V_{s_d}| \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} |\mathrm{d}V_M|,$$

in the sense of the weak convergence of measures. That is, $\mathrm{d}\nu$-almost surely,

$$\forall \phi \in \mathcal{C}^0(M), \qquad d^{-\frac{r}{2}} \langle |\mathrm{d}V_{s_d}| \, , \phi \rangle \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)} \left(\int_M \phi \, |\mathrm{d}V_M| \right).$$

Related works and novelty of the main results. This paper extends the results of [22], by the first-named author. In [22, Thm. 1.6], our main result (Thm. 1.6 above) was proved for $r < n$ and $\alpha \in (0, \alpha_0)$, where $\alpha_0 \in (0,1)$ is some explicit constant depending on n and r. The main novelty in Thm. 1.6 is that it covers the case of maximal codimension ($r = n$), that is the case where Z_d is almost surely a finite subset of M. This case was not considered in [22] because of additional singularities arising in the course of the proof, which caused it to fail when $r = n$.

An important contribution of the present paper is that we prove new estimates (see Lem. 5.26, 5.28 and 5.29) for operators related to the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$, which is the correlation kernel of the random field $(s_d(x))_{x \in M}$. These estimates are one of the key improvements that allow us to prove Thm. 1.6 in the case $r = n$. They also allow us to consider $\alpha \in (0,1)$ instead of $\alpha \in (0, \alpha_0)$. Finally, the use of these estimates greatly clarifies the proof of Thm. 1.6 in the case $r < n$, compared to the proof given in [22]. For this reason, we give the proof of Thm. 1.6 in the general case $r \in \{1, \dots, n\}$ and not only for $r = n$. This does not lengthen the proof.

The second main contribution of this article is the proof of the positivity of the leading constant in Thm. 1.6 (cf. Thm. 1.8). This result did not appear in [22]. Since the leading constant in Thm. 1.6 is universal, when $r = n$ one can deduce Thm. 1.8 from results of Dalmao [11] (if $r = n = 1$) and Armentano–Azaïs–Dalmao–León [2] (if $r = n \geqslant 2$). In [2, 11], the authors proved Thm. 1.6 in the special case where Z_d is the zero set in $\mathbb{RP}^n$ of n independent Kostlan–Shub–Smale polynomials (see Sect. 6.1 below). Their results include the positivity of the leading constant, hence implies Thm.1.8 in this case. To the best of our knowledge, Thm. 1.8 is completely new for $r < n$.

Note that when $n = r = 1$, our setting covers the case of the binomial polynomials on $\mathbb{C}$ with standard Gaussian coefficients. Much more is known in this case, including variance estimates for the number of real zeros of non-Gaussian ensembles of real polynomials (see [34]).

Our proof of Thm. 1.8 uses the Wiener–Itō expansion of the linear statistics associated with the field $(s_d(x))_{x \in M}$. This kind of expansion has been studied by Slud [33] and Kratz–León [18, 19]. It was used in a random geometry context in [2, 11, 12, 25]. In [12, 25], the authors used these Wiener chaos techniques to prove Central Limit Theorems for the volume of the zero set of Arithmetic Random Waves on the two-dimensional flat torus (see also [11] in an algebraic setting). In [2, 11], these methods where used to prove Thm. 1.8 when $r = n$.

In the related setting of Riemannian Random Waves, Canzani and Hanin [8] obtained recently an asymptotic upper bound for the variance of the linear statistics. To the best of our knowledge, in this Riemannian setting, the precise asymptotics of the variance of the volume of random submanifolds are known only when the ambient manifold is $\mathbb{S}^2$ (cf. [35]) or $\mathbb{T}^2$ (cf. [12, 20]). We refer to the introduction of [22] for more details about related works.

About the proofs. The proof of Thm. 1.6 broadly follows the lines of the proof of [22, Thm. 1.6]. The random section s_d defines a centered Gaussian field $(s_d(x))_{x \in M}$, whose correlation kernel is E_d, the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$ (see Sect. 2.4). Thanks to results of Dai–Liu–Ma [10] and Ma–Marinescu [24], we know that this kernel decreases exponentially fast outside of the diagonal $\Delta = \{(x,y) \in M^2 \mid x = y\}$ and that it admits a universal local scaling limit close to Δ (see Sect. 3 for details).

By an application of Kac–Rice formulas (cf. Thm. 5.1 and 5.5), we can express the covariance of $\langle |\mathrm{d}V_d| \, , \phi_1 \rangle$ and $\langle |\mathrm{d}V_d| \, , \phi_2 \rangle$ as a double integral over $M \times M$ of $\phi_1(x)\phi_2(y)$ times a density function $\mathcal{D}_d(x,y)$ that depends only on E_d. Our main concern is to understand the asymptotics of the integral of $\mathcal{D}_d(x,y)$, as $d \to +\infty$.

Thanks to the exponential decay of the Bergman kernel, we can show that the leading term in our asymptotics is given by the integral of $\mathcal{D}_d$ over a neighborhood Δ_d of Δ, of typical size $\frac{1}{\sqrt{d}}$ (see Prop. 5.22). Changing variables so that we integrate on a domain of typical size independent of d leads to the apparition of a factor $d^{-\frac{n}{2}}$. Besides, $\mathcal{D}_d$ takes values of order d^r on Δ_d (see Prop. 5.25). This explains why the asymptotic variance is of order $d^{r-\frac{n}{2}}$ in Thm. 1.6.

The behavior of E_d allows to prove that $\mathcal{D}_d$ admits a universal local scaling limit on Δ_d. The main difficulty in our proof is to show that the convergence to this scaling limit is uniform on Δ_d (see Prop. 5.25 for a precise statement). This difficulty comes from the fact that $\mathcal{D}_d$ is singular along Δ, just like almost everything in this problem. This is where our proof differs from [22]. In [22], the uniform convergence of $\mathcal{D}_d$ to its scaling limit on Δ_d is not established, and one has to work around this lack of uniformity. This yields a complicated proof that fails when $r = n$. Here,

we manage to prove this uniform convergence, thanks to some new key estimates (see Lem. 5.26, 5.28 and 5.29) that form the technical core of the paper. This allows us to both improve on the results of [22] and simplify their proof.

As we explained, our proof relies on two properties of the Bergman kernel E_d: namely, the existence of a scaling limit around any point at scale $\frac{1}{\sqrt{d}}$, and its exponential decay outside of the diagonal. These features are also exhibited by Bergman kernels in other settings such as those of [4] or [5], so one might hope to generalize our results to these settings, at least in the bulk. Unfortunately, we also need a precise understanding of the scaling limit of E_d, which is possible in our framework because it is universal (it only depends on n) and invariant under isometries (see Sect. 4 for more details). As far as we know, it is much more complicated to study this scaling limit in other settings (such as those of [4] and [5]), so we do not pursue this line of inquiry in the present paper and leave it for future research.

Let us now discuss the proof of Thm. 1.8. One would expect to prove this by computing a good lower bound for $\mathcal{I}_{n,r}$, directly from its expression (see Eq. (1.2)). To the best of our knowledge this approach fails, and we have to use subtler techniques.

Since the leading constant in (1.1) only depends on n and r, we can focus on the case of the volume of Z_d (where $\phi_1 = \mathbf{1} = \phi_2$) in a particular geometric setting. We consider the common real zero set of r independent Kostlan–Shub–Smale polynomials in $\mathbb{RP}^n$ (see Sect. 6.1 for details). This allows for explicit computations since the Bergman kernel is explicitly known in this setting. Moreover, the distribution of these polynomials is invariant under the action of $O_{n+1}(\mathbb{R})$ on $\mathbb{RP}^n$, which leads to useful independence proprieties that are not satisfied in general.

In this framework, we adapt the strategy of [2, 11] to the case $r < n$. First, we compute the Wiener–Itō expansion of the volume of Z_d. That is, we expand $\mathrm{Vol}\,(Z_d)$ as $\sum_{q\in\mathbb{N}} \mathrm{Vol}\,(Z_d)\,[q]$, where the convergence is in the space of L^2 random variables on our probability space, and $\mathrm{Vol}\,(Z_d)\,[q]$ denotes the q-th chaotic component of $\mathrm{Vol}\,(Z_d)$. In particular, $\mathrm{Vol}\,(Z_d)\,[0]$ is the expectation of $\mathrm{Vol}\,(Z_d)$ and the $(\mathrm{Vol}\,(Z_d)\,[q])_{q\in\mathbb{N}}$ are pairwise orthogonal L^2 random variables. Hence,

$$\mathrm{Var}(\mathrm{Vol}\,(Z_d)) = \sum_{q\geqslant 1} \mathrm{Var}(\mathrm{Vol}\,(Z_d)\,[q])\,.$$

The chaotic components of odd order of $\mathrm{Vol}\,(Z_d)$ are zero, but we prove that $\mathrm{Var}(\mathrm{Vol}\,(Z_d)\,[2])$ is equivalent to $d^{r-\frac{n}{2}}C$ as $d \to +\infty$, where $C > 0$ (see Lem. 6.17). This implies Thm. 1.8.

Outline of the paper. In Sect. 2 we describe precisely our framework and the construction of the random measures $|\mathrm{d}V_d|$. We also introduce the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$ and prove that it is the correlation kernel of $(s_d(x))_{x\in M}$. In Sect. 3, we recall estimates for the Bergman kernel, and its scaling limit. Sect. 4 is dedicated to the study of the Bargmann–Fock process, that is the Gaussian centered random process on $\mathbb{R}^n$ whose correlation function is:

$$(w,z) \longmapsto \exp\left(-\frac{1}{2}\left\|w-z\right\|^2\right).$$

This field is the local scaling limit of the random field $(s_d(x))_{x\in M}$, in a sense to be made precise below. Sect. 5 and 6 are concerned with the proofs of Thm. 1.6 and Thm. 1.8 respectively. Note that in Sect. 6 we have to study in details the model of Kostlan–Shub–Smale polynomials, which is the simplest example of our general real algebraic setting. We conclude this paper by two appendices, App. A and App. B, in which we gathered the proofs of the technical lemmas of Sect. 4 and Sect. 5 respectively.

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Contents

2 Random real algebraic submanifolds

In this section, we introduce the main objects we will be studying throughout this paper. We first recall some basic definitions in Sect. 2.1. In Sect. 2.2, we introduce our geometric framework. In Sect. 2.3, we describe our model of random real algebraic submanifolds. Finally, we relate these random submanifolds to Bergman kernels in Sect. 2.4.

2.1 Random vectors

Let us recall some facts about random vectors (see for example [21, appendix A]). In this paper, we only consider centered random vectors, so we give the following definitions in this setting.

Let X_1 and X_2 be centered random vectors taking values in Euclidean (or Hermitian) vector spaces V_1 and V_2 respectively, then we define their *covariance operator* as:

$$\mathrm{Cov}(X_1, X_2) : v \longmapsto \mathbb{E}[X_1 \langle v\,, X_2\rangle]$$

from V_2 to V_1. For all $v \in V_2$, we set $v^* = \langle \cdot\,, v\rangle \in V_2^*$. Then $\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 \otimes X_2^*]$ is an element of $V_1 \otimes V_2^*$. Let X be a centered random vector in a Euclidean space V. The *variance*

operator of X is defined as $\mathrm{Var}(X) = \mathrm{Cov}(X,X) = \mathbb{E}[X \otimes X^*] \in V \otimes V^*$. Let Λ be a non-negative self-adjoint operator on $(V, \langle \cdot\,, \cdot \rangle)$, we denote by $X \sim \mathcal{N}(\Lambda)$ the fact that X is a centered Gaussian vector with variance operator Λ. This means that the characteristic function of X is $\xi \mapsto \exp\left(-\frac{1}{2}\langle \Lambda\xi\,, \xi\rangle\right)$. Finally, we say that $X \in V$ is a *standard* Gaussian vector if $X \sim \mathcal{N}(\mathrm{Id})$, where Id is the identity operator on V.

If Λ is positive, the distribution of $X \sim \mathcal{N}(\Lambda)$ admits the density:

$$x \mapsto \frac{1}{\sqrt{2\pi}^N \sqrt{\det(\Lambda)}} \exp\left(-\frac{1}{2}\langle \Lambda^{-1}x\,, x\rangle\right) \tag{2.1}$$

with respect to the normalized Lebesgue measure of V, where $N = \dim(V)$. Otherwise, X takes values in $\ker(\Lambda)^\perp$ almost surely, and it admits a similar density as a variable in $\ker(\Lambda)^\perp$.

2.2 General setting

Let us introduce more precisely our geometric framework. Let $\mathcal{X}$ be a smooth complex projective manifold of positive complex dimension n. Let $c_\mathcal{X}$ be a real structure on $\mathcal{X}$, i.e. an anti-holomorphic involution. The real locus of $(\mathcal{X}, c_\mathcal{X})$ is the set M of fixed points of $c_\mathcal{X}$. In the following, we assume that M is non-empty. It is known that M is a smooth closed submanifold of $\mathcal{X}$ of real dimension n (see [32, chap. 1]).

Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle of rank $r \in \{1, \dots, n\}$, we denote by $\pi_\mathcal{E}$ its bundle projection. Let $c_\mathcal{E}$ be a real structure on $\mathcal{E}$, compatible with $c_\mathcal{X}$ in the sense that $c_\mathcal{X} \circ \pi_\mathcal{E} = \pi_\mathcal{E} \circ c_\mathcal{E}$ and $c_\mathcal{E}$ is fiberwise $\mathbb{C}$-anti-linear. Let $h_\mathcal{E}$ be a Hermitian metric on $\mathcal{E}$ such that $c_\mathcal{E}^\star(h_\mathcal{E}) = \overline{h_\mathcal{E}}$. A Hermitian metric satisfying this condition is said to be *real*. Similarly, let $\mathcal{L} \to \mathcal{X}$ be an ample holomorphic line bundle equipped with a compatible real structure $c_\mathcal{L}$ and a real Hermitian metric $h_\mathcal{L}$.

We assume that $(\mathcal{L}, h_\mathcal{L})$ has positive curvature, that is its curvature form ω is Kähler. Recall that, if ζ_0 is a local non-vanishing holomorphic section of $\mathcal{L}$, then $\omega = \frac{1}{2i}\partial\overline{\partial} \ln\left(h_\mathcal{L}(\zeta_0, \zeta_0)\right)$ locally. This Kähler form defines a Riemannian metric g on $\mathcal{X}$ (see [16, Sect. 0.2] for example). In turn, g induces a Riemannian volume measure on $\mathcal{X}$ and on any smooth submanifold of $\mathcal{X}$. We denote by $\mathrm{d}V_\mathcal{X} = \frac{\omega^n}{n!}$ the Riemannian volume form on $(\mathcal{X}, g)$. Similarly, let $|\mathrm{d}V_M|$ denote the Riemannian measure on (M, g).

Let $d \in \mathbb{N}$, we endow $\mathcal{E} \otimes \mathcal{L}^d$ with the real structure $c_d = c_\mathcal{E} \otimes (c_\mathcal{L})^d$, which is compatible with $c_\mathcal{X}$, and the real Hermitian metric $h_d = h_\mathcal{E} \otimes h_\mathcal{L}^d$. Let $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$, we define a Hermitian inner product on $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ by:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \qquad \langle s_1\,, s_2\rangle = \int_\mathcal{X} h_d(s_1(x), s_2(x))\, \mathrm{d}V_\mathcal{X}. \tag{2.2}$$

Remark 2.1. In this paper, $\langle \cdot\,, \cdot\rangle$ will either denote the inner product of a Euclidean (or Hermitian) space or the duality pairing between a Banach space and its topological dual. Which one should be clear from the context.

We say that a section $s \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is *real* if it is equivariant for the real structures, that is: $c_d \circ s = s \circ c_\mathcal{X}$. We denote by $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ the real vector space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$. The restriction of $\langle \cdot\,, \cdot\rangle$ to $\mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ is a Euclidean inner product. Note that, despite their name, real sections are defined on the whole complex locus $\mathcal{X}$ and that the Euclidean inner product is defined by integrating over $\mathcal{X}$, not just M.

Let $x \in M$, then the fiber $(\mathcal{E} \otimes \mathcal{L}^d)_x$ is a dimension r complex vector space and the restriction of c_d to this space is a $\mathbb{C}$-anti-linear involution. We denote by $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ the set of fixed points of this involution, which is a real r-dimensional vector space. Then, $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$ is a rank r real vector bundle and, for any $s \in \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$, the restriction of s to M is a smooth section of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \to M$.

Let $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This space is known to be finite-dimensional (compare [23, Thm. 1.4.1]). Let N_d denote the complex dimension of $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. We denote by:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \;\middle|\; c_d \circ s = s \circ c_\mathcal{X} \right\}$$

the space of global real holomorphic sections of $\mathcal{E}\otimes\mathcal{L}^d$. The restriction of the inner product (2.2) to $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ (resp. $H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$) makes it into a Euclidean (resp. Hermitian) space of real (resp. complex) dimension N_d.

2.3 Random real algebraic submanifolds

This section is concerned with the definition of the random submanifolds we consider and some related random variables.

Let $d\in\mathbb{N}$ and $s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, we denote by Z_s the real zero set $s^{-1}(0)\cap M$ of s. If the restriction of s to M vanishes transversally, then Z_s is a smooth closed submanifold of codimension r of M (note that this includes the case where Z_s is empty). In this case, we denote by $|\mathrm{d}V_s|$ the Riemannian volume measure on Z_s induced by g. In the following, we consider $|\mathrm{d}V_s|$ as the continuous linear form on $(\mathcal{C}^0(M),\|\cdot\|_\infty)$ defined by:

$$\forall\phi\in\mathcal{C}^0(M),\qquad \langle|\mathrm{d}V_s|\,,\phi\rangle=\int_{x\in Z_s}\phi(x)\,|\mathrm{d}V_s|\,.$$

Definition 2.2 (compare [27])**.** We say that $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-*ample* if, for any $x\in M$, the evaluation map $\mathrm{ev}_x^d: s\mapsto s(x)$ from $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ to $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$ is surjective.

Lemma 2.3. *There exists $d_1\in\mathbb{N}$, depending only on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$, such that for all $d\geqslant d_1$, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample.*

Proof. This can be deduced from the Riemann–Roch formula, for example. It is also a by-product of the computations of the present paper and will be proved later on, see Cor. 5.11 below. □

Let us now consider a random section in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Recall that $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, endowed with the inner product (2.2), is a Euclidean inner product of dimension N_d. Let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$.

Lemma 2.4. *For every $d\geqslant d_1$, Z_{s_d} is almost surely a smooth closed codimension r submanifold of M.*

Proof. Since $d\geqslant d_1$, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample. By a transversality argument (see [21, Sect. 2.6] for details), this implies that the restriction of s to M vanishes transversally for almost every $s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ (with respect to the Lebesgue measure). Thus, almost surely, s_d restricted to M vanishes transversally, and its zero set is a smooth closed submanifold of codimension r. □

From now on, we only consider the case $d\geqslant d_1$, so that Z_{s_d} is almost surely a random smooth closed submanifold of M of codimension r. For simplicity, we denote $Z_d=Z_{s_d}$ and $|\mathrm{d}V_d|=|\mathrm{d}V_{s_d}|$. Let $\phi\in\mathcal{C}^0(M)$ and $d\geqslant d_1$, the real random variable $\langle|\mathrm{d}V_d|\,,\phi\rangle$ is called the *linear statistic* of degree d associated with ϕ. For example, $\langle|\mathrm{d}V_d|\,,\mathbf{1}\rangle$ is the volume of Z_d.

2.4 The correlation kernel

For any $d\in\mathbb{N}$, the random section $s_d\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ defines a centered Gaussian process $(s_d(x))_{x\in\mathcal{X}}$. In this section, we recall the relation between the distribution of this process and the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$.

Recall that $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$ stands for the bundle $P_1^\star\left(\mathcal{E}\otimes\mathcal{L}^d\right)\otimes P_2^\star\left(\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*\right)$ over $\mathcal{X}\times\mathcal{X}$, where P_1 (resp. P_2) denotes the projection from $\mathcal{X}\times\mathcal{X}$ onto the first (resp. second) $\mathcal{X}$ factor. The distribution of $(s_d(x))_{x\in\mathcal{X}}$ is characterized by its covariance kernel, that is the section of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$ defined by: $(x,y)\mapsto\mathrm{Cov}(s_d(x),s_d(y))=\mathbb{E}[s_d(x)\otimes s_d(y)^*]$.

Definition 2.5. Let E_d denote the *Bergman kernel* of $\mathcal{E}\otimes\mathcal{L}^d\to\mathcal{X}$, that is the Schwartz kernel of the orthogonal projection from $\Gamma(\mathcal{E}\otimes\mathcal{L}^d)$ onto $H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$.

Let $(s_{1,d},\dots,s_{N_d,d})$ be an orthonormal basis of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, then it is also an orthonormal basis of $H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Recall that, E_d is given by:

$$E_d : (x,y) \longmapsto \sum_{i=1}^{N_d} s_{i,d}(x)\otimes s_{i,d}(y)^*.$$

This shows that E_d is a real global holomorphic section of $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$. The following proves that the distribution of $(s_d(x))_{x\in\mathcal{X}}$ is totally described by E_d.

Proposition 2.6 (compare [22], Prop. 2.6)**.** *Let $d\in\mathbb{N}$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Then, for all x and $y\in\mathcal{X}$, we have:* $\mathrm{Cov}(s_d(x),s_d(y)) = E_d(x,y)$.

Thus, the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$ gives the correlations between the values of the random section s_d. By taking partial derivatives of this relation, we obtain the correlations between the values of s_d and its derivatives. More details about what follows can be found in [22, Sect. 2.3].

Let ∇^d be a metric connection on $\mathcal{E}\otimes\mathcal{L}^d$, it induces a dual connection $(\nabla^d)^*$ on $(\mathcal{E}\otimes\mathcal{L}^d)^*$, which is compatible with the metric (cf. [16, Sect. 0.5]). We can then define a natural metric connection ∇^d_1 on $P_1^\star(\mathcal{E}\otimes\mathcal{L}^d)\to\mathcal{X}\times\mathcal{X}$ whose partial derivatives are: ∇^d with respect to the first variable, and the trivial connection with respect to the second. Similarly, $(\nabla^d)^*$ induces a metric connection ∇^d_2 on $P_2^\star\left((\mathcal{E}\otimes\mathcal{L}^d)^*\right)$ and $\nabla^d_1\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^d_2$ is a metric connection on $(\mathcal{E}\otimes\mathcal{L}^d)\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)^*$.

We denote by ∂_x (resp. ∂_y) the partial derivative of $\nabla^d_1\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^d_2$ with respect to the first (resp. second) variable. Let $\partial^\sharp_y E_d(x,y)\in T_y\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*_y$ be defined by:

$$\forall w\in T_y\mathcal{X},\qquad \partial^\sharp_y E_d(x,y)\cdot w^* = \partial_y E_d(x,y)\cdot w.$$

Similarly, let $\partial_x\partial^\sharp_y E_d(x,y)\in T^*_x\mathcal{X}\otimes T_y\mathcal{X}\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\otimes\left(\mathcal{E}\otimes\mathcal{L}^d\right)^*_y$ be defined by:

$$\forall (v,w)\in T_x\mathcal{X}\times T_y\mathcal{X},\qquad \partial_x\partial^\sharp_y E_d(x,y)\cdot(v,w^*) = \partial_x\partial_y E_d(x,y)\cdot(v,w).$$

The following corollary was proved in [22, Cor. 2.13].

Corollary 2.7. *Let $d\in\mathbb{N}$, let ∇^d be a metric connection on $\mathcal{E}\otimes\mathcal{L}^d$ and let s_d be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Then, for all x and $y\in\mathcal{X}$, we have:*

$$\begin{aligned}
\mathrm{Cov}\big(\nabla^d_x s, s(y)\big) &= \mathbb{E}\big[\nabla^d_x s\otimes s(y)^*\big] = \partial_x E_d(x,y),\\
\mathrm{Cov}\big(s(x),\nabla^d_y s\big) &= \mathbb{E}\Big[s(x)\otimes\big(\nabla^d_y s\big)^*\Big] = \partial^\sharp_y E_d(x,y),\\
\mathrm{Cov}\big(\nabla^d_x s,\nabla^d_y s\big) &= \mathbb{E}\Big[\nabla^d_x s\otimes\big(\nabla^d_y s\big)^*\Big] = \partial_x\partial^\sharp_y E_d(x,y).
\end{aligned}$$

3 Estimates for the Bergman kernel

In this section, we recall useful estimates for the Bergman kernels. In Sect. 3.1 we recall the definition of a preferred trivialization of $\mathcal{E}\otimes\mathcal{L}^d\to\mathcal{X}$. Then we state near-diagonal and off-diagonal estimates for a scaled version of E_d in Sect. 3.2 and 3.3.

3.1 Real normal trivialization

Notation 3.1. In the following, $B_A(a,R)$ denotes the open ball of center a and radius R in the metric space A.

Let $d\in\mathbb{N}$, let us define a preferred trivialization of $\mathcal{E}\otimes\mathcal{L}^d$ in a neighborhood of any point of M. The properties of this trivialization were studied in [22, Sect. 3.1]. Recall that the metric g on $\mathcal{X}$ is induced by the curvature of $(\mathcal{L},h_\mathcal{L})$. Since, $h_\mathcal{L}$ is compatible with the real structures, $c_\mathcal{X}$ is an isometry of $(\mathcal{X},g)$ (see [22, Sect. 2.1] for details).

Let $R>0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. Let $x\in M$, then the exponential map $\exp_x : B_{T_x\mathcal{X}}(0,2R)\to B_\mathcal{X}(x,2R)$ defines a chart around x such that:

$$d_x c_\mathcal{X} = (\exp_x)^{-1}\circ c_\mathcal{X}\circ\exp_x.$$

In particular, since $T_xM = \ker(d_xc_{\mathcal{X}} - \mathrm{Id})$, we have that $\exp_x$ induces a diffeomorphism from $B_{T_xM}(0, 2R)$ to $B_M(x, 2R)$ that coincides with the exponential map of (M, g) at x.

We can now trivialize $\mathcal{E} \otimes \mathcal{L}^d$ over $B_{\mathcal{X}}(x, 2R)$, by identifying each fiber with $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ by parallel transport along geodesics, with respect to the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$. This defines a bundle map

$$\varphi_x : B_{T_x\mathcal{X}}(0, 2R) \times \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x \to \left(\mathcal{E} \otimes \mathcal{L}^d\right)_{/B_{\mathcal{X}}(x,2R)}$$

that covers $\exp_x$. We call this trivialization the *real normal trivialization* of $\mathcal{E} \otimes \mathcal{L}^d$ around x.

Definition 3.2. A connection ∇^d on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ is said to be *real* if for every smooth section s we have:

$$\forall y \in \mathcal{X}, \qquad \nabla^d_y \left(c_d \circ s \circ c_{\mathcal{X}}\right) = c_d \circ \nabla^d_{c_{\mathcal{X}}(y)} s \circ d_y c_{\mathcal{X}}.$$

Such a connection induces a connection on $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big) \to M$ by restriction.

Recall that c_d denotes the real structure of $\mathcal{E} \otimes \mathcal{L}^d$. Let $c_{d,x}$ denote its restriction to $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$, then $(d_xc_{\mathcal{X}}, c_{d,x})$ is a $\mathbb{C}$-anti-linear involution of $B_{T_x\mathcal{X}}(0, 2R) \times \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ which is compatible with the real structure on the first factor. Since the Chern connection of $\mathcal{E} \otimes \mathcal{L}^d$ is real (see [22, Lem. 3.4]), the real normal trivialization is well-behaved with respect to the real structures, in the sense that for all $z \in B_{T_x\mathcal{X}}(0, 2R)$ and $\zeta \in \left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$,

$$c_d(\varphi_x(z, \zeta)) = \varphi_x \left(d_xc_{\mathcal{X}} \cdot z, c_{d,x}(\zeta)\right).$$

Thus, φ_x can be restricted to a bundle map

$$B_{T_xM}(0, 2R) \times \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x \to \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_{/B_M(x,2R)}$$

that covers $\exp_x$.

Finally, it is known (cf. [22, Sect. 3.1]) that φ_x is a unitary trivialization, i.e. its restriction to each fiber is an isometry. Similarly, its restriction to the real locus is an orthogonal trivialization of $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_{/B_M(x,2R)}$.

The point is the following. The usual differentiation for maps from $T_x\mathcal{X}$ to $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ defines locally a connection $\widetilde{\nabla}^d$ on $\left(\mathcal{E} \otimes \mathcal{L}^d\right)_{/B_{\mathcal{X}}(x,2R)}$ via the real normal trivialization. Since this trivialization is well-behaved with respect to both the real and the metric structures, $\widetilde{\nabla}^d$ is a real metric connection. Then, by a partition of unity argument, there exists a global real metric connection ∇^d on $\mathcal{E} \otimes \mathcal{L}^d$ that agrees with $\widetilde{\nabla}^d$ on $B_{\mathcal{X}}(x, R)$, i.e. ∇^d is trivial in the real normal trivialization, in a neighborhood of x. The existence of such a connection will be useful in the proof of our main theorem.

3.2 Near-diagonal estimates

In this section, we state estimates for a scaled version of the Bergman kernel in a neighborhood of the diagonal of $M \times M$. As in the previous section, let $R > 0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. Let $x \in M$, then the real normal trivialization φ_x induces a trivialization of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ over $B_{\mathcal{X}}(x, 2R) \times B_{\mathcal{X}}(x, 2R)$ that covers $\exp_x \times \exp_x$. This trivialization agrees with the real normal trivialization of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ around (x, x).

In the normal chart $\exp_x$, the Riemannian measure $\mathrm{d}V_{\mathcal{X}}$ admits a positive density with respect to the Lebesgue measure of $(T_x\mathcal{X}, g_x)$, denoted by $\kappa : B_{T_x\mathcal{X}}(0, 2R) \to \mathbb{R}_+$. Then, in the chart $\exp_x : B_{T_xM}(0, 2R) \to B_M(x, 2R)$, the density of $|\mathrm{d}V_M|$ with respect to the Lebesgue measure of (T_xM, g_x) is $\sqrt{\kappa}$.

Let us identify E_d with its expression in the real normal trivialization of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$ around (x, x). Thus, the restriction of E_d to the real locus is a map from $T_xM \times T_xM$ to $\mathrm{End}\left(\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x\right)$. Then, by [10, Thm. 4.18'] (see also [22, Thm. 3.5] for a statement with the same notations as the present paper) we get the following estimate for E_d and its derivatives of order at most 6.

Theorem 3.3 (Dai–Liu–Ma). *There exist C_1 and $C_2 > 0$, such that $\forall k \in \{0, 1, \dots, 6\}$, $\forall d \in \mathbb{N}^*$, $\forall x \in M$, $\forall w, z \in B_{T_xM}(0, R)$,*

$$\left\| D^k_{(w,z)} \left(E_d(w,z) - \left(\frac{d}{\pi}\right)^n \frac{\exp\left(-\frac{d}{2}\|z-w\|^2\right)}{\sqrt{\kappa(w)}\sqrt{\kappa(z)}} \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right) \right\|$$
$$\leqslant C_1 d^{n+\frac{k}{2}-1} \left(1 + \sqrt{d}(\|w\| + \|z\|)\right)^{2n+12} \exp\left(-C_2\sqrt{d}\,\|z-w\|\right) + O\big(d^{-\infty}\big),$$

where D^k is the k-th differential for a map from $T_xM \times T_xM$ to $\mathrm{End}\left(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\right)$, the norm on T_xM is induced by g_x and the norm on $(T^*_xM)^{\otimes k} \otimes \mathrm{End}\left(\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\right)$ is induced by g_x and $(h_d)_x$.

The notation $O(d^{-\infty})$ means that, for any $l \in \mathbb{N}$, this term is $O\big(d^{-l}\big)$ with a constant that does not depend on x, w, z nor d.

Recall that x is fixed. We denote by the e_d the following scaled version of the Bergman kernel around x:

$$\forall w, z \in B_{T_xM}\left(0, 2R\sqrt{d}\right), \qquad e_d(w,z) = \left(\frac{\pi}{d}\right)^n E_d\left(\exp_x\left(\frac{w}{\sqrt{d}}\right), \exp_x\left(\frac{z}{\sqrt{d}}\right)\right). \tag{3.1}$$

We consider e_d as a map with values in $\mathrm{End}\left(\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x\right)$ using the real normal trivialization around x. Note that e_d highly depends on x, even if this is not reflected in the notation. In the following, the base point x will always be clear from the context.

Let $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $\xi(w,z) = \exp\left(-\frac{1}{2}\|w-z\|^2\right)$, where $\|\cdot\|$ is the usual Euclidean norm. Let $x \in M$, any isometry from T_xM to $\mathbb{R}^n$ allows us to see ξ as a map from $T_xM \times T_xM$ to $\mathbb{R}$. Let b_n be a positive constant depending only on n and whose value will be chosen later on. Then, using the same notations as in Thm. 3.3 we get the following.

Proposition 3.4. *Let $\alpha \in (0,1)$, then there exists $C > 0$, depending only on α, n and the geometry of $\mathcal{X}$, such that $\forall k \in \{0, 1, \dots, 6\}$, $\forall d \in \mathbb{N}^*$, $\forall x \in M$, $\forall w, z \in B_{T_xM}(0, b_n \ln d)$, we have:*

$$\left\| D^k_{(w,z)} e_d - \left(D^k_{(w,z)}\xi\right) \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right\| \leqslant C d^{-\alpha}.$$

Proof. First, we apply Thm. 3.3 for the scaled kernel e_d. This yields that $\forall k \in \{0, 1, \dots, 6\}$, $\forall d \in \mathbb{N}^*$, $\forall x \in M$, $\forall w, z \in B_{T_xM}(0, b_n \ln d)$:

$$\left\| D^k_{(w,z)} \left(e_d(w,z) - \frac{\xi(w,z)}{\sqrt{\widetilde{\kappa}(w)\widetilde{\kappa}(z)}} \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x} \right) \right\| \leqslant \frac{C_1}{d} \left(1 + 2b_n \ln d\right)^{2n+12} + O\big(d^{-\infty}\big) = O\big(d^{-\alpha}\big),$$

where $\widetilde{\kappa} : z \mapsto \kappa\left(\frac{z}{\sqrt{d}}\right)$. Since we used normal coordinates to define κ, the following relations hold uniformly on $B_{T_x\mathcal{X}}(0, R)$:

$$\kappa(z) = 1 + O\Big(\|z\|^2\Big), \qquad D_z\kappa = O(\|z\|) \qquad \text{and} \qquad \forall k \in \{2, \dots, 6\},\ D^k_z\kappa = O(1).$$

By compactness, these estimates can be made independent of $x \in M$. Then, we get the following estimates for $\widetilde{\kappa}$ and its derivatives, uniformly in $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$:

$$\widetilde{\kappa}(z) = 1 + O\left(\frac{(b_n \ln d)^2}{d}\right), \quad D_z\widetilde{\kappa} = O\left(\frac{b_n \ln d}{d}\right) \quad \text{and} \quad \forall k \in \{2, \dots, 6\},\ D^k_z\widetilde{\kappa} = O\left(\frac{1}{d}\right).$$

Therefore, $\forall k \in \{0, 1, \dots, 6\}$, $\forall d \in \mathbb{N}^*$, $\forall x \in M$, $\forall w, z \in B_{T_xM}(0, b_n \ln d)$:

$$\left\| D^k_{(w,z)} \left(\frac{\xi(w,z)}{\sqrt{\widetilde{\kappa}(w)\widetilde{\kappa}(z)}} \right) - D^k_{(w,z)}\xi \right\| = O\big(d^{-\alpha}\big). \qquad \square$$

We will use the expressions of some of the partial derivatives of ξ. Let us choose any orthonormal basis of T_xM and denote by ∂_{x_i} (resp. ∂_{y_i}) the partial derivative with respect to the i-th component of the the first (resp. second) variable.

Lemma 3.5. *Let $i, j \in \{1, \dots, n\}$, for all $w = (w_1, \dots, w_n)$ and $z = (z_1, \dots, z_n) \in T_xM$ we have:*

$$\partial_{x_i}\xi(w,z) = -(w_i - z_i)\exp\left(-\frac{1}{2}\|w-z\|^2\right),$$

$$\partial_{y_j}\xi(x,y) = (w_j - z_j)\exp\left(-\frac{1}{2}\|w-z\|^2\right),$$

$$\text{and} \qquad \partial_{x_i}\partial_{y_j}\xi(x,y) = (\delta_{ij} - (w_i - z_i)(w_j - z_j))\exp\left(-\frac{1}{2}\|w-z\|^2\right),$$

where δ_{ij} equals 1 if $i = j$ and 0 otherwise.

Proof. This is given by a direct computation. □

3.3 Off-diagonal estimates

Finally, let us recall estimates quantifying the long range decay of the Bergman kernel E_d. These estimates were proved by Ma and Marinescu in [24, Thm. 5].

Let S be a smooth section of $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big) \boxtimes \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)^*$ and let $x, y \in M$. We denote by $\|S(x,y)\|_{\mathcal{C}^k}$ the maximum of the norms of S and its derivatives of order at most k at $(x,y) \in M\times M$, where the derivatives of S are computed with respect to the connection induced by the Chern connection of $\mathcal{E}\otimes\mathcal{L}^d$ and the Levi–Civita connection on M. The norms are the natural ones induced by h_d and g.

Theorem 3.6 (Ma–Marinescu)**.** *There exist $d_0 \in \mathbb{N}^*$, and positive constants C_1 and C_2 such that, for all $k \in \{0,1,2\}$, $\forall d \geqslant d_0$, $\forall x, y \in M$, we have:*

$$\|E_d(x,y)\|_{\mathcal{C}^k} \leqslant C_1 d^{n+\frac{k}{2}} \exp\left(-C_2\sqrt{d}\,\rho_g(x,y)\right),$$

where $\rho_g(\cdot,\cdot)$ denotes the geodesic distance in (M,g).

4 Properties of the limit distribution

The estimates of Sect. 3.2 show that the family of random fields $(s_d(x))_{x\in M}$ has a local scaling limit around any point $x \in M$, as d goes to infinity. Moreover, the limit field does not depend on x. The limit is a Gaussian centered random process from $\mathbb{R}^n$ to $\mathbb{R}^r$ whose correlation kernel is $e_\infty : (w,z) \mapsto \xi(w,z)I_r$, where I_r stands for the identity of $\mathbb{R}^r$ and ξ was defined in Sect. 3.2. This limit process is known as the Bargmann–Fock process.

The goal of this section is to establish some properties of the Bargmann–Fock process. These results will be useful in the next section to prove that, for d large enough, the local behavior of s_d around any given $x \in M$ is the same as that of the limit process.

In the following, we denote by $(s(z))_{z\in\mathbb{R}^n}$ a copy of the Bargmann–Fock process. Since ξ is smooth, we can assume the trajectories of s to be smooth. Note that s is both stationary and isotropic. Moreover, since $e_\infty = \xi I_r$, the field s is just a tuple of r independent identically distributed centered Gaussian fields whose correlation kernel is ξ.

4.1 Variance of the values

The first thing we want to understand about s is the distribution of $(s(0), s(z)) \in \mathbb{R}^r \oplus \mathbb{R}^r$ for any $z \in \mathbb{R}^n$. In the following, we canonically identify $\mathbb{R}^r \oplus \mathbb{R}^r$ with $\mathbb{R}^2 \otimes \mathbb{R}^r$.

Let $z \in \mathbb{R}^n$, then $(s(0), s(z))$ is a centered Gaussian vector in $\mathbb{R}^2 \otimes \mathbb{R}^r$ whith variance operator:

$$\Theta(z) = \begin{pmatrix} e_\infty(0,0) & e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z) \end{pmatrix} = \begin{pmatrix} \xi(0,0)I_r & \xi(0,z)I_r \\ \xi(z,0)I_r & \xi(z,z)I_r \end{pmatrix} = \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes I_r. \tag{4.1}$$

Let $Q = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in O_2(\mathbb{R})$ denote the rotation of angle $\frac{\pi}{4}$ in $\mathbb{R}^2$. We can explicitly diagonalize $\Theta(z)$ as follows.

Lemma 4.1. *For any $z \in \mathbb{R}^n$ we have the following:*

$$(Q \otimes I_r)\,\Theta(z)\,(Q \otimes I_r)^{-1} = \begin{pmatrix} 1 - e^{-\frac{1}{2}\|z\|^2} & 0 \\ 0 & 1 + e^{-\frac{1}{2}\|z\|^2} \end{pmatrix} \otimes I_r.$$

Proof. Since $(Q \otimes I_r)^{-1} = Q^{\mathrm{t}} \otimes I_r$, by Eq. (4.1), it is enough to notice that:

$$Q \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} Q^{\mathrm{t}} = \begin{pmatrix} 1 - e^{-\frac{1}{2}\|z\|^2} & 0 \\ 0 & 1 + e^{-\frac{1}{2}\|z\|^2} \end{pmatrix}. \qquad \square$$

Lemma 4.2. *For all $z \in \mathbb{R}^n$, $\det(\Theta(z)) = \left(1 - e^{-\|z\|^2}\right)^r$. In particular, the distribution of $(s(0), s(z))$ is non-degenerate for all $z \in \mathbb{R}^n \setminus \{0\}$.*

Proof. We take the determinant of both sides in Eq. (4.1). $\square$

4.2 Variance of the 1-jets

Let us now study the variance structure of the 1-jets of s. For any $z \in \mathbb{R}^n$, we know that $(s(0), s(z), d_0 s, d_z s)$ is a centered Gaussian vector in:

$$\mathbb{R}^r \oplus \mathbb{R}^r \oplus ((\mathbb{R}^n)^* \otimes \mathbb{R}^r) \oplus ((\mathbb{R}^n)^* \otimes \mathbb{R}^r) \simeq (\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r.$$

Our goal in this section is to better understand its variance operator $\Omega(z)$. In the following, we write $\Omega(z)$ by blocks according to the previous splitting. Let ∂_x (resp. ∂_y) denote the partial derivative with respect to the first (resp. second) variable for maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathrm{End}(\mathbb{R}^r)$. Let us also define $\partial_y^\sharp$ as in Sect. 2.4. Then, we have:

$$\Omega(z) = \left(\begin{array}{cc|cc} e_\infty(0,0) & e_\infty(0,z) & \partial_y^\sharp e_\infty(0,0) & \partial_y^\sharp e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z) & \partial_y^\sharp e_\infty(z,0) & \partial_y^\sharp e_\infty(z,z) \\ \hline \partial_x e_\infty(0,0) & \partial_x e_\infty(0,z) & \partial_x\partial_y^\sharp e_\infty(0,0) & \partial_x\partial_y^\sharp e_\infty(0,z) \\ \partial_x e_\infty(z,0) & \partial_x e_\infty(z,z) & \partial_x\partial_y^\sharp e_\infty(z,0) & \partial_x\partial_y^\sharp e_\infty(z,z) \end{array}\right) = \Omega'(z) \otimes I_r, \tag{4.2}$$

where

$$\Omega'(z) = \left(\begin{array}{cc|cc} \xi(0,0) & \xi(0,z) & \partial_y^\sharp \xi(0,0) & \partial_y^\sharp \xi(0,z) \\ \xi(z,0) & \xi(z,z) & \partial_y^\sharp \xi(z,0) & \partial_y^\sharp \xi(z,z) \\ \hline \partial_x \xi(0,0) & \partial_x \xi(0,z) & \partial_x\partial_y^\sharp \xi(0,0) & \partial_x\partial_y^\sharp \xi(0,z) \\ \partial_x \xi(z,0) & \partial_x \xi(z,z) & \partial_x\partial_y^\sharp \xi(z,0) & \partial_x\partial_y^\sharp \xi(z,z) \end{array}\right).$$

Let $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ be any orthonormal basis of $\mathbb{R}^n$ such that $z = \|z\| \frac{\partial}{\partial x_1}$ and let $(dx_1, \ldots, dx_n)$ denote its dual basis. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$, we denote by $\mathcal{B}_z$ the following orthonormal basis of $\mathbb{R}^2 \otimes (\mathbb{R} \oplus (\mathbb{R}^n)^*) \simeq \mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$:

$$\mathcal{B}_z = (e_1 \otimes 1, e_2 \otimes 1, e_1 \otimes dx_1, e_2 \otimes dx_1, \ldots, e_1 \otimes dx_n, e_2 \otimes dx_n).$$

Lemma 4.3. *For any $z \in \mathbb{R}^n$, the matrix of $\Omega'(z)$ in the basis $\mathcal{B}_z$ is:*

$$\left(\begin{array}{c|c} \widetilde{\Omega}(\|z\|^2) & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right),$$

where I_{n-1} is the identity matrix of size $n-1$ and, for all $t \geqslant 0$, we set:

$$\widetilde{\Omega}(t) = \begin{pmatrix} 1 & e^{-\frac{1}{2}t} & 0 & -\sqrt{t}e^{-\frac{1}{2}t} \\ e^{-\frac{1}{2}t} & 1 & \sqrt{t}e^{-\frac{1}{2}t} & 0 \\ 0 & \sqrt{t}e^{-\frac{1}{2}t} & 1 & (1-t)e^{-\frac{1}{2}t} \\ -\sqrt{t}e^{-\frac{1}{2}t} & 0 & (1-t)e^{-\frac{1}{2}t} & 1 \end{pmatrix}. \tag{4.3}$$

Proof. A direct computation yields the result, using the fact that $z = (\|z\|, 0, \dots, 0)$ in the basis $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Recall that the partial derivatives of ξ are given by Lem. 3.5. □

Let $z \in \mathbb{R}^n$, and recall that $z^* \in (\mathbb{R}^n)^*$ was defined as $z^* = \langle \cdot, z \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical scalar product of $\mathbb{R}^n$. We denote by $z^* \otimes z \in \mathrm{End}((\mathbb{R}^n)^*)$ the map $\eta \mapsto \eta(z)z^*$. Then Lem. 4.3 shows that, as an operator on $\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$:

$$\Omega'(z) = \left(\begin{array}{cc|cc} 1 & e^{-\frac{1}{2}\|z\|^2} & 0 & -e^{-\frac{1}{2}\|z\|^2} z \\ e^{-\frac{1}{2}\|z\|^2} & 1 & e^{-\frac{1}{2}\|z\|^2} z & 0 \\ \hline 0 & e^{-\frac{1}{2}\|z\|^2} z^* & I_n & e^{-\frac{1}{2}\|z\|^2}(I_n - z^* \otimes z) \\ -e^{-\frac{1}{2}\|z\|^2} z^* & 0 & e^{-\frac{1}{2}\|z\|^2}(I_n - z^* \otimes z) & I_n \end{array}\right), \tag{4.4}$$

where z^* is to be understood as the constant map $t \mapsto z^*$ from $\mathbb{R}$ to $(\mathbb{R}^n)^*$, z is to be understood as the evaluation at the point z from $(\mathbb{R}^n)^*$ to $\mathbb{R}$ and I_n is the identity of $\mathbb{R}^n$. Indeed both sides of (4.4) have the same matrix in the basis $\mathcal{B}_z$.

We will now diagonalize explicitly $\Omega(z)$, as we did for $\Theta(z)$ in the previous section. The main step is to diagonalize $\widetilde{\Omega}(z)$.

Definitions 4.4. We denote by v_1, v_2, v_3, v_4 and a the following functions from $[0, +\infty)$ to $\mathbb{R}$:

$$v_1 : t \longmapsto 1 - e^{-\frac{1}{2}t}\left(\frac{t}{2} - \sqrt{1 + \left(\frac{t}{2}\right)^2}\right), \qquad v_2 : t \longmapsto 1 - e^{-\frac{1}{2}t}\left(\frac{t}{2} + \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$v_3 : t \longmapsto 1 + e^{-\frac{1}{2}t}\left(\frac{t}{2} - \sqrt{1 + \left(\frac{t}{2}\right)^2}\right), \qquad v_4 : t \longmapsto 1 + e^{-\frac{1}{2}t}\left(\frac{t}{2} + \sqrt{1 + \left(\frac{t}{2}\right)^2}\right),$$

$$a : t \longmapsto \frac{1 - \frac{t}{2}}{\sqrt{1 + \left(\frac{t}{2}\right)^2}}.$$

Note that, for all $t \geqslant 0$, $|a(t)| \leqslant 1$, so that the following makes sense.

Definitions 4.5. Let $b_+ : t \mapsto \sqrt{1 + a(t)}$ and $b_- : t \mapsto \sqrt{1 - a(t)}$ from $[0, +\infty)$ to $\mathbb{R}$. For all $t \geqslant 0$, let us also denote:

$$P(t) = \frac{1}{2}\begin{pmatrix} b_-(t) & -b_-(t) & -b_+(t) & -b_+(t) \\ b_+(t) & -b_+(t) & b_-(t) & b_-(t) \\ b_-(t) & b_-(t) & -b_+(t) & b_+(t) \\ b_+(t) & b_+(t) & b_-(t) & -b_-(t) \end{pmatrix}.$$

One can check that, for all $t \geqslant 0$, $P(t)$ is an orthogonal matrix.

Lemma 4.6. *For every $t \in [0, +\infty)$, we have:*

$$P(t)\widetilde{\Omega}(t)P(t)^{-1} = \begin{pmatrix} v_1(t) & 0 & 0 & 0 \\ 0 & v_2(t) & 0 & 0 \\ 0 & 0 & v_3(t) & 0 \\ 0 & 0 & 0 & v_4(t) \end{pmatrix}.$$

Proof. See Appendix A. □

Corollary 4.7. *Let $z \in \mathbb{R}^n$, identifying $\Omega'(z)$ with its matrix in $\mathcal{B}_z$, we have:*

$$\left(\begin{array}{c|c} P(\|z\|^2) & 0 \\ \hline 0 & Q \otimes I_{n-1} \end{array}\right) \Omega'(z) \left(\begin{array}{c|c} P(\|z\|^2) & 0 \\ \hline 0 & Q \otimes I_{n-1} \end{array}\right)^{-1}$$

$$= \left(\begin{array}{cccc|c} v_1(\|z\|^2) & 0 & 0 & 0 & \\ 0 & v_2(\|z\|^2) & 0 & 0 & \\ 0 & 0 & v_3(\|z\|^2) & 0 & 0 \\ 0 & 0 & 0 & v_4(\|z\|^2) & \\ \hline & 0 & & & \begin{pmatrix} 1 - e^{-\frac{1}{2}\|z\|^2} & 0 \\ 0 & 1 + e^{-\frac{1}{2}\|z\|^2} \end{pmatrix} \otimes I_{n-1} \end{array}\right).$$

By Eq. (4.2), we get a diagonalization of $\Omega(z)$ by tensoring each factor by I_r in Corollary 4.7.

Lemma 4.8. *For all $z \in \mathbb{R}^n \setminus \{0\}$, we have $\det(\Omega(z)) > 0$, that is the distribution of the random vector $(s(0), s(z), d_0 s, d_z s)$ is non-degenerate.*

Proof. See Appendix A. □

4.3 Conditional variance of the derivatives

The next step is to study the conditonal distribution of $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$, for any $z \in \mathbb{R}^n \setminus \{0\}$. Recall that $(s(0), s(z), d_0 s, d_z s)$ is a centered Gaussian vector with variance $\Omega(z)$ (see Eq. (4.2)). Moreover, if $z \neq 0$, the distribution of $(s(0), s(z))$ is non-degenerate by Lem. 4.2.

Thus, $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$ is a centered Gaussian vector in $((\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r$ with variance operator:

$$\Lambda(z) = \begin{pmatrix} \partial_x \partial_y^\sharp e_\infty(0,0) & \partial_x \partial_y^\sharp e_\infty(0,z) \\ \partial_x \partial_y^\sharp e_\infty(z,0) & \partial_x \partial_y^\sharp e_\infty(z,z) \end{pmatrix} - \begin{pmatrix} \partial_x e_\infty(0,0) & \partial_x e_\infty(0,z) \\ \partial_x e_\infty(z,0) & \partial_x e_\infty(z,z) \end{pmatrix} \begin{pmatrix} e_\infty(0,0) & e_\infty(0,z) \\ e_\infty(z,0) & e_\infty(z,z) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp e_\infty(0,0) & \partial_y^\sharp e_\infty(0,z) \\ \partial_y^\sharp e_\infty(z,0) & \partial_y^\sharp e_\infty(z,z) \end{pmatrix}.$$

By Equations (4.2) and (4.4), for all $z \in \mathbb{R}^n \setminus \{0\}$, we have $\Lambda(z) = \Lambda'(z) \otimes I_r$, where:

$$\Lambda'(z) = \begin{pmatrix} I_n - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z & e^{-\frac{1}{2}\|z\|^2} \left(I_n - \frac{1}{1-e^{-\|z\|^2}} z^* \otimes z \right) \\ e^{-\frac{1}{2}\|z\|^2} \left(I_n - \frac{1}{1-e^{-\|z\|^2}} z^* \otimes z \right) & I_n - \frac{e^{-\|z\|^2}}{1-e^{-\|z\|^2}} z^* \otimes z \end{pmatrix}. \tag{4.5}$$

As in the previous section, let us denote by $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$ an orthonormal basis of $\mathbb{R}^n$ such that $z = \|z\| \frac{\partial}{\partial x_1}$ and let $(dx_1, \dots, dx_n)$ denote its dual basis. Let (e_1, e_2) denote the canonical basis of $\mathbb{R}^2$, we define $\mathcal{B}'_z$ to be the following orthonormal basis of $\mathbb{R}^2 \otimes (\mathbb{R}^n)^* \simeq (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*$:

$$\mathcal{B}'_z = (e_1 \otimes dx_1, e_2 \otimes dx_1, \dots, e_1 \otimes dx_n, e_2 \otimes dx_n).$$

Lemma 4.9. *For any $z \in \mathbb{R}^n \setminus \{0\}$, the matrix of $\Lambda'(z)$ in the basis $\mathcal{B}'_z$ is:*

$$\left(\begin{array}{c|c} \widetilde{\Lambda}(\|z\|^2) & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array} \right),$$

where, for all $t > 0$, we set:

$$\widetilde{\Lambda}(t) = \begin{pmatrix} 1 - \frac{te^{-t}}{1-e^{-t}} & e^{-\frac{1}{2}t} \left(1 - \frac{t}{1-e^{-t}} \right) \\ e^{-\frac{1}{2}t} \left(1 - \frac{t}{1-e^{-t}} \right) & 1 - \frac{te^{-t}}{1-e^{-t}} \end{pmatrix}. \tag{4.6}$$

Proof. Since $z = \|z\| \frac{\partial}{\partial x_1}$, we have $z^* \otimes z = \|z\|^2 dx_1 \otimes \frac{\partial}{\partial x_1}$. Hence, the matrix of $z^* \otimes z$ in $(dx_1, \dots, dx_n)$ is:

$$\left(\begin{array}{c|c} \|z\|^2 & 0 \\ \hline 0 & 0 \end{array} \right).$$

Then the conclusion follows from Eq. (4.5). □

Remark 4.10. We can extend continuously $\widetilde{\Lambda}$ at $t = 0$ by setting $\widetilde{\Lambda}(0) = 0$. This yields continuous extensions of Λ' and Λ at $z = 0$. Note that $\Lambda(0)$ is not the variance operator of $(d_0 s, d_0 s)$ given that $s(0) = 0$.

Definitions 4.11. Let u_1 and u_2 denote the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$u_1 : t \longmapsto \frac{1 - e^{-t} + te^{-\frac{1}{2}t}}{1 + e^{-\frac{1}{2}t}} \qquad u_2 : t \longmapsto \begin{cases} \dfrac{1 - e^{-t} - te^{-\frac{1}{2}t}}{1 - e^{-\frac{1}{2}t}} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Once again, we will need an explicit diagonalization of $\Lambda(z)$. Such a diagonalization is given by the following lemma, once we tensor each factor by I_r.

Lemma 4.12. *Let $z \in \mathbb{R}^n$, identifying $\Lambda'(z)$ with its matrix in $\mathcal{B}'_z$, we have:*

$$(Q \otimes I_n)\,\Lambda'(z)\,(Q \otimes I_n)^{-1} = \left(\begin{array}{cc|c} u_1(\|z\|^2) & 0 & \\ 0 & u_2(\|z\|^2) & 0 \\ \hline & & \begin{pmatrix} 1-e^{-\frac{1}{2}\|z\|^2} & 0 \\ 0 & 1+e^{-\frac{1}{2}\|z\|^2} \end{pmatrix} \otimes I_{n-1} \end{array}\right).$$

Proof. By Lem. 4.9, we only need to check that, for all $t \geqslant 0$,

$$Q\widetilde{\Lambda}(t)Q^{\mathrm{t}} = \begin{pmatrix} u_1(t) & 0 \\ 0 & u_2(t) \end{pmatrix}. \qquad \square$$

Lemma 4.13. *For all $z \in \mathbb{R}^n \setminus \{0\}$, we have $\det(\Lambda(z)) > 0$, i.e. the distribution of $(d_0 s, d_z s)$ given that $s(0) = 0 = s(z)$ is non-degenerate.*

Proof. See Appendix A. $\square$

By Lem. 4.8, $\Omega(z)$ is a positive self-adjoint operator on $(\mathbb{R} \oplus \mathbb{R} \oplus (\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r$, for all $z \in \mathbb{R}^n \setminus \{0\}$, and so is its inverse. Hence, $\Omega(z)^{-1}$ admits a unique positive square root, that we denote by $\Omega(z)^{-\frac{1}{2}}$. Similarly, by Lem. 4.13, $\Lambda(z)$ is a positive self-adjoint operator on $((\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^*) \otimes \mathbb{R}^r$ and we denote by $\Lambda(z)^{\frac{1}{2}}$ its positive square root.

Lemma 4.14. *The map $z \longmapsto \begin{pmatrix} 0 & \Lambda(z)^{\frac{1}{2}} \end{pmatrix} \Omega(z)^{-\frac{1}{2}}$ is bounded on $\mathbb{R}^n \setminus \{0\}$.*

Proof. See Appendix A. $\square$

4.4 Finiteness of the leading constant

The goal of this section, it to prove that the constant $\mathcal{I}_{n,r}$ defined by Eq. (1.2) and appearing in Thm. 1.6 is well-defined and finite.

Definition 4.15. Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, for every $t > 0$ we set:

$$D_{n,r}(t) = \frac{\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right| \left|\det{}^{\perp}(Y(t))\right|\right]}{(1-e^{-t})^{\frac{r}{2}}} - (2\pi)^r \left(\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\right)^2,$$

where $(X(t), Y(t))$ is the centered Gaussian vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ defined in Def. 1.5.

By the definition of $\mathcal{I}_{n,r}$ (see Eq. (1.2)), we have:

$$\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} D_{n,r}(t)\, t^{\frac{n-2}{2}}\,\mathrm{d}t.$$

Hence, we have to prove that $t \mapsto D_{n,r}(t)t^{\frac{n-2}{2}}$ is integrable on $(0, +\infty)$, which boils down to computing the asymptotic expansions of $\mathbb{E}\left[\left|\det{}^{\perp}(X(t))\right| \left|\det{}^{\perp}(Y(t))\right|\right]$ as $t \to 0$ and as $t \to +\infty$.

Let us now relate this to the Bargmann–Fock process $(s(z))_{z \in \mathbb{R}^n}$.

Lemma 4.16. *Let $z \in \mathbb{R}^n \setminus \{0\}$. Let $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ be an orthonormal basis of $\mathbb{R}^n$ such that $z = \|z\|\frac{\partial}{\partial x_1}$ and let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{R}^r$. Then, the matrices of $d_0 s$ and $d_z s$ in these bases, given that $s(0) = 0 = s(z)$, form a random vector in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$ which is distributed as $\left(X(\|z\|^2), Y(\|z\|^2)\right)$.*

Proof. Let us denote by $\widetilde{X}(z)$ and $\widetilde{Y}(z)$ the matrices of $d_0 s$ and $d_z s$ in the bases $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and $(\zeta_1, \dots, \zeta_r)$, given that $s(0) = 0 = s(z)$. We denote by $\widetilde{X}_{ij}(z)$ (resp. $\widetilde{Y}_{ij}(z)$) the coefficients of $\widetilde{X}$ (resp. $\widetilde{Y}$) for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$. By Lem. 4.9, the couples $\left(\widetilde{X}_{ij}, \widetilde{Y}_{ij}\right)$ are centered Gaussian vectors in $\mathbb{R}^2$ which are independant from one another. Moreover, the variance matrix of $\left(\widetilde{X}_{ij}(z), \widetilde{Y}_{ij}(z)\right)$ equals $\widetilde{\Lambda}(\|z\|^2)$ if $j = 1$ and $\begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix}$ otherwise. By Def. 1.5, this is exactly saying that $\left(\widetilde{X}(z), \widetilde{Y}(z)\right)$ is distributed as $\left(X(\|z\|^2), Y(\|z\|^2)\right)$. $\square$

Lemma 4.17. *Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$. Then, as $t \to 0$, we have the following:*

$$\mathbb{E}\big[\big|\det^\perp(X(t))\big|\,\big|\det^\perp(Y(t))\big|\big] \sim \begin{cases} \dfrac{(n-1)!}{(n-r-1)!} & \text{if } r < n, \\ \dfrac{n!}{2}t & \text{if } r = n. \end{cases}$$

Proof. See Appendix A. □

Lemma 4.18. *For all $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, we have the following as $t \to +\infty$:*

$$\mathbb{E}\big[\big|\det^\perp(X(t))\big|\,\big|\det^\perp(Y(t))\big|\big] = (2\pi)^r \left(\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\right)^2 + O\Big(te^{-\frac{t}{2}}\Big)$$

Proof. See Appendix A. □

Lem. 4.17 and 4.18 and the definition of $D_{n,r}$ (Def. 4.15) allows to derive the following.

Corollary 4.19. *Let $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$, then we have:*

$$t^{\frac{n-2}{2}} D_{n,r}(t) = \begin{cases} O\Big(\dfrac{1}{\sqrt{t}}\Big) & \text{as } t \to 0, \\ O\Big(e^{-\frac{t}{4}}\Big) & \text{as } t \to +\infty. \end{cases}$$

In particular, $\mathcal{I}_{n,r} = \frac{1}{2}\int_0^{+\infty} D_{n,r}(t) t^{\frac{n-2}{2}}\,\mathrm{d}t$ is well-defined and finite.

5 Proof of Theorem 1.6

This section is concerned with the proof of our main result (Thm. 1.6). Recall that $\mathcal{X}$ is a compact Kähler manifold of complex dimension $n \geqslant 1$ defined over the reals and that M denotes its real locus, assumed to be non-empty. Let $\mathcal{E} \to \mathcal{X}$ be a rank $r \in \{1, \dots, n\}$ real Hermitian vector bundle and $\mathcal{L} \to \mathcal{X}$ be a real Hermitian line bundle whose curvature form is ω, the Kähler form of $\mathcal{X}$. We assume that $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. For all $d \in \mathbb{N}$, we still denote by E_d the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Finally, let s_d denote a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, whose real zero set is denoted by Z_d, and let $|\mathrm{d}V_d|$ denote the Riemannian volume measure on Z_d.

In Sect. 5.1 we recall Kac–Rice formulas and use them to derive an integral expression of $\mathrm{Var}(|\mathrm{d}V_d|)$. Sect. 5.2 is concerned with the study of some relevant random variables related to $(s_d(x))_{x \in M}$. Finally, we conclude the proof in two steps, in Sect. 5.3 and 5.4.

5.1 Kac–Rice formulas

In this section, we use Kac–Rice formulas to derive an integral expression of $\mathrm{Var}(|\mathrm{d}V_d|)$. Classical references for this material are [1, chap. 11.5] and [3, Thm. 6.3]. Since our probability space is the finite-dimensional vector space $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, it is possible to derive Kac–Rice formulas under weaker hypothesis than those given in [1] and [3]. This uses Federer's coaera formula and the so-called double fibration trick, see [21, App. C] and the references therein. The first Kac–Rice formula we state (Thm. 5.1 below) was proved in [21, Thm. 5.3] and the second (Thm. 5.5 below) was proved in [22, Thm. 4.4].

Recall that the Jacobian $\big|\det^\perp(L)\big|$ of an operator L was defined in Def. 1.4, that d_1 was defined in Lem. 2.3 and that a connection is said to be real if it satisfies the condition given in Def. 3.2.

Theorem 5.1 (Kac–Rice formula 1)**.** *Let $d \geqslant d_1$, let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$ and let $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then for all $\phi \in \mathcal{C}^0(M)$ we have:*

$$\mathbb{E}\left[\int_{x \in Z_d} \phi(x)\,|\mathrm{d}V_d|\right] = (2\pi)^{-\frac{r}{2}} \int_{x \in M} \frac{\phi(x)}{|\det^\perp(\mathrm{ev}_x^d)|} \mathbb{E}\Big[\big|\det^\perp\big(\nabla_x^d s_d\big)\big| \,\Big|\, s_d(x) = 0\Big]\,|\mathrm{d}V_M|\,. \tag{5.1}$$

The expectation on the right-hand side of Eq. (5.1) is to be understood as the conditional expectation of $\big|\det^\perp\big(\nabla_x^d s_d\big)\big|$ given that $s_d(x) = 0$.

Notation 5.2. Let $\Delta = \{(x,y) \in M^2 \mid x = y\}$ denote the diagonal of M^2.

Definition 5.3. Let $d \in \mathbb{N}$ and let $(x,y) \in M^2 \setminus \Delta$ we denote by $\mathrm{ev}^d_{x,y}$ the evaluation map:

$$\begin{array}{rccc} \mathrm{ev}^d_{x,y}: & \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d) & \longrightarrow & \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_y . \\ & s & \longmapsto & (s(x), s(y)) . \end{array}$$

Lemma 5.4. *There exists $d_2 \in \mathbb{N}$, such that for all $(x,y) \in M^2 \setminus \Delta$, $\mathrm{ev}^d_{x,y}$ is surjective.*

This was proved in [22, Prop. 4.2] in the case $r < n$, using Kodaira's embedding theorem. The proof can be adapted verbatim to the case $r \leqslant n$. We will give an alternative proof using only estimates on the Bergman kernel (Lem. 5.23 and 5.26), see p. 24 below.

Theorem 5.5 (Kac–Rice formula 2)**.** *Let $d \geqslant d_2$, let ∇^d be any real connection on $\mathcal{E}\otimes\mathcal{L}^d$ and let $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$. Then for all ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$ we have:*

$$\begin{aligned} &\mathbb{E}\left[\int_{(x,y)\in(Z_d)^2\setminus\Delta} \phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2\right] = \\ &\quad\frac{1}{(2\pi)^r}\int_{(x,y)\in M^2\setminus\Delta} \frac{\phi_1(x)\phi_2(y)}{\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right|}\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right|\,\Big|\,\mathrm{ev}^d_{x,y}(s_d)=0\Big]\,|\mathrm{d}V_M|^2 . \end{aligned} \tag{5.2}$$

Here, $|\mathrm{d}V_M|^2$ (resp. $|\mathrm{d}V_d|^2$) stands for the product measure on M^2 (resp. $(Z_d)^2$) induced by $|\mathrm{d}V_M|$ (resp. $|\mathrm{d}V_d|$). The expectation on the right-hand side of Eq. (5.2) *is the conditional expectation of $\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right|$ given that $\mathrm{ev}^d_{x,y}(s_d) = 0$.*

Proof. This formula was proved in [22, Thm. 4.4] in the case $1 \leqslant r < n$. The hypothesis $r < n$ does not appear in the proof and can be changed to $r \leqslant n$ without any other modification. □

Definition 5.6. Let $d \geqslant \max(d_1, d_2)$, and let ∇^d be any real connection on $\mathcal{E}\otimes\mathcal{L}^d$. We denote by $\mathcal{D}_d : M^2\setminus\Delta \to \mathbb{R}$ the map defined by:

$$\begin{aligned} \mathcal{D}_d(x,y) = \Bigg(&\frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right|\,\Big|\,s_d(x)=0=s_d(y)\Big]}{\left|\det{}^\perp\left(\mathrm{ev}^d_{x,y}\right)\right|} \\ &- \frac{\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_x s_d\right)\right|\,\Big|\,s_d(x)=0\Big]\,\mathbb{E}\Big[\left|\det{}^\perp\left(\nabla^d_y s_d\right)\right|\,\Big|\,s_d(y)=0\Big]}{\left|\det{}^\perp(\mathrm{ev}^d_x)\right|\left|\det{}^\perp\left(\mathrm{ev}^d_y\right)\right|}\Bigg) . \end{aligned}$$

Remark 5.7. Note that $\mathcal{D}_d$ does not depend on the choice of ∇^d. Indeed, we only consider derivatives of s_d at points where it vanishes.

Proposition 5.8. *For all $d \geqslant \max(d_1, d_2)$, we have for any $\phi_1, \phi_2 \in \mathcal{C}^0(M)$:*

$$\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2) = \frac{1}{(2\pi)^r}\int_{M^2\setminus\Delta}\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2 + \delta_{rn}\;\mathbb{E}[\langle|\mathrm{d}V_d|\;,\phi_1\phi_2\rangle] ,$$

where δ_{rn} equals 1 if $r = n$ and 0 otherwise.

Proof. This was proved in [22, Sect. 4.2] for $r < n$, the case $r = n$ requires an extra argument. The following proof if valid for any $r \in \{1,\dots,n\}$. Let ϕ_1 and $\phi_2 \in \mathcal{C}^0(M)$, we have:

$$\mathrm{Var}(|\mathrm{d}V_d|)\,(\phi_1,\phi_2) = \mathbb{E}[\langle|\mathrm{d}V_d|\;,\phi_1\rangle\,\langle|\mathrm{d}V_d|\;,\phi_2\rangle] - \mathbb{E}[\langle|\mathrm{d}V_d|\;,\phi_1\rangle]\,\mathbb{E}[\langle|\mathrm{d}V_d|\;,\phi_2\rangle] . \tag{5.3}$$

Since Z_d has almost surely dimension $n-r$, the diagonal in $Z_d\times Z_d$ is negligible if and only if $r < n$. Moreover, if $r = n$ then both $|\mathrm{d}V_d|$ and $|\mathrm{d}V_d|^2$ are counting measures. Then,

$$\begin{aligned} \langle|\mathrm{d}V_d|\;,\phi_1\rangle\,\langle|\mathrm{d}V_d|\;,\phi_2\rangle &= \int_{(x,y)\in(Z_d)^2}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2 \\ &= \int_{(x,y)\in(Z_d)^2\setminus\Delta}\phi_1(x)\phi_2(y)\,|\mathrm{d}V_d|^2 + \delta_{rn}\int_{x\in Z_d}\phi_1(x)\phi_2(x)\,|\mathrm{d}V_d| , \end{aligned}$$

almost surely. Hence

$$\mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_1 \rangle \langle |\mathrm{d}V_d| \,, \phi_2 \rangle] = \mathbb{E}\left[\int_{(x,y)\in (Z_d)^2 \setminus \Delta} \phi_1(x)\phi_2(y) \left|\mathrm{d}V_d\right|^2 \right] + \delta_{rn} \mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_1\phi_2 \rangle] \,. \quad (5.4)$$

We apply Thm. 5.5 to the first term on the right-hand side of Eq. (5.4). Similarly, we apply Thm. 5.1 to $\mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_i \rangle]$ for $i \in \{1,2\}$. This yields the result by Eq. (5.3). □

By Thm. 1.2, if $r = n$, for all $\phi_1, \phi_2 \in \mathcal{C}^0(M)$ we have:

$$\mathbb{E}[\langle |\mathrm{d}V_d| \,, \phi_1\phi_2 \rangle] = d^{\frac{n}{2}} \frac{2}{\mathrm{Vol}\,(\mathbb{S}^n)} \left(\int_M \phi_1\phi_2 \left|\mathrm{d}V_M\right| \right) + \|\phi_1\|_\infty \|\phi_2\|_\infty O\big(d^{\frac{n}{2}-1}\big) \,.$$

Hence, in order to prove Thm. 1.6, we have to show that, for any $n \in \mathbb{N}^*$ and $r \in \{1, \dots, n\}$:

$$\begin{aligned} \int_{M^2\setminus\Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y) \left|\mathrm{d}V_M\right|^2 = d^{r-\frac{n}{2}} \left(\int_M \phi_1\phi_2 \left|\mathrm{d}V_M\right| \right) \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \mathcal{I}_{n,r} \\ + \|\phi_1\|_\infty \|\phi_2\|_\infty O\big(d^{r-\frac{n}{2}-\alpha}\big) + \|\phi_1\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O\big(d^{r-\frac{n}{2}}\big) \,, \quad (5.5) \end{aligned}$$

where α, β, C_β and $\mathcal{I}_{n,r}$ are as in Thm. 1.6.

This is done in two steps. The mass of the integral on the left-hand side of Eq. (5.5) concentrates in a neighborhood of Δ of typical size $\frac{1}{\sqrt{d}}$. More precisely, let us now fix the value of the constant b_n appearing in Prop. 3.4.

Definitions 5.9. We set $b_n = \frac{1}{C_2}\left(\frac{n}{2}+1\right)$, where $C_2 > 0$ is the constant appearing in the exponential in Thm. 3.6. Moreover, for all $d \in \mathbb{N}^*$, we denote:

$$\Delta_d = \left\{ (x,y) \in M^2 \;\middle|\; \rho_g(x,y) < b_n \frac{\ln d}{\sqrt{d}} \right\},$$

where, ρ_g is the geodesic distance in (M,g).

In Sect. 5.3 below, we show that, in Eq. (5.5), the integral over $M^2 \setminus \Delta_d$ only contributes what turns out to be an error term. We refer to this term as the *far off-diagonal term*. In Sect. 5.4 we complete the proof of (5.5) by studying the *near-diagonal term*, that is the integral of $\phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)$ over $\Delta_d \setminus \Delta$. This turns out to be the leading term.

5.2 Expression of some covariances

In order to prove (5.5), we need to study the distribution of the random variables appearing in the definition of $\mathcal{D}_d$ (see Def. 5.6). The purpose of this section is to introduce several variance operators that will appear in the proof. In the following ∇^d denotes a real connection on $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$.

5.2.1 Uncorrelated terms

First of all, let us consider the distribution of $s_d(x)$ for any $x \in M$. Since $s_d \sim \mathcal{N}(\mathrm{Id})$ and ev_x^d is linear (see Def. 2.2), $s_d(x)$ is a centered Gaussian vector in $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x$ with variance operator $\mathrm{ev}_x^d(\mathrm{ev}_x^d)^* = E_d(x,x)$.

Lemma 5.10. *For all $x \in M$, we have $\left|\det{}^\perp\left(\mathrm{ev}_x^d\right)\right| = \left(\frac{d}{\pi}\right)^{\frac{rn}{2}} (1 + O(d^{-1}))$, where the error term is independent of x.*

Proof. We have $\left|\det{}^\perp\left(\mathrm{ev}_x^d\right)\right| = \det(E_d(x,x))^{\frac{1}{2}}$, and by Thm. 3.3 $E_d(x,x) = \left(\frac{d}{n}\right)^n \left(\mathrm{Id} + O(d^{-1})\right)$. □

Corollary 5.11. *There exists $d_1 \in \mathbb{N}$, such that for all $d \geqslant d_1$, for all $x \in M$, ev_x^d is surjective, that is $(s_d(x))$ is non-degenerate.*

Then, let $d \in \mathbb{N}$ and $x \in M$, we denote by $j^d_x : s \mapsto (s(x), \nabla^d_x s)$ the evaluation of the 1-jet of a section at the point x. The distribution of the random vector $(s_d(x), \nabla^d_x s_d)$ is a centered Gaussian in $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T^*_x M\right)$ with variance operator:

$$j^d_x(j^d_x)^* = \mathbb{E}\left[j^d_x(s_d)\otimes j^d_x(s_d)^*\right] = \begin{pmatrix} \mathbb{E}[s_d(x)\otimes s_d(x)^*] & \mathbb{E}\left[s_d(x)\otimes(\nabla^d_x s_d)^*\right] \\ \mathbb{E}\left[(\nabla^d_x s_d)\otimes s_d(x)^*\right] & \mathbb{E}\left[(\nabla^d_x s_d)\otimes(\nabla^d_x s_d)^*\right]\end{pmatrix}$$
$$= \begin{pmatrix} E_d(x,x) & \partial^\sharp_y E_d(x,x) \\ \partial_x E_d(x,x) & \partial_x\partial^\sharp_y E_d(x,x)\end{pmatrix}.$$

If $d \geqslant d_1$, then $s_d(x)$ is non-degenerate and the distribution of $\nabla^d_x s_d$ given that $s_d(x) = 0$ is a centered Gaussian whose variance equals

$$\partial_x\partial^\sharp_y E_d(x,x) - \partial_x E_d(x,x)\left(E_d(x,x)\right)^{-1}\partial^\sharp_y E_d(x,x).$$

By Thm. 3.3, this variance equals

$$\frac{d^{n+1}}{\pi^n}\left(\mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes T^*_x M} + O(d^{-1})\right)$$

as d goes to infinity and the error does not depend on x.

Remark 5.12. If (s,x) is such that $s(x) = 0$, then $\nabla^d_x s$ does not depend on the connection ∇^d. This explains why the distribution of $\nabla^d_x s_d$ given that $s_d(x) = 0$, in particular its variance, does not depend on ∇^d.

Lemma 5.13. *For every $x \in M$, we have:*

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla^d_x s_d\right)\right| \,\middle|\, s_d(x) = 0\right] = \left(\frac{d^{n+1}}{\pi^n}\right)^{\frac{r}{2}} (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\left(1 + O\left(d^{-1}\right)\right),$$

where the error term is independent of x.

Proof. This was proved in [22, Lem. 4.7] for $r < n$. The proof is the same here. □

5.2.2 Correlated terms far from the diagonal

Let us now focus on variables where non trivial correlations may appear in the limit. Let $d \in \mathbb{N}$, for all $(x,y) \in M^2 \setminus \Delta$, the random vector $\mathrm{ev}^d_{x,y}(s_d) = (s_d(x), s_d(y))$ is a centered Gaussian vector with variance operator:

$$\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^* = \mathbb{E}\left[\mathrm{ev}^d_{x,y}(s_d)\otimes\mathrm{ev}^d_{x,y}(s_d)^*\right] = \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix}, \tag{5.6}$$

where we decomposed this operator according to the direct sum $\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y$.

Definition 5.14. For all $d \in \mathbb{N}$, for all $(x,y) \in M^2 \setminus \Delta$, we denote by:

$$\Theta_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y)\end{pmatrix},$$

the variance of the centered Gaussian vector $\left(\frac{\pi}{d}\right)^{\frac{n}{2}}(s_d(x), s_d(y))$.

Note that, by Lem. 5.4, for all $d \geqslant d_2$, $\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*$ is non-singular, i.e. $(s_d(x), s_d(y))$ is non-degenerate and $\Theta_d(x,y)$ is non-singular.

Let $d \in \mathbb{N}$ and $(x,y) \in M^2 \setminus \Delta$, we denote by $j^d_{x,y} : s \mapsto \left(s(x), s(y), \nabla^d_x s, \nabla^d_y s\right)$ the evaluation of the 1-jets at (x,y). Then $j^d_{x,y}(s_d) = \left(s_d(x), s_d(y), \nabla^d_x s_d, \nabla^d_y s_d\right)$ is a centered Gaussian vector in:

$$\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \oplus \left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \otimes T^*_x M\right) \oplus \left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_y \otimes T^*_y M\right),$$

whose variance operator $j^d_{x,y}(j^d_{x,y})^*$ equals:

$$\mathbb{E}\left[j^d_{x,y}(s_d)\otimes\left(j^d_{x,y}(s_d)\right)^*\right] = \begin{pmatrix} E_d(x,x) & E_d(x,y) & \partial^\sharp_y E_d(x,x) & \partial^\sharp_y E_d(x,y) \\ E_d(y,x) & E_d(y,y) & \partial^\sharp_y E_d(y,x) & \partial^\sharp_y E_d(y,y) \\ \partial_x E_d(x,x) & \partial_x E_d(x,y) & \partial_x\partial^\sharp_y E_d(x,x) & \partial_x\partial^\sharp_y E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) & \partial_x\partial^\sharp_y E_d(y,x) & \partial_x\partial^\sharp_y E_d(y,y)\end{pmatrix}.$$

Definition 5.15. For all $d \in \mathbb{N}$, for all $(x,y) \in M^2 \setminus \Delta$, we denote by:

$$\Omega_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & E_d(x,y) & d^{-\frac{1}{2}}\partial_y^\sharp E_d(x,x) & d^{-\frac{1}{2}}\partial_y^\sharp E_d(x,y) \\ E_d(y,x) & E_d(y,y) & d^{-\frac{1}{2}}\partial_y^\sharp E_d(y,x) & d^{-\frac{1}{2}}\partial_y^\sharp E_d(y,y) \\ d^{-\frac{1}{2}}\partial_x E_d(x,x) & d^{-\frac{1}{2}}\partial_x E_d(x,y) & d^{-1}\partial_x\partial_y^\sharp E_d(x,x) & d^{-1}\partial_x\partial_y^\sharp E_d(x,y) \\ d^{-\frac{1}{2}}\partial_x E_d(y,x) & d^{-\frac{1}{2}}\partial_x E_d(y,y) & d^{-1}\partial_x\partial_y^\sharp E_d(y,x) & d^{-1}\partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix},$$

the variance operator of the centered Gaussian vector: $\left(\frac{\pi}{d}\right)^{\frac{n}{2}} \left(s_d(x), s_d(y), \frac{1}{\sqrt{d}}\nabla_x^d s_d, \frac{1}{\sqrt{d}}\nabla_y^d s_d\right)$.

Let us now assume that $d \geqslant d_2$, so that the distribution of $(s_d(x), s_d(y))$ is non-degenerate. Then the distribution of $\left(\nabla_x^d s, \nabla_y^d s\right)$ given that $s_d(x) = 0 = s_d(y)$ is a centered Gaussian with variance operator:

$$\begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y) \\ \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix} - \begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix}.$$

Definition 5.16. For all $d \geqslant d_2$, for all $(x,y) \in M^2 \setminus \Delta$, we set:

$$\begin{aligned}\Lambda_d(x,y) =& \frac{\pi^n}{d^{n+1}} \left(\begin{pmatrix} \partial_x\partial_y^\sharp E_d(x,x) & \partial_x\partial_y^\sharp E_d(x,y) \\ \partial_x\partial_y^\sharp E_d(y,x) & \partial_x\partial_y^\sharp E_d(y,y) \end{pmatrix} \right. \\ & \left. - \begin{pmatrix} \partial_x E_d(x,x) & \partial_x E_d(x,y) \\ \partial_x E_d(y,x) & \partial_x E_d(y,y) \end{pmatrix} \begin{pmatrix} E_d(x,x) & E_d(x,y) \\ E_d(y,x) & E_d(y,y) \end{pmatrix}^{-1} \begin{pmatrix} \partial_y^\sharp E_d(x,x) & \partial_y^\sharp E_d(x,y) \\ \partial_y^\sharp E_d(y,x) & \partial_y^\sharp E_d(y,y) \end{pmatrix} \right),\end{aligned}$$

which is the variance of the Gaussian vector $\left(\frac{\pi^n}{d^{n+1}}\right)^{\frac{1}{2}} \left(\nabla_x^d s_d, \nabla_y^d s_d\right)$ given that $s_d(x) = 0 = s_d(y)$.

Remark 5.17. Once again, $\Lambda_d(x,y)$ is independent of the choice of ∇^d, and so is the distribution of $(\nabla_x^d s_d, \nabla_y^d s_d)$ given that $s_d(x) = 0 = s_d(y)$. On the other hand, the distribution of $\left(s_d(x), s_d(y), \nabla_x^d s_d, \nabla_y^d s_d\right)$ heavily depends on ∇^d, and so does $\Omega_d(x,y)$. Hence, we will need to specify a choice of ∇^d at some point when dealing with Ω_d.

5.2.3 Correlated terms close to the diagonal

Finally, we need to consider the distribution of the 1-jets of s_d at x and $y \in M$, when the distance between x and y is of order $\frac{1}{\sqrt{d}}$. As in Sect. 3, let $R > 0$ be such that $2R$ is less than the injectivity radius of $\mathcal{X}$. There exists $d_3 \in \mathbb{N}$ such that, for all $d \geqslant d_3$, $b_n \frac{\ln d}{\sqrt{d}} \leqslant R$.

Let $d \geqslant d_3$ and let $(x,y) \in \Delta_d \setminus \Delta$. Using the real normal trivialization of $\mathcal{E} \otimes \mathcal{L}^d$ around x (see Sect. 3.1), we can see $\left(\frac{\pi}{d}\right)^{\frac{n}{2}} (s_d(x), s_d(y))$ as a random vector in $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$. Since the distance from x to y is smaller than the injectivity radius of M, we can write y as $\exp_x\left(\frac{z}{\sqrt{d}}\right)$ for some $z \in T_x M$. Moreover $\|z\| = \sqrt{d}\,\rho_g(x,y) < b_n \ln d$.

Definition 5.18. Let $d \geqslant d_3$, let $x \in M$ and let $z \in B_{T_x M}(0, b_n \ln d) \setminus \{0\}$, we set:

$$\Theta_d(z) = \Theta_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x \oplus \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ via the real normal trivialization centered at x.

Remark 5.19. Beware that $\Theta_d(z)$ depends on x, even if this is not reflected in the notation. However, we will show that the limit of $\Theta_d(z)$ as $d \to +\infty$ does not depend on x.

Recall that e_d was defined by Eq. (3.1) as a map from $T_x M \times T_x M$ to $\mathrm{End}\left(\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x\right)$. The definitions of $\Theta_d(x,y)$ (Def. 5.14) and e_d show that, for all $d \geqslant d_3$, for all $x \in M$ and for all $z \in B_{T_x M}(0, b_n \ln d) \setminus \{0\}$:

$$\Theta_d(z) = \begin{pmatrix} e_d(0,0) & e_d(0,z) \\ e_d(z,0) & e_d(z,z) \end{pmatrix}. \tag{5.7}$$

We can define $\Omega_d(z)$ and $\Lambda_d(z)$ similarly and express them in terms of e_d and its derivatives.

Definition 5.20. Let $d \geqslant d_3$, let $x \in M$ and let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, we set:

$$\Omega_d(z) = \Omega_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $(\mathbb{R} \oplus \mathbb{R} \oplus T_x^*M \oplus T_x^*M) \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ via the real normal trivialization centered at x.

Let ∇^d be a real connection on $\mathcal{E} \otimes \mathcal{L}^d$ such that, in the real normal trivialization around x, this connection coincides over the ball $B_{T_x\mathcal{X}}(0, R)$ with the usual differentiation for maps from $T_x\mathcal{X}$ to $\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$. The existence of such a connection was established at the end of Sect. 3.1. Then, by Def. 5.15 and 5.20, we have for all $d \geqslant d_3$, for all $x \in M$ and for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$:

$$\Omega_d(z) = \begin{pmatrix} e_d(0,0) & e_d(0,z) & \partial_y^\sharp e_d(0,0) & \partial_y^\sharp e_d(0,z) \\ e_d(z,0) & e_d(z,z) & \partial_y^\sharp e_d(z,0) & \partial_y^\sharp e_d(z,z) \\ \partial_x e_d(0,0) & \partial_x e_d(0,z) & \partial_x\partial_y^\sharp e_d(0,0) & \partial_x\partial_y^\sharp e_d(0,z) \\ \partial_x e_d(z,0) & \partial_x e_d(z,z) & \partial_x\partial_y^\sharp e_d(z,0) & \partial_x\partial_y^\sharp e_d(z,z) \end{pmatrix}. \tag{5.8}$$

Definition 5.21. Let $d \geqslant \max(d_2, d_3)$, let $x \in M$ and let $z \in B_{T_x^*M}(0, b_n \ln d) \setminus \{0\}$, we set:

$$\Lambda_d(z) = \Lambda_d\left(x, \exp_x\left(\frac{z}{\sqrt{d}}\right)\right),$$

seen as an operator on $(T_x^*M \oplus T_x^*M) \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ via the real normal trivialization around x.

Then, for all $d \geqslant \max(d_2, d_3)$, for all $x \in M$ and all $z \in B_{T_x^*M}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\Lambda_d(z) = \begin{pmatrix} \partial_x\partial_y^\sharp e_d(0,0) & \partial_x\partial_y^\sharp e_d(0,z) \\ \partial_x\partial_y^\sharp e_d(z,0) & \partial_x\partial_y^\sharp e_d(z,z) \end{pmatrix} - \begin{pmatrix} \partial_x e_d(0,0) & \partial_x e_d(0,z) \\ \partial_x e_d(z,0) & \partial_x e_d(z,z) \end{pmatrix} \Theta_d(z)^{-1} \begin{pmatrix} \partial_y^\sharp e_d(0,0) & \partial_y^\sharp e_d(0,z) \\ \partial_y^\sharp e_d(z,0) & \partial_y^\sharp e_d(z,z) \end{pmatrix}.$$

5.3 Far off-diagonal term

In this section, we state that the far off-diagonal term in Eq. (5.5) only contributes an error term. This was already proved in [22] for $r < n$. The proof is the same for $r = n$, so we refer to [22] for the proof. Lem. 5.23 below is used in the proof of Prop. 5.22 but is also of independent interest for our purpose.

Proposition 5.22. *Let $\phi_1, \phi_2 \in \mathcal{C}^0(M)$, then we have the following as $d \to +\infty$:*

$$\int_{M^2 \setminus \Delta_d} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2 = \|\phi_1\|_\infty \|\phi_2\|_\infty O\big(d^{r-\frac{n}{2}-1}\big),$$

where the error term is independent of (ϕ_1, ϕ_2).

Lemma 5.23. *For every $(x, y) \in M^2 \setminus \Delta_d$, we have:*

$$\Theta_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} \left(\mathrm{Id} + O\big(d^{-\frac{n}{2}-1}\big)\right),$$

where the error term is independent of $(x, y) \in M^2 \setminus \Delta_d$.

Proof. Since $(x, y) \in M^2 \setminus \Delta_d$, we have $\rho_g(x, y) \geqslant b_n \frac{\ln d}{\sqrt{d}}$. With our choice of b_n (see Def. 5.9), the error term in Thm. 3.6 is then $O(d^{\frac{n-k}{2}-1})$, uniformly on $M^2 \setminus \Delta_d$. Thus, by Thm. 3.6,

$$\Theta_d(x,y) = \left(\frac{\pi}{d}\right)^n \begin{pmatrix} E_d(x,x) & 0 \\ 0 & E_d(y,y) \end{pmatrix} + O\big(d^{-\frac{n}{2}-1}\big).$$

The results follows from the fact that the leading term is $\mathrm{Id} + O(d^{-1})$, by Thm. 3.3. □

5.4 Near-diagonal term

In this section, we conclude the proof of Thm. 1.6, up to the technical lemmas whose proofs were postponed until Appendices A and B.

Definition 5.24. Let $d \geqslant \max(d_1, d_2, d_3)$, let $x \in M$ and let $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, we define:

$$D_d(x,z) = d^{-r} \mathcal{D}_d \left(x, \exp_x \left(\frac{z}{\sqrt{d}} \right) \right).$$

Recall that $D_{n,r}$ was defined by Def. 4.15. The main result of this section is the following.

Proposition 5.25. *Let $\alpha \in (0,1)$, then for all $x \in M$, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:*

$$D_d(x,z) = D_{n,r}\left(\|z\|^2 \right) \left(1 + O\big(d^{-\alpha}\big) \right) + O\big(d^{-\alpha}\big),$$

where the error terms do not depend on (x,z).

First, let us prove that Prop. 5.8, 5.22 and 5.25 together imply Thm. 1.6.

Proof of Thm. 1.6. The main point is to compute the asymptotics of the near-diagonal term in Eq. (5.5). Let us fix $\alpha \in (0,1)$, $\beta \in \left(0, \frac{1}{2}\right)$ and $\phi_1, \phi_2 \in \mathcal{C}^0(M)$. Let $x \in M$, recall that $\sqrt{\kappa}$ is the density of $|\mathrm{d}V_M|$ with respect to the Lebesgue measure, in the exponential chart centered at x, where κ was introduced in Sect. 3.2. Then, by a change of variable $y = \exp_x \left(\frac{z}{\sqrt{d}} \right)$, we have:

$$\begin{aligned} \int_{\Delta_d \setminus \Delta} \phi_1(x)\phi_2(y) \mathcal{D}_d(x,y) \left| \mathrm{d}V_M \right|^2 = \\ d^{r-\frac{n}{2}} \int_{x \in M} \phi_1(x) \int_{z \in B_{T_xM}(0, b_n \ln d)} \phi_2 \left(\exp_x \left(\frac{z}{\sqrt{d}} \right) \right) D_d(x,z) \kappa \left(\frac{z}{\sqrt{d}} \right)^{\frac{1}{2}} \mathrm{d}z \left| \mathrm{d}V_M \right|. \end{aligned} \tag{5.9}$$

As we already discussed in Sect. 3.2, $\kappa(z) = 1 + O(\|z\|^2)$ and the error term is independent of x. Hence, $\kappa \left(\frac{z}{\sqrt{d}} \right)^{\frac{1}{2}} = 1 + O\Big(\frac{(\ln d)^2}{d} \Big)$, and by Prop. 5.25, for all $\gamma \in (\alpha, 1)$,

$$\begin{aligned} \int_{z \in B_{T_xM}(0, b_n \ln d)} \phi_2 \left(\exp_x \left(\frac{z}{\sqrt{d}} \right) \right) D_d(x,z) \kappa \left(\frac{z}{\sqrt{d}} \right)^{\frac{1}{2}} \mathrm{d}z = \\ \left(\int_{z \in B_{T_xM}(0, b_n \ln d)} \phi_2 \left(\exp_x \left(\frac{z}{\sqrt{d}} \right) \right) D_{n,r}(\|z\|^2) \, \mathrm{d}z \right) \left(1 + O\big(d^{-\gamma}\big) \right) + \|\phi_2\|_\infty O\left(\frac{(\ln d)^n}{d^\gamma} \right). \end{aligned} \tag{5.10}$$

Since $\gamma > \alpha$, $(\ln d)^n d^{-\gamma} = O(d^{-\alpha})$. Similarly, there exists $C_\beta > 0$ such that $b_n \frac{\ln d}{\sqrt{d}} \leqslant C_\beta d^{-\beta}$ for all $d \in \mathbb{N}^*$. Then we have:

$$\begin{aligned} \left| \int_{z \in B_{T_xM}(0, b_n \ln d)} \left(\phi_2 \left(\exp_x \left(\frac{z}{\sqrt{d}} \right) \right) - \phi_2(x) \right) D_{n,r}(\|z\|^2) \, \mathrm{d}z \right| \\ \leqslant \varpi_{\phi_2}\left(C_\beta d^{-\beta} \right) \int_{z \in B_{T_xM}(0, b_n \ln d)} \left| D_{n,r}(\|z\|^2) \right| \mathrm{d}z, \end{aligned} \tag{5.11}$$

where ϖ_{ϕ_2} is the continuity modulus of ϕ_2 (see Def. 1.3). Besides, by Cor. 4.19,

$$\begin{aligned} \int_{z \in B_{T_xM}(0, b_n \ln d)} \left| D_{n,r}(\|z\|^2) \right| \mathrm{d}z &= \mathrm{Vol}\left(\mathbb{S}^{n-1} \right) \frac{1}{2} \int_{t=0}^{(b_n \ln d)^2} D_{n,r}(t) t^{\frac{n-2}{2}} \, \mathrm{d}t \\ &= \mathrm{Vol}\left(\mathbb{S}^{n-1} \right) \left(\mathcal{I}_{n,r} + O\Big(e^{-\frac{1}{4}(b_n \ln d)^2} \Big) \right), \end{aligned} \tag{5.12}$$

and the error term is $O(d^{-1})$, since $\frac{1}{4}(b_n \ln d)^2 \geqslant \ln d$ for d large enough. By Eq. (5.10), (5.11) and (5.12), the innermost integral on the right-hand side of Eq. (5.9) equals:

$$\phi_2(x) \operatorname{Vol}\left(\mathbb{S}^{n-1}\right) \mathcal{I}_{n,r} + \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O(1) + \|\phi_2\|_\infty O(d^{-\alpha}),$$

and the error terms are independent of $x \in M$ and (ϕ_1, ϕ_2). Finally, by Eq. (5.9),

$$\int_{\Delta_d \setminus \Delta} \phi_1(x)\phi_2(y)\mathcal{D}_d(x,y) \left|\mathrm{d}V_M\right|^2 = d^{r-\frac{n}{2}} \left(\int_M \phi_1\phi_2 \left|\mathrm{d}V_M\right|\right) \operatorname{Vol}\left(\mathbb{S}^{n-1}\right) \mathcal{I}_{n,r} + \|\phi_1\|_\infty \varpi_{\phi_2}\left(C_\beta d^{-\beta}\right) O(d^{r-\frac{n}{2}}) + \|\phi_1\|_\infty \|\phi_2\|_\infty O(d^{r-\frac{n}{2}-\alpha}).$$

We conclude the proof by combining this last relation with Prop. 5.8, 5.22 and, in the case $r = n$, Thm. 1.2 for $\phi_1\phi_2$. □

The remainder of this section is mostly dedicated to the proof of Prop. 5.25. We will deduce this proposition from several technical lemmas stated below.

Let $x \in M$, then, any choice of an isometry between T_xM and $\mathbb{R}^n$ and an isometry between $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ and $\mathbb{R}^r$ allows us to see the Bargmann–Fock process $(s(z))_{z\in\mathbb{R}^n}$, studied in Sect. 4, as a smooth Gaussian process from T_xM to $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$. The distribution of this process does not depend on our choice of isometries. Thus, in the following, we can consider $\Theta(z)$ and $\Theta_d(z)$ (resp. $\Omega(z)$ and $\Omega_d(z)$, resp. $\Lambda(z)$ and $\Lambda_d(z)$) as operators on the same space.

Lemma 5.26. *Let $\alpha \in (0,1)$, then for all $x \in M$, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:*

$$\Theta(z)^{-\frac{1}{2}} \Theta_d(z) \Theta(z)^{-\frac{1}{2}} = \mathrm{Id} + O\big(d^{-\alpha}\big),$$

where the error term does not depend on (x, z).

Proof. See Appendix B. □

Remark 5.27. One could wonder why Lem. 5.26 does not state that $\Theta_d(z) = \Theta(z)\left(\mathrm{Id} + O(d^{-\alpha})\right)$, which would be somewhat simpler. First, note that this statement is not equivalent to Lem. 5.26, since some of the eigenvalues of $\Theta(z)$ converge to 0 as $z \to 0$. In fact, this alternative statement turns out to be false in general. Moreover, even if $\Theta_d(z)$ is a linear map, it represents a variance, i.e. something intrinsically bilinear. It is then quite natural to consider $\Theta(z)^{-\frac{1}{2}} \Theta_d(z) \Theta(z)^{-\frac{1}{2}}$ since this is how $\Theta_d(z)$ transforms if we act on $\mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ by $\Theta(z)^{\frac{1}{2}}$. This remark also applies to Lem. 5.28 and 5.29 below.

Let us forget about the proof of Prop. 5.25 for a minute, and prove the existence of d_2 (see Lem. 5.4) as a corollary of Lem. 5.23 and 5.26. Note that the proofs of these lemmas only rely on the estimates of Sect. 3, so there is no logical loop here.

Proof of Lem. 5.4. We want to prove that, as soon as d is large enough, $\mathrm{ev}^d_{x,y}$ is surjective for all $(x, y) \in M^2 \setminus \Delta$, that is $\det\left(\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*\right) \neq 0$. By Eq. (5.6) and the definition of Θ_d (Def. 5.14),

$$\det\left(\mathrm{ev}^d_{x,y}(\mathrm{ev}^d_{x,y})^*\right) = \left(\frac{d}{\pi}\right)^{2rn} \det\left(\Theta_d(x,y)\right),$$

so we have to show that $\det\left(\Theta_d(x,y)\right)$ does not vanish on $M^2 \setminus \Delta$, for d large enough. By Lem. 5.23 and Thm. 3.3,

$$\det\left(\Theta_d(x,y)\right) = 1 + O\big(d^{-1}\big), \tag{5.13}$$

uniformly on $M^2 \setminus \Delta_d$. Let $(x, y) \in \Delta_d \setminus \Delta$ and let us assume that $d \geqslant d_3$ so that we can write y as $\exp_x\left(\frac{z}{\sqrt{d}}\right)$ with $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Then, by Lem. 5.26 and 4.8,

$$\det\left(\Theta_d(x,y)\right) = \det\left(\Theta_d(z)\right) = \det\left(\Theta(z)\right)\left(1 + O\Big(\sqrt{d}\Big)\right) = \left(1 - e^{-\|z\|^2}\right)^r \left(1 + O\Big(\sqrt{d}\Big)\right), \tag{5.14}$$

uniformly on $\Delta_d \setminus \Delta$. The result follows from Eq. (5.13) and (5.14). □

We can now go back to the proof of Prop. 5.25.

Lemma 5.28. *Let $\alpha \in (0,1)$, then for all $x \in M$, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:*

$$\Omega(z)^{-\frac{1}{2}} \Omega_d(z) \Omega(z)^{-\frac{1}{2}} = \mathrm{Id} + O\big(d^{-\alpha}\big)\,,$$

where the error term does not depend on (x,z).

Proof. See Appendix B. □

Lemma 5.29. *Let $\alpha \in (0,1)$, then for all $x \in M$, for all $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:*

$$\Lambda(z)^{-\frac{1}{2}} \Lambda_d(z) \Lambda(z)^{-\frac{1}{2}} = \mathrm{Id} + O\big(d^{-\alpha}\big)\,,$$

where the error term does not depend on (x,z).

Proof. Let $\alpha \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. By Def. 5.20, $\Omega_d(z)$ is an operator on:

$$\big(\mathbb{R}^2 \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x\big) \oplus \big((T^*_x M)^2 \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x\big)\,.$$

Using this splitting, we write $\Omega_d(z)$ by blocks as:

$$\Omega_d(z) = \begin{pmatrix} \Theta_d(z) & \Omega^1_d(z)^* \\ \Omega^1_d(z) & \Omega^2_d(z) \end{pmatrix},$$

thus defining $\Omega^1_d(z)$ and $\Omega^2_d(z)$. For d large enough, $\Theta_d(z)$ is invertible and its Schur complement is $\Lambda_d(z) = \Omega^2_d(z) - \Omega^1_d(z)\Theta_d(z)^{-1}\Omega^1_d(z)^*$. It is then known that $\Lambda_d(z)^{-1}$ is the bottom-right block of $\Omega_d(z)^{-1}$, that is:

$$\Lambda_d(z)^{-1} = \begin{pmatrix} 0 & \mathrm{Id} \end{pmatrix} \Omega_d(z)^{-1} \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \end{pmatrix} \Big(\Omega(z)^{-1} + \Omega(z)^{-\frac{1}{2}} O\big(d^{-\alpha}\big)\, \Omega(z)^{-\frac{1}{2}} \Big) \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix},$$

where the second equality is given by Lem. 5.28 and the error term is independent of (x,z). Similarly, $\Lambda(z)$ is the Schur complement of $\Theta(z)$ in $\Omega(z)$, so that

$$\Lambda(z)^{-1} = \begin{pmatrix} 0 & \mathrm{Id} \end{pmatrix} \Omega(z)^{-1} \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix}.$$

Since by Lem. 4.14, $\begin{pmatrix} 0 & \Lambda(z)^{\frac{1}{2}} \end{pmatrix} \Omega_d(z)^{-\frac{1}{2}}$ is bounded, $\Lambda(z)^{\frac{1}{2}} \Lambda_d(z)^{-1} \Lambda(z)^{\frac{1}{2}} = \mathrm{Id} + O(d^{-\alpha})$, and the error term still does not depend on (x,z). □

Lemma 5.30. *Let $\alpha \in (0,1)$, let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. Let $X_d(z)$ and $Y_d(z)$ be random vectors in $T^*_x M \otimes \mathbb{R}\big(\mathcal{E} \otimes \mathcal{L}^d\big)_x$ such that $(X_d(z), Y_d(z)) \sim \mathcal{N}(\Lambda_d(z))$. Then we have:*

$$\mathbb{E}\big[\big|\det{}^{\perp}(X_d(z))\big|\,\big|\det{}^{\perp}(Y_d(z))\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big]\,\big(1 + O\big(d^{-\alpha}\big)\big)$$

where $(X_\infty(z), Y_\infty(z)) \sim \mathcal{N}(\Lambda(z))$ and the error term does not depend on (x,z).

Proof. See Appendix B. □

We conclude this section with the proof of Prop. 5.25. Recall the definitions of $D_{n,r}(t)$ (Def. 4.15), $\mathcal{D}_d(x,y)$ (Def. 5.6) and $D_d(x,z)$ (Def. 5.24).

Proof of Prop. 5.25. Let us fix $\alpha \in (0,1)$. Let $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$, we set $y = \exp_x\Big(\frac{z}{\sqrt{d}}\Big)$. We have defined $\Theta_d(z)$ and $\Lambda_d(z)$ so that:

$$\frac{1}{d^r} \frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_x s_d\big)\big|\big|\det{}^{\perp}\big(\nabla^d_y s_d\big)\big|\,\Big|\, s_d(x) = 0 = s_d(y)\Big]}{\big|\det{}^{\perp}\big(\mathrm{ev}^d_{x,y}\big)\big|} = \frac{\mathbb{E}\big[\big|\det{}^{\perp}(X_d(z))\big|\,\big|\det{}^{\perp}(Y_d(z))\big|\big]}{\det\left(\Theta_d(z)\right)^{\frac{1}{2}}},$$

where $(X_d(z), Y_d(z)) \sim \mathcal{N}(\Lambda_d(z))$. By Lem. 5.26 and 5.30, this equals:

$$\frac{\mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big]}{\det\left(\Theta(z)\right)^{\frac{1}{2}}}\left(1+O\big(d^{-\alpha}\big)\right),$$

where $(X_\infty(z), Y_\infty(z)) \sim \mathcal{N}(\Lambda(z))$ and the error term does not depend on (x, z). By the definition of $\Lambda(z)$ (cf. Sect. 4.3) $(X_\infty(z), Y_\infty(z))$ is distributed as $(d_0 s, d_z s)$, where s is a copy of the Bargmann–Fock process from T_xM to $\mathbb{R}\big(\mathcal{E}\otimes\mathcal{L}^d\big)_x$. Then, by Lem. 4.16,

$$\mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big] = \mathbb{E}\left[\left|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\right|\left|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\right|\right],$$

where $\Big(X(\|z\|^2), Y(\|z\|^2)\Big)$ was defined by Def. 1.5. Besides, $\det\left(\Theta(z)\right) = \Big(1-e^{-\|z\|^2}\Big)^r$ by Lem. 4.1, so that:

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_x s_d\big)\big|\big|\det{}^{\perp}\big(\nabla^d_y s_d\big)\big|\,\Big|\,s_d(x)=0=s_d(y)\Big]}{\big|\det{}^{\perp}\big(\mathrm{ev}^d_{x,y}\big)\big|} =$$
$$\left(D_{n,r}\left(\|z\|^2\right)+(2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\right)^2\right)\left(1+O\big(d^{-\alpha}\big)\right).$$

Besides, Lem. 5.10 and 5.13,

$$\frac{1}{d^r}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_x s_d\big)\big|\,\Big|\,s_d(x)=0\Big]}{\big|\det{}^{\perp}(\mathrm{ev}^d_x)\big|}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\big(\nabla^d_y s_d\big)\big|\,\Big|\,s_d(y)=0\Big]}{\big|\det{}^{\perp}(\mathrm{ev}^d_x)\big|} = (2\pi)^r\left(\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}\right)^2+O\big(d^{-1}\big)$$

This yields the desired relation. □

6 Proof of Theorem 1.8

The goal of this section is to prove that the leading constant in Thm. 1.6 is positive. Sect. 6.1 is concerned with the definition and proprieties of Kostlan–Shub–Smale polynomials. In Sect. 6.2 we recall some facts about Wiener chaoses. In Sect. 6.3 we compute the chaotic expansion of the linear statistics $\langle|\mathrm{d}V_d|\,,\phi\rangle$. Finally, we conclude the proof of Thm. 1.8 in Sect. 6.4.

6.1 Kostlan–Shub–Smale polynomials

In this section, we describe a special case of our real algebraic framework. This is the setting we will be considering throughout the proof of Thm. 1.8. A good reference for the complex algebraic material of this section is [16].

6.1.1 Definition

We choose $\mathcal{X}$ to be the complex projective space $\mathbb{CP}^n$ with the real structure induced by the complex conjugation in $\mathbb{C}^{n+1}$. The real locus of $\mathcal{X}$ is the real projective space $\mathbb{RP}^n$. We set $\mathcal{L} = \mathcal{O}(1) \to \mathbb{CP}^n$ as the *hyperplane line bundle*, that is the dual of the tautological line bundle:

$$\mathcal{O}(-1) = \big\{(\zeta, x) \in \mathbb{C}^{n+1}\times\mathbb{CP}^n \;\big|\; \zeta \in x\big\} \longrightarrow \mathbb{CP}^n.$$

Recall that ample line bundles on $\mathbb{CP}^n$ are of the form $\mathcal{O}(d) = (\mathcal{O}(1))^{\otimes d}$ with $d \in \mathbb{N}^*$ (see [16, Sect. 1.1]). The complex conjugation and the usual Hermitian inner product of $\mathbb{C}^{n+1}$ induce compatible real and metric structures on $\mathcal{O}(-1)$, hence on $\mathcal{O}(1)$ by duality. The resulting Hermitian metric on $\mathcal{O}(1)$ is positive and its curvature is the Fubini–Study Kähler form on $\mathbb{CP}^n$. With our choice of normalization, the induced Riemannian metric is the quotient of the Euclidean metric on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Finally, we choose $\mathcal{E}$ to be the rank r trivial bundle $\mathbb{C}^r \times \mathbb{CP}^n \to \mathbb{CP}^n$, with the compatible real and metric structures inherited from the standard ones in $\mathbb{C}^r$.

Notations 6.1. Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we denote its *length* by $|\alpha| = \alpha_0 + \cdots + \alpha_n$. We also define $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ and $\alpha! = \alpha_0! \cdots \alpha_n!$. Finally, if $|\alpha| = d$, we denote by $\binom{d}{\alpha}$ the multinomial coefficient $\frac{d!}{\alpha!}$.

It is well-known (cf. [6, 7, 17, 22]) that $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ is the space $\left(\mathbb{R}_d^{\hom}[X_0, \dots, X_n]\right)^r$ of tuples of real homogeneous polynomials of degree d in $n+1$ variables. The r terms $\mathbb{R}_d^{\hom}[X_0, \dots, X_n]$ in $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ are pairwise orthogonal for the inner product (2.2). Besides, in restriction to one of these terms, (2.2) equals:

$$(P, Q) \mapsto \int_{x \in \mathbb{CP}^n} h_d(P(x), Q(x)) \, |\mathrm{d}V_{\mathbb{CP}^n}| = \frac{1}{\pi(d+n)!} \int_{z \in \mathbb{C}^{n+1}} P(z)\overline{Q(z)} e^{-\|z\|^2} \, \mathrm{d}z. \tag{6.1}$$

An orthonormal basis of $\mathbb{R}_d^{\hom}[X_0, \dots, X_n]$ is then $\left(\sqrt{\frac{(d+n)!}{\pi^n d!}} \sqrt{\binom{d}{\alpha}} X^\alpha\right)_{|\alpha|=d}$. Hence, a standard Gaussian in $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d))$ is a r-tuple of independent random polynomials of the form:

$$\sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha, \tag{6.2}$$

where the coefficients $(a_\alpha)_{|\alpha|=d}$ are independent real standard Gaussian variables. Such a random polynomial is called a *Kostlan–Shub–Smale polynomial* (KSS for short).

6.1.2 Correlation kernel

In this section, we study the distribution of the KSS polynomial (see Eq. (6.2)). In the setting of Sect. 6.1.1, E_d is the Bergman kernel of $\mathbb{C}^r \otimes \mathcal{O}(d) \to \mathbb{CP}^n$. Since the first factor is trivial, we have $E_d = I_r \otimes E'_d$, where I_r is the identity of $\mathbb{C}^r$ and E'_d is the Bergman kernel of $\mathcal{O}(d) \to \mathbb{CP}^n$. Note that E'_d is the correlation kernel of the field s'_d defined by one KSS polynomial, seen as a random section of $\mathcal{O}(d)$. By Eq. (6.2) we have:

$$E'_d(x, y) = \mathbb{E}[s'_d(x) \otimes s'_d(y)^*] = \frac{(d+n)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} X^\alpha(x) \otimes X^\alpha(y)^*$$

Note that (6.1) is invariant under the action of the orthogonal group $O_{n+1}(\mathbb{R})$ on the right. Hence, the distribution of KSS polynomials (6.2) and E'_d are equivariant under this action. Since $O_{n+1}(\mathbb{R})$ acts transitively on the couples of points of $\mathbb{RP}^n$ at a given distance, $E'_d(x, y)$ only depends on the geodesic distance $\rho_g(x, y)$, and the same holds for derivatives. Loosely speaking, this implies the following, where derivatives are computed with respect to the Chern connection.

1. The variance of $s'_d(x)$ does not depend on $x \in \mathbb{RP}^n$.
2. For all $x \in \mathbb{RP}^n$, $s'_d(x)$ and $\nabla_x^d s'_d$ are independent.
3. If $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is any orthonormal basis of $T_x\mathbb{RP}^n$, then $\frac{\partial s'_d}{\partial x_i}(x)$ and $\frac{\partial s'_d}{\partial x_j}(x)$ are independent if $i \neq j$. Moreover, the variance of $\frac{\partial s'_d}{\partial x_i}(x)$ does not depend on i, nor on our choice of orthonormal basis, nor on $x \in \mathbb{RP}^n$.

These properties are very specific of the case of KSS polynomials. They will be useful in Sect. 6.3, to compute the Wiener–Itō expansion of $\langle |\mathrm{d}V_d| \, , \phi \rangle$. We do not give more details here, since Properties 1, 2 and 3 can also be deduced from the expression of E'_d in local coordinates that we derive below.

6.1.3 Local expression of the kernel

Let $x \in \mathbb{RP}^n$, we want to compute the expression of E'_d in some good coordinates around x. We could use the real normal trivialization, but the computations would be cumbersome. Instead, we use a slightly different trivialization. Since E'_d is equivariant under the action $O_{n+1}(\mathbb{R})$, we can assume that $x = [1 : 0 : \cdots : 0]$.

We have a chart $\psi_x : (z_1, \dots, z_n) \longmapsto [1 : z_1 : \cdots : z_n]$ from $\mathbb{R}^n$ to $B_{\mathbb{RP}^n}\left(x, \frac{\pi}{2}\right)$. We trivialize $\mathcal{O}(d)$ over $B_{\mathbb{RP}^n}\left(x, \frac{\pi}{2}\right)$ by identifying each fiber with $\mathcal{O}(d)_x$, by parallel transport with respect to the Chern connection ∇^d along curves of the form $t \mapsto \psi_x(tz)$ with $z \in \mathbb{R}^n$. Thanks to this trivialization, we can consider E'_d as a map taking values in $\mathbb{R}$. Recall that we defined a scaled version e_d of the Bergman kernel E_d by Eq. (3.1). The following is related without being an exact analogue. For all $w, z \in \mathbb{R}^n$, we set:

$$\xi_d(w, z) = \frac{\pi^n d!}{(d+n)!} E'_d\left(\psi_x(w), \psi_x(z)\right). \tag{6.3}$$

A computation in local coordinates yields the following lemma. The Chern connection ∇^d coincides at the origin with the usual differential in our trivialization. Hence taking the values at $(0, 0)$ of the following expressions proves that s'_d satisfies Properties 1, 2 and 3 (cf. Sect. 6.1.2).

Lemma 6.2. *Let $d \in \mathbb{N}^*$ and let $i, j \in \{1, \dots, n\}$. Then for all $w, z \in \mathbb{R}^n$ we have:*

$$\xi_d(w, z) = \left(\frac{1 + \langle w\,, z\rangle}{\sqrt{1 + \|w\|^2}\sqrt{1 + \|z\|^2}}\right)^d,$$

$$\partial_{x_i}\xi_d(w, z) = d\xi_d(w, z)\left(\frac{z_i}{1 + \langle w\,, z\rangle} - \frac{w_i}{1 + \|w\|^2}\right),$$

$$\partial_{y_j}\xi_d(w, z) = d\xi_d(w, z)\left(\frac{w_j}{1 + \langle w\,, z\rangle} - \frac{z_j}{1 + \|z\|^2}\right),$$

$$\partial_{x_i}\partial_{y_j}\xi_d(w, z) = \xi_d(w, z)\left(\frac{d\delta_{ij}}{1 + \langle w\,, z\rangle} - \frac{d^2 w_i w_j}{(1 + \langle w\,, z\rangle)(1 + \|w\|^2)} - \frac{d^2 z_i z_j}{(1 + \langle w\,, z\rangle)(1 + \|z\|^2)} + \frac{d^2 w_i z_j}{(1 + \|w\|^2)(1 + \|z\|^2)} + \frac{(d^2 - d) z_i w_j}{(1 + \langle w\,, z\rangle)^2}\right),$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

6.2 Hermite polynomials and Wiener chaos

In the setting of KSS polynomials, we consider $\mathbb{R}H^0(\mathbb{CP}^n, \mathbb{C}^r \otimes \mathcal{O}(d)) = \left(\mathbb{R}^{\mathrm{hom}}_d[X_0, \dots, X_n]\right)^r$, equipped with the inner product (6.1). For simplicty, in this section and the following, we denote by V_d this Euclidean space and by $\mathrm{d}\nu_d$ its standard Gaussian measure. With these notations, $(V_d, \mathrm{d}\nu_d)$ is our probability space and we denote by $L^1(\mathrm{d}\nu_d)$ (resp. $L^2(\mathrm{d}\nu_d)$) the space of integrable (resp. square integrable) random variables on this space. Thm. 1.6 shows that for d large enough, for all $\phi \in \mathcal{C}^0(\mathbb{RP}^n)$, $\langle |\mathrm{d}V_d|\,, \phi\rangle \in L^2(\mathrm{d}\nu_d)$. The proof given in Sect. 5 shows that this is true for any $d \geqslant \max(d_0, d_1, d_2, d_3)$, in this framework this is true for any $d \in \mathbb{N}^*$. The idea of this section is to find a nice orthogonal decomposition of $L^2(\mathrm{d}\nu_d)$. We will study $\langle |\mathrm{d}V_d|\,, \phi\rangle$ thanks to this decomposition in Sect. 6.3. These techniques were already used in a similar context in [2, 11, 12, 25], for example. See [28] for background on the following material.

Definition 6.3. For all $k \in \mathbb{N}$, we denote by H_k the k-th *Hermite polynomial*. These polynomials are defined recursively by: $H_0 = 1$, $H_1 = X$ and, for all $k \in \mathbb{N}^*$, $H_{k+1}(X) = XH_k(X) - kH_{k-1}(X)$.

Remark 6.4. Equivalently, one can define H_k by: $H_0 = 1$ and $\forall k \in \mathbb{N}$, $H_{k+1} = XH_k - H'_k$.

Lemma 6.5. *Let $k \in \mathbb{N}$, then H_k is a polynomial of degree k which is even if k is even and odd if k is odd. Moreover,*

$$H_{2k}(0) = (-1)^k \frac{(2k)!}{2^k k!} \qquad \text{and} \qquad H_{2k+1}(0) = 0.$$

Proof. This is proved by induction, using the recursive definition of H_k. □

Let us denote by $\mathrm{d}\mu_N$ the standard Gaussian measure on $\mathbb{R}^N$. We also denote by $L^2(\mathrm{d}\mu_N)$ the space of square integrable functions with respect to $\mathrm{d}\mu_N$. Recall that the family $\left(\frac{1}{\sqrt{k!}}H_k\right)_{k\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mathrm{d}\mu_1)$ (see [28, Prop. 1.4.2]). Similarly, in dimension N, the family:

$$\left\{\prod_{i=1}^{N}\frac{1}{\sqrt{\alpha_i!}}H_{\alpha_i}(X_i)\;\middle|\;\alpha\in\mathbb{N}^N\right\}$$

is a Hilbert basis of $L^2(\mathrm{d}\mu_N)$. The result in dimension 1 shows that this family is orthonormal. Then, one only needs to check that the space of polynomials in N variables is dense in $L^2(\mathrm{d}\mu_N)$. For $N=1$ this is proved in [28, Prop. 1.1.5], and the same proof works in any dimension.

As in Sect. 5, we denote by s_d a generic element of $(V_d,\mathrm{d}\nu_d)$, that we think of as a standard Gaussian vector in V_d. Let $\eta\in V_d^*$, then $\eta(s_d)\in L^2(\mathrm{d}\nu_d)$ is a real centered Gaussian variable. Moreover, for any $\eta,\eta'\in V_d^*$, we have $\mathbb{E}[\eta(s_d)\eta'(s_d)]=\langle\eta\,,\eta'\rangle$. Thus, V_d^* is canonically isometric to a subspace of $L^2(\mathrm{d}\nu_d)$, via $\eta\mapsto\eta(s_d)$. From now on, we identify V_d^* with its image, so that $V_d^*\subset L^2(\mathrm{d}\nu_d)$ is a centered Gaussian Hilbert space.

Definition 6.6. Let $(\eta_1,\dots,\eta_{N_d})$ denote an orthonormal basis of V_d^*, that is the $(\eta_i(s_d))_{i\in\{1,\dots,N_d\}}$ are independent real standard Gaussian variables. For all $q\in\mathbb{N}$, we define $C_d[q]$, the *q-th Wiener chaos* of the field s_d, as the subspace of $L^2(\mathrm{d}\nu_d)$ spanned by the orthogonal family:

$$\left\{\prod_{i=1}^{N_d}H_{\alpha_i}(\eta_i)\;\middle|\;\alpha\in\mathbb{N}^{N_d},|\alpha|=q\right\}.$$

Remarks 6.7.
- $C_d[0]$ is the space of constant random variables in $L^2(\mathrm{d}\nu_d)$ and $C_d[1]=V_d^*$.
- We do not need to take the closure in the definition of $C_d[q]$ since it is finite-dimensional.

Lemma 6.8. *The Wiener chaoses $(C_d[q])_{q\in\mathbb{N}}$ of s_d do not depend on the choice of the orthonormal basis $(\eta_1,\dots,\eta_{N_d})$ appearing in Def. 6.6.*

Proof. Let $(\eta_1,\dots,\eta_{N_d})$ and $(\eta'_1,\dots,\eta'_{N_d})$ be two orthonormal basis of V_d^*. There exists an orthogonal transformation U of V_d^* such that, for all $i\in\{1,\dots,N_d\}$, $\eta'_i=U(\eta_i)$. The situation being symmetric, we only have to prove that, for any $\beta\in\mathbb{N}^{N_d}$ such that $|\beta|=q$, $\prod_{i=1}^{N_d}H_{\beta_i}(\eta_i)$ is a linear combination of elements of the family: $\left\{\prod_{i=1}^{N_d}H_{\alpha_i}(U(\eta_i))\;\middle|\;\alpha\in\mathbb{N}^{N_d},|\alpha|=q\right\}$. Dropping the dependence on d, this amounts to proving that: if $X=(X_1,\dots,X_N)\in\mathbb{R}^N$ and $U\in O_N(\mathbb{R})$ then, for all $\beta\in\mathbb{N}^N$ such that $|\beta|=q$, $\prod_{i=1}^{N}H_{\beta_i}(X_i)$ is a linear combination of the $\left(\prod_{i=1}^{N}H_{\alpha_i}(U(X_i))\right)_{|\alpha|=q}$.

By [28, Prop. 1.4.2], we have:

$$\forall t\in\mathbb{R}^n,\qquad\sum_{\alpha\in\mathbb{N}^N}\left(\prod_{i=1}^{N}H_{\alpha_i}(X_i)\right)\frac{t^\alpha}{\alpha!}=\exp\left(\langle t\,,X\rangle-\frac{\|t\|^2}{2}\right),$$

where $\langle\cdot\,,\cdot\rangle$ is the standard inner product of $\mathbb{R}^N$ and $\|\cdot\|$ the associated norm. The right-hand side being invariant under orthogonal transformation, we have:

$$\forall t\in\mathbb{R}^n,\qquad\sum_{\alpha\in\mathbb{N}^N}\left(\prod_{i=1}^{N}H_{\alpha_i}(U(X_i))\right)\frac{(U(t))^\alpha}{\alpha!}=\sum_{\alpha\in\mathbb{N}^N}\left(\prod_{i=1}^{N}H_{\alpha_i}(X_i)\right)\frac{t^\alpha}{\alpha!}.$$

Now, the components of $U(t)$ are homogeneous polynomials of degree 1 in $(t_1,\dots,t_N)$. Hence $(U(t))^\alpha$ can only contribute terms of degree $|\alpha|$ to the sum. We conclude by identifying the coefficients of these power series of the variable t. □

Lemma 6.9. *For all $d\in\mathbb{N}^*$, $\bigoplus_{q\in\mathbb{N}}C_d[q]$ is dense in $L^2(\mathrm{d}\nu_d)$. Moreover, the terms of this direct sum are pairwise orthogonal.*

Proof. The family $\left(\prod_{i=1}^{N_d} H_{\alpha_i}(X_i)\right)_{\alpha\in\mathbb{N}^{N_d}}$ being orthogonal, the $(C_d[q])_{q\in\mathbb{N}}$ are pairwise orthogonal by definition. Let $(s_{1,d},\dots,s_{N_d,d})$ be an orthonormal basis of V_d. We have $s_d=\sum a_i s_{i,d}$, where the a_i are independent $\mathcal{N}(1)$. For any $i\in\{1,\dots,N_d\}$, let $\eta_i=\langle\cdot\,,s_{i,d}\rangle$, so that $\eta_i(s_d)=a_i$. Then, for any $q\in\mathbb{N}$, $C_d[q]$ is spanned by the random variables $\left(\prod_{i=1}^{N_d} H_{\alpha_i}(a_i)\right)_{|\alpha|=q}$.

Any square integrable functional of s_d can be written as $F(a_1,\dots,a_{N_d})$, with $F\in L^2(\mathrm{d}\mu_{N_d})$. The conclusion follows, since the span of $\left\{\prod_{i=1}^{N_d} H_{\alpha_i}(X_i)\;\middle|\;\alpha\in\mathbb{N}^{N_d}\right\}$ is dense in $L^2(\mathrm{d}\mu_{N_d})$. □

Notation 6.10. Let $d\in\mathbb{N}^*$ and let $A\in L^2(\mathrm{d}\nu_d)$. For any $q\in\mathbb{N}$, we denote by $A[q]$ the *q-th chaotic component* of A, that is its projection onto $C_d[q]$. Then, we have $A=\sum_{q\in\mathbb{N}} A[q]$ in $L^2(\mathrm{d}\nu_d)$.

By definition, $A[0]=\mathbb{E}[A]$. Moreover, the $C_d[q]$ being pairwise orthogonal, we have $\mathbb{E}[A[q]]=0$ for any $q\in\mathbb{N}^*$, and $\mathrm{Var}(A)=\sum_{q\in\mathbb{N}^*}\mathrm{Var}(A[q])$.

6.3 Wiener–Itō expansion of the linear statistics

Recall that we consider a standard Gaussian section $s_d\in V_d=\left(\mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n]\right)^r$ and that $|\mathrm{d}V_d|$ denotes the Riemannian measure of integration over its real zero set. Let us fix $d\in\mathbb{N}^*$ and $\phi\in\mathcal{C}^0(\mathbb{RP}^n)$. By Thm. 1.6, $\langle|\mathrm{d}V_d|\,,\phi\rangle\in L^2(\mathrm{d}\nu_d)$. The goal of this section is to compute the chaotic expansion of these variables. For all $q\in\mathbb{N}$, we denote $\langle|\mathrm{d}V_d|\,[q],\phi\rangle$ for $(\langle|\mathrm{d}V_d|\,,\phi\rangle)\,[q]$.

Since $\langle|\mathrm{d}V_d|\,,\phi\rangle\in L^2(\mathrm{d}\nu_d)$, for any $A\in L^2(\mathrm{d}\nu_d)$ we have $(A\,\langle|\mathrm{d}V_d|\,,\phi\rangle)\in L^1(\mathrm{d}\nu_d)$ and:

$$\mathbb{E}[A\,\langle|\mathrm{d}V_d|\,,\phi\rangle]=\mathbb{E}\left[\int_{x\in Z_d}\phi(x)A(s_d)\,|\mathrm{d}V_d|\right].$$

Even if A depends on s_d, we can apply a Kac–Rice formula (cf. [22, Thm. 5.3]). Thus, we have:

$$\mathbb{E}[A\,\langle|\mathrm{d}V_d|\,,\phi\rangle]=(2\pi)^{-\frac{r}{2}}\int_{x\in\mathbb{RP}^n}\frac{\phi(x)}{|\det^{\perp}(\mathrm{ev}_x^d)|}\mathbb{E}\Big[A\left|\det^{\perp}(\nabla_x^d s_d)\right|\,\Big|\,s_d(x)=0\Big]\,|\mathrm{d}V_{\mathbb{RP}^n}|\,,$$

Recall that s_d, is a tuple of independent KSS polynomials and that $E_d=I_r\otimes E'_d$, where E'_d is the correlation kernel of one KSS polynomial. By Eq. (6.3) and Lem. 6.2 we have:

$$\left|\det^{\perp}\left(\mathrm{ev}_x^d\right)\right|=\det(E_d(x,x))^{\frac12}=\det(E'_d(x,x))^{\frac{r}{2}}=\left(\frac{(d+n)!}{\pi^n d!}\right)^{\frac{r}{2}}.$$

Denoting $\left(d\frac{(d+n)!}{\pi^n d!}\right)^{-\frac12}\nabla_x^d s_d$ by $L_d(x)$ and $\left(\frac{(d+n)!}{\pi^n d!}\right)^{-\frac12}s_d(x)$ by $t_d(x)$, we get:

$$\mathbb{E}[A\,\langle|\mathrm{d}V_d|\,,\phi\rangle]=\left(\frac{d}{2\pi}\right)^{\frac{r}{2}}\int_{x\in\mathbb{RP}^n}\phi(x)\mathbb{E}\Big[A\left|\det^{\perp}(L_d(x))\right|\,\Big|\,t_d(x)=0\Big]\,|\mathrm{d}V_{\mathbb{RP}^n}|\,.\tag{6.4}$$

Let $x\in\mathbb{RP}^n$, without loss of generality we can assume that the coordinates on $\mathbb{R}^{n+1}$ are such that $x=[1:0:\cdots:0]$. Let $\zeta_0(x)$ be one of the two unit vectors in $\mathbb{R}\mathcal{O}(d)_x$, the other one being $-\zeta_0(x)$. This gives an isomorphism between $(\mathbb{R}\oplus T_x^*(\mathbb{RP}^n))\otimes\mathcal{O}(d)_x$ and $\mathbb{R}\oplus T_x^*(\mathbb{RP}^n)$, so that we can consider $(t_d(x),L_d(x))$ as an element of $\mathbb{R}^r\oplus(T_x^*\mathbb{RP}^n)^r$. We denote by $(t_d^{(1)}(x),\dots,t_d^{(r)}(x))$ the components of $t_d(x)$ and by $(L_d^{(1)}(x),\dots,L_d^{(r)}(x))$ those of $L_d(x)$. The couples $(t_d^{(i)}(x),L_d^{(i)}(x))$ are centered Gaussian vectors in $\mathbb{R}\oplus T_x^*\mathbb{RP}^n$ that are independent from one another. Moreover, by Lem. 6.2, for all $i\in\{1,\dots,r\}$, the variance operator of $(t_d^{(i)}(x),L_d^{(i)}(x))$ is Id.

Let us choose any orthonormal basis of $T_x^*\mathbb{RP}^n$, and denote by $(L_d^{i1}(x),\dots,L_d^{in}(x))$ the coordinates of $L_d^{(i)}(x)$ in this basis, so that $\left(L_d^{ij}(x)\right)_{\substack{1\leqslant i\leqslant r\\1\leqslant j\leqslant n}}$ is the matrix of $L_d(x)$. Then,

$$\left\{t_d^{(i)}(x)\;\middle|\;1\leqslant i\leqslant r\right\}\sqcup\left\{L_d^{ij}(x)\;\middle|\;1\leqslant i\leqslant r,1\leqslant j\leqslant n\right\}$$

is a family of independent real standard Gaussian variables in $L^2(\mathrm{d}\nu_d)$ and we can complete it into an orthonormal basis of $C_d[1]$. We denote by $\{S_d^{(i)}(x)\mid r(n+1)<i\leqslant N_d\}$, the last elements of such a basis. Below, we will work in the Hilbert basis of $L^2(\mathrm{d}\nu_d)$ obtained by considering Hermite polynomials in these variables.

Remark 6.11. We just used the fact that our random field satifies Properties 1, 2 and 3 of Sect. 6.1.2. This is what makes this computation specific to the case of KSS polynomials.

Notation 6.12. Let $\alpha \in \mathbb{N}^r$, $\beta \in \mathbb{N}^r \times \mathbb{N}^n$ and $\gamma \in \mathbb{N}^{N_d - r(n+1)}$, we will use the following notations:

$$H_\alpha(t_d(x)) = \prod_{i=1}^{r} H_{\alpha_i}\left(t_d^{(i)}(x)\right), \qquad \widetilde{H}_\beta(L_d(x)) = \prod_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n}} H_{\beta_{ij}}\left(L_d^{ij}(x)\right),$$

$$\text{and} \qquad \widehat{H}_\gamma(S_d(x)) = \prod_{i=r(n+1)+1}^{N_d} H_{\gamma_i}\left(S_d^{(i)}(x)\right).$$

We first expand $\left|\det{}^\perp(L_d(x))\right|$ in $L^2(\mathrm{d}\nu_d)$. Since $\left|\det{}^\perp(L_d(x))\right|$ only depends on the variables $\{L_d^{ij}(x) \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$, we have:

$$\left|\det{}^\perp(L_d(x))\right| = \sum_{\beta \in \mathbb{N}^r \times \mathbb{N}^n} B_\beta \frac{\widetilde{H}_\beta(L_d(x))}{\sqrt{\beta!}},$$

where $B_\beta = \frac{1}{\sqrt{\beta!}} \mathbb{E}\Big[\left|\det{}^\perp(L_d(x))\right| \widetilde{H}_\beta(L_d(x))\Big]$ for all $\beta \in \mathbb{N}^r \times \mathbb{N}^n$. The coefficient B_β only depends on the distribution of $L_d(x)$, which is a standard Gaussian for all $x \in \mathbb{RP}^n$. Hence B_β is independent of x. These coefficients have several symmetries. Note that $\left|\det{}^\perp\left(\left(L_d^{ij}(x)\right)_{i,j}\right)\right|$ is invariant under the following operations:

- multiplying a whole column or a whole row by -1;
- permuting the rows or permuting the columns.

Since the Hermite polynomials of odd degrees are odd (cf. Lem. 6.5), the first point shows that $B_\beta = 0$ whenever there exists $i \in \{1, \dots, r\}$ such that $\sum_{j=1}^n \beta_{ij}$ is odd or there exists $j \in \{1, \dots, n\}$ such that $\sum_{i=1}^r \beta_{ij}$ is odd. We denote by I the set of multi-indices $\beta \in \mathbb{N}^r \times \mathbb{N}^n$ such that for all $i \in \{1, \dots, r\}$, $\sum_{j=1}^n \beta_{ij}$ is even and, for all $j \in \{1, \dots, n\}$, $\sum_{i=1}^r \beta_{ij}$ is even.

If $|\beta| = 2$, then the only way for β to belong to I is that there exists (i, j) such that $\beta_{ij} = 2$, the other components of β being zero. The second point above shows that, in this case, the value of B_β does not depend on the index (i, j) such that $\beta_{ij} = 2$.

Notation 6.13. Let B_2 denote the common value of the B_β for $\beta \in I$ such that $|\beta| = 2$.

We can also expand $A \in L^2(\mathrm{d}\nu_d)$ as:

$$A = \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma}(x) \frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}} \frac{\widetilde{H}_\beta(L_d(x))}{\sqrt{\beta!}} \frac{\widehat{H}_\gamma(S_d(x))}{\sqrt{\gamma!}},$$

where $A_{\alpha,\beta,\gamma}(x) = \mathbb{E}\Big[A \frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}} \frac{\widetilde{H}_\beta(L_d(x))}{\sqrt{\beta!}} \frac{\widehat{H}_\gamma(S_d(x))}{\sqrt{\gamma!}}\Big]$. Then, using the orthonormality properties of the Hermite polynomials, we get:

$$\mathbb{E}\Big[A \left|\det{}^\perp(L_d(x))\right| \Big| t_d(x) = 0\Big] = \sum_{\alpha,\beta} A_{\alpha,\beta,0}(x) B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}}, \tag{6.5}$$

where the sum runs over multi-indices such that $\alpha \in 2\mathbb{N}^r$ (see Lem.6.5), and $\beta \in I$. Then, by Eq. (6.4) and (6.5), for any $A \in C_d[q]$, we have:

$$\mathbb{E}[A \langle |\mathrm{d}V_d| \, , \phi \rangle] = \left(\frac{d}{2\pi}\right)^{\frac{r}{2}} \sum_{\substack{\alpha \in 2\mathbb{N}^r, \beta \in I \\ |\alpha| + |\beta| = q}} B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}} \mathbb{E}\left[A \int_{x \in \mathbb{RP}^n} \phi(x) \frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}} \frac{\widetilde{H}_\beta(L_d(x))}{\sqrt{\beta!}} \, |\mathrm{d}V_{\mathbb{RP}^n}|\right]$$

We have proved the following proposition.

Proposition 6.14. *For all $d \in \mathbb{N}^*$, for all $\phi \in \mathcal{C}^0(\mathbb{RP}^n)$, for all $q \in \mathbb{N}$, $\langle |\mathrm{d}V_d|\,[2q+1], \phi\rangle = 0$ and*

$$\langle |\mathrm{d}V_d|\,[2q], \phi\rangle = \left(\frac{d}{2\pi}\right)^{\frac{r}{2}} \int_{x\in\mathbb{RP}^n} \phi(x) \sum_{\substack{\alpha\in 2\mathbb{N}^r,\, \beta\in I\\ |\alpha|+|\beta|=2q}} B_\beta \frac{H_\alpha(0)}{\sqrt{\alpha!}} \frac{H_\alpha(t_d(x))}{\sqrt{\alpha!}} \frac{\widetilde{H}_\beta(L_d(x))}{\sqrt{\beta!}} \,|\mathrm{d}V_{\mathbb{RP}^n}|\,. \quad (6.6)$$

Remarks 6.15.

- Recall that the values of the $t_d^{(i)}(x)$ and $L_d^{ij}(x)$ depend on the choice of the unit vector $\zeta_0(x)$, that we used to trivialize $\mathcal{O}(d)_x$. The only other choice of such a unit vector is $-\zeta_0(x)$. Changing $\zeta_0(x)$ to $-\zeta_0(x)$ changes $t_d^{(i)}(x)$ to $-t_d^{(i)}(x)$ and $L_d^{ij}(x)$ to $-L_d^{ij}(x)$. Since we only consider multi-indices (α,β) such that $|\alpha|+|\beta|$ is even, the monomials appearing with a non-zero coefficient in $H_\alpha(t_d(x))\widetilde{H}_\beta(L_d(x))$ have even total degree. Hence, the value of $H_\alpha(t_d(x))\widetilde{H}_\beta(L_d(x))$ does not depend on the choice of $\zeta_0(x)$.
- Since $\sum_{\beta\in I,\,|\beta|=p} B_\beta \frac{1}{\sqrt{\beta!}}\widetilde{H}_\beta(L_d(x))$ is the p-th chaotic component of $\left|\det{}^\perp(L_d(x))\right|$, it does not depend on our choice of an orthonormal basis of $T_x^*\mathbb{RP}^n$. Hence, neither does the value of the sum on the right-hand side of Eq. (6.6), for any given $x \in \mathbb{RP}^n$.
- By [21, Lem. A.14], we have:

$$B_0 = \mathbb{E}\left[\left|\det{}^\perp(L_x(d))\right|\right] = (2\pi)^r \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}.$$

  Then, Prop. 6.14 for $q=0$ shows that, in the setting of KSS polynomials, for all $\phi \in \mathcal{C}^0(\mathbb{RP}^n)$,

$$\mathbb{E}[\langle |\mathrm{d}V_d|\ , \phi\rangle] = d^{\frac{r}{2}} \left(\int_{\mathbb{RP}^n} \phi\,|\mathrm{d}V_{\mathbb{RP}^n}|\right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}.$$

  That is, in this case, the error term in Thm. 1.2 is zero for any $d \in \mathbb{N}^*$.

Let us conclude this section by writing $\langle |\mathrm{d}V_d|\,[2], \phi\rangle$ in a more explicit way.

Lemma 6.16. *For all $d \in \mathbb{N}^*$, for all $\phi \in \mathcal{C}^0(\mathbb{RP}^n)$,*

$$\langle |\mathrm{d}V_d|\,[2], \phi\rangle = d^{\frac{r}{2}} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{2n\,\mathrm{Vol}\,(\mathbb{S}^n)} \int_{x\in\mathbb{RP}^n} \phi(x) \left(\|L_d(x)\|^2 - n\,\|t_d(x)\|^2\right) |\mathrm{d}V_{\mathbb{RP}^n}|\,.$$

Proof. By Prop. 6.14 and Lem. 6.5, we have:

$$\langle |\mathrm{d}V_d|\,[2], \phi\rangle = \left(\frac{d}{2\pi}\right)^{\frac{r}{2}} \int_{x\in\mathbb{RP}^n} \phi(x) \left(-\frac{B_0}{2}(\|t_d(x)\|^2 - r) + \frac{B_2}{\sqrt{2}}(\|L_d(x)\|^2 - nr)\right) |\mathrm{d}V_{\mathbb{RP}^n}|\,, \quad (6.7)$$

where B_2 is defined by Ntn. 6.13. Since $H_2 = X^2 - 1$, we have:

$$n\sqrt{2}B_2 = \sum_{j=1}^n \mathbb{E}\left[\left|\det{}^\perp(L_d(x))\right| H_2(L_d^{1j}(x))\right] = \mathbb{E}\left[\left|\det{}^\perp(L_d(x))\right| \left\|L_d^{(1)}(x)\right\|^2\right] - nB_0.$$

It was proved in [21, App. B] that $\left|\det{}^\perp(L_d(x))\right|$ is distributed as $\left\|L_d^{(1)}(x)\right\| \|Z_{n-1}\| \dots \|Z_{n-r+1}\|$, where $\left(L_d^{(1)}(x), Z_{n-1}, \dots, Z_{n-r+1}\right)$ are globally independent and, for all $p \in \{n-r+1, \dots, n-1\}$, Z_p is a standard Gaussian vector in $\mathbb{R}^p$. Since $L_d^{(1)}(x) \sim \mathcal{N}(\mathrm{Id})$ in a Euclidean space of dimension n, we have:

$$B_0 = \mathbb{E}\left[\left|\det{}^\perp(L_d(x))\right|\right] = \mathbb{E}\left[\left\|L_d^{(1)}(x)\right\|\right] \prod_{p=n-r+1}^{n-1} \mathbb{E}[\|Z_p\|] = (2\pi)^{\frac{r}{2}} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)},$$

$$\text{and} \quad B_2 = \frac{1}{n\sqrt{2}} \mathbb{E}\left[\left\|L_d^{(1)}(x)\right\|^3\right] \prod_{p=n-r+1}^{n-1} \mathbb{E}[\|Z_p\|] - \frac{B_0}{\sqrt{2}} = \frac{B_0}{\sqrt{2}} \left(\frac{2\pi}{n} \frac{\mathrm{Vol}\,(\mathbb{S}^n)}{\mathrm{Vol}\,(\mathbb{S}^{n+2})} - 1\right) = \frac{B_0}{n\sqrt{2}}.$$

We plug these relations in Eq. (6.7) and this yields the result. ∎

6.4 Conclusion of the proof

In this section, we finally prove Thm. 1.8. The key point is the following.

Lemma 6.17. *Let Z_d be the common zero set of r independent Kostlan–Shub–Smale polynomials in $\mathbb{RP}^n$ then we have the following as d goes to infinity:*

$$\operatorname{Var}(\operatorname{Vol}(Z_d)[2]) \sim d^{r-\frac{n}{2}} r \left(1+\frac{2}{n}\right) \pi^{\frac{n}{2}} \frac{\operatorname{Vol}(\mathbb{S}^{n-r})^2}{16 \operatorname{Vol}(\mathbb{S}^n)}.$$

Let us prove that this lemma implies Thm. 1.8.

Proof of Thm. 1.8. Let us consider the common zero set Z_d of r independent KSS polynomials in $\mathbb{RP}^n$, and denote by $|\mathrm{d}V_d|$ the Riemannian volume measure on Z_d. Let $\mathbf{1}$ be the unit constant function on $\mathbb{RP}^n$, we have $\langle |\mathrm{d}V_d|, \mathbf{1}\rangle = \operatorname{Vol}(Z_d)$ and, by Thm. 1.6,

$$d^{-r+\frac{n}{2}} \operatorname{Var}(\operatorname{Vol}(Z_d)) = \operatorname{Vol}(\mathbb{RP}^n) \left(\frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r} \mathcal{I}_{n,r} + \delta_{rn} \frac{2}{\operatorname{Vol}(\mathbb{S}^n)} \right) + o(1).$$

On the other hand, as we explained as the end of Sect. 6.2,

$$d^{-r+\frac{n}{2}} \operatorname{Var}(\operatorname{Vol}(Z_d)) = d^{-r+\frac{n}{2}} \sum_{q\in\mathbb{N}^*} \operatorname{Var}(\operatorname{Vol}(Z_d)[q]) \geqslant d^{-r+\frac{n}{2}} \operatorname{Var}(\operatorname{Vol}(Z_d)[2]).$$

By Lem. 6.17, we get:

$$\left(\frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{(2\pi)^r} \mathcal{I}_{n,r} + \delta_{rn} \frac{2}{\operatorname{Vol}(\mathbb{S}^n)} \right) \geqslant \frac{r}{8} \left(1+\frac{2}{n}\right) \pi^{\frac{n}{2}} \left(\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^n)} \right)^2 > 0. \qquad \square$$

Remark 6.18. In [11], Dalmao proved that for $n = r = 1$, we have $\operatorname{Var}(\operatorname{Vol}(Z_d)) \sim \sigma^2 \sqrt{d}$ with $\sigma^2 \simeq 0.57\cdots$. What we just said shows that $\sigma^2 = 1 + \mathcal{I}_{1,1}$, and the lower bound we get for this term in the proof of Thm. 1.8 is $\frac{3}{8\sqrt{\pi}} \simeq 0.21\cdots$. Thus, asymptotically, chaotic components of order greater than 4 must contribute to the leading term of $\operatorname{Var}(\operatorname{Vol}(Z_d))$.

We conclude this section by the proof of Lem. 6.17.

Proof of Lem. 6.17. Recall that $|\mathrm{d}V_d|$ is the Riemannian volume measure on Z_d and that $\mathbf{1}$ is the unit constant function on $\mathbb{RP}^n$. By Lem. 6.16, we have:

$$\operatorname{Vol}(Z_d)[2] = \langle |\mathrm{d}V_d|[2], \mathbf{1}\rangle = d^{\frac{r}{2}} \frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{2n \operatorname{Vol}(\mathbb{S}^n)} \int_{x\in\mathbb{RP}^n} \left(\|L_d(x)\|^2 - n \|t_d(x)\|^2 \right) |\mathrm{d}V_{\mathbb{RP}^n}|.$$

Since this is a centered variable, its variance equals:

$$d^r \left(\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{2n \operatorname{Vol}(\mathbb{S}^n)} \right)^2 \int_{(x,y)\in(\mathbb{RP}^n)^2} \mathbb{E}\Big[\left(\|L_d(x)\|^2 - n \|t_d(x)\|^2 \right) \left(\|L_d(y)\|^2 - n \|t_d(y)\|^2 \right) \Big] |\mathrm{d}V_{\mathbb{RP}^n}|^2.$$

Using the invariance of the distribution of s_d under isometries, we get that:

$$\operatorname{Var}(\operatorname{Vol}(Z_d)[2]) = d^r \frac{\operatorname{Vol}(\mathbb{S}^{n-r})^2}{8n^2 \operatorname{Vol}(\mathbb{S}^n)} \mathcal{J}_{n,r}(d), \tag{6.8}$$

where, setting $x_0 = [1:0:\cdots:0]$,

$$\mathcal{J}_{n,r}(d) = \int_{y\in\mathbb{RP}^n} \mathbb{E}\Big[\left(\|L_d(x_0)\|^2 - n \|t_d(x_0)\|^2 \right) \left(\|L_d(y)\|^2 - n \|t_d(y)\|^2 \right) \Big] |\mathrm{d}V_{\mathbb{RP}^n}|.$$

Since $B_{\mathbb{RP}^n}\left(x_0, \frac{\pi}{2}\right) = \{[1:z_1:\cdots:z_n] \in \mathbb{RP}^n \mid z \in \mathbb{R}^n\}$ has full measure in $\mathbb{RP}^n$, we can restrict the above integral to this ball and use the local coordinates introduced in Sect. 6.1.3. These coordinates

are centered at x_0. Moreover, the density of $|\mathrm{d}V_{\mathbb{RP}^n}|$ with respect to the Lebesgue measure in this chart is $z \mapsto (1+\|z\|^2)^{-\frac{n+1}{2}}$ (cf. [16, p. 30]). By a change of variable $y = [1 : z_1 : \cdots : z_n]$, we have:

$$\mathcal{J}_{n,r}(d) = \int_{z\in\mathbb{R}^n} \mathcal{F}_d(z)(1+\|z\|^2)^{-\frac{n+1}{2}}\,\mathrm{d}z, \tag{6.9}$$

where

$$\mathcal{F}_d(z) = \mathbb{E}\Big[\Big(\|L_d(0)\|^2 - n\,\|t_d(0)\|^2\Big)\Big(\|L_d(z)\|^2 - n\,\|t_d(z)\|^2\Big)\Big].$$

Here, we denoted $t_d(z)$ instead of $t_d([1 : z_1 : \cdots : z_n])$ and $L_d(z)$ instead of $L_d([1 : z_1 : \cdots : z_n])$.

Let us fix $z \in \mathbb{R}^n$ and compute $\mathcal{F}_d(z)$. Using once again the invariance under the action of $O_{n+1}(\mathbb{R})$ on $\mathbb{RP}^n$, we can assume that $z = (\|z\|, 0, \ldots, 0)$. Let $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ denote the basis of the tangent space of $\mathbb{RP}^n$ at $[1 : \|z\| : 0 : \cdots : 0]$ given by the partial derivatives in our chart ψ_{x_0} (see Sect. 6.1.3). This basis is orthogonal, but $\left\|\frac{\partial}{\partial x_1}\right\| = (1+\|z\|^2)^{-1}$ and $\left\|\frac{\partial}{\partial x_j}\right\| = (1+\|z\|^2)^{-\frac{1}{2}}$ for all $j \in \{2, \ldots, n\}$.

The random vectors $\left(t_d^{(i)}(0), t_d^{(i)}(z), L_d^{i1}(0), L_d^{i1}(z), \ldots, L_d^{in}(0), L_d^{in}(z)\right)$ for $i \in \{1, \ldots, r\}$ are independent equidistributed centered Gaussian vector in $\mathbb{R}^{2n+2}$. The previous relations, together with Lem. 6.2, show that their common variance matrix, by blocks of size 2×2, is:

$$\begin{pmatrix} A_d(\|z\|^2) & \left(B_d(\|z\|^2)\right)^{\mathrm{t}} & 0 & \cdots & 0 \\ B_d(\|z\|^2) & D_d(\|z\|^2) & 0 & \cdots & 0 \\ 0 & 0 & C_d(\|z\|^2) & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & C_d(\|z\|^2) \end{pmatrix}, \tag{6.10}$$

where, for all $t \geqslant 0$,

$$A_d(t) = \begin{pmatrix} 1 & (1+t)^{-\frac{d}{2}} \\ (1+t)^{-\frac{d}{2}} & 1 \end{pmatrix}, \qquad B_d(t) = \begin{pmatrix} 0 & \sqrt{dt}(1+t)^{-\frac{d}{2}} \\ -\sqrt{dt}(1+t)^{-\frac{d}{2}} & 0 \end{pmatrix},$$
$$C_d(t) = \begin{pmatrix} 1 & (1+t)^{\frac{1-d}{2}} \\ (1+t)^{\frac{1-d}{2}} & 1 \end{pmatrix} \quad \text{and } D_d(t) = \begin{pmatrix} 1 & (1+t-dt)(1+t)^{-\frac{d}{2}} \\ (1+t-dt)(1+t)^{-\frac{d}{2}} & 1 \end{pmatrix}. \tag{6.11}$$

Using the independence and equidistribution of the couples $\left(t_d^{(i)}(x), L_d^{(i)}(x)\right)$, we have:

$$\mathcal{F}_d(z) = r\left(\sum_{j,l}\mathbb{E}\left[\left(L_d^{1j}(0)\right)^2\left(L_d^{1l}(z)\right)^2\right] - n\sum_{l}\mathbb{E}\left[\left(t_d^{(1)}(0)\right)^2\left(L_d^{1l}(z)\right)^2\right]\right.$$
$$\left. -n\sum_{j}\mathbb{E}\left[\left(L_d^{1j}(0)\right)^2\left(t_d^{(1)}(z)\right)^2\right] + n^2\mathbb{E}\left[\left(t_d^{(1)}(0)\right)^2\left(t_d^{(1)}(z)\right)^2\right]\right).$$

If (X, Y) is a centered Gaussian vector in $\mathbb{R}^2$ such that $\mathrm{Var}(X) = 1 = \mathrm{Var}(Y)$, then by Wick's formula (cf. [1, Lem. 11.6.1]) we have: $\mathbb{E}\big[X^2Y^2\big] = 9 + 2\mathbb{E}[XY]^2$. We apply this relation to each term of the previous sum. Then, by Eq. (6.10) and Eq. (6.11), we have $\mathcal{F}_d(z) = 2rF_d(d\,\|z\|^2)$, where F_d is defined by:

$$\forall t \in \mathbb{R}, \qquad F_d(t) = \left(1+\frac{t}{d}\right)^{-d}\left(\left(1+\frac{t}{d}-t\right)^2 + (n-1)\left(1+\frac{t}{d}\right) - 2nt + n^2\right).$$

Then, by a change of variable $t = d\,\|z\|^2$ in Eq. (6.9),

$$\mathcal{J}_{n,r}(d) = d^{-\frac{n}{2}} r\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\int_{t=0}^{+\infty} F_d(t)t^{\frac{n-2}{2}}\left(1+\frac{t}{d}\right)^{-\frac{n+1}{2}}\mathrm{d}t. \tag{6.12}$$

Let $t \geqslant 0$, we have:

$$F_d(t)t^{\frac{n-2}{2}}\left(1+\frac{t}{d}\right)^{-\frac{n+1}{2}} \xrightarrow[d\to+\infty]{} \left(t^2-2t(n+1)+n(n+1)\right)t^{\frac{n-2}{2}}e^{-t}.$$

Moreover, for all $d \in \mathbb{N}^*$,

$$\left|F_d(t)t^{\frac{n-2}{2}}\left(1+\frac{t}{d}\right)^{-\frac{n+1}{2}}\right| \leqslant \left(1+\frac{t}{d}\right)^{-d} t^{\frac{n-2}{2}}\left(4t^2+(n+1)(3t+n)\right).$$

Let $d_0 > \frac{n}{2}+2$. Since $\left(1+\frac{t}{d}\right)^{-d}$ is a non-increasing sequence of d, for all $d \geqslant d_0$,

$$\left|F_d(t)t^{\frac{n-2}{2}}\left(1+\frac{t}{d}\right)^{-\frac{n+1}{2}}\right| \leqslant \left(1+\frac{t}{d_0}\right)^{-d_0} t^{\frac{n-2}{2}}\left(4t^2+(n+1)(3t+n)\right),$$

and the right-hand side is integrable as a function of t. By Lebesgue's Theorem, we have:

$$\int_{t=0}^{+\infty}\frac{F_d(t)t^{\frac{n-2}{2}}}{\left(1+\frac{t}{d}\right)^{\frac{n+1}{2}}}\,\mathrm{d}t \xrightarrow[d\to+\infty]{} \int_0^{+\infty}\left(t^2-2t(n+1)+n(n+1)\right)t^{\frac{n-2}{2}}e^{-t}\,\mathrm{d}t = \Gamma\left(\frac{n}{2}+2\right), \quad (6.13)$$

where Γ is Euler's Gamma function. The conclusion follows from Eq. (6.8), (6.12) and (6.13). □

A Technical computations for Section 4

Before proving the technical lemmas of Sect. 4, we state several estimates that will be useful in this section and the next. Recall Def. 4.4, 4.5 and 4.11. The following hold as t goes to infinity.

$$a(t) \xrightarrow[t\to+\infty]{} -1, \qquad b_+(t) \xrightarrow[t\to+\infty]{} 0 \qquad \text{and} \qquad b_-(t) \xrightarrow[t\to+\infty]{} \sqrt{2}. \quad (A.1)$$

$$\forall i \in \{1,2\},\ u_i(t) \xrightarrow[t\to+\infty]{} 1 \qquad \text{and} \qquad \forall i \in \{1,2,3,4\},\ v_i(t) \xrightarrow[t\to+\infty]{} 1. \quad (A.2)$$

The following hold as t goes to 0.

$$a(t) = 1-\frac{t}{2}-\frac{t^2}{8}+O(t^3), \qquad (b_+(t))^2 = 2-\frac{t}{2}+O(t^2), \quad (A.3)$$

$$b_+(t)b_-(t) = \sqrt{t}\left(1+O(t^2)\right), \qquad (b_-(t))^2 = \frac{t}{2}+\frac{t^2}{8}+O(t^3), \quad (A.4)$$

$$u_1(t) = t+O(t^2), \qquad u_2(t) = \frac{t^2}{12}+O(t^3), \quad (A.5)$$

$$v_1(t) = 2+O(t), \qquad v_2(t) = \frac{t^3}{48}+O(t^4), \quad (A.6)$$

$$v_3(t) = t+O(t^2), \qquad v_4(t) = 2+O(t). \quad (A.7)$$

Proof of Lem. 4.6. Recall that P is defined by Def. 4.5 and $\widetilde{\Omega}$ by Eq. (4.3). One can check by a direct computation that, for any $t \in [0,+\infty)$, $P(t) = (A(t)\otimes I_2)\sigma(Q\otimes I_2)$, where

$$A(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} b_-(t) & -b_+(t) \\ b_+(t) & b_-(t)\end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \qquad \text{and} \qquad Q = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}.$$

Moreover, these three matrices are orthogonal. Then we have:

$$\begin{aligned}\sigma(Q\otimes I_2)\widetilde{\Omega}(t)(Q^{\mathrm{t}}\otimes I_2)\sigma^{\mathrm{t}} &= \sigma\begin{pmatrix} 1-e^{-\frac{1}{2}t} & 0 & 0 & -\sqrt{t}e^{-\frac{1}{2}t} \\ 0 & 1+e^{-\frac{1}{2}t} & \sqrt{t}e^{-\frac{1}{2}t} & 0 \\ 0 & \sqrt{t}e^{-\frac{1}{2}t} & 1-(1-t)e^{-\frac{1}{2}t} & 0 \\ -\sqrt{t}e^{-\frac{1}{2}t} & 0 & 0 & 1+(1-t)e^{-\frac{1}{2}t}\end{pmatrix}\sigma^{\mathrm{t}} \\ &= \begin{pmatrix} 1-e^{-\frac{1}{2}t} & -\sqrt{t}e^{-\frac{1}{2}t} & 0 & 0 \\ -\sqrt{t}e^{-\frac{1}{2}t} & 1+(1-t)e^{-\frac{1}{2}t} & 0 & 0 \\ 0 & 0 & 1+e^{-\frac{1}{2}t} & \sqrt{t}e^{-\frac{1}{2}t} \\ 0 & 0 & \sqrt{t}e^{-\frac{1}{2}t} & 1-(1-t)e^{-\frac{1}{2}t}\end{pmatrix} \\ &= I_4 + e^{-\frac{t}{2}}\left(\begin{pmatrix}1 & \sqrt{t} \\ \sqrt{t} & t-1\end{pmatrix}\otimes\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}\right),\end{aligned}$$

where I_4 stands for identity matrix of size 4. Recalling the definitions of $(v_i(t))_{1\leqslant i\leqslant 4}$, $b_+(t)$ and $b_-(t)$ (see Def. 4.4 and 4.5). We conclude the proof by checking that:

$$A(t)\begin{pmatrix}1 & \sqrt{t} \\ \sqrt{t} & t-1\end{pmatrix}(A(t))^{\mathrm{t}} = \begin{pmatrix}\frac{t}{2}-\sqrt{1+\left(\frac{t}{2}\right)^2} & 0 \\ 0 & \frac{t}{2}+\sqrt{1+\left(\frac{t}{2}\right)^2}\end{pmatrix}. \qquad \square$$

Proof of Lem. 4.8. Let $z\in\mathbb{R}^n\setminus\{0\}$, by Eq. (4.2) and Lem. 4.3, we have:

$$\det\left(\Omega(z)\right) = \det\left(\Omega'(z)\right)^r = \det\left(\widetilde{\Omega}(\|z\|^2)\right)^r\left(1-e^{-\|z\|^2}\right)^{r(n-1)}$$

and it is enough to prove that $\det\left(\widetilde{\Omega}(t)\right)>0$ whenever $t>0$. By Lem. 4.6, we have:

$$\forall t\geqslant 0,\qquad \det\left(\widetilde{\Omega}(t)\right) = v_1(t)v_2(t)v_3(t)v_4(t) = 1-(t^2+2)e^{-t}+e^{-2t} = f(t), \tag{A.8}$$

where the last equality defines $f:[0,+\infty)\to\mathbb{R}$. We have $f(0)=0$ and for all $t>0$, $f'(t)=e^{-t}g(t)$ where $g(t)=t^2-2t+2-2e^{-t}$. Then $g(0)=0$ and $\forall t>0$, $g'(t)=2(e^{-t}-1+t)>0$. Thus g is positive on $(0,+\infty)$ and so is f. Finally, we have $\forall t>0$, $\det\left(\widetilde{\Omega}(t)\right)>0$. $\square$

Proof of Lem. 4.13. Let $z\in\mathbb{R}^n\setminus\{0\}$, as above we have:

$$\det\left(\Lambda(z)\right) = \det\left(\Lambda'(z)\right)^r = \det\left(\widetilde{\Lambda}(\|z\|^2)\right)^r\left(1-e^{-\|z\|^2}\right)^{r(n-1)}$$

and it is enough to prove that $\det\left(\widetilde{\Lambda}(t)\right)>0$ whenever $t>0$. By Lem. 4.12, we have:

$$\forall t>0,\qquad \det\left(\widetilde{\Lambda}(t)\right) = u_1(t)u_2(t) = \frac{1-(t^2+2)e^{-t}+e^{-2t}}{1-e^{-t}} = \frac{\det\left(\widetilde{\Omega}(t)\right)}{1-e^{-t}},$$

by Eq. (A.8). We just proved that $\det\left(\widetilde{\Omega}(t)\right)$ is positive for every positive t. Hence the result. $\square$

Proof of Lem. 4.14. First, recall that $\Omega(z)=\Omega'(z)\otimes I_r$ (see Eq. (4.2)) and $\Lambda(z)=\Lambda'(z)\otimes I_r$ (see Eq. (4.5)). Hence, we only need to prove that the map $z\longmapsto\begin{pmatrix}0 & \Lambda'(z)^{\frac{1}{2}}\end{pmatrix}\Omega'(z)^{-\frac{1}{2}}$ is bounded on $\mathbb{R}^n\setminus\{0\}$. Then, let $z\in\mathbb{R}^n\setminus\{0\}$, the matrix of $\Omega'(z)$ in the orthonormal basis $\mathcal{B}_z$ of $\mathbb{R}^2\otimes(\mathbb{R}\oplus\mathbb{R}^n)$ (see Sect. 4.2) is given by Lem. 4.3, and the matrix of $\Omega'(z)^{-\frac{1}{2}}$ in $\mathcal{B}_z$ is:

$$\left(\begin{array}{c|c} \widetilde{\Omega}(\|z\|^2)^{-\frac{1}{2}} & 0 \\ \hline 0 & \begin{pmatrix}1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1\end{pmatrix}^{-\frac{1}{2}}\otimes I_{n-1}\end{array}\right).$$

Similarly, by Lem. 4.9, the matrix of $\begin{pmatrix} 0 & \Lambda'(z)^{\frac{1}{2}} \end{pmatrix}$ in $\mathcal{B}_z$ is:

$$\left(\begin{array}{cc|c} 0 & \widetilde{\Lambda}(\|z\|^2)^{\frac{1}{2}} & 0 \\ \hline 0 & & \begin{pmatrix} 1 & e^{-\frac{1}{2}\|z\|^2} \\ e^{-\frac{1}{2}\|z\|^2} & 1 \end{pmatrix}^{\frac{1}{2}} \otimes I_{n-1} \end{array}\right).$$

Hence, our problem reduces to proving that: $t \longmapsto \begin{pmatrix} 0 & \widetilde{\Lambda}(t)^{\frac{1}{2}} \end{pmatrix} \widetilde{\Omega}(t)^{-\frac{1}{2}}$ is bounded on $(0, +\infty)$.

Recall that, for all $t \in [0, +\infty)$, $P(t) \in O_4(\mathbb{R})$ was defined by Def. 4.5. By Lem. 4.6 and 4.12, for all $t \in (0, +\infty)$ we have:

$$\begin{aligned} &\begin{pmatrix} 0 & \widetilde{\Lambda}(t)^{\frac{1}{2}} \end{pmatrix} \widetilde{\Omega}(t)^{-\frac{1}{2}} \\ &\qquad = \left(\begin{array}{c|c} 0 & Q^{\mathrm{t}} \begin{pmatrix} u_1(t)^{\frac{1}{2}} & 0 \\ 0 & u_2(t)^{\frac{1}{2}} \end{pmatrix} Q \end{array} \right) P(t)^{\mathrm{t}} \begin{pmatrix} v_1(t)^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & v_2(t)^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & v_3(t)^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & v_4(t)^{-\frac{1}{2}} \end{pmatrix} P(t) \\ &\qquad = \begin{pmatrix} m_1(t) & m_3(t) & m_5(t) & m_6(t) \\ m_2(t) & m_4(t) & m_6(t) & m_5(t) \end{pmatrix}, \end{aligned}$$

where:

$$m_1 = \frac{b_+ b_-}{4}\left(-\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_3}} + \sqrt{\frac{u_1}{v_4}}\right), \quad m_2 = \frac{b_+ b_-}{4}\left(-\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_3}} - \sqrt{\frac{u_1}{v_4}}\right),$$
$$m_3 = \frac{b_+ b_-}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_3}} + \sqrt{\frac{u_1}{v_4}}\right), \quad m_4 = \frac{b_+ b_-}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_3}} - \sqrt{\frac{u_1}{v_4}}\right),$$

$$m_5 = \frac{(b_+)^2}{4}\left(\sqrt{\frac{u_2}{v_1}} + \sqrt{\frac{u_1}{v_3}}\right) + \frac{(b_-)^2}{4}\left(\sqrt{\frac{u_2}{v_2}} + \sqrt{\frac{u_1}{v_4}}\right),$$
$$m_6 = \frac{(b_+)^2}{4}\left(\sqrt{\frac{u_2}{v_1}} - \sqrt{\frac{u_1}{v_3}}\right) + \frac{(b_-)^2}{4}\left(\sqrt{\frac{u_2}{v_2}} - \sqrt{\frac{u_1}{v_4}}\right).$$

By Lem. 4.8, for all $t > 0$, the $(v_i(t))_{1 \leqslant i \leqslant 4}$ are the eigenvalues of a symmetric positive operator, hence are positive. Similarly for all $t > 0$, $u_1(t) > 0$ and $u_2(t) > 0$ by Lem. 4.13. Thus the $(m_i)_{1 \leqslant i \leqslant 6}$ are well-defined continuous maps from $(0, +\infty)$ to $\mathbb{R}$. By Eq. (A.1) and (A.2),

$$\forall i \in \{1, 2, 3, 4, 6\}, \quad m_i(t) \xrightarrow[t \to +\infty]{} 0 \qquad \text{and} \qquad m_5(t) \xrightarrow[t \to +\infty]{} 1.$$

Moreover, by Eq. (A.3)–(A.7), for all $i \in \{1, 2, 5, 6\}$, $m_i(t) = 1/2 + O(\sqrt{t})$ as to goes to 0 and, for any $i \in \{3, 4\}$, $m_i(t) = -1/2 + O(\sqrt{t})$ as to goes to 0. Hence for all $i \in \{1, \ldots, 6\}$, m_i is a bounded function from $(0, +\infty)$ to $\mathbb{R}$, which concludes the proof. □

Proof of Lem. 4.17. Recall that, for all $t > 0$, the couples $(X_{ij}(t), Y_{ij}(t))$ are independent centered Gaussian vectors in $\mathbb{R}^2$. We denote by $\Lambda_{ij}(t)$ the variance matrix of $(X_{ij}(t), Y_{ij}(t))$, which equals $\widetilde{\Lambda}(t)$ if $j = 1$ and

$$\begin{pmatrix} 1 & e^{-\frac{1}{2}t^2} \\ e^{-\frac{1}{2}t^2} & 1 \end{pmatrix}$$

otherwise (see Def. 1.5, Lem. 4.16 and Lem. 4.9).

For all $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, n\}$ and $t > 0$, we can write:

$$\begin{pmatrix} X_{ij}(t) \\ Y_{ij}(t) \end{pmatrix} = \sqrt{\Lambda_{ij}(t)} \begin{pmatrix} A_{ij} \\ B_{ij} \end{pmatrix},$$

where the (A_{ij}) and (B_{ij}) are globally independent real standard Gaussian variables, not depending on t. Note that by Lem. 4.13, the $\Lambda_{ij}(t)$ are positive for any $t > 0$. We deduce from Lem. 4.12 that for any $i \in \{1, \dots, r\}$, for all $t > 0$:

$$\sqrt{\Lambda_{i1}(t)} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \alpha(t) \end{pmatrix} \qquad \text{and} \quad \forall j \in \{2, \dots, n\}, \qquad \sqrt{\Lambda_{ij}(t)} = \begin{pmatrix} \gamma(t) & \delta(t) \\ \delta(t) & \gamma(t) \end{pmatrix},$$

where

$$\alpha(t) = \frac{1}{2}\left(\sqrt{u_2(t)} + \sqrt{u_1(t)}\right), \qquad \gamma(t) = \frac{1}{2}\left(\sqrt{1 + e^{-\frac{1}{2}t^2}} + \sqrt{1 - e^{-\frac{1}{2}t^2}}\right), \tag{A.9}$$

$$\beta(t) = \frac{1}{2}\left(\sqrt{u_2(t)} - \sqrt{u_1(t)}\right), \qquad \delta(t) = \frac{1}{2}\left(\sqrt{1 + e^{-\frac{1}{2}t^2}} - \sqrt{1 - e^{-\frac{1}{2}t^2}}\right). \tag{A.10}$$

We denote $A_j = (A_{1j}, \dots, A_{rj})^{\mathrm{t}}$ the j-th column of A and similarly $B_j = (B_{1j}, \dots, B_{rj})^{\mathrm{t}}$. Then, $\mathbb{E}\big[\big|\det^{\perp}(X(t))\big|\,\big|\det^{\perp}(Y(t))\big|\big] = \mathbb{E}[\Psi(t, A, B)]$, where

$$\Psi(t, A, B) = \big|\det^{\perp}(\alpha(t)A_1 + \beta(t)B_1, \gamma(t)A_2 + \delta(t)B_2, \dots, \gamma(t)A_n + \delta(t)B_n)\big| \\ \big|\det^{\perp}(\beta(t)A_1 + \alpha(t)B_1, \delta(t)A_2 + \gamma(t)B_2, \dots, \delta(t)A_n + \gamma(t)B_n)\big|. \tag{A.11}$$

By (A.5), $\alpha(t) = \frac{1}{2}\sqrt{t} + O(t)$ and $\beta(t) = -\frac{1}{2}\sqrt{t} + O(t)$. We extend continuously α, β, γ and δ by $\alpha(0) = 0 = \beta(0)$ and $\gamma(0) = \frac{1}{\sqrt{2}} = \delta(0)$. The function Ψ also extend continuously at $t = 0$.

Then α, β, γ and δ are bounded functions on $(0, 1]$ and Ψ is the square root of a polynomial of degree $4r$ in (A, B) whose coefficients are bounded functions of t. In particular, for all $t \in (0, 1]$, $\Psi(t, A, B)$ is dominated by a polynomial in (A, B) whose coefficients are independent of t. By Lebesgue's Theorem,

$$\mathbb{E}\big[\big|\det^{\perp}(X(t))\big|\,\big|\det^{\perp}(Y(t))\big|\big] \xrightarrow[t\to 0]{} \mathbb{E}[\Psi(0, A, B)]\,. \tag{A.12}$$

Let $j \in \{2, \dots, n\}$, we define $X_j = (X_{1j}, \dots, X_{rj})^{\mathrm{t}}$ by $X_j = \gamma(0)A_j + \delta(0)B_j = \frac{1}{\sqrt{2}}(A_j + B_j)$. Then the (X_{ij}) with $i \in \{1, \dots, r\}$ and $j \in \{2, \dots, n\}$ are independent real standard Gaussian variables. Setting $X_1 = (X_{11}, \dots, X_{r1})^{\mathrm{t}} = 0$, we have:

$$\Psi(0, A, B) = \big|\det^{\perp}(X_1, X_2, \dots, X_n)\big|^2 = \det\big((X_1, X_2, \dots, X_n)(X_1, X_2, \dots, X_n)^{\mathrm{t}}\big) \\ = \sum_{1 \leqslant k_1 < \cdots < k_r \leqslant n} \det\big((X_{ik_j})_{1 \leqslant i,j \leqslant r}\big)^2,$$

by the Cauchy–Binet formula. Let $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n$. If $k_1 = 1$, the first column of $(X_{ik_j})_{1 \leqslant i,j \leqslant r}$ is zero and its determinant equals 0. Otherwise,

$$\mathbb{E}\Big[\det\big((X_{ik_j})_{1 \leqslant i,j \leqslant r}\big)^2\Big] = \sum_{\sigma,\tau \in \mathfrak{S}_r} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^{r} \mathbb{E}\big[X_{ik_{\sigma(i)}} X_{ik_{\tau(i)}}\big] = r!. \tag{A.13}$$

Hence, if $r < n$, we have

$$\mathbb{E}[\Psi(0, A, B)] = r!\binom{n-1}{r} = \frac{(n-1)!}{(n-r-1)!},$$

and by Eq. (A.12), we proved Lem. 4.17 in this case. If $r = n$, we have $\mathbb{E}[\Psi(0, A, B)] = 0$ and we must be more precise.

Let us now assume that $r = n$. Then, X and Y are square matrices and their Jacobians are simply the absolute values of their determinants. For all $t > 0$, we have:

$$\Psi(t, A, B) = \frac{t}{2}\left|\det\left(\sqrt{\frac{2}{t}}\alpha(t)A_1 + \sqrt{\frac{2}{t}}\beta(t)B_1, \gamma(t)A_2 + \delta(t)B_2, \dots, \gamma(t)A_n + \delta(t)B_n\right)\right| \\ \left|\det\left(\sqrt{\frac{2}{t}}\beta(t)A_1 + \sqrt{\frac{2}{t}}\alpha(t)B_1, \delta(t)A_2 + \gamma(t)B_2, \dots, \delta(t)A_n + \gamma(t)B_n\right)\right|. \tag{A.14}$$

By Eq. (A.5), we have: $\sqrt{\frac{2}{t}}\alpha(t) = \frac{1}{\sqrt{2}} + O(\sqrt{t})$ and $\sqrt{\frac{2}{t}}\beta(t) = -\frac{1}{\sqrt{2}} + O(\sqrt{t})$. We can apply the same kind of argument as above. By Lebesgue's Theorem:

$$\frac{2}{t}\mathbb{E}[\Psi(t,A,B)] \xrightarrow[t\to 0]{} \mathbb{E}[|\det(Y_1,X_2,\dots,X_n)|\,|\det(-Y_1,X_2,\dots,X_n)|] = \mathbb{E}\big[\det(Y_1,X_2,\dots,X_n)^2\big]\,,$$

where $Y_1 = (Y_{11},\dots,Y_{r1})^{\mathrm{t}} = \frac{1}{\sqrt{2}}(A_1 - B_1)$. Since $Y_1, X_2, \dots, X_n$ are independent $\mathcal{N}(\mathrm{Id})$ in $\mathbb{R}^r$, the same computation as Eq. (A.13) shows that: $\mathbb{E}\big[\det(Y_1,X_2,\dots,X_n)^2\big] = r! = n!$. Hence, if $r = n$, we have: $\mathbb{E}\big[\big|\det{}^{\perp}(X(t))\big|\,\big|\det{}^{\perp}(Y(t))\big|\big] = \mathbb{E}[\Psi(t,A,B)] \sim \frac{n!}{2}t$, as $t \to 0$. □

Proof of Lem. 4.18. For any $t > 0$, let us denote by:

$$\widehat{\Lambda}(t) = \left(\begin{array}{c|c} \widetilde{\Lambda}(t) & 0 \\ \hline 0 & \begin{pmatrix} 1 & e^{-\frac{1}{2}t^2} \\ e^{-\frac{1}{2}t^2} & 1 \end{pmatrix} \otimes I_{n-1} \end{array}\right) \otimes I_r, \tag{A.15}$$

the variance matrix of $(X(t), Y(t))$.

In the following, we denote by $L = (X, Y)$ a generic element of $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$. We have:

$$\mathbb{E}\big[\big|\det{}^{\perp}(X(t))\big|\,\big|\det{}^{\perp}(Y(t))\big|\big] = \frac{1}{(2\pi)^{rn}}\det\left(\widehat{\Lambda}(t)\right)^{-\frac{1}{2}}\int |\det{}^{\perp}(X)|\,|\det{}^{\perp}(Y)|\exp\left(-\frac{1}{2}\left\langle\widehat{\Lambda}(t)^{-1}L\,,L\right\rangle\right)\mathrm{d}L. \tag{A.16}$$

By Lem. 4.12, we have $\widehat{\Lambda}(t) = \mathrm{Id} + O\Big(te^{-\frac{t}{2}}\Big)$ as $t \to +\infty$. Then, $\det\left(\widehat{\Lambda}(t)\right)^{-\frac{1}{2}} = 1 + O\Big(te^{-\frac{t}{2}}\Big)$. Moreover, by the Mean Value Theorem,

$$\begin{aligned}\left|\exp\left(-\frac{1}{2}\left\langle\widehat{\Lambda}(t)^{-1}L\,,L\right\rangle\right) - e^{-\frac{1}{2}\|L\|^2}\right| &= e^{-\frac{1}{2}\|L\|^2}\left|\exp\left(-\frac{1}{2}\left\langle\left(\widehat{\Lambda}(t)^{-1} - \mathrm{Id}\right)L\,,L\right\rangle\right) - 1\right| \\ &\leqslant e^{-\frac{1}{2}\|L\|^2}\frac{\|L\|^2}{2}\left\|\widehat{\Lambda}(t)^{-1} - \mathrm{Id}\right\|\exp\left(\frac{\|L\|^2}{2}\left\|\widehat{\Lambda}(t)^{-1} - \mathrm{Id}\right\|\right).\end{aligned}$$

Then, since $\widehat{\Lambda}(t)^{-1} = \mathrm{Id} + O\Big(te^{-\frac{t}{2}}\Big)$, this last term is smaller than $e^{-\frac{1}{4}\|L\|^2}\frac{\|L\|^2}{2}\left\|\widehat{\Lambda}(t)^{-1} - \mathrm{Id}\right\|$ for all t large enough. Hence,

$$\begin{aligned}&\int |\det{}^{\perp}(X)|\,|\det{}^{\perp}(Y)|\left|\exp\left(-\frac{1}{2}\left\langle\widehat{\Lambda}(t)^{-1}L\,,L\right\rangle\right) - e^{-\frac{1}{2}\|L\|^2}\right|\mathrm{d}L \\ &\qquad\leqslant \frac{1}{2}\left\|\widehat{\Lambda}(t)^{-1} - \mathrm{Id}\right\|\int |\det{}^{\perp}(X)|\,|\det{}^{\perp}(Y)|\,\|L\|^2\,e^{-\frac{1}{4}\|L\|^2}\,\mathrm{d}L = O\Big(te^{-\frac{t}{2}}\Big).\end{aligned}$$

Thanks to this relation and Eq. (A.16), we get that:

$$\mathbb{E}\big[\big|\det{}^{\perp}(X(t))\big|\,\big|\det{}^{\perp}(Y(t))\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}(X(\infty))\big|\,\big|\det{}^{\perp}(Y(\infty))\big|\big] + O\Big(te^{-\frac{t}{2}}\Big)\,,$$

where $(X(\infty), Y(\infty)) \sim \mathcal{N}(\mathrm{Id})$ in $\mathcal{M}_{rn}(\mathbb{R}) \times \mathcal{M}_{rn}(\mathbb{R})$. Finally, by [21, Lem. A.14],

$$\mathbb{E}\big[\big|\det{}^{\perp}(X(\infty))\big|\,\big|\det{}^{\perp}(Y(\infty))\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}(X(\infty))\big|\big]^2 = (2\pi)^r\left(\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}\right)^2.$$ □

B Technical computations for Section 5

Proof of Lem. 5.26. Let $\alpha \in (0,1)$, we want to prove that $\Theta(z)^{-\frac{1}{2}}\Theta_d(z)\Theta(z)^{-\frac{1}{2}} - \mathrm{Id} = O(d^{-\alpha})$ uniformly for $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d)$. Recall that $Q = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Since $Q \in O_2(\mathbb{R})$, it is equivalent to prove that:

$$\Big(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\Big)\,\Theta(z)^{-\frac{1}{2}}\left(\Theta_d(z) - \Theta(z)\right)\Theta(z)^{-\frac{1}{2}}\Big(Q \otimes \mathrm{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\Big)^{-1} = O(d^{-\alpha}). \tag{B.1}$$

Recall that e_d was defined by Eq. (3.1) and that $e_\infty = \xi \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$ (see Sect. 4). For any $d \in \mathbb{N}$, for all $x \in M$ and for all $w, z \in B_{T_xM}(0, b_n \ln d)$ we set: $\varepsilon_d(w,z) = e_d(w,z) - e_\infty(w,z)$. By Eq. (4.1) and (5.7) we have:

$$\Theta_d(z) - \Theta(z) = \begin{pmatrix} \varepsilon_d(0,0) & \varepsilon_d(0,z) \\ \varepsilon_d(z,0) & \varepsilon_d(z,z) \end{pmatrix}.$$

Then, by Lem. 4.1, for all $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$ we have:

$$\left(Q \otimes \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right) \Theta(z)^{-\frac{1}{2}} \left(\Theta_d(z) - \Theta(z)\right) \Theta(z)^{-\frac{1}{2}} \left(Q \otimes \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}\right)^{-1} = \begin{pmatrix} a_d(z) & b_d(z)^* \\ b_d(z) & c_d(z) \end{pmatrix},$$

where

$$\begin{aligned} a_d(z) &= \frac{1}{2}\left(1 - e^{-\frac{1}{2}\|z\|^2}\right)^{-1} (\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)), \\ b_d(z) &= -\frac{1}{2}\left(1 - e^{-\|z\|^2}\right)^{-1/2} (\varepsilon_d(z,z) - \varepsilon_d(z,0) + \varepsilon_d(0,z) - \varepsilon_d(0,0)), \\ \text{and} \qquad c_d(z) &= \frac{1}{2}\left(1 + e^{-\frac{1}{2}\|z\|^2}\right)^{-1} (\varepsilon_d(z,z) + \varepsilon_d(z,0) + \varepsilon_d(0,z) + \varepsilon_d(0,0)). \end{aligned}$$

Let $\beta \in (\alpha, 1)$, by Prop. 3.4 we have $\left\| D^2_{(w,z)} \varepsilon_d \right\| \leqslant C d^{-\beta}$, where C is independent of $x \in M$ and $w, z \in B_{T_xM}(0, b_n \ln d)$. Then, a second order Taylor expansion around $(0,0)$ gives:

$$\|\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0)\| \leqslant C \|z\|^2 d^{-\beta}.$$

Since we consider $z \in B_{T_xM}(0, b_n \ln d)$ and $1 - e^{-\frac{1}{2}\|z\|^2} \sim \frac{\|z\|^2}{2}$ as $z \to 0$, we have:

$$\|a_d(z)\| \leqslant \frac{C \|z\|^2 d^{-\beta}}{2\left(1 - e^{-\frac{1}{2}\|z\|^2}\right)} = O\left((\ln d)^2 d^{-\beta}\right) = O(d^{-\alpha}),$$

where the error term does not depend on (x, z). We obtain Eq. (B.1) by reasonning similarly for $b_d(z)$ and $c_d(z)$. □

Proof of Lem. 5.28. The idea of the proof is the same as that of Lem. 5.26 above. Let $\alpha \in (0,1)$, we want to prove that:

$$\Omega(z)^{-\frac{1}{2}} \left(\Omega_d(z) - \Omega(z)\right) \Omega(z)^{-\frac{1}{2}} = O\big(d^{-\alpha}\big). \tag{B.2}$$

Recall that we defined: $\varepsilon_d(w,z) = e_d(w,z) - e_\infty(w,z)$ for any $x \in M$ and $w, z \in B_{T_xM}(0, b_n \ln d)$. We can express $\Omega_d(z) - \Omega(z)$ in terms of ε_d and its derivatives. Then we write the matrix of the left-hand side of Eq. (B.2) in an orthonormal basis that diagonalizes $\Omega(z)$. The coefficients of this matrix are linear combinations of ε_d and its derivatives. We will prove that they are $O(d^{-\alpha})$ using Taylor expansions and the estimates of Sect. 3.3.

The details are longer than in the proof of Lem. 5.26 for two reasons. First, the basis in which $\Omega(z)$ is diagonal now depends on z. Second, some of the eigenvalues of $\Omega(z)$ are $O(\|z\|^6)$ as $z \to 0$, so that we need to consider Taylor expansions of order 6 for some coefficients. In addition, the matrices involved are less easily described than in the proof of Lem. 5.26.

Recall that e_d was defined by Eq. (3.1) and that $e_\infty = \xi \operatorname{Id}_{\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x}$ (see Sect. 4). We expressed $\Omega(z)$ in terms of e_∞ in Eq. (4.2) and $\Omega_d(z)$ in terms of e_d in Eq. (5.8). As an operator on:

$$\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x \oplus \left(T^*_xM \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right) \oplus \left(T^*_xM \otimes \mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right),$$

we have:

$$\Omega_d(z) - \Omega(z) = \left(\begin{array}{cc|cc} \varepsilon_d(0,0) & \varepsilon_d(0,z) & \partial^\sharp_y \varepsilon_d(0,0) & \partial^\sharp_y \varepsilon_d(0,z) \\ \varepsilon_d(z,0) & \varepsilon_d(z,z) & \partial^\sharp_y \varepsilon_d(z,0) & \partial^\sharp_y \varepsilon_d(z,z) \\ \hline \partial_x \varepsilon_d(0,0) & \partial_x \varepsilon_d(0,z) & \partial_x \partial^\sharp_y \varepsilon_d(0,0) & \partial_x \partial^\sharp_y \varepsilon_d(0,z) \\ \partial_x \varepsilon_d(z,0) & \partial_x \varepsilon_d(z,z) & \partial_x \partial^\sharp_y \varepsilon_d(z,0) & \partial_x \partial^\sharp_y \varepsilon_d(z,z) \end{array}\right).$$

Let us choose an orthonormal basis $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$ of T_xM such that $z=\|z\|\frac{\partial}{\partial x_1}$. We denote by $(dx_1,\dots,dx_n)$ its dual basis. We can then define a basis of $\mathbb{R}^2\otimes(\mathbb{R}\oplus T_x^*M)$ similar to $\mathcal{B}_z$ (see Sect. 4.2). For any $i\in\{1,\dots,n\}$, we denote by ∂_{x_i} (resp. ∂_{y_i}) the partial derivative with respect to the i-th component of first (resp. second) variable for maps from $T_xM\times T_xM$ to $\mathrm{End}\left(\mathbb{R}\left(\mathcal{E}\otimes\mathcal{L}^d\right)_x\right)$. Then we can split $\Omega_d(z)-\Omega(z)$ according to the previous basis in the following way:

$$\Omega_d(z)-\Omega(z)=\begin{pmatrix} A_d(z) & B_d^{(1)}(z)^* & \cdots & B_d^{(n)}(z)^* \\ B_d^{(1)}(z) & C_d^{(11)}(z) & \cdots & C_d^{(1n)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ B_d^{(n)}(z) & C_d^{(n1)}(z) & \cdots & C_d^{(nn)}(z)\end{pmatrix}, \tag{B.3}$$

where,

$$A_d(z)=\begin{pmatrix}\varepsilon_d(0,0) & \varepsilon_d(0,z)\\ \varepsilon_d(z,0) & \varepsilon_d(z,z)\end{pmatrix}, \tag{B.4}$$

$$\forall i\in\{1,\dots,n\},\qquad B_d^{(i)}(z)=\begin{pmatrix}\partial_{x_i}\varepsilon_d(0,0) & \partial_{x_i}\varepsilon_d(0,z)\\ \partial_{x_i}\varepsilon_d(z,0) & \partial_{x_i}\varepsilon_d(z,z)\end{pmatrix}, \tag{B.5}$$

$$\forall i,j\in\{1,\dots,n\},\qquad C_d^{(ij)}(z)=\begin{pmatrix}\partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(0,0) & \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(0,z)\\ \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(z,0) & \partial_{x_i}\partial_{y_j}^\sharp\varepsilon_d(z,z)\end{pmatrix}. \tag{B.6}$$

Let us denote by $\mathcal{P}(z)$ the operator whose matrix in our basis is:

$$\left(\begin{array}{c|c} P(\|z\|^2) & 0\\ \hline 0 & Q\otimes I_{n-1}\end{array}\right)\otimes I_r,$$

where P was defined by Def. 4.5 and $Q=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$. Since $\mathcal{P}(z)$ is orthogonal, (B.2) is equivalent to the following:

$$\mathcal{P}(z)\Omega(z)^{-\frac{1}{2}}\left(\Omega_d(z)-\Omega(z)\right)\Omega(z)^{-\frac{1}{2}}\mathcal{P}(z)^{-1}=O\left(d^{-\alpha}\right). \tag{B.7}$$

By Lem. 4.7, the matrix of $\mathcal{P}(z)\Omega(z)^{-\frac{1}{2}}\mathcal{P}(z)^{-1}$ is $\left(\begin{smallmatrix}V(z) & 0\\ 0 & N(z)\otimes I_{n-1}\end{smallmatrix}\right)$, where

$$V(z)=\begin{pmatrix}v_1(\|z\|^2)^{-\frac{1}{2}} & 0 & 0 & 0\\ 0 & v_2(\|z\|^2)^{-\frac{1}{2}} & 0 & 0\\ 0 & 0 & v_3(\|z\|^2)^{-\frac{1}{2}} & 0\\ 0 & 0 & 0 & v_4(\|z\|^2)^{-\frac{1}{2}}\end{pmatrix}\otimes I_r$$

$$\text{and}\quad N(z)=\begin{pmatrix}\left(1-e^{-\frac{1}{2}\|z\|^2}\right)^{-\frac{1}{2}} & 0\\ 0 & \left(1+e^{-\frac{1}{2}\|z\|^2}\right)^{-\frac{1}{2}}\end{pmatrix}\otimes I_r.$$

On the other hand, by Eq. (B.3),

$$\mathcal{P}(z)\left(\Omega_d(z)-\Omega(z)\right)\mathcal{P}(z)^{-1}=\begin{pmatrix}\widetilde{A}_d(z) & \widetilde{B}_d^{(1)}(z)^* & \cdots & \widetilde{B}_d^{(n)}(z)^*\\ \widetilde{B}_d^{(1)}(z) & \widetilde{C}_d^{(11)}(z) & \cdots & \widetilde{C}_d^{(1n)}(z)\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{B}_d^{(n)}(z) & \widetilde{C}_d^{(n1)}(z) & \cdots & \widetilde{C}_d^{(nn)}(z)\end{pmatrix},$$

where

$$\begin{pmatrix}\widetilde{A}_d(z) & \widetilde{B}_d^{(1)}(z)^*\\ \widetilde{B}_d^{(1)}(z) & \widetilde{C}_d^{(11)}(z)\end{pmatrix}=\left(P(\|z\|^2)\otimes\mathrm{Id}\right)\begin{pmatrix}A_d(z) & B_d^{(1)}(z)^*\\ B_d^{(1)}(z) & C_d^{(11)}(z)\end{pmatrix}\left(P(\|z\|^2)^{\mathrm{t}}\otimes\mathrm{Id}\right), \tag{B.8}$$

$$\forall i\in\{2,\dots,n\},\quad \begin{pmatrix}\widetilde{B}_d^{(i)}(z) & \widetilde{C}_d^{(i1)}(z)\end{pmatrix}=(Q\otimes\mathrm{Id})\begin{pmatrix}B_d^{(i)}(z) & C_d^{(i1)}(z)\end{pmatrix}\left(P(\|z\|^2)^{\mathrm{t}}\otimes\mathrm{Id}\right), \tag{B.9}$$

$$\forall i,j\in\{2,\dots,n\},\quad \widetilde{C}_d^{(ij)}(z)=(Q\otimes\mathrm{Id})C_d^{(ij)}(z)(Q^{\mathrm{t}}\otimes\mathrm{Id}). \tag{B.10}$$

Then, in order to prove Eq. (B.7), we have to prove that:

$$V(z)\begin{pmatrix}\widetilde{A}_d(z) & \widetilde{B}_d^{(1)}(z)^*\\ \widetilde{B}_d^{(1)}(z) & \widetilde{C}_d^{(11)}(z)\end{pmatrix}V(z) = O(d^{-\alpha}), \tag{B.11}$$

$$\forall i \in \{2,\dots,n\}, \qquad N(z)\begin{pmatrix}\widetilde{B}_d^{(i)}(z) & \widetilde{C}_d^{(i1)}(z)\end{pmatrix}V(z) = O(d^{-\alpha}), \tag{B.12}$$

$$\forall i,j \in \{2,\dots,n\}, \qquad N(z)\widetilde{C}_d^{(ij)}(z)N(z) = O(d^{-\alpha}). \tag{B.13}$$

Since these are heavy computations, we do not reproduce them in totality here. In the following, we give some details about the proof of (B.11), which is the most difficult of these three relations to establish. The proofs of (B.12) and (B.13) are similar and left to the fearless reader.

Let us focus on the proof of (B.11). We denote

$$V(z)\begin{pmatrix}\widetilde{A}_d(z) & \widetilde{B}_d^{(1)}(z)^*\\ \widetilde{B}_d^{(1)}(z) & \widetilde{C}_d^{(11)}(z)\end{pmatrix}V(z) = \begin{pmatrix} a_d^{(1)} & a_d^{(2)*} & b_d^{(1)*} & b_d^{(2)*}\\ a_d^{(2)} & a_d^{(3)} & b_d^{(3)*} & b_d^{(4)*}\\ b_d^{(1)} & b_d^{(3)} & c_d^{(1)} & c_d^{(2)*}\\ b_d^{(2)} & b_d^{(4)} & c_d^{(2)} & c_d^{(3)}\end{pmatrix}.$$

Then by Def. 4.5 and Eq. (B.4), (B.5), (B.6) and (B.8), we have:

$$a_d^{(1)}(z) = \frac{1}{4}\left(v_1(\|z\|^2)\right)^{-1}\times \begin{pmatrix}(b_+(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ +\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)+\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right)\\ +\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ +\,(b_-(\|z\|^2))^2\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)-\varepsilon_d(0,z)+\varepsilon_d(0,0)\right)\end{pmatrix}, \tag{B.14}$$

$$a_d^{(2)}(z) = \frac{1}{4}\left(v_1(\|z\|^2)v_2(\|z\|^2)\right)^{-\frac{1}{2}}\times \begin{pmatrix}-\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ -\,(b_-(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)+\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right)\\ +\,(b_+(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ +\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)-\varepsilon_d(0,z)+\varepsilon_d(0,0)\right)\end{pmatrix}, \tag{B.15}$$

$$a_d^{(3)}(z) = \frac{1}{4}\left(v_2(\|z\|^2)\right)^{-1}\times \begin{pmatrix}(b_-(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ -\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)+\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right)\\ -\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ +\,(b_+(\|z\|^2))^2\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)-\varepsilon_d(0,z)+\varepsilon_d(0,0)\right)\end{pmatrix}, \tag{B.16}$$

$$b_d^{(1)}(z) = \frac{1}{4}\left(v_1(\|z\|^2)v_3(\|z\|^2)\right)^{-\frac{1}{2}}\times \begin{pmatrix}-\,(b_+(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ -\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\right)\\ -\,b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{y_1}^\sharp\varepsilon_d(0,0)\right)\\ -\,(b_-(\|z\|^2))^2\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\right)\end{pmatrix}, \tag{B.17}$$

$$b_d^{(2)}(z) = \frac{1}{4}\left(v_1(\|z\|^2)v_4(\|z\|^2)\right)^{-\frac{1}{2}} \times$$
$$\begin{pmatrix} b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +(b_-(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\right) \\ -(b_+(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\right) \end{pmatrix}, \quad \text{(B.18)}$$

$$b_d^{(3)}(z) = \frac{1}{4}\left(v_2(\|z\|^2)v_3(\|z\|^2)\right)^{-\frac{1}{2}} \times$$
$$\begin{pmatrix} b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -(b_+(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\right) \\ +(b_-(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\right) \end{pmatrix}, \quad \text{(B.19)}$$

$$b_d^{(4)}(z) = \frac{1}{4}\left(v_2(\|z\|^2)v_4(\|z\|^2)\right)^{-\frac{1}{2}} \times$$
$$\begin{pmatrix} -(b_-(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)-\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)+\partial_{x_1}\varepsilon_d(0,0)\right) \\ +b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)+\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -(b_+(\|z\|^2))^2\left(\varepsilon_d(z,z)-\varepsilon_d(z,0)+\varepsilon_d(0,z)-\varepsilon_d(0,0)\right) \end{pmatrix}, \quad \text{(B.20)}$$

$$c_d^{(1)}(z) = \frac{1}{4}\left(v_3(\|z\|^2)\right)^{-1} \times$$
$$\begin{pmatrix} (b_+(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)+\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right) \\ +b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +(b_-(\|z\|^2))^2\left(\varepsilon_d(z,z)+\varepsilon_d(z,0)+\varepsilon_d(0,z)+\varepsilon_d(0,0)\right) \end{pmatrix}, \quad \text{(B.21)}$$

$$c_d^{(2)}(z) = \frac{1}{4}\left(v_3(\|z\|^2)v_4(\|z\|^2)\right)^{-\frac{1}{2}} \times$$
$$\begin{pmatrix} -b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -(b_-(\|z\|^2))^2\left(\partial_{x_1}\varepsilon_d(z,z)+\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right) \\ +(b_+(\|z\|^2))^2\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +b_+(\|z\|^2)b_-(\|z\|^2)\left(\varepsilon_d(z,z)+\varepsilon_d(z,0)+\varepsilon_d(0,z)+\varepsilon_d(0,0)\right) \end{pmatrix}, \quad \text{(B.22)}$$

$$c_d^{(3)}(z) = \frac{1}{4}\left(v_4(\|z\|^2)\right)^{-1} \times$$
$$\begin{pmatrix} (b_-(\|z\|^2))^2\left(\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(z,0)-\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,z)+\partial_{x_1}\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ -b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{x_1}\varepsilon_d(z,z)+\partial_{x_1}\varepsilon_d(z,0)-\partial_{x_1}\varepsilon_d(0,z)-\partial_{x_1}\varepsilon_d(0,0)\right) \\ -b_+(\|z\|^2)b_-(\|z\|^2)\left(\partial_{y_1}^\sharp\varepsilon_d(z,z)-\partial_{y_1}^\sharp\varepsilon_d(z,0)+\partial_{y_1}^\sharp\varepsilon_d(0,z)-\partial_{y_1}^\sharp\varepsilon_d(0,0)\right) \\ +(b_+(\|z\|^2))^2\left(\varepsilon_d(z,z)+\varepsilon_d(z,0)+\varepsilon_d(0,z)+\varepsilon_d(0,0)\right) \end{pmatrix}. \quad \text{(B.23)}$$

We need to prove that each one of the terms (B.14) to (B.23) is a $O(d^{-\alpha})$, where the constant involved in this notation is independent of (x, z). The main difficulty comes the fact that v_2 and v_3 converge to 0 as $z \to 0$ (see Eq. (A.6) and (A.7)).

The term with the worst apparent singularity at $z = 0$ is $a_d^{(3)}$ (see (B.16)). We will show below that $a_d^{(3)}(z) = O(d^{-\alpha})$ uniformly in (x, z). The proofs that the other nine coefficients are $O(d^{-\alpha})$ follow the same lines, and they are strictly easier technically. We leave them to the reader.

By Eq. (A.6), $v_2(\|z\|^2) \sim \frac{\|z\|^6}{48}$ as $z \to 0$. Hence, we have to expand the second factor in (B.16) up to a $O(\|z\|^6)$. Let $\beta \in (\alpha, 1)$, recall that, by Prop. 3.4, the partial derivatives of ε_d of order up to 6 are $O(d^{-\beta})$ uniformly on $B_{T_xM}(0, b_n \ln d) \times B_{T_xM}(0, b_n \ln d)$. Recall also that we chose our coordinates so that $z = (\|z\|, 0, \dots, 0)$. Using Taylor expansions around $(0, 0)$ for ε_d and its derivatives, we get:

$$\begin{aligned}
&\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,z) + \partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(z,0) + \partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,z) + \partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) = \\
&\qquad 4\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + 2\|z\|\left(\partial^2_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \partial_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^2\left(\partial^3_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \partial_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0)\right) \\
&+ \|z\|^3\left(\frac{1}{3}\partial^4_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{2}\partial^3_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{2}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0) + \frac{1}{3}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^4\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^4 O(d^{-\beta}), \quad \text{(B.24)}
\end{aligned}$$

$$\begin{aligned}
&\partial_{x_1}\varepsilon_d(z,z) - \partial_{x_1}\varepsilon_d(z,0) + \partial_{x_1}\varepsilon_d(0,z) - \partial_{x_1}\varepsilon_d(0,0) = \\
&\qquad 2\|z\|\,\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \|z\|^2\left(\partial^2_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \partial_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^3\left(\frac{1}{2}\partial^3_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{2}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{3}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0)\right) \\
&+ \|z\|^4\left(\frac{1}{6}\partial^4_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{4}\partial^3_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{6}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0) + \frac{1}{12}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^4\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^5 O(d^{-\beta}), \quad \text{(B.25)}
\end{aligned}$$

$$\begin{aligned}
&\partial^\sharp_{y_1}\varepsilon_d(z,z) + \partial^\sharp_{y_1}\varepsilon_d(z,0) - \partial^\sharp_{y_1}\varepsilon_d(0,z) - \partial^\sharp_{y_1}\varepsilon_d(0,0) = \\
&\qquad 2\|z\|\,\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \|z\|^2\left(\partial^2_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \partial_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^3\left(\frac{1}{3}\partial^3_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{2}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{2}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0)\right) \\
&+ \|z\|^4\left(\frac{1}{12}\partial^4_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{6}\partial^3_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{4}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0) + \frac{1}{6}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^4\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^5 O(d^{-\beta}), \quad \text{(B.26)}
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_d(z,z) - \varepsilon_d(z,0) - \varepsilon_d(0,z) + \varepsilon_d(0,0) = \\
&\qquad \|z\|^2\,\partial_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \|z\|^3\left(\frac{1}{2}\partial^2_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{2}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^4\left(\frac{1}{6}\partial^3_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{4}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{6}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0)\right) \\
&+ \|z\|^5\left(\frac{1}{24}\partial^4_{x_1}\partial^\sharp_{y_1}\varepsilon_d(0,0) + \frac{1}{12}\partial^3_{x_1}\left(\partial^\sharp_{y_1}\right)^2\varepsilon_d(0,0) + \frac{1}{12}\partial^2_{x_1}\left(\partial^\sharp_{y_1}\right)^3\varepsilon_d(0,0) + \frac{1}{24}\partial_{x_1}\left(\partial^\sharp_{y_1}\right)^4\varepsilon_d(0,0)\right) \\
&\qquad + \|z\|^6 O(d^{-\beta}). \quad \text{(B.27)}
\end{aligned}$$

Now, we can combine Eq. (B.24), (B.25),(B.26) and (B.27) with the expansions around 0 of $(b_+(\|z\|^2))^2$ (cf. Eq. (A.3)), $(b_-(\|z\|^2))^2$ and $b_+(\|z\|^2)b_-(\|z\|^2)$ (cf. Eq. (A.4)). Using Prop. 3.4

once again, we obtain:

$$a_d^{(3)}(z) = \frac{1}{4v_2(\|z\|^2)} \|z\|^6 O(d^{-\beta}) = O((\ln d)^6 d^{-\beta}) = O(d^{-\alpha}),$$

where we used Eq. (A.2), (A.6) and the fact that $\|z\| \leqslant b_n \ln d$. This concludes the proof for $a_d^{(3)}$. As we already explained, we proceed similarly for the other nine coefficients to get (B.11), and the same kind of computations yields (B.12) and (B.13). □

Proof of Lem. 5.30. Let $\alpha \in (0,1)$, $x \in M$ and $z \in B_{T_xM}(0, b_n \ln d) \setminus \{0\}$. We will denote by $L = (X,Y)$ a generic element of $\mathbb{R}^2 \otimes T_x^*M \otimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$. We also set $\chi(L) = \left|\det{}^\perp(X)\right| \left|\det{}^\perp(Y)\right|$. We have:

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^\perp(X_d(z))\right| \left|\det{}^\perp(Y_d(z))\right|\right] &= \frac{1}{(2\pi)^{rn}} \det\left(\Lambda_d(z)\right)^{-\frac{1}{2}} \int \chi(L) \exp\left(-\frac{1}{2}\left\langle \Lambda_d(z)^{-1} L\,,\, L\right\rangle\right) \mathrm{d}L \\ &= \frac{1}{(2\pi)^{rn}} \left(\frac{\det \Lambda(z)}{\det \Lambda_d(z)}\right)^{\frac{1}{2}} \int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) \exp\left(-\frac{1}{2}\left\langle \Lambda(z)^{\frac{1}{2}} \Lambda_d(z)^{-1} \Lambda(z)^{\frac{1}{2}} L\,,\, L\right\rangle\right) \mathrm{d}L, \quad \text{(B.28)}\end{aligned}$$

by a change of variable. And, by Lem. 5.29, we have $\det \Lambda_d(z) = (\det \Lambda(z))\left(1 + O(d^{-\alpha})\right)$.

If we set $\Xi_d(z) = \Lambda(z)^{\frac{1}{2}} \Lambda_d(z)^{-1} \Lambda(z)^{\frac{1}{2}} - \mathrm{Id}$, then $\Xi_d(z) = O(d^{-\alpha})$, and these estimates are uniform in (x,z). As in the proof of Lem. 4.18, by the Mean Value Theorem, for all L we have:

$$\left|\exp\left(-\frac{1}{2}\left\langle \Xi_d(z) L\,,\, L\right\rangle\right) - 1\right| \leqslant \frac{1}{2} \|L\|^2 \|\Xi_d(z)\| \exp\left(\frac{1}{2} \|L\|^2 \|\Xi_d(z)\|\right).$$

Since $\Xi_d(z) = O(d^{-\alpha})$, for d large enough $\|\Xi_d(z)\| \leqslant \frac{1}{2}$. Hence,

$$\begin{aligned}\int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) e^{-\frac{1}{2}\|L\|^2} \left|\exp\left(-\frac{1}{2}\left\langle \Xi_D(z) L\,,\, L\right\rangle\right) - 1\right| \mathrm{d}L& \\ \leqslant \frac{\|\Xi_d(z)\|}{2} \int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) \|L\|^2 e^{-\frac{1}{4}\|L\|^2} \mathrm{d}L. \quad \text{(B.29)}&\end{aligned}$$

Recall that, by Lem. 4.12, the eigenvalues of the positive symmetric operator $\Lambda(z)$ are $u_1(\|z\|^2)$, $u_2(\|z\|^2)$, $1 + \exp\left(-\frac{1}{2}\|z\|^2\right)$ and $1 - \exp\left(-\frac{1}{2}\|z\|^2\right)$, with some multiplicities. These are bounded functions of z (see Eq. (A.2) and (A.5)). Hence, $\chi\left(\Lambda(z)^{\frac{1}{2}} L\right)$ is the square root of a polynomial in L whose coefficients are bounded functions of z. Thus, the integral on the right-hand side of Eq. (B.29) is bounded, independently of (x,z). We get:

$$\begin{aligned}\int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) \exp\left(-\frac{1}{2}\left\langle \Lambda(z)^{\frac{1}{2}} \Lambda_d(z)^{-1} \Lambda(z)^{\frac{1}{2}} L\,,\, L\right\rangle\right) \mathrm{d}L& \\ = \int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) e^{-\frac{1}{2}\|L\|^2} \mathrm{d}L + O\big(d^{-\alpha}\big)& \\ = (2\pi)^{rn} \mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right] + O\big(d^{-\alpha}\big).&\end{aligned}$$

Finally, by Eq. (B.28), we find

$$\mathbb{E}\left[\left|\det{}^\perp(X_d(z))\right| \left|\det{}^\perp(Y_d(z))\right|\right] = \mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right] + O\big(d^{-\alpha}\big). \quad \text{(B.30)}$$

By Lem. 4.13, for all $z \neq 0$, $\Lambda(z)$ is non-singular. Hence $\mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right]$, is a positive function of z. By Lem. 4.16,

$$\mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right] = \mathbb{E}\left[\left|\det{}^\perp\left(X(\|z\|^2)\right)\right| \left|\det{}^\perp\left(Y(\|z\|^2)\right)\right|\right],$$

and by Lem. 4.17 and 4.18, if $r < n$, this quantity admits positive limits when $\|z\|$ goes to 0 or $\|z\|$ goes to $+\infty$. Thus, in this case, $\mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right]$ is bounded from below by positive constant, independent of (x,z). Then, Eq. (B.30) shows that:

$$\mathbb{E}\left[\left|\det{}^\perp(X_d(z))\right| \left|\det{}^\perp(Y_d(z))\right|\right] = \mathbb{E}\left[\left|\det{}^\perp(X_\infty(z))\right| \left|\det{}^\perp(Y_\infty(z))\right|\right] \left(1 + O\big(d^{-\alpha}\big)\right)$$

and this concludes the proof for $r < n$.

If $r = n$, the leading term in Eq. (B.30) goes to 0 as $\|z\| \to 0$, so that we need to be more precise. From now on, we assume that $r = n$. Let us assume for now that, in this case, we have:

$$\int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) \|L\|^2 e^{-\frac{1}{4}\|L\|^2} \,\mathrm{d}L = O\Big(\|z\|^2\Big) \tag{B.31}$$

as $z \to 0$, where the constant involved in the $O\Big(\|z\|^2\Big)$ is uniform in (x, z). Then, proceeding as we did in the case $r < n$, we get the following equivalent of Eq. (B.30):

$$\mathbb{E}\big[\big|\det{}^{\perp}(X_d(z))\big|\,\big|\det{}^{\perp}(Y_d(z))\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big] + O\Big(\|z\|^2 d^{-\alpha}\Big)\,.$$

By Lem. 4.17,

$$\mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big] = \mathbb{E}\Big[\Big|\det{}^{\perp}\Big(X(\|z\|^2)\Big)\Big|\,\Big|\det{}^{\perp}\Big(Y(\|z\|^2)\Big)\Big|\Big] \sim \frac{n!}{2}\,\|z\|^2\,,$$

as $z \to 0$. Hence,

$$\mathbb{E}\big[\big|\det{}^{\perp}(X_d(z))\big|\,\big|\det{}^{\perp}(Y_d(z))\big|\big] = \mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big]\,\big(1 + O\big(d^{-\alpha}\big)\big)$$

uniformly for $x \in M$ and $\|z\| \leqslant 1$. In the domain $\|z\| \geqslant 1$, $\mathbb{E}\big[\big|\det{}^{\perp}(X_\infty(z))\big|\,\big|\det{}^{\perp}(Y_\infty(z))\big|\big]$ is bounded from below by a positive constant independent of (x, z), and we proceed as in the case $r < n$, using Eq. (B.30). This yields the result for $r = n$.

To conclude the proof, we still have to prove that (B.31) holds when $r = n$. Let us write $L = (A, B)$ and $\Lambda(z)^{\frac{1}{2}} L = (X(z), Y(z))$ with $A, B, X(z)$ and $Y(z) \in T_x^*M \otimes \mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$. We choose any orthonormal basis of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ and an orthonormal basis of T_xM such that the coordinates of z are $(\|z\|, 0, \dots, 0)$. We denote by (A_{ij}), (B_{ij}), $(X_{ij}(z))$ and $(Y_{ij}(z)) \in \mathcal{M}_{rn}(\mathbb{R})$ the matrices of $A, B, X(z)$ and $Y(z)$ in these bases.

The matrix of $\Lambda(z)$ in the basis defined by $\mathcal{B}'_z$ (see Sect. 4.3) and our basis of $\mathbb{R}\left(\mathcal{E} \otimes \mathcal{L}^d\right)_x$ is $\widehat{\Lambda}(\|z\|^2)$, where $\widehat{\Lambda}$ was defined by Eq. (A.15). That is, using the same notations as in the proof of Lem. 4.17 (see Eq. (A.9) and (A.10)), for all $i \in \{1, \dots, r\}$:

$$\begin{pmatrix} X_{i1} \\ Y_{i1} \end{pmatrix} = \begin{pmatrix} \alpha(\|z\|^2) & \beta(\|z\|^2) \\ \beta(\|z\|^2) & \alpha(\|z\|^2) \end{pmatrix} \begin{pmatrix} A_{i1} \\ B_{i1} \end{pmatrix} \quad \text{and } \forall j \geqslant 2, \quad \begin{pmatrix} X_{ij} \\ Y_{ij} \end{pmatrix} = \begin{pmatrix} \gamma(\|z\|^2) & \delta(\|z\|^2) \\ \delta(\|z\|^2) & \gamma(\|z\|^2) \end{pmatrix} \begin{pmatrix} A_{ij} \\ B_{ij} \end{pmatrix}.$$

Hence, we have:

$$\chi\left(\Lambda(z)^{\frac{1}{2}} L\right) = \chi(X(z), Y(z)) = \big|\det{}^{\perp}(X(z))\big|\,\big|\det{}^{\perp}(Y(z))\big| = \Psi\left(\|z\|^2, (A_{ij}), (B_{ij})\right),$$

where Ψ was defined by Eq. (A.11). Recall that Ψ satisfies (A.14) when $r = n$. As in the proof of Lem. 4.17 (cf. App. A), by Lebesgue's Theorem we have:

$$\begin{aligned} \frac{2}{\|z\|^2} \int \chi\left(\Lambda(z)^{\frac{1}{2}} L\right) \|L\|^2 e^{-\frac{1}{4}\|L\|^2} \,\mathrm{d}L &= \int \frac{2}{\|z\|^2} \Psi\left(\|z\|^2, (A_{ij}), (B_{ij})\right) \|L\|^2 e^{-\frac{1}{4}\|L\|^2} \,\mathrm{d}L \\ &\xrightarrow[\|z\| \to 0]{} \int \det\left(\frac{A_1 - B_1}{\sqrt{2}}, \frac{A_2 + B_2}{\sqrt{2}}, \dots, \frac{A_n + B_n}{\sqrt{2}}\right)^2 \|L\|^2 e^{-\frac{1}{4}\|L\|^2} \,\mathrm{d}L, \end{aligned}$$

where A_j (resp. B_j) denotes the j-th column of the matrix of A (resp. B) and $L = (A, B)$. This limit is finite, which proves that (B.31) is satisfied and concludes the proof. □

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