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CYCLICITY IN ℓ^p SPACES AND ZERO SETS OF THE FOURIER TRANSFORMS

FLORIAN LE MANACH

ABSTRACT. We study the cyclicity of vectors u in $\ell^p(\mathbb{Z})$. It is known that a vector u is cyclic in $\ell^2(\mathbb{Z})$ if and only if the zero set, $\mathcal{Z}(\hat{u})$, of its Fourier transform, \hat{u} , has Lebesgue measure zero and $\log|\hat{u}| \notin L^1(\mathbb{T})$, where \mathbb{T} is the unit circle. Here we show that, unlike $\ell^2(\mathbb{Z})$, there is no characterization of the cyclicity of u in $\ell^p(\mathbb{Z})$, $1 < p < 2$, in terms of $\mathcal{Z}(\hat{u})$ and the divergence of the integral $\int_{\mathbb{T}} \log|\hat{u}|$. Moreover we give both necessary conditions and sufficient conditions for u to be cyclic in $\ell^p(\mathbb{Z})$, $1 < p < 2$.

1. INTRODUCTION AND MAIN RESULTS

In this paper we will study cyclic vectors in ℓ^p spaces for $1 < p < 2$. A vector $u = (u_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ is called *cyclic* in $\ell^p(\mathbb{Z})$ if the linear span of $\{(u_{n-k})_{n \in \mathbb{Z}}, k = 0, 1, 2, \dots\}$ is dense in $\ell^p(\mathbb{Z})$ and u is called *bicyclic* in $\ell^p(\mathbb{Z})$ if the linear span of $\{(u_{n-k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$ is dense in $\ell^p(\mathbb{Z})$. Note that the bicyclicity in this paper is defined as cyclicity in [13] and [14].

We denote by \mathbb{T} the unit circle of \mathbb{C} . The Fourier transform of $u = (u_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ is given by

$$\hat{u} : \zeta \in \mathbb{T} \mapsto \sum_{n \in \mathbb{Z}} u_n \zeta^n$$

and we denote by $\mathcal{Z}(\hat{u})$ the zero set of \hat{u} on \mathbb{T} :

$$\mathcal{Z}(\hat{u}) = \{\zeta \in \mathbb{T}, \hat{u}(\zeta) = 0\}.$$

Note that if $u \in \ell^1(\mathbb{Z})$ then \hat{u} is continuous and $\mathcal{Z}(\hat{u})$ is well defined, while if $u \in \ell^2(\mathbb{Z})$ then $\hat{u} \in L^2(\mathbb{T})$ and $\mathcal{Z}(\hat{u})$ is defined modulo null sets.

Wiener showed in [18] that the bicyclicity of a sequence u in $\ell^1(\mathbb{Z})$ and $\ell^2(\mathbb{Z})$ is characterized only in terms of $\mathcal{Z}(\hat{u})$: u is bicyclic in $\ell^1(\mathbb{Z})$ if and only if $\mathcal{Z}(\hat{u}) = \emptyset$ and u is bicyclic in $\ell^2(\mathbb{Z})$ if and only if $\mathcal{Z}(\hat{u})$ has Lebesgue measure zero. Wiener asked if the same phenomenon holds for $u \in \ell^p(\mathbb{Z})$, $1 < p < 2$.

Lev and Olevskii, in [14], answered this question negatively. We cannot characterize the bicyclicity of $u \in \ell^1(\mathbb{Z})$ only in terms of the zero set of \hat{u} : for $1 < p < 2$ there exist $u, v \in \ell^1(\mathbb{Z})$ such that $\mathcal{Z}(\hat{u}) = \mathcal{Z}(\hat{v})$ and one is bicyclic in $\ell^p(\mathbb{Z})$ and the other is not. Here we consider a similar problem for cyclicity in $\ell^p(\mathbb{Z})$, $1 < p < 2$.

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It follows from Wiener characterization of bicyclic vectors and Szegő infimum that a vector u is cyclic in $\ell^2(\mathbb{Z})$ if and only if $\mathcal{Z}(\hat{u})$ has Lebesgue measure zero and $\log |\hat{u}| \notin L^1(\mathbb{T})$ (see [16, Corollary 4.1.2, Theorem 4.1.4]). Furthermore when u is cyclic in $\ell^p(\mathbb{Z})$ then u is cyclic in $\ell^q(\mathbb{Z})$ for $q \geq p$. A direct consequence of this fact is that, for $u \in \ell^1(\mathbb{Z})$, if u is cyclic in $\ell^p(\mathbb{Z})$, $1 \leq p \leq 2$, then $\mathcal{Z}(\hat{u})$ has Lebesgue measure zero and $\log |\hat{u}| \notin L^1(\mathbb{T})$. It follows from this and Wiener characterization of bicyclic vectors in $\ell^1(\mathbb{Z})$ that there is no cyclic vectors in $\ell^1(\mathbb{Z})$ (see also [1] for a different proof).

The existence of cyclic vectors on $\ell^p(\mathbb{Z})$, for $p > 1$, was established by Abakumov, Atzmon and Grivaux in [1, Corollary 1] in 2007. Note that this result was first proved by Olevskii in an unpublished manuscript in 1998 (see [1, Remark 1]).

We show that for sequences $u \in \ell^1(\mathbb{Z})$ such that $\log |\hat{u}| \notin L^1(\mathbb{T})$, there is no characterization of the cyclic vectors u in $\ell^p(\mathbb{Z})$ only in terms of $\mathcal{Z}(\hat{u})$.

Theorem 1.1. *Let $1 < p < 2$. There exist u and v in $\ell^1(\mathbb{Z})$ such that $\mathcal{Z}(\hat{u}) = \mathcal{Z}(\hat{v})$, $\log |\hat{u}| \notin L^1(\mathbb{T})$, $\log |\hat{v}| \notin L^1(\mathbb{T})$, with u non-cyclic in $\ell^p(\mathbb{Z})$ and v cyclic in $\ell^p(\mathbb{Z})$.*

Until now we did not know more than the existence of cyclic vectors in $\ell^p(\mathbb{Z})$. Here we give, when \hat{u} is smooth, both necessary conditions and sufficient conditions for u to be cyclic in $\ell^p(\mathbb{Z})$, $1 < p < 2$, and in a more general case: the weighted ℓ^p spaces. In particular this allows to construct explicit examples of cyclic vectors in weighted ℓ^p spaces. Here are some results that we get: for $u \in \ell^1(\mathbb{Z})$ and $1 < p < 2$,

- if $\hat{u} \in C^\infty(\mathbb{T})$, $\dim(\mathcal{Z}(\hat{u})) < 2(p-1)/p$ and $\log |\hat{u}| \notin L^1(\mathbb{T})$ then u is cyclic in $\ell^p(\mathbb{Z})$,
- if $\hat{u} \in \text{Lip}_\delta(\mathbb{T})$ for $\delta > 1/p - 1/2$, $\dim(\mathcal{Z}(\hat{u})) < 2(p-1)/p$ and $\log(d(\cdot, \mathcal{Z}(\hat{u}))) \notin L^1(\mathbb{T})$ then u is cyclic in $\ell^p(\mathbb{Z})$,

where \dim is the Hausdorff dimension (see [10, Chap II]), $d(\cdot, \mathcal{Z}(\hat{u}))$ is the Euclidean distance to $\mathcal{Z}(\hat{u})$ and $f \in \text{Lip}_\delta(\mathbb{T})$ means that there exists $C > 0$ such that

$$|f(\zeta) - f(\zeta')| \leq C|\zeta - \zeta'|^\delta, \quad \zeta, \zeta' \in \mathbb{T}. \quad (1.1)$$

Let E be a closed subset of \mathbb{T} . The set E is called a Carleson set if $\log d(\cdot, E) \in L^1(\mathbb{T})$ (see [4]). This condition is equivalent to the fact that E has a Lebesgue measure zero and $\sum_n |I_n| \log |I_n| > -\infty$, where (I_n) are the complementary intervals of E . Note that if $\hat{u} \in \text{Lip}_\delta(\mathbb{T})$ such that $\mathcal{Z}(\hat{u})$ is not a Carleson set then $\log |\hat{u}| \notin L^1(\mathbb{T})$.

The paper is organized as follows: in section 2 we give preliminary results, in section 3 we prove Theorem 1.1 and section 4 is devoted to establishing both necessary conditions and sufficient conditions for cyclicity in weighted $\ell^p(\mathbb{Z})$.

We use the following notations:

- $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2. PRELIMINARIES

We denote by \mathbb{N}_0 the set of all non-negative integers and by \mathbb{N} the set of all positive integers. Let X be a metric linear space of complex functions on \mathbb{T} such that the shift operator S , given by

$$S(f)(\zeta) = \zeta f(\zeta)$$

is a topological isomorphism of X onto itself. Let $f \in X$ and $\Lambda \subset \mathbb{Z}$, we denote by $[f]_{\Lambda}^X$ the closed linear span of $\{z^n f, n \in \Lambda\}$. We say that $f \in X$ is *cyclic* in X if $[f]_{\mathbb{N}_0}^X = X$ and f is *bicyclic* in X if $[f]_{\mathbb{Z}}^X = X$. Note that f is cyclic in X if and only if f is bicyclic in X and $[f]_{\mathbb{N}_0}^X = [f]_{\mathbb{Z}}^X$. Moreover

$$[f]_{\mathbb{N}_0}^X = [f]_{\mathbb{Z}}^X \iff f \in [f]_{\mathbb{N}}^X \quad (2.1)$$

Indeed, if $[f]_{\mathbb{N}_0}^X = [f]_{\mathbb{Z}}^X$ then we have $\bar{z}f \in [f]_{\mathbb{N}_0}^X$ and we obtain that $f \in [f]_{\mathbb{N}}^X$, since the multiplication by z is continuous in X . Conversely, we assume that $f \in [f]_{\mathbb{N}}^X$. Then we see easily that $\bar{z}f \in [f]_{\mathbb{N}_0}^X$. Moreover if $\bar{z}^n f \in [f]_{\mathbb{N}_0}^X$ for some $n \geq 1$ then $\bar{z}^{n+1}f = \lim \bar{z}P_k f$ with $P_k \in \mathcal{P}_+(\mathbb{T})$. Since $P_k \bar{z}f \in [f]_{\mathbb{N}_0}^X$ and $[f]_{\mathbb{N}_0}^X$ is closed in X , we obtain $\bar{z}^{n+1}f \in [f]_{\mathbb{N}_0}^X$. Thus by induction, for all $n \geq 1$, $\bar{z}^n f \in [f]_{\mathbb{N}_0}^X$ and then $[f]_{\mathbb{N}_0}^X = [f]_{\mathbb{Z}}^X$.

We denote by $\mathcal{P}(\mathbb{T})$ the set of trigonometric polynomials on \mathbb{T} and by $\mathcal{P}_+(\mathbb{T})$ the set of analytic polynomials on \mathbb{T} . Now we suppose, in addition, that X is a Banach space such that $\mathcal{P}(\mathbb{T})$ is dense in X . Then $f \in X$ is bicyclic in X if and only if there exists a sequence (P_n) of trigonometric polynomials such that

$$\lim_{n \rightarrow \infty} \|1 - P_n f\|_X = 0. \quad (2.2)$$

So we obtain, from this and (2.1), the following proposition:

Proposition 2.1. *Let $f \in X$. The function f is cyclic in X if and only if there exist $P_n \in \mathcal{P}(\mathbb{T})$ and $Q_n \in \mathcal{P}_+(\mathbb{T})$ satisfying*

$$\|1 - P_n f\|_X \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \|f - zQ_n f\|_X \xrightarrow{n \rightarrow \infty} 0.$$

Now we study the weighted ℓ^p spaces. We denote by $\mathcal{D}'(\mathbb{T})$ the set of distributions on \mathbb{T} and $\mathcal{M}(\mathbb{T})$ the set of measures on \mathbb{T} . For $S \in \mathcal{D}'(\mathbb{T})$ and $n \in \mathbb{Z}$, we denote by $\widehat{S}(n)$ the n th Fourier coefficient of S defined by $\widehat{S}(n) = \langle S, z^{-n} \rangle$ and we write $S = \sum_{n \in \mathbb{Z}} \widehat{S}(n) z^n$. If $f \in L^1(\mathbb{T})$ we have

$$\widehat{f}(n) = \int_{\mathbb{T}} f(\zeta) \zeta^{-n} \frac{d\zeta}{2\pi}.$$

For $p \geq 1$ and $\beta \in \mathbb{R}$, we define the Banach spaces

$$\ell_{\beta}^p(\mathbb{Z}) = \left\{ (u_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad \|(u_n)\|_{\ell_{\beta}^p}^p = \sum_{n \in \mathbb{Z}} |u_n|^p (1 + |n|)^{p\beta} < \infty \right\}$$

and

$$A_{\beta}^p(\mathbb{T}) = \left\{ S \in \mathcal{D}'(\mathbb{T}), \quad \|S\|_{A_{\beta}^p}^p = \sum_{n \in \mathbb{Z}} |\widehat{S}(n)|^p (1 + |n|)^{p\beta} < \infty \right\}.$$

We will write $A^p(\mathbb{T})$ for the space $A_0^p(\mathbb{T})$ and $A(\mathbb{T})$ for the space $A^1(\mathbb{T})$ which is the Wiener algebra. The Fourier transformation is an isometric isomorphism between $\ell_\beta^p(\mathbb{Z})$ and $A_\beta^p(\mathbb{T})$ since for $S \in A_\beta^p(\mathbb{T})$ we have

$$\|S\|_{A_\beta^p} = \|(\widehat{S}(n))_{n \in \mathbb{Z}}\|_{\ell_\beta^p}.$$

Moreover, for a sequence $(u_n)_{n \in \mathbb{Z}}$ and $k \in \mathbb{Z}$, we have $\widehat{(u_{n-k})} = z^k \widehat{(u_n)}$, so u is cyclic in $\ell_\beta^p(\mathbb{Z})$ if and only if \widehat{u} is cyclic in $A_\beta^p(\mathbb{T})$. From now we will state and prove our results in $A_\beta^p(\mathbb{T})$. Note that if $1 \leq p \leq 2$ and $\beta \geq 0$, $A_\beta^p(\mathbb{T})$ is a subset $L^2(\mathbb{T})$.

For $1 \leq p < \infty$ and $\beta \geq 0$ we define the product of $f \in A_\beta^1(\mathbb{T})$ and $S \in A_\beta^p(\mathbb{T})$ by

$$fS = \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \widehat{f}(k) \widehat{S}(n-k) \right) z^n,$$

and we see that $\|fS\|_{A_\beta^p} \leq \|f\|_{A_\beta^1} \|S\|_{A_\beta^p}$.

The following lemma gives us different inclusions between the $A_\beta^p(\mathbb{T})$ spaces.

Lemma 2.2 ([13]). *Let $1 \leq r, s < \infty$ and $\beta, \gamma \in \mathbb{R}$.*

- (1) *If $r \leq s$ then $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \iff \gamma \leq \beta$.*
- (2) *If $r > s$ then $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \iff \beta - \gamma > \frac{1}{s} - \frac{1}{r}$.*

For $p \neq 1$, the dual space of $A_\beta^p(\mathbb{T})$ can be identified with $A_{-\beta}^q(\mathbb{T})$ ($q = \frac{p}{p-1}$) by the following formula

$$\langle S, T \rangle = \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{T}(-n), \quad S \in A_\beta^p(\mathbb{T}), T \in A_{-\beta}^q(\mathbb{T}).$$

A distribution S on \mathbb{T} is said to vanish on an open set V , if $\langle S, \varphi \rangle = 0$ for all functions $\varphi \in C^\infty(\mathbb{T})$ with support in V . The support of the distribution S , denoted by $\text{supp}(S)$, is the complement of the maximal open set on which the distribution S vanishes. When f is continuous, we denote by $\mathcal{Z}(f)$ the zero set of the function f . For $\delta \geq 0$ we recall that $\text{Lip}_\delta(\mathbb{T})$ denotes the set of functions f defined on \mathbb{T} which satisfies (1.1).

We need the following proposition which gives a necessary condition for a function $f \in \text{Lip}_\delta(\mathbb{T})$ to be bicyclic in $A^p(\mathbb{T})$.

Proposition 2.3 ([9]). *Let $1 < p < \infty$, $q = p/(p-1)$ and $f \in A(\mathbb{T})$. If there exists $\delta > 0$ such that $f \in \text{Lip}_\delta(\mathbb{T})$ and if there exists $S \in A^q(\mathbb{T}) \setminus \{0\}$ such that $\text{supp}(S) \subset \mathcal{Z}(f)$ then f is not bicyclic in $A^p(\mathbb{T})$.*

3. NO CHARACTERIZATION OF CYCLICITY IN TERMS OF ZERO SET

The following Theorem is a reformulation of Theorem 1.1 in $A^p(\mathbb{T})$.

Theorem 3.1. *Let $1 < p < 2$. There exist f and g in $A(\mathbb{T})$ such that $\mathcal{Z}(f) = \mathcal{Z}(g)$, $\log |f| \notin L^1(\mathbb{T})$, $\log |g| \notin L^1(\mathbb{T})$ with f non-cyclic in $A^p(\mathbb{T})$ and g cyclic in $A^p(\mathbb{T})$.*

For the proof we need some notations and definitions. Let E be a closed subset of \mathbb{T} . We set

$$\mathcal{I}(E) = \{f \in A(\mathbb{T}), f|_E = 0\}$$

and we denote by $\mathcal{J}(E)$ the closure in $A(\mathbb{T})$ of the set of functions in $A(\mathbb{T})$ which vanish on a neighborhood of E . Clearly $\mathcal{J}(E) \subset \mathcal{I}(E)$. We say that E is a set of synthesis if $\mathcal{J}(E) = \mathcal{I}(E)$. Since $\{1\}$ is a set of synthesis (see [10, Theorem IV] p. 123 or [8, Appendix 3] pp. 416-418), we have $\mathcal{I}(\{1\}) = \mathcal{J}(\{1\})$. Unlike the space $A(\mathbb{T})$, $\mathcal{I}(\{1\})$ contains cyclic vectors. Moreover, by [1, Proposition 2], the set of cyclic vectors in $\mathcal{I}(\{1\})$ is a dense G_δ subset of $\mathcal{I}(\{1\})$.

Lemma 3.2. *Let $f \in \mathcal{I}(\{1\})$. If f is cyclic in $\mathcal{I}(\{1\})$ then f is cyclic in $A^p(\mathbb{T})$ for all $p > 1$.*

Proof. Let $f \in \mathcal{I}(\{1\})$ cyclic in $\mathcal{I}(\{1\})$ and $\varepsilon > 0$. We denote by h_k , for $k \in \mathbb{N}$, the function defined on \mathbb{T} by

$$h_k(z) = \frac{z - 1}{z - 1 - 1/k}.$$

Clearly $h_k \in \mathcal{I}(\{1\})$. Moreover for $z \in \mathbb{T}$,

$$h_k(z) = 1 - \frac{1}{k+1} \sum_{n=0}^{\infty} \frac{z^n}{(1+1/k)^n}.$$

Thus for $p > 1$, we have

$$\begin{aligned} \|1 - h_k\|_{A^p}^p &= \frac{1}{(k+1)^p} \sum_{n=0}^{\infty} \frac{1}{(1+1/k)^{pn}} \\ &= \frac{1}{(k+1)^p} \frac{1}{1 - \frac{1}{(1+1/k)^p}} \\ &= \frac{1}{(k+1)^p - k^p} \underset{k \rightarrow \infty}{\sim} \frac{1}{pk^{p-1}}. \end{aligned}$$

So for fixed $p > 1$, we choose $k \in \mathbb{N}$ such that $\|1 - h_k\|_{A^p} < \varepsilon$. Since f is cyclic in $\mathcal{I}(\{1\})$, there exist P and Q in $\mathcal{P}_+(\mathbb{T})$ such that

$$\|h_k - Pf\|_{A^1} < \varepsilon \quad \text{and} \quad \|f - zQf\|_{A^1} < \varepsilon.$$

We obtain that $\|f - zQf\|_{A^p} < \varepsilon$ and

$$\|1 - Pf\|_{A^p} \leq \|1 - h_k\|_{A^p} + \|h_k - Pf\|_{A^1} < 2\varepsilon.$$

This proves, by Proposition 2.1, that f is cyclic in $A^p(\mathbb{T})$ for all $p > 1$. \square

To prove our main theorem of this section, we need to introduce the notion of Helson sets.

Definition 3.3 ([10], Chap. XI). *A compact set $K \subset \mathbb{T}$ is called a Helson set if it satisfies any one of the following equivalent conditions:*

(i) *if $f \in \mathcal{C}(K)$ then there exists $g \in A(\mathbb{T})$ such that $g|_K = f$,*

(ii) there exists $\delta_1 > 0$ such that for all $\mu \in \mathcal{M}(\mathbb{T})$ satisfying $\text{supp}(\mu) \subset K$ we have

$$\sup_{n \in \mathbb{Z}} |\hat{\mu}(n)| \geq \delta_1 \int_{\mathbb{T}} |d\mu|,$$

(iii) there exists $\delta_2 > 0$ such that for all $\mu \in \mathcal{M}(\mathbb{T})$ satisfying $\text{supp}(\mu) \subset K$ we have

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n)| \geq \delta_2 \int_{\mathbb{T}} |d\mu|.$$

By (iii), a Helson set does not support any non zero measure μ such that $\hat{\mu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$. On the other hand Körner shows, in [12], that there exists a Helson set which support a non zero distribution S such that $\hat{S}(n) \rightarrow 0$ as $|n| \rightarrow \infty$ (see also Kaufman in [11] for a simple proof). Lev and Olevskii extend significantly this result by showing the following theorem which is the key of their result about bicyclicity in $\ell^p(\mathbb{Z})$.

Theorem 3.4 ([14], Theorem 3). *For $q > 2$, there exists a Helson set K which supports a non-zero distribution $S \in A^q(\mathbb{T})$.*

We need also the following lemma obtained by Lev and Olevskii.

Lemma 3.5 ([14], Lemma 10). *Let K be a Helson set. For $\varepsilon > 0$, $p > 1$ and $f \in \mathcal{C}(K)$, there exists $g \in A(\mathbb{T})$ such that*

$$\begin{cases} g|_K = f, \\ \|g\|_{A^1} \leq \frac{1}{\delta_2} \|f\|_{\infty}, \\ \|g\|_{A^p} < \varepsilon \end{cases}$$

where δ_2 is a constant which satisfies (iii) in the definition 3.3.

The main result, Theorem 3.1, is a consequence of the following theorem which is an analogue of Lev and Olevskii result.

Theorem 3.6. *Let K be a Helson set on \mathbb{T} . There exists $g \in A(\mathbb{T})$ such that g vanishes on K and such that g is cyclic in $A^p(\mathbb{T})$ for all $p > 1$.*

Proof. Let K be a Helson set on \mathbb{T} . We recall that

$$\mathcal{I}(K) = \{g \in A(\mathbb{T}), g|_K = 0\}$$

which, endowed with the norm $\|\cdot\|_{A^1}$, is a Banach space. We define

$$\mathcal{G}(K) = \{g \in \mathcal{I}(K), g \text{ is cyclic in } A^p(\mathbb{T}), \forall p > 1\}.$$

We will show that $\mathcal{G}(K)$ is a dense G_δ subset of $\mathcal{I}(K)$. For $\varepsilon > 0$ and $p > 1$, we consider the set

$$G(\varepsilon, p) = \{g \in \mathcal{I}(K), \exists P \in \mathcal{P}(\mathbb{T}), \exists Q \in \mathcal{P}_+(\mathbb{T}), \|1 - Pg\|_{A^p} < \varepsilon \text{ and } \|g - zQg\|_{A^p} < \varepsilon\}.$$

Let $\varepsilon_n = \frac{1}{n}$ and $p_n = 1 + \frac{1}{n}$. To use a Baire Category argument, we will show that

- (i) $\bigcap_{n=1}^{\infty} G(\varepsilon_n, p_n) = \mathcal{G}(K)$,
- (ii) for $\varepsilon > 0$ and $p > 1$, $G(\varepsilon, p)$ is an open subset of $\mathcal{I}(K)$,
- (iii) for $\varepsilon > 0$ and $p > 1$, $G(\varepsilon, p)$ is a dense subset of $\mathcal{I}(K)$.

(i): Let $g \in \mathcal{I}(K)$ such that for all $n \in \mathbb{N}$, there exist $P_n \in \mathcal{P}(\mathbb{T})$ and $Q_n \in \mathcal{P}_+(\mathbb{T})$ satisfying $\|1 - P_n g\|_{A^{p_n}} < \varepsilon_n$ and $\|g - zQ_n g\|_{A^{p_n}} < \varepsilon_n$. Since for all $p > 1$, there exists N such that for all $n \geq N$ we have $p > p_n$,

$$\|1 - P_n g\|_{A^p} \leq \|1 - P_n g\|_{A^{p_n}} < \varepsilon_n \rightarrow 0, \quad n \rightarrow \infty$$

and

$$\|g - zQ_n g\|_{A^p} \leq \|g - zQ_n g\|_{A^{p_n}} < \varepsilon_n \rightarrow 0, \quad n \rightarrow \infty.$$

By Proposition 2.1, g is cyclic in $A^p(\mathbb{T})$ for all $p > 1$ and thus we have $g \in \mathcal{G}(K)$. The other inclusion is clear.

(ii) follows from the fact that, for $f \in A(\mathbb{T})$ and $g \in A^p(\mathbb{T})$,

$$\|fg\|_{A^p} \leq \|f\|_{A^1} \|g\|_{A^p}.$$

(iii): Let $g_0 \in \mathcal{I}(K)$. We will show that for all $\eta > 0$, there exists $g \in G(\varepsilon, p)$ such that $\|g - g_0\|_{A^1} < \eta$. Without loss of generality, we suppose that $1 \in K$. So we have $g_0 \in \mathcal{I}(\{1\})$. Since K is a Helson set, we consider δ_2 the constant satisfying the condition (iii) of Definition 3.3. Since the set of cyclic vectors in $\mathcal{I}(\{1\})$ is dense in $\mathcal{I}(\{1\})$ (see [1]), there exists a function h cyclic in $\mathcal{I}(\{1\})$ such that

$$\|h - g_0\|_{A^1} < \frac{\delta_2}{1 + \delta_2} \eta.$$

Hence for all $t \in K$,

$$|h(t)| = |h(t) - g_0(t)| \leq \|h - g_0\|_{\infty} \leq \|h - g_0\|_{A^1} < \frac{\delta_2}{1 + \delta_2} \eta.$$

Since h is cyclic in $\mathcal{I}(\{1\})$, by Lemma 3.2, h is also cyclic in $A^p(\mathbb{T})$. By Proposition 2.1, there exist $P \in \mathcal{P}(\mathbb{T})$ and $Q \in \mathcal{P}_+(\mathbb{T})$ satisfying $\|1 - Ph\|_{A^p} < \frac{\varepsilon}{2}$ and $\|h - zQh\|_{A^p} < \frac{\varepsilon}{2}$. By Lemma 3.5, there exists $f \in A(\mathbb{T})$ satisfying

$$\begin{cases} f|_K = h|_K, \\ \|f\|_{A^1} \leq \frac{\|h|_K\|_{\infty}}{\delta_2} < \frac{\eta}{1 + \delta_2}, \\ \|f\|_{A^p} < \min\left(\frac{\varepsilon}{2\|P\|_{A^1}}, \frac{\varepsilon}{2\|1 - zQ\|_{A^1}}\right). \end{cases}$$

Let $g = h - f$. We have $g \in \mathcal{I}(K)$ since $f|_K = h|_K$. Moreover $g \in G(\varepsilon, p)$ since

$$\|1 - Pg\|_{A^p} \leq \|1 - Ph\|_{A^p} + \|P\|_{A^1} \|f\|_{A^p} < \varepsilon$$

and

$$\|g - zQg\|_{A^p} \leq \|h - zQh\|_{A^p} + \|1 - zQ\|_{A^1} \|f\|_{A^p} < \varepsilon.$$

Finally we have

$$\|g - g_0\|_{A^1} \leq \|h - g_0\|_{A^1} + \|f\|_{A^1} < \frac{\delta_2}{1 + \delta_2} \eta + \frac{\eta}{1 + \delta_2} < \eta.$$

Hence $G(\varepsilon, p)$ is a dense subset of $\mathcal{I}(K)$.

By the Baire Category Theorem, $\bigcap_{n=1}^{\infty} G(\varepsilon_n, p_n)$ is a dense G_δ subset of $\mathcal{I}(K)$. In particular $\mathcal{G}(K)$ is not empty. \square

Proof. (of Theorem 3.1) We recall that $1 < p < 2$. Let K be a Helson set given by Theorem 3.4. By Theorem 3.6, there exists $g \in A(\mathbb{T})$ such that g vanishes on K and such that g is cyclic in $A^p(\mathbb{T})$. This implies in particular that $\log |g| \notin L^1(\mathbb{T})$. Moreover there exists $f \in C^1(\mathbb{T})$ such that $\mathcal{Z}(f) = \mathcal{Z}(g)$ and $\log |f| \notin L^1(\mathbb{T})$ (take $f(\zeta) = e^{-1/d(\zeta, \mathcal{Z}(g))}$ if $\zeta \notin \mathcal{Z}(g)$ and 0 otherwise). By Proposition 2.3 and Theorem 3.4, f is not bicyclic in $A^p(\mathbb{T})$ so it is not cyclic in $A^p(\mathbb{T})$. \square

Remark 3.7. *In the construction of f and g in Theorem 3.1, one is bicyclic and the other one is not. A natural question arises: Is it possible to construct two bicyclic functions with the same properties as in Theorem 3.1?*

4. CYCLICITY IN WEIGHTED ℓ^p SPACES

In the following, we denote $q = p/(p-1)$. In this section, we give both necessary conditions and sufficient conditions for f to be cyclic in $A_\beta^p(\mathbb{T})$, when f is smooth, for $1 < p < 2$ and $0 \leq \beta q \leq 1$. When $\beta q > 1$, there is no cyclic vector in $A_\beta^p(\mathbb{T})$. Indeed, for $1 \leq p < \infty$ and $\beta \geq 0$, $A_\beta^p(\mathbb{T})$ is a Banach algebra of continuous functions if and only if $\beta q > 1$ (see [7]). So, if f is cyclic in $A_\beta^p(\mathbb{T})$, when $\beta q > 1$, then $\mathcal{Z}(f) = \emptyset$ and $\log |f| \notin L^1(\mathbb{T})$ which is impossible since f is continuous. In particular the conditions obtained in this section allow us to construct explicit examples of cyclic vectors. We also study the density of the set of cyclic vectors in weighted $\ell^p(\mathbb{Z})$.

4.1. Conditions for the cyclicity. We denote by H^∞ the space of functions $f \in L^\infty(\mathbb{T})$ satisfying $\widehat{f}(n) = 0$ for all $n < 0$. It is well known that H^∞ can be identified with the space of bounded holomorphic functions on the disk $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$. Recall that outer functions are given by

$$f(z) = \exp \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D},$$

where φ is a positive function such that $\log \varphi \in L^1(\mathbb{T})$. We denote by $f(\zeta)$ the radial limit of f at $\zeta \in \mathbb{T}$ if it exists. We have $|f| = \varphi$ a.e. on \mathbb{T} . Moreover if $\varphi \in L^\infty(\mathbb{T})$ then $f \in H^\infty$.

Proposition 4.1. *Let $1 < p < 2$, $\beta \geq 0$ and $f \in A_\alpha^2(\mathbb{T})$ with $\alpha > 1/p - 1/2 + \beta$. The following are equivalent.*

- (1) $[f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})}$,
- (2) $\inf \left\{ \|Pf\|_{A_\beta^p}, P \in H^\infty, Pf \in A_\alpha^2(\mathbb{T}), P(0) = 1 \right\} = 0$.

Note that, by Lemma 2.2, $\alpha > 1/p - 1/2 + \beta$ implies that $A_\alpha^2(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$.

Proof. We suppose first $[f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})}$. By Proposition 2.1, there exists $P_n \in \mathcal{P}_+(\mathbb{T})$ such that $\lim \|f - zP_n f\|_{A_\beta^p} = 0$ as $n \rightarrow \infty$. Since $1 - zP_n \in H^\infty$, we have

$$\inf \left\{ \|Pf\|_{A_\beta^p}, P \in H^\infty, Pf \in A_\alpha^2(\mathbb{T}), P(0) = 1 \right\} = 0.$$

Now suppose that (2) holds. Let $P_n \in H^\infty$, for $n \in \mathbb{N}$, such that $P_n(0) = 1$, $P_n f \in A_\alpha^2(\mathbb{T})$ and $\lim \|P_n f\|_{A_\beta^p} = 0$ as $n \rightarrow \infty$. We write $P_n = 1 - zQ_n$ with $Q_n \in H^\infty$. So we have, $\lim \|zQ_n f - f\|_{A_\beta^p} = 0$. Recall that $[f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})}$ holds if and only if $f \in [f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$. Thus it suffices to show that $zQ_n f \in [f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$ since $[f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$ is a closed subset of $A_\beta^p(\mathbb{T})$. Note that $zQ_n f \in A_\alpha^2(\mathbb{T})$ since $P_n f$ and f are in $A_\alpha^2(\mathbb{T})$. So we will show the following claim: if $Q \in H^\infty$ verifies $zQf \in A_\alpha^2(\mathbb{T})$ then $zQf \in [f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$.

We note, for $0 < r < 1$, $Q_r(z) = Q(rz)$. We have $zQ_r f \in [f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$ since for all $\varepsilon > 0$, there exists $\tilde{Q}_r \in \mathcal{P}_+(\mathbb{T})$ such that $\|Q_r - \tilde{Q}_r\|_{A_\beta^1} < \varepsilon$ and $\|zQ_r f - z\tilde{Q}_r f\|_{A_\beta^p} \leq \|Q_r - \tilde{Q}_r\|_{A_\beta^1} \|zf\|_{A_\beta^p} < \varepsilon \|zf\|_{A_\beta^p}$. Moreover, by the Hahn-Banach theorem, $[f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$ is weakly closed in $A_\beta^p(\mathbb{T})$. So it suffices to show that $zQ_r f \rightarrow zQf$ weakly in $A_\beta^p(\mathbb{T})$ as $r \rightarrow 1$. Let $g \in A_{-\beta}^q(\mathbb{T})$ with $1/p + 1/q = 1$. There exists a sequence (g_n) in $\mathcal{P}(\mathbb{T})$ such that $\lim \|g - g_n\|_{A_{-\beta}^q} = 0$, as $n \rightarrow \infty$. So we have

$$\begin{aligned} |\langle zQ_r f - zQf, g \rangle| &\leq |\langle zQ_r f - zQf, g - g_n \rangle| + |\langle zQ_r f - zQf, g_n \rangle| \\ &\leq \|zQ_r f - zQf\|_{A_\beta^p} \|g - g_n\|_{A_{-\beta}^q} + \\ &\quad \left| \int_{\mathbb{T}} (zQ_r f - zQf)(\zeta) g_n(\zeta) |d\zeta| \right|. \end{aligned}$$

Since $\|zQ_r f - zQf\|_{A_\beta^p} \leq \|zQ_r f - zQf\|_{A_\alpha^2}$, we can show, with the same calculus that in [17, proposition 3.4] and [5], that $\|zQ_r f - zQf\|_{A_\beta^p}$ remains bounded as $r \rightarrow 1$. Moreover $\|g - g_n\|_{A_{-\beta}^q} \rightarrow 0$ and, by the dominated convergence theorem, $\int_{\mathbb{T}} |(zQ_r f - zQf)(\zeta) g_n(\zeta)| |d\zeta| \rightarrow 0$ as $r \rightarrow 1$ since $f \in L^1(\mathbb{T})$. So $zQ_r f \rightarrow zQf$ weakly in $A_\beta^p(\mathbb{T})$ as $r \rightarrow 1$ and $zQf \in [f]_{\mathbb{N}}^{A_\beta^p(\mathbb{T})}$. This completes the proof. \square

We also need the following lemma, see the proof of Theorem 1.3 in [2].

Lemma 4.2. *Let E be a closed subset of \mathbb{T} with Lebesgue measure zero and such that $\log(d(\cdot, E)) \notin L^1(\mathbb{T})$. Let $\delta' > 1/2$ and $\gamma > 0$ such that $2\delta' - \gamma - 1 \geq 0$. Let F_ε be the outer function F_ε satisfying*

$$|F_\varepsilon(\zeta)| = (d(\zeta, E)^\gamma + \varepsilon)^{1/2} \quad \text{a.e. on } \mathbb{T}$$

and

$$M_\varepsilon = \frac{1}{2} \int_{\mathbb{T}} \log \left(\frac{1}{d(\zeta, E)^\gamma + \varepsilon} \right) d\zeta.$$

For all $\varepsilon > 0$ we have

$$\iint_{\mathbb{T}^2} d(\zeta', E)^{2(\delta' - \gamma)} \frac{|F_\varepsilon(\zeta) - F_\varepsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \lesssim M_\varepsilon.$$

The following result was obtained in [2] for $p = 2$ and $\beta = 1/2$.

Theorem 4.3. *Let $1 < p < 2$ and $\beta > 0$ such that $\beta q < 1$. Let $f \in \text{Lip}_\delta(\mathbb{T})$, where $\delta > \beta + 1/p - 1/2$. We suppose that $\mathcal{Z}(f)$ has Lebesgue measure zero. If*

$$\int_{\mathbb{T}} \log d(\zeta, \mathcal{Z}(f)) |d\zeta| = -\infty$$

then

$$[f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})}.$$

Furthermore, if $f \in A_\beta^1(\mathbb{T})$ and $\dim(\mathcal{Z}(f)) < \frac{q}{2}(1 - \beta q)$ then f is cyclic in $A_\beta^p(\mathbb{T})$.

Proof. Let $\alpha > 0$ such that $\beta + 1/p - 1/2 < \alpha < \min(\delta, 1/2)$. First we have $f \in \text{Lip}_\delta(\mathbb{T}) \subset A_\alpha^2(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$. Indeed by Douglas' formula (see [17, section 5]), $f \in A_\alpha^2(\mathbb{T})$ if and only if $f \in L^2(\mathbb{T})$ and

$$\mathcal{D}_\alpha(f) = \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| < \infty.$$

Moreover we have

$$\|f\|_{A_\alpha^2}^2 \asymp \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}_\alpha(f). \quad (4.1)$$

Thus if $f \in \text{Lip}_\delta(\mathbb{T})$ then

$$\mathcal{D}_\alpha(f) \lesssim \iint_{\mathbb{T}^2} \frac{1}{|\zeta - \zeta'|^{1+2\alpha-2\delta}} |d\zeta| |d\zeta'| < \infty$$

and we get $f \in A_\alpha^2(\mathbb{T})$. On the other hand, by Proposition 2.2, we have $A_\alpha^2(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$.

By Proposition 4.1, it suffices to show that

$$\inf \left\{ \|Pf\|_{A_\beta^p}, P \in H^\infty, Pf \in A_\alpha^2(\mathbb{T}), P(0) = 1 \right\} = 0$$

to complete the proof. Let $\varepsilon > 0$ and $\gamma > 0$ such that $\gamma < \delta - \alpha$. We define p_ε to be the outer function satisfying

$$|p_\varepsilon(\zeta)| = \frac{e^{-M_\varepsilon}}{(d(\zeta, E)^\gamma + \varepsilon)^{1/2}} \quad a.e.$$

where $E = \mathcal{Z}(f)$ and

$$M_\varepsilon = \frac{1}{2} \int_{\mathbb{T}} \log \left(\frac{1}{d(\zeta, E)^\gamma + \varepsilon} \right) |d\zeta|$$

so that $p_\varepsilon(0) = 1$. Since $\log d(\cdot, E) \notin L^1(\mathbb{T})$, $M_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

We will show that

$$\|p_\varepsilon f\|_{A_\beta^p} \leq \|p_\varepsilon f\|_{A_\alpha^2} \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By (4.1), we have

$$\|p_\varepsilon f\|_{A_\alpha^2}^2 \asymp \|p_\varepsilon f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}_\alpha(p_\varepsilon f).$$

First, since $\gamma < 2\delta$,

$$\|p_\varepsilon f\|_{L^2(\mathbb{T})}^2 \lesssim e^{-2M_\varepsilon} \int_{\mathbb{T}} \frac{d(\zeta, E)^{2\delta}}{d(\zeta, E)^\gamma} |d\zeta| \lesssim e^{-2M_\varepsilon} \rightarrow 0.$$

Then we have

$$\mathcal{D}_\alpha(p_\varepsilon f) = \iint_{\mathbb{T}^2} \frac{|p_\varepsilon f(\zeta) - p_\varepsilon f(\zeta')|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'|.$$

Note that for all $(\zeta, \zeta') \in \mathbb{T}^2$ we have

$$|p_\varepsilon f(\zeta) - p_\varepsilon f(\zeta')|^2 \leq 2|p_\varepsilon(\zeta)|^2 |f(\zeta) - f(\zeta')|^2 + 2|f(\zeta')|^2 |p_\varepsilon(\zeta) - p_\varepsilon(\zeta')|^2.$$

So if we denote $\Gamma = \{(\zeta, \zeta') \in \mathbb{T}^2, |d(\zeta', E)| \leq |d(\zeta, E)|\}$, we obtain

$$\begin{aligned} \mathcal{D}_\alpha(p_\varepsilon f) &\lesssim \iint_{\Gamma} |p_\varepsilon(\zeta)|^2 \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| + \iint_{\Gamma} |f(\zeta')|^2 \frac{|p_\varepsilon(\zeta) - p_\varepsilon(\zeta')|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| \\ &= A_\varepsilon + B_\varepsilon. \end{aligned}$$

Let η such that $\frac{\gamma}{2\delta} \leq \eta < \frac{2\delta-2\alpha}{2\delta}$. Then, using the fact that for $(\zeta, \zeta') \in \Gamma$, $|f(\zeta')| \leq d(\zeta, E)^\delta$, we get

$$\begin{aligned} A_\varepsilon &= e^{-2M_\varepsilon} \iint_{\Gamma} \left(\frac{|f(\zeta) - f(\zeta')|^{2\eta}}{d(\zeta, E)^\gamma + \varepsilon} \right) \left(\frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^{1+2\alpha}} \right) |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\varepsilon} \iint_{\mathbb{T}^2} \frac{d(\zeta, E)^{2\eta\delta}}{d(\zeta, E)^\gamma} \left(\frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^{1+2\alpha}} \right) |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\varepsilon} \iint_{\mathbb{T}^2} \frac{|\zeta - \zeta'|^{(2-2\eta)\delta}}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\varepsilon} \rightarrow 0. \end{aligned}$$

To estimate B_ε , we consider the outer function F_ε given in Lemma 4.2. Let

$$\Gamma_1 = \{(\zeta, \zeta') \in \Gamma, d(\zeta', E) \leq |\zeta - \zeta'|\},$$

$$\Gamma_2 = \{(\zeta, \zeta') \in \Gamma, d(\zeta', E) > |\zeta - \zeta'|\}.$$

We have

$$\begin{aligned} B_\varepsilon &= \iint_{\Gamma} 2|f(\zeta')|^2 |p_\varepsilon(\zeta)p_\varepsilon(\zeta')|^2 \frac{|1/p_\varepsilon(\zeta') - 1/p_\varepsilon(\zeta)|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\varepsilon} \iint_{\Gamma} \frac{d(\zeta', E)^{2\delta}}{(d(\zeta, E)^\gamma + \varepsilon)(d(\zeta', E)^\gamma + \varepsilon)} \frac{|F_\varepsilon(\zeta) - F_\varepsilon(\zeta')|^2}{|\zeta - \zeta'|^{1+2\alpha}} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\varepsilon} \iint_{\Gamma_1} |\zeta - \zeta'|^{2\delta-2\gamma-2\alpha-1} |F_\varepsilon(\zeta) - F_\varepsilon(\zeta')|^2 |d\zeta| |d\zeta'| \\ &\quad + e^{-2M_\varepsilon} \iint_{\Gamma_2} d(\zeta', E)^{2\delta-2\gamma-2\alpha+1} \frac{|F_\varepsilon(\zeta) - F_\varepsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &= B_\varepsilon^1 + B_\varepsilon^2. \end{aligned}$$

Since $\gamma < \delta - \alpha$ and $|F_\varepsilon|$ is bounded on \mathbb{T} , we have $B_\varepsilon^1 \lesssim e^{-2M\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. On the other hand, applying Lemma 4.2 with $\delta' = \delta - \alpha + 1/2$, we obtain $B_\varepsilon^2 \lesssim M_\varepsilon e^{-2M\varepsilon}$. Thus $B_\varepsilon^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore we have $\lim \|p_\varepsilon f\|_{A_\beta^p} = 0$ and, by Proposition 4.1, $[f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})}$. Furthermore, by [13, Theorem A], if $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ then f is bicyclic in $A_\beta^p(\mathbb{T})$, so f is cyclic in $A_\beta^p(\mathbb{T})$. \square

Remark. Let $1 < p < 2$ and $\beta > 0$ such that $\beta q < 1$. Using the fact that $A_{1/2}^2(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$ and [2, Theorem 1.2] we obtain, for $f \in A_\beta^p(\mathbb{T})$ such that $|f| \in C^1(\mathbb{T})$ and $|f'| \in \text{Lip}_\delta(\mathbb{T})$ for $\delta > 0$, the following result: if $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ and $\log |f| \notin L^1(\mathbb{T})$ then f^2 is cyclic in $A_\beta^p(\mathbb{T})$.

4.2. Cyclicity for smooth functions. When $f \in C^\infty(\mathbb{T})$, Makarov proves in [15], that $[f]_{\mathbb{N}_0}^{C^\infty(\mathbb{T})} = [f]_{\mathbb{Z}}^{C^\infty(\mathbb{T})}$ if and only if $\log |f| \notin L^1(\mathbb{T})$. As a consequence of Makarov's Theorem and some results obtained in [13], we get the following result:

Theorem 4.4. *Let $1 < p < 2$, $\beta \geq 0$ such that $\beta q \leq 1$. We have the following assertions:*

- (1) *If $f \in C^\infty(\mathbb{T})$, $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ and $\log |f| \notin L^1(\mathbb{T})$ then f is cyclic in $A_\beta^p(\mathbb{T})$.*
- (2) *If $f \in A_\beta^1(\mathbb{T})$, $\dim(\mathcal{Z}(f)) > 1 - \beta q$ then f is not cyclic in $A_\beta^p(\mathbb{T})$.*
- (3) *For $\frac{2}{q}(1 - \beta q) \leq \alpha \leq 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) \subset E$ is not cyclic in $A_\beta^p(\mathbb{T})$.*
- (4) *If $p = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}$ then there exists a closed set $E \subset \mathbb{T}$ such that $\dim(E) = 1 - \beta q$ and every $f \in C^\infty(\mathbb{T})$ satisfying $\mathcal{Z}(f) \subset E$ and $\log |f| \notin L^1(\mathbb{T})$ is cyclic in $A_\beta^p(\mathbb{T})$.*

Note that, for $1 < p < 2$ and $\beta \geq 0$ such that $\beta q \leq 1$, if f is cyclic in $A_\beta^p(\mathbb{T})$ then $\log |f| \notin L^1(\mathbb{T})$.

Proof. For (2) and (3), we use [13, Theorem 3.4 and Theorem 4.3] since if f is not bicyclic then f is not cyclic in $A_\beta^p(\mathbb{T})$.

For (1) and (4), suppose $\log |f| \notin L^1(\mathbb{T})$. By [15], we have $[f]_{\mathbb{N}_0}^{C^\infty(\mathbb{T})} = [f]_{\mathbb{Z}}^{C^\infty(\mathbb{T})}$. Since the embedding $C^\infty(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$ is continuous, we obtain $[f]_{\mathbb{N}_0}^{A_\beta^p(\mathbb{T})} = [f]_{\mathbb{Z}}^{A_\beta^p(\mathbb{T})}$. Moreover, by [13, Theorem 3.4], f is bicyclic in $A_\beta^p(\mathbb{T})$. Therefore f is cyclic in $A_\beta^p(\mathbb{T})$. \square

In the following, we denote, for $E \subset \mathbb{T}$, $|E|$ the Lebesgue measure of E .

Examples. Let E be a closed subset of \mathbb{T} and Λ be a continuous, positive, decreasing function defined on $(0, \infty)$ such that $\lim \Lambda(t) = \infty$ and $\lim t\Lambda(t) = 0$ as $t \rightarrow 0$. We consider the function f defined on \mathbb{T} by

$$f(\zeta) = \exp(-\Lambda(d(\zeta, E))) \tag{4.2}$$

if $\zeta \notin E$ and $f(\zeta) = 0$ otherwise. So, by [6, Proposition A.1], we have

$$\int_{\mathbb{T}} \log |f(\zeta)| |d\zeta| = - \int_{\mathbb{T}} \Lambda(d(\zeta, E)) |d\zeta| = -\infty \Leftrightarrow \int_{\mathbb{T}} |E_t| d\Lambda(t) = -\infty,$$

where $E_t = \{\zeta \in \mathbb{T}, d(\zeta, E) \leq t\}$. Let $N_E(t)$ be the smallest number of closed arcs of length $2t$ that cover E . We have, by [6, Lemma A.3], that $tN_E(t) \leq |E_t| \leq 4tN_E(t)$. Thus, by [3, Theorem 2, p. 30], we have for $0 < \alpha < 1$,

$$\int_{\mathbb{T}} \frac{1}{t^{\alpha+1} N_E(t)} dt = \infty \Leftrightarrow \int_{\mathbb{T}} \frac{1}{t^\alpha |E_t|} dt = \infty \Rightarrow \dim(E) < \alpha.$$

Now we suppose that $\Lambda(t) = 1/t^\gamma$ for $\gamma > 0$. The function f defined by (4.2) is infinitely differentiable on \mathbb{T} . Let $1 < p < 2$, $\beta \geq 0$ such that $\beta q \leq 1$. We can construct a Cantor set E such that

$$\int_{\mathbb{T}} \frac{|E_t|}{t^{\gamma+1}} dt = \infty \quad \text{and} \quad \int_{\mathbb{T}} \frac{1}{t^\alpha |E_t|} dt = \infty$$

for $\alpha < \frac{2}{q}(1 - \beta q)$. Then, by Theorem 4.4, the function f is cyclic in $A_\beta^p(\mathbb{T})$. Moreover if $\gamma \geq 1$ we have $\log |f| \notin L^1(\mathbb{T})$. So, by Theorem 4.4,

- (1) if $\dim(E) < \frac{2}{q}(1 - \beta q)$ then f is cyclic in $A_\beta^p(\mathbb{T})$;
- (2) if $\dim(E) > 1 - \beta q$ then f is not cyclic in $A_\beta^p(\mathbb{T})$;
- (3) for $\frac{2}{q}(1 - \beta q) \leq \alpha \leq 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and f is not cyclic in $A_\beta^p(\mathbb{T})$;
- (4) if $p = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}$ then there exists a closed set $E \subset \mathbb{T}$ such that $\dim(E) = 1 - \beta q$ and f is cyclic in $A_\beta^p(\mathbb{T})$.

4.3. Density of cyclic vectors set. When $\beta q \leq 1$, we have seen, in subsection 4.2, that there exist cyclic vectors in $A_\beta^p(\mathbb{T})$. Here we show that the set of cyclic vectors in $A_\beta^p(\mathbb{T})$ is a dense G_δ subset of $A_\beta^p(\mathbb{T})$.

Proposition 4.5. *Let $f \in A(\mathbb{T})$. If $\mathcal{Z}(f)$ is finite then f is bicyclic in $A_\beta^p(\mathbb{T})$ for $p > 1$ and $\beta \geq 0$ such that $\beta q \leq 1$.*

Proof. Let $S \in A_\beta^q(\mathbb{T})$ such that $\langle S, z^n f \rangle = 0$ for all $n \in \mathbb{Z}$. It suffices to prove that $S = 0$. First we have $\text{supp}(S) \subset \mathcal{Z}(f)$ (see [13, Lemma 2.4]) and since $\mathcal{Z}(f)$ is finite, we obtain that

$$S = \sum_{k=1}^N \sum_{n=0}^M \lambda_{k,n} \delta_{z_k}^{(n)}$$

where $\mathcal{Z}(f) = \{z_k, k \in \llbracket 1, N \rrbracket\} \subset \mathbb{T}$, $\lambda_{k,n} \in \mathbb{C}$ and where $\delta_{z_k}^{(n)}$ is the n th derivative of the Dirac delta measure concentrated at the point z_k .

Suppose that $S \neq 0$. There exist k_0 and n_0 such that $\lambda_{k_0, n_0} \neq 0$. We define

$$T = \prod_{\substack{k=1 \\ k \neq k_0}}^N (z - z_k)^{M+1} S.$$

So $T \in A_{-\beta}^q(\mathbb{T})$ and $T \neq 0$. Hence there exists $(c_n) \in \mathbb{C}^{M+1}$ such that for every test function $\varphi \in \mathcal{D}(\mathbb{T})$,

$$\langle T, \varphi \rangle = \langle S, \prod_{\substack{k=1 \\ k \neq k_0}}^N (z - z_k)^{M+1} \varphi \rangle = \sum_{n=0}^M \lambda_{k_0, n} c_n \varphi^{(n)}(z_{k_0}).$$

If we denote $n_1 = \max\{n, c_n \lambda_{k_0, n} \neq 0\}$, then we have for all $l \in \mathbb{Z}$

$$|\langle T, z^l \rangle| = \left| \sum_{n=0}^N \lambda_{k_0, n} c_n (il)^n z_{k_0}^l \right| \underset{l \rightarrow \infty}{\sim} |\lambda_{k_0, n_1}| |c_{n_1}| |l|^{n_1}$$

(here $z^l = e^{il\theta}$). This is not possible since $T \in A_{-\beta}^q(\mathbb{T})$ and $\beta q \leq 1$. \square

Lemma 4.6. *Let $1 < p \leq 2$ and $\beta \geq 0$ such that $\beta q \leq 1$ where $1/p + 1/q = 1$. If $f \in A_{\beta}^p(\mathbb{T})$ is cyclic in $A_{\beta}^p(\mathbb{T})$ and if $P \in \mathcal{P}(\mathbb{T})$ is not zero then Pf is cyclic in $A_{\beta}^p(\mathbb{T})$.*

Proof. Let $\varepsilon > 0$. By Lemma 4.5, P is bicyclic in $A_{\beta}^p(\mathbb{T})$ since $\mathcal{Z}(P)$ is finite. So there exists $P_1 \in \mathcal{P}(\mathbb{T})$ such that $\|1 - P_1 P\|_{A_{\beta}^p} < \varepsilon$. Moreover since f is bicyclic, there exists $P_2 \in \mathcal{P}(\mathbb{T})$ such that

$$\|1 - P_2 f\|_{A_{\beta}^p} < \frac{\varepsilon}{\|P_1 P\|_{A_{\beta}^1}}.$$

Therefore we have

$$\|1 - P_1 P_2 P f\|_{A_{\beta}^p} \leq \|1 - P_1 P\|_{A_{\beta}^p} + \|P_1 P\|_{A_{\beta}^1} \|1 - P_2 f\|_{A_{\beta}^p} < 2\varepsilon.$$

So Pf is bicyclic in $A_{\beta}^p(\mathbb{T})$.

Moreover, since f is cyclic in $A_{\beta}^p(\mathbb{T})$, there exists $Q \in \mathcal{P}_+(\mathbb{T})$ such that $\|f - zQf\|_{A_{\beta}^p} < \frac{\varepsilon}{\|P\|_{A_{\beta}^1}}$. So we have

$$\|Pf - zQPf\|_{A_{\beta}^p} \leq \|P\|_{A_{\beta}^1} \|f - zQf\|_{A_{\beta}^p} < \varepsilon.$$

Hence, by Proposition 2.1, Pf is cyclic in $A_{\beta}^p(\mathbb{T})$. \square

Theorem 4.7. *Let $1 < p \leq 2$ and $\beta \geq 0$ such that $\beta q \leq 1$ where $1/p + 1/q = 1$. The set of cyclic vectors in $A_{\beta}^p(\mathbb{T})$ is a dense G_{δ} subset of $A_{\beta}^p(\mathbb{T})$.*

Proof. Let $\varepsilon > 0$. We consider the set

$$G_{\varepsilon} = \{g \in A_{\beta}^p(\mathbb{T}), \exists P \in \mathcal{P}(\mathbb{T}), \exists Q \in \mathcal{P}_+(\mathbb{T}), \|1 - Pg\|_{A_{\beta}^p} < \varepsilon \text{ and } \|g - zQg\|_{A_{\beta}^p} < \varepsilon\}.$$

To use a Baire Category argument, we will show that

- (i) $\bigcap_{n=1}^{\infty} G_{1/n} = \{f \in A_{\beta}^p(\mathbb{T}) \text{ cyclic in } A_{\beta}^p(\mathbb{T})\}$,
- (ii) for $\varepsilon > 0$, G_{ε} is an open subset of $A_{\beta}^p(\mathbb{T})$,
- (iii) for $\varepsilon > 0$, G_{ε} is a dense subset of $A_{\beta}^p(\mathbb{T})$.

(i): Let $g \in A_\beta^p(\mathbb{T})$ such that for all $n \in \mathbb{N}$, there exist $P_n \in \mathcal{P}(\mathbb{T})$ and $Q_n \in \mathcal{P}_+(\mathbb{T})$ satisfying $\|1 - P_n g\|_{A_\beta^p} < 1/n$ and $\|g - zQ_n g\|_{A_\beta^p} < 1/n$. By Proposition 2.1, g is cyclic in $A_\beta^p(\mathbb{T})$. The other inclusion is clear.

(ii) follows from the fact that, for $f \in A_\beta^1(\mathbb{T})$ and $g \in A_\beta^p(\mathbb{T})$,

$$\|fg\|_{A_\beta^p} \leq \|f\|_{A_\beta^1} \|g\|_{A_\beta^p}.$$

(iii): Since for $\varepsilon > 0$, $\{f \in A_\beta^p(\mathbb{T}) \text{ cyclic in } A_\beta^p(\mathbb{T})\} \subset G_\varepsilon$, we will show that $\{f \in A_\beta^p(\mathbb{T}) \text{ cyclic in } A_\beta^p(\mathbb{T})\}$ is dense in $A_\beta^p(\mathbb{T})$. Let $h \in A_\beta^p(\mathbb{T})$ and $\eta > 0$ such that $\eta < \|h\|_{A_\beta^p}$. Consider $f \in A_\beta^p(\mathbb{T})$ which is cyclic in $A_\beta^p(\mathbb{T})$. So there exists a non-zero polynomial $P \in \mathcal{P}_+(\mathbb{T})$ such that $\|h - Pf\|_{A_\beta^p} < \eta$ and by Lemma 4.6, Pf is cyclic in $A_\beta^p(\mathbb{T})$. Therefore this shows that G_ε is dense in $A_\beta^p(\mathbb{T})$. \square

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