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Hadamard well-posedness for a class of nonlinear shallow shell problems

John Cagnol^{a,*}, Irena Lasiecka^b, Catherine Lebedzik^c, Richard Marchand^d

^a *Pôle Universitaire Léonard de Vinci, ESILV, DER CS, 92916 Paris La Défense, France*

^b *University of Virginia, Department of Mathematics, Charlottesville, VA 22904, USA*

^c *Wayne State University, Department of Mathematics, Detroit, MI 48201, USA*

^d *Slippery Rock University, Department of Mathematics, Slippery Rock, PA 16057, USA*

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Abstract

This paper is concerned with the nonlinear shallow shell model introduced in 1966 by W.T. Koiter in [On the nonlinear theory of thin elastic shells. III, *Nederl. Akad. Wetensch. Proc. Ser. B* 69 (1966) 33–54, Section 11] and later studied in [M. Bernadou, J.T. Oden, An existence theorem for a class of nonlinear shallow shell problems, *J. Math. Pures Appl.* (9) 60(3) (1981) 285–308]. We consider a version of this model which is based upon the intrinsic shell modeling techniques introduced by Michel Delfour and Jean-Paul Zolésio. We show existence and uniqueness of both regular and weak solutions to the dynamical model and that the solutions are continuous with respect to the initial data. While existence and uniqueness of *regular solutions* to nonlinear dynamic shell equations has been known, full Hadamard well-posedness of *weak solutions*, as shown in this paper, is a new result which solves an old open problem in the field.

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1. Introduction

In [21], Koiter classifies shell problems according to the order of magnitude of the deflection. He arranges the classes in an order of increasing specialization, as follows: *Large Deflections*, characterized by the absence of restriction as to the magnitude of the displacement; *Moderate Deflections*, characterized by small values of all displacement gradients with respect to unity, although nothing is assumed *a priori* about their relative orders of magnitude; *Small Finite Deflections*, characterized by small displacement gradients and by rotations whose squares do not exceed the middle surface strain in order of magnitude; *Infinitesimal Deflections*, characterized by small displacement gradients none of which exceed the middle surface strains in order of magnitude.

* Corresponding author. Tel.: +33 1 41 16 71 88; fax: +33 1 41 16 71 71.

E-mail address: john.cagnol@devinci.fr (J. Cagnol).

The infinitesimal deflections hypothesis leads to the well-known linear theory. Instead, in this article we shall consider the small finite deflections case, which is studied in [21, Section 11]. This corresponds to the “medium bending case” in the sense of Naghdi (see [28]) and to the “approximation of small strains and moderately small rotations” in the sense of Lyell and Sanders (see [29]). This case leads to the geometrically nonlinear model discussed in [3] and summarized in [2, Section 7.3]. In [3], Bernadou and Oden have proven that a solution to the static model exists and that it is unique provided the load is sufficiently small. Philippe G. Ciarlet proposed in [8] a model where the exact change of curvature tensor is modified and Liliana Gratie generalized this latter model to the case of non-constant thickness in [17]. See [10] for the linear and nonlinear theories of shells.

This paper is concerned with the dynamical model. Dynamical systems of nonlinear elasticity have attracted much interest over the years [32,9,22,14,18,13]. Numerous physical phenomena can be described by these models, which mathematically take the form of a set of coupled partial differential equations. The shell model considered here is a nonlinear, strongly coupled system of hyperbolic equations with clamped (Dirichlet) boundary conditions. Its counterpart for a flat geometry is the full von Kármán system [32,9], which couples a plate-like equation for the normal displacement variable with wave equations for the tangential (in-plane) displacements.

Existence of *weak* solutions for such models follows from standard Faedo–Galerkin method [27]. However, the issue of well-posedness (uniqueness and continuous dependence on initial data) is much more subtle. The nonlinear term in the equation is neither locally Lipschitz nor bounded in the finite energy space. In addition, the problem is not monotone and weak solutions do not exhibit any additional regularity properties (unlike parabolic-like problems). Thus typical or known methods used for proving well-posedness of weak solutions are no longer applicable.

It is the main goal set in this paper to provide an affirmative answer to full Hadamard well-posedness of weak (often referred to as finite energy) solutions. More specifically, the main contribution of this work is threefold:

- Existence and uniqueness of weak solutions (*i.e.* finite energy solutions).
- Continuous dependence of solutions with respect to initial data measured in finite energy norm.
- Regularity of weak solutions.

We shall use the intrinsic geometry method of Michel Delfour and Jean-Paul Zolésio, which relies on the oriented distance function to describe the geometry [12,11]. This technique takes advantage of the intrinsic geometric properties of the shell. Here, the shell is described in terms of tangential differential operators which are defined by means of the oriented boundary distance function in \mathbb{R}^3 . Sobolev spaces, Green’s formula, and key inequalities such as Poincaré and Korn’s inequalities are all well defined. Delfour and Zolésio have constructed models under a variety of assumptions. A linear version of the model developed in this paper was introduced in [5,6].

This model is a geometrically expanded version of the full (vectorial) von Kármán system, which has been rather extensively studied. Much literature however deals with a static or semi-static model for which the mathematical issues are quite different (both in the case of the plate [9] and in that of the shell [3]). In the dynamic case, there are many results available for the modified (scalar) von Kármán model, which does not account for the in-plane displacements. This modification results in a scalar model where the nonlinearity appears non-locally via the Airy stress function. The scalar von Kármán model has been extensively studied in the literature with many results on well-posedness and stability available [7,4,15,19,22]. It is worth saying that much of the rich theory developed for scalar von Kármán evolutions owes its success to the rather unexpected *regularity* of the Airy stress function [15]. In the full (vectorial) von Kármán system the Airy stress function plays no role. Thus, the benefits of additional regularity of the nonlinear term are not available. However, the simplification of ignoring the in-plane displacements does not make sense in the case of a shell. Thus, in the case of nonlinear shells the relevant dynamic model is that of vectorial structure. The full vectorial system is much more complicated than the modified version, due to the strong coupling between hyperbolic equations. The real difficulty lies in the lack of regularity of the nonlinear terms in the finite energy norms, at least in the two-dimensional case (the one-dimensional case was treated in [23]). The two-dimensional case with homogeneous Dirichlet boundary data was considered in [30] where uniqueness of both regular and finite energy solutions was derived. A related result is [31] which relies on Strichartz-type estimates and covers the cases of the whole plane and of special geometries such as hinged rectangular plates. A full well-posedness theory for these equations was finally provided in [26] which proves existence and uniqueness of weak, intermediate and regular solutions under a variety of boundary conditions including free boundary conditions with nonlinear boundary dissipation. This proof is based in part on the method introduced by Sedenko [30] for the Marguerre–Vlasov equations.

In addition, Hadamard well-posedness is proved in [18] by application of a finite difference method which allows for the rigorous derivation of an appropriate energy identity for the system.

It is the aim of this work to further extend the techniques developed in the case of “flat” geometry [26,18] to the curved case required by a shell model. The final result is full Hadamard well-posedness of finite energy solutions governing nonlinear dynamic shells.

Throughout this paper we will denote by \mathcal{S}_l the shell and by Γ its mid-surface. We shall make the following hypothesis:

- (i) The shell is assumed to be made of an isotropic and homogeneous material, so that the Lamé coefficients $\lambda > 0$ and $\mu > 0$ are constant. The density of the material will be denoted as $\rho > 0$.
- (ii) (shallowness assumption) The thickness l of the shell is small enough to accommodate the curvatures H and K , *i.e.* the product of the thickness and the curvatures is small as compared to 1. In addition, the shell is shallow in the sense that the second fundamental form (here given in terms of the oriented distance function as D^2b) and its derivative (D^3b) are small. This assumption allows us to neglect certain terms in the strain energy in comparison to the model in [5,6] and yields a model in which the coupling between the normal and tangential displacements is of the first (rather than third) order. For a detailed justification of these assumptions we refer the reader to Koiter [20,21].
- (iii) (Kirchhoff hypothesis) In the classical thin plate theory named after Kirchhoff, the displacement vectors of the shell \mathcal{S}_l and of the mid-surface Γ are related by the hypothesis that the filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed surface, and undergo neither contraction nor extension. We generalize this hypothesis to the case of a shell using the intrinsic geometry in [5].

For the sake of self-containment, we provide background material on intrinsic modeling in Section 2. In Section 3.3, we state the main findings of this paper. Section 5 is concerned with establishing preliminary results that will be used in the subsequent sections. Section 4 is concerned with describing the model [2, Section 7.3] and subsequently justifying the investigation of the equations presented in (12). We prove the existence result in Section 6, the uniqueness result in Section 7, and the continuous dependence with respect to the initial data in Section 8.

2. Intrinsic modeling

In this section we present a brief overview of the oriented distance function and the intrinsic tangential calculus that forms the basis of our shell model.

2.1. The oriented distance function and the intrinsic geometry

In order to improve readability we here include a brief discussion of the oriented distance function and the intrinsic geometric methods of Delfour and Zolésio. Since by necessity this overview will lack detail, the reader is referred to [12,11] for a definitive exposition on this topic.

Consider a domain $\mathcal{O} \subset \mathbb{R}^3$ whose non-empty boundary $\partial\mathcal{O}$ is a C^1 two-dimensional submanifold of \mathbb{R}^3 . Define the oriented (or signed) distance function to \mathcal{O} as

$$b(x) = d_{\mathcal{O}}(x) - d_{\mathbb{R}^3 \setminus \mathcal{O}}(x) \tag{1}$$

where d is the Euclidean distance from the point x to the domain \mathcal{O} . In other words, $b(x)$ is simply the positive or negative distance to the boundary $\partial\mathcal{O}$, depending on whether we are outside or inside the domain \mathcal{O} . It can be shown that for every $x \in \partial\mathcal{O}$, there exists a neighborhood where the function $\nabla b = \nu$, the unit outward external normal to $\partial\mathcal{O}$ [11].

Consider a subset $\Gamma \subseteq \partial\mathcal{O}$ which will eventually become the mid-surface of our shell. We define the projection $p(x)$ of a point x onto Γ as $p(x) = x - b(x)\nabla b(x)$. Then, we define a shell \mathcal{S}_l of thickness l as (see Fig. 1)

$$\mathcal{S}_l(\Gamma) \equiv \left\{ x \in \mathbb{R}^3 : p(x) \in \Gamma, |b(x)| < l/2 \right\}. \tag{2}$$

A natural curvilinear coordinate system (X, z) is thus induced on the shell \mathcal{S}_l , where the coordinate vector X gives the position of a point on the mid-surface Γ , and $z \in (-\frac{l}{2}, \frac{l}{2})$ gives the vertical (normal) distance from the mid-surface.

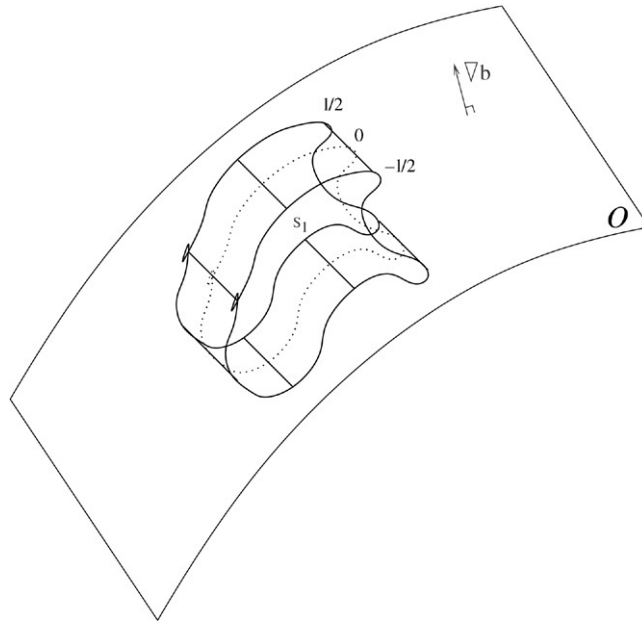


Fig. 1. The shell.

Using this notation, we also define the “flow mapping” $T_z(X)$ as

$$T_z(X) = X + z\nabla b(X) \tag{3}$$

for all X and z in S_l . This allows us to reconstruct the action at a given height z of the shell, once we know the action of the mid-surface Γ . Define as Γ^z the surface $T_z(\Gamma)$ at the ‘altitude’ z . Then, one can also describe the shell S_l as

$$S_l = \bigcup_{z=-1/2}^{1/2} \Gamma_z.$$

The curvatures of the shell will be denoted as H and K . These can be reconstructed from the boundary distance function $b(x)$ by noting that at any point (X, z) , the matrix D^2b has eigenvalues $0, \lambda_1, \lambda_2$. The curvatures are then given by $\text{tr}(D^2b) = 2H = \lambda_1 + \lambda_2$ and $K = \lambda_1\lambda_2$.

2.2. Tangential differential calculus

Next, we mention briefly some useful aspects of the tangential differential calculus. Given $f \in C^1(\Gamma)$, we define the tangential gradient $\nabla_\Gamma f$ of the scalar function f by means of the projection as

$$\nabla_\Gamma f \equiv \nabla(f \circ p)(x)|_\Gamma. \tag{4a}$$

This notion of the tangential gradient is equivalent to the classical definition using an extension F of f in the neighborhood of Γ , i.e. $\nabla_\Gamma f = \nabla F|_\Gamma - \frac{\partial F}{\partial \nu} \nu$ [11]. Following the same idea we can define the tangential Jacobian matrix of a vector function $v \in C^1(\Gamma)^3$ as

$$D_\Gamma v \equiv D(v \circ p)|_\Gamma \quad \text{or} \quad (D_\Gamma v)_{ij} = (\nabla_\Gamma v_i)_j, \tag{4b}$$

the tangential divergence as

$$\text{div}_\Gamma v \equiv \text{div}(v \circ p)|_\Gamma, \tag{4c}$$

the Hessian $D_\Gamma^2 f$ of $f \in C^2(\Gamma)$ as

$$D_\Gamma^2 f = D_\Gamma(\nabla_\Gamma f), \tag{4d}$$

the Laplace–Beltrami operator of $f \in C^2(\Gamma)$ as

$$\Delta_\Gamma f \equiv \operatorname{div}_\Gamma (\nabla_\Gamma f) = \Delta(f \circ p)|_\Gamma, \tag{4e}$$

the tangential linear strain tensor of elasticity as

$$\varepsilon_\Gamma(v) \equiv \frac{1}{2} (D_\Gamma v + {}^*D_\Gamma v) = \varepsilon(v \circ p)|_\Gamma \tag{4f}$$

and the tangential vectorial divergence of a second-order tensor A as

$$\operatorname{div}_\Gamma A \equiv \operatorname{div}(A \circ p)|_\Gamma = \operatorname{div}_\Gamma A_i. \tag{4g}$$

Using the definitions given in (4) one can derive Green’s formula in the tangential calculus [11]:

$$\int_\Gamma f \operatorname{div}_\Gamma v \, d\Gamma + \int_\Gamma \langle \nabla_\Gamma f, v \rangle \, d\Gamma = \int_\Upsilon \langle f v, v \rangle \, d\Upsilon + 2 \int_\Gamma f H \langle v, \nabla b \rangle \, d\Gamma \tag{5}$$

where v is the outward unit normal to the curve Υ . We use the definition of $\varepsilon_\Gamma(u)$ and of the tangential vectorial divergence of a second-order tensor A as

$$\operatorname{div}_\Gamma A \equiv \operatorname{div}(A \circ p)|_\Gamma = \operatorname{div}_\Gamma A_i \tag{6}$$

to derive the integration formula

$$\int_\Gamma \operatorname{tr}(\varepsilon_\Gamma(u)A) \, d\Gamma = \int_{\partial\Gamma} \langle u, Av \rangle \, d(\partial\Gamma) - \int_\Gamma \langle u, \operatorname{div}_\Gamma A \rangle \, d\Gamma + \int_\Gamma 2H \langle u, A \nabla b \rangle \, d\Gamma \tag{7}$$

for A a symmetric matrix. Finally, from [11,12] we have that

$$\langle \nabla_\Gamma w, \nabla b \rangle = 0, \quad D_\Gamma v \nabla b = 0 \tag{8}$$

by definition for any scalar w and vector v . In addition, if we consider a purely tangent vector $v = v_\Gamma$, *i.e.* $\langle v_\Gamma, \nabla b \rangle = 0$, we can take the tangential gradient of both sides of this expression and derive the following useful formula

$$D^2b v_\Gamma + {}^*D_\Gamma v_\Gamma \nabla b = 0. \tag{9}$$

In the case of tangential v_Γ , the last term of Eq. (5) is also zero. We conclude this section with some remarks about notation. We shall adopt the following notation:

$$|w|_{s,\Gamma} \equiv |w|_{H^s(\Gamma)}; \quad (u, v)_\Gamma \equiv \int_\Gamma uv \, d\Gamma$$

and use $\langle \cdot, \cdot \rangle$ to denote the scalar product of two vectors. Throughout this paper the conventions of [16] concerning tensors are used. For instance, we will make no distinction between a second-order tensor and a matrix, nor will we make a distinction between a first-order tensor and a vector. Consequently we will not distinguish simple contraction and multiplication. Finally, the notation $\partial_t \varphi$ is used to denote the partial derivative of φ with respect to the time variable, and the notation $A \cdot \cdot B$ denotes the double contraction of two matrices — *i.e.* $A \cdot \cdot B = \operatorname{tr}(AB)$.

2.3. Some tangential operators and spaces

Let \mathbf{e} be a vector function; we denote by e_Γ and w the tangential component of \mathbf{e} and its algebraic magnitude on the normal part respectively, *i.e.*

$$w = \langle \mathbf{e}, \nabla b \rangle \tag{10a}$$

$$e_\Gamma = \mathbf{e} - w \nabla b. \tag{10b}$$

Vector function \mathbf{e} will physically represent the displacement of the mid-surface; its components e_Γ and w will represent the in-place displacement and the normal deflection respectively. Following the definitions introduced in [5], we will use the subsequent notation.

Definition 1. We define

$$\begin{aligned}
 G_\Gamma w &= \frac{1}{2}((\nabla b \otimes \nabla_\Gamma w)D^2b + D^2b(\nabla_\Gamma w \otimes \nabla b)) \\
 V_\Gamma e_\Gamma &= \frac{1}{2}((D^2b e_\Gamma) \otimes \nabla b + \nabla b \otimes (D^2b e_\Gamma)) \\
 S_\Gamma w &= \frac{1}{2}(D_\Gamma^2 w + {}^*D_\Gamma^2 w).
 \end{aligned}$$

Operator G_Γ is a first-order tangential operator, V_Γ is a zero-order tangential operator, and S_Γ is the symmetrization of the Hessian (which is *not* symmetric in the tangential calculus [11]).

Definition 2. Let us define

$$\begin{aligned}
 B_M(\mathbf{e}) &= \varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma + wD^2b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w \\
 B_F(\mathbf{e}) &= S_\Gamma w + G_\Gamma w.
 \end{aligned}$$

Operators B_M and B_F will correspond to the membrane and bending (flexural) strains respectively. In the rest of this paper the following functional space will be useful.

Definition 3. For $m \in \mathbb{N}$, define the spaces V^m as

$$V^m(\Gamma) = \left\{ \mathbf{e} \in [H^m(\Gamma)]^2 \times H^{m+1}(\Gamma) \mid e_\Gamma = 0, w = 0, \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial\Gamma \right\}. \tag{11}$$

3. Statement of the main results

3.1. Equation to be studied

Let $\gamma = l^2/12$, and for a matrix A define operator \mathcal{C} by

$$\mathcal{C}(A) = \lambda(\text{tr } A)I + 2\mu A.$$

We consider the following equation in the variable $\mathbf{e} = [e_\Gamma, w]$, where e_Γ and w are respectively the tangential and vertical displacements and are defined above in Eqs. (10):

$$\left\{ \begin{aligned}
 &\rho \partial_{tt} w - \rho \gamma \Delta_\Gamma \partial_{tt} w + (\lambda + 2\mu)\gamma \Delta_\Gamma^2 w + \rho \frac{\gamma}{2} \text{div}_\Gamma(D^2b \partial_{tt} e_\Gamma) + \text{tr}(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)D^2b) \\
 &\quad + \text{tr}\left(\mathcal{C}\left(wD^2b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w\right)D^2b\right) - \text{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)\nabla_\Gamma w) \\
 &\quad - \text{div}_\Gamma\left(\mathcal{C}\left(wD^2b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w\right)\nabla_\Gamma w\right) + 2\mu\gamma \text{div}_\Gamma(K\nabla_\Gamma w) + 2\mu\gamma \text{div}_\Gamma((D^2b)^2\nabla w) = 0 \\
 &\rho(I + \gamma(D^2b)^2)\partial_{tt} e_\Gamma - \rho \frac{\gamma}{2} D^2b \partial_{tt} w - \text{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)) \\
 &\quad - \text{div}_\Gamma\left(\mathcal{C}\left(wD^2b + \frac{1}{2}\nabla_\Gamma w \otimes \nabla_\Gamma w\right)\right) = 0
 \end{aligned} \right. \tag{12}$$

where the derivatives are to be understood in the distributional sense.

Let $k = \lambda H^2 + \mu(H^2 - 2K)$ be a positive real number and $\mathbf{m}(\mathbf{e}, \hat{\mathbf{e}})$, $\mathbf{a}(\mathbf{e}, \hat{\mathbf{e}})$, $\mathbf{n}(\mathbf{e}, \hat{\mathbf{e}})$ be defined by

$$\begin{aligned}
 \mathbf{m}(\mathbf{e}, \hat{\mathbf{e}}) &= \rho[2(e_\Gamma, \hat{e}_\Gamma)_\Gamma + 2\gamma((D^2b)e_\Gamma, (D^2b)\hat{e}_\Gamma)_\Gamma - \gamma(\nabla_\Gamma w, (D^2b)\hat{e}_\Gamma)_\Gamma \\
 &\quad - \gamma((D^2b)e_\Gamma, \nabla_\Gamma \hat{w})_\Gamma + 2(w, \hat{w})_\Gamma + 2\gamma(\nabla_\Gamma w, \nabla_\Gamma \hat{w})_\Gamma] \\
 \mathbf{a}(\mathbf{e}, \hat{\mathbf{e}}) &= 8\lambda(kw, k\hat{w})_\Gamma + 4\lambda(H \text{div}_\Gamma e_\Gamma, \hat{w})_\Gamma + 4\lambda(Hw, \text{div}_\Gamma \hat{e}_\Gamma)_\Gamma \\
 &\quad + 2\lambda(\text{div}_\Gamma e_\Gamma, \text{div}_\Gamma \hat{e}_\Gamma)_\Gamma + 4\mu \int_\Gamma \text{tr}((\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)(\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma))
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 &+ 4\mu \int_{\Gamma} w \operatorname{tr}((\varepsilon_{\Gamma}(\hat{e}_{\Gamma}) + V_{\Gamma} \hat{e}_{\Gamma}) D^2 b) + 4\mu \int_{\Gamma} \hat{w} \operatorname{tr}((\varepsilon_{\Gamma}(e_{\Gamma}) + V_{\Gamma} e_{\Gamma}) D^2 b) \\
 &+ 2\gamma \lambda (\Delta_{\Gamma} w, \Delta_{\Gamma} \hat{w})_{\Gamma} + 4\gamma \mu \int_{\Gamma} \operatorname{tr}((S_{\Gamma} w + G_{\Gamma} w)(S_{\Gamma} \hat{w} + G_{\Gamma} \hat{w}))
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 -\mathbf{n}(\mathbf{e}, \hat{\mathbf{e}}) &= 2\lambda(H\|\nabla_{\Gamma} w\|^2, \hat{w})_{\Gamma} + \lambda(\|\nabla_{\Gamma} w\|^2, \operatorname{div}_{\Gamma} \hat{e}_{\Gamma})_{\Gamma} \\
 &+ 4\lambda(Hw \nabla_{\Gamma} w, \nabla_{\Gamma} \hat{w})_{\Gamma} + 2\lambda(\operatorname{div}_{\Gamma} e_{\Gamma} \nabla_{\Gamma} w, \nabla_{\Gamma} \hat{w})_{\Gamma} \\
 &+ (\lambda + 2\mu)(\|\nabla_{\Gamma} w\|^2 \nabla_{\Gamma} w, \nabla_{\Gamma} \hat{w})_{\Gamma} + 2\mu \int_{\Gamma} \operatorname{tr}(\varepsilon_{\Gamma}(\hat{e}_{\Gamma})(\nabla_{\Gamma} w \otimes \nabla_{\Gamma} w)) \\
 &+ 2\mu \int_{\Gamma} \hat{w} \operatorname{tr}(D^2 b(\nabla_{\Gamma} w \otimes \nabla_{\Gamma} w)) + 4\mu \int_{\Gamma} w \langle D^2 b \nabla_{\Gamma} w, \nabla_{\Gamma} \hat{w} \rangle \\
 &+ 4\mu \int_{\Gamma} \operatorname{tr}((\nabla_{\Gamma} w \otimes \nabla_{\Gamma} \hat{w})(\varepsilon_{\Gamma}(e_{\Gamma}) + V_{\Gamma} e_{\Gamma})).
 \end{aligned} \tag{15}$$

The variational problem associated with (12) is, for each $t \in [0, \tau[$ and all test functions $\hat{\mathbf{e}} \in V^1(\Gamma)$,

$$\mathbf{m}(\partial_{tt} \mathbf{e}, \hat{\mathbf{e}}) + \mathbf{a}(\mathbf{e}, \hat{\mathbf{e}}) - \mathbf{n}(\mathbf{e}, \hat{\mathbf{e}}) = 0 \tag{16}$$

where $\mathbf{e}(0) = \mathbf{e}^0$ and $\partial_t \mathbf{e}(0) = \mathbf{e}^1$.

Definition 4 (Weak Solutions). We say that $\mathbf{e} \in C^0([0, \tau[, V^1(\Gamma)) \cap C^1([0, \tau[, V^0(\Gamma))$ is a weak solution to the small finite deflections shell problem if and only if (16) is satisfied.

Definition 5 (Regular Solutions). We say that \mathbf{e} is a regular solution to the small finite deflections shell problem if and only if it is a weak solution (i.e. satisfies Eq. (16)) and $\mathbf{e} \in C^0([0, T[, V^2(\Gamma)) \cap C^1([0, \tau[, V^1(\Gamma)) \cap C^2([0, \tau[, V^0(\Gamma))$.

Remark 1. In the special case of a plate, the curvature tensor $D^2 b$ vanishes so all the terms including it, like $G_{\Gamma}, V_{\Gamma}, K$, vanish as well; we obtain:

$$\begin{cases} \rho \partial_{tt} w - \rho \gamma \Delta_{\Gamma} \partial_{tt} w + (\lambda + 2\mu) \gamma \Delta_{\Gamma}^2 w - \operatorname{div}_{\Gamma} \left(\mathcal{C} \left(\varepsilon_{\Gamma} e_{\Gamma} + \frac{1}{2} \nabla_{\Gamma} w \otimes \nabla_{\Gamma} w \right) \nabla_{\Gamma} w \right) = 0 \\ \rho \partial_{tt} e_{\Gamma} - \operatorname{div}_{\Gamma} \left(\mathcal{C}(\varepsilon_{\Gamma} e_{\Gamma}) + \frac{1}{2} \nabla_{\Gamma} w \otimes \nabla_{\Gamma} w \right) = 0 \end{cases}$$

which is the full von Kármán system; see [24].

3.2. Operators

Definition 6. Let \mathcal{M}, \mathcal{A} be the linear operators defined by

$$(\mathcal{M}\mathbf{e}, \hat{\mathbf{e}}) = \mathbf{m}(\mathbf{e}, \hat{\mathbf{e}}); \quad (\mathcal{A}\mathbf{e}, \hat{\mathbf{e}}) = \mathbf{a}(\mathbf{e}, \hat{\mathbf{e}}),$$

let

$$\mathcal{H} = \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}^{1/2}) = V^1(\Gamma) \times V^0(\Gamma) = [H^1(\Gamma)]^2 \times H^2(\Gamma) \times [(L^2(\Gamma))]^2 \times H^1(\Gamma) \tag{17}$$

and let $\mathbb{A} : \mathcal{H} \rightarrow \mathcal{H}$ be defined by

$$\begin{aligned}
 \mathbb{A} &= \begin{pmatrix} 0 & -I \\ \mathcal{M}^{-1} \mathcal{A} & 0 \end{pmatrix} \\
 \mathcal{D}(\mathbb{A}) &= \{(e_1, e_2) \in \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}), \mathcal{A}e_1 \in [\mathcal{D}(\mathcal{M}^{1/2})]'\}.
 \end{aligned} \tag{18}$$

Remark 2. A more compact but less explicit expression of the weak formulation can be given by

$$\frac{1}{2} (\mathcal{M} \partial_{tt} \mathbf{e}, \hat{\mathbf{e}})_{\Gamma} + \int_{\Gamma} (\mathcal{C} B_M(\mathbf{e})) \cdot \cdot B'_M(\mathbf{e}, \hat{\mathbf{e}}) + \gamma \int_{\Gamma} (\mathcal{C} B_F(\mathbf{e})) \cdot \cdot B_F(\hat{\mathbf{e}}) = 0, \quad \forall \hat{\mathbf{e}} \in V^1(\Gamma), \tag{19}$$

where

$$B'_M(\mathbf{e}, \hat{\mathbf{e}}) = \left. \frac{\partial B_M(\mathbf{e} + \theta \hat{\mathbf{e}})}{\partial \theta} \right|_{\theta=0} = \varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma + \hat{w} D^2 b + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w} + \frac{1}{2} \nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w.$$

This compact expression (19) will be used in the proof of the existence of regular solutions (Section 6), while the more detailed expression will be used in the proof of the uniqueness (Section 7).

Definition 7. Let N be the nonlinear operator defined by $(N(\mathbf{e}), \hat{\mathbf{e}}) = \mathbf{n}(\mathbf{e}, \hat{\mathbf{e}})$ and N_Γ, N_n be respectively the vector function and scalar function defined by

$$\begin{aligned} N_\Gamma(w) &= -\lambda \nabla_\Gamma (\|\nabla_\Gamma w\|^2) + 2\mu \operatorname{div}_\Gamma (\nabla_\Gamma w \otimes \nabla_\Gamma w) \\ N_n(e_\Gamma, w) &= -2\lambda H \|\nabla_\Gamma w\|^2 + 4\lambda \operatorname{div}_\Gamma (H w \nabla_\Gamma w) + 4\mu \operatorname{div}_\Gamma (w D^2 b \nabla_\Gamma w) \\ &\quad + (\lambda + 2\mu) \operatorname{div}_\Gamma (\|\nabla_\Gamma w\|^2 \nabla_\Gamma w) - 2\mu (D^2 b \nabla_\Gamma w, \nabla_\Gamma w) \\ &\quad - 2\lambda \operatorname{div}_\Gamma (\operatorname{div}_\Gamma e_\Gamma \nabla_\Gamma w) - 4\mu \operatorname{div}_\Gamma ((\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) \nabla_\Gamma w) \end{aligned}$$

so we have $N(\mathbf{e}) = N_\Gamma(w) + N_n(e_\Gamma, w) \nabla b$.

It follows, from (16), that:

$$\partial_t \begin{pmatrix} \mathbf{e} \\ \partial_t \mathbf{e} \end{pmatrix} + \mathbb{A} \begin{pmatrix} \mathbf{e} \\ \partial_t \mathbf{e} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{M}^{-1} N(\mathbf{e}) \end{pmatrix}. \tag{20}$$

3.3. Main results

Our main results are:

Theorem 1 (Regular Solutions). We consider Eq. (16). Assume that $\mathbf{e}(0) \in V^2(\Gamma)$ and $\partial_t \mathbf{e}(0) \in V^1(\Gamma)$. Then, there exists a unique regular global solution

$$\mathbf{e} \in C([0, \tau[; V^2(\Gamma)) \cap C^1([0, \tau[; V^1(\Gamma)) \cap C^2([0, \tau[; V^0(\Gamma)) \tag{21}$$

where $\tau > 0$ is arbitrary.

Theorem 2 (Existence and Uniqueness of Weak Solutions). We consider Eq. (16). Assume that $\mathbf{e}(0) \in V^1(\Gamma)$ and $\partial_t \mathbf{e}(0) \in V^0(\Gamma)$. Then there exists a weak solution

$$\mathbf{e} \in C([0, \tau[; V^1(\Gamma)) \cap C^1([0, \tau[; V^0(\Gamma)) \tag{22}$$

which is unique, where $\tau > 0$ is arbitrary.

Theorem 3 (Continuous Dependence w.r.t. Initial Data). Weak solutions to Eq. (16) depend continuously on the initial data in the finite energy norm. That is to say, for all $\tau > 0$ and all sequences of initial data such that

$$\begin{aligned} \mathbf{e}^n(t) &\rightarrow \mathbf{e}^0 \quad \text{in } V^1(\Gamma) \\ \partial_t \mathbf{e}^n(t) &\rightarrow \mathbf{e}^1 \quad \text{in } V^0(\Gamma) \end{aligned}$$

the corresponding solutions $\mathbf{e}^n(t) \in C([0, \tau[; V^1(\Gamma))$ and $\mathbf{e}(t) \in C([0, \tau[; V^1(\Gamma))$ satisfy:

$$\begin{aligned} \mathbf{e}^n &\rightarrow \mathbf{e} \quad \text{in } C([0, \tau[; V^1(\Gamma)) \\ \partial_t \mathbf{e}^n &\rightarrow \partial_t \mathbf{e} \quad \text{in } C([0, \tau[; V^0(\Gamma)). \end{aligned}$$

As a consequence we obtain an energy identity for weak solutions.

The result presented in Theorem 1 is standard for an equation of the type of (16), though no references seem to be available for the proof this particular result. Theorems 2 and 3 are new and solve an old problem in the field.

The reminder of the paper is devoted to the proofs of the three results stated above. Theorem 1 is proved in Section 6. Theorem 2 is proved in Section 7. Theorem 3 is proved in Section 8.

4. Derivation of the model

This section is devoted to deriving the model that leads to Eq. (12), which justifies the interest raised by this equation in the first place. The model will be obtained using Hamilton’s principle. The next two subsections are devoted to computing the energies. In Section 4.3 Hamilton’s principle is used to get the weak formulation, and in Section 4.4 the PDE formulation is derived. Finally in Section 4.5, the total energy of the system is computed and the system is proved to be conservative (Proposition 1).

The reader solely interested in the Hadamard well-posedness of (12) may skip this section as notation introduced herein is explicitly referenced in the next sections and Proposition 1 is fairly standard.

Since the deflections are not infinitesimal, the strain–displacement relationship is not assumed to be purely linear, and instead is of the form given in Eq. (27) (see [9,22]). The displacement of the shell T and the displacement of the mid-surface e are related by the Kirchhoff hypothesis, that is that the filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed surface, and undergo neither contraction nor extension. Following [5], we have

$$T = \mathbf{e} \circ p - b (*D_{\Gamma} \mathbf{e} \nabla b) \circ p \tag{23}$$

which leads to

$$T = e_{\Gamma} \circ p + (w \nabla b) \circ p + b (D^2 b e_{\Gamma} - \nabla_{\Gamma} w) \circ p. \tag{24}$$

4.1. The kinetic energy

The kinetic energy is given by

$$\mathcal{E}_k(\mathbf{e}) = \frac{\rho}{2} \int_{S_t} |\partial_t T|^2. \tag{25}$$

Following [5], the kinetic energy of the system at any time t is given by

$$\mathcal{E}_k(\mathbf{e}) = \frac{\rho l}{2} \int_{\Gamma} |\partial_t e_{\Gamma}|^2 + |\partial_t w|^2 + \frac{\rho l \gamma}{2} \int_{\Gamma} |D^2 b \partial_t e_{\Gamma}|^2 + |\nabla_{\Gamma} \partial_t w|^2 - \langle D^2 b \partial_t e_{\Gamma}, \nabla_{\Gamma} \partial_t w \rangle. \tag{26}$$

4.2. The elastic energy

The key hypothesis, and main difference from the model studied in [5], is the strain–displacement relations

$$\varepsilon(T) = \frac{1}{2} (*DT + DT) + \frac{1}{2} *DT DT. \tag{27}$$

Following [21, Section 11], several parts of the nonlinear term $*DT DT$ will be neglected; this will in particular be the case for all the terms involving $*D_{\Gamma} e_{\Gamma}$ that appear in $*DT DT$. This is consistent with the shallowness assumption (Hypothesis (ii), Section 1), as these nonlinear terms appear in combination with higher order derivatives of the oriented distance function b . Eqs. (23) and (27) yield

$$\varepsilon(T) = \left(\varepsilon_{\Gamma}(e_{\Gamma}) + w D^2 b + V_{\Gamma} e_{\Gamma} + \frac{1}{2} \nabla_{\Gamma} w \otimes \nabla_{\Gamma} w + \frac{1}{2} \|\nabla_{\Gamma} w\|^2 \nabla b \otimes \nabla b \right) \circ p - b (S_{\Gamma} w + G_{\Gamma} w) \circ p. \tag{28}$$

From now on, and when no confusion is possible, we will write ε instead of $\varepsilon(T)$.

Assuming isotropy and homogeneity of the material, one can apply Hooke’s law to give the elastic energy:

$$\mathcal{E}_p(\mathbf{e}) = \frac{1}{2} \int_{S_t} \mathcal{C}(\varepsilon) \cdot \cdot \varepsilon. \tag{29}$$

At this point in the computation of the elastic energy, it is customary to impose the hypothesis of plane stresses: $\sigma \cdot \cdot (\nabla b \otimes \nabla b) = 0$ (which in local coordinates is denoted as $\sigma_{33} = 0$). In order to satisfy this requirement, we change $\varepsilon(T)$ to

$$\varepsilon(T) = \left(\varepsilon_\Gamma(e_\Gamma) + wD^2b + V_\Gamma e_\Gamma + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) \circ p - b (S_\Gamma w + G_\Gamma w) \circ p. \tag{30}$$

With the notation of **Definition 2** we have

$$\varepsilon(T) = (B_M(\mathbf{e})) \circ p + b (B_F(\mathbf{e})) \circ p.$$

This new form of $\varepsilon(T)$ is in line with [2, page 38, Section 7.3]. Eq. (29) yields

$$\begin{aligned} \mathcal{E}_p(\mathbf{e}) &= \frac{1}{2} \int_{S_l} [\mathcal{C}(B_M(\mathbf{e})) \cdot \cdot B_M(\mathbf{e})) \circ p + \frac{1}{2} \int_{S_l} b[\mathcal{C}(B_M(\mathbf{e})) \cdot \cdot B_F(\mathbf{e})) + \mathcal{C}(B_F(\mathbf{e})) \cdot \cdot B_M(\mathbf{e})) \circ p \\ &\quad + \frac{1}{2} \int_{S_l} b^2[\mathcal{C}(B_F(\mathbf{e})) \cdot \cdot B_F(\mathbf{e})) \circ p. \end{aligned} \tag{31}$$

We define

$$\mathcal{E}_p^M(\mathbf{e}) = \frac{1}{2} \int_{S_l} [\mathcal{C}(B_M(\mathbf{e})) \cdot \cdot B_M(\mathbf{e})) \circ p \tag{32}$$

$$\mathcal{E}_p^F(\mathbf{e}) = \frac{1}{2} \int_{S_l} b^2[\mathcal{C}(B_F(\mathbf{e})) \cdot \cdot B_F(\mathbf{e})) \circ p \tag{33}$$

as the membrane and bending (flexural) energy, respectively. Since the second integral of the right hand side of (31) vanishes, we have

$$\mathcal{E}_p(\mathbf{e}) = \mathcal{E}_p^M(\mathbf{e}) + \mathcal{E}_p^F(\mathbf{e}).$$

We compute the explicit form of the bending energy:

$$\begin{aligned} \mathcal{E}_p^F(\mathbf{e}) &= \frac{1}{2} \int_{S_l} b^2[\mathcal{C}(B_F(\mathbf{e})) \cdot \cdot B_F(\mathbf{e})) \circ p \\ &= \frac{\lambda}{2} \int_{S_l} b^2(\text{tr}(B_F(\varepsilon)))^2 \circ p + \mu \int_{S_l} b^2 \text{tr}((B_F \varepsilon)^2) \circ p \\ &= \frac{\lambda}{2} \int_{-l/2}^{l/2} z^2 \int_{\Gamma^z} (\text{tr}(B_F(\varepsilon)))^2 + \mu \int_{-l/2}^{l/2} z^2 \int_{\Gamma^z} \text{tr}((B_F \varepsilon)^2) \\ &= \frac{\lambda l^3}{24} \int_\Gamma (\text{tr}(B_F(\varepsilon)))^2 + \frac{\mu l^3}{12} \int_\Gamma \text{tr}((B_F \varepsilon)^2). \end{aligned}$$

Using $\text{tr } G_\Gamma w = 0$ and $\text{tr } S_\Gamma w = \Delta_\Gamma w$ we get

$$\mathcal{E}_p^F(\mathbf{e}) = \frac{\lambda l^3}{24} \|\Delta_\Gamma w\|_{L_2(\Gamma)}^2 + \frac{\mu l^3}{12} \int_\Gamma \text{tr}((S_\Gamma w + G_\Gamma w)^2). \tag{34}$$

Analogous computations can be carried out with $\mathcal{E}_p^M(\mathbf{e})$. We use that $D^2b \nabla b = 0$ and that for any vector u we have $\text{tr } V_\Gamma u = 0$ and $\text{tr}(V_\Gamma u D^2b) = 0$ for any vector u . We also use that $\text{tr}(\nabla_\Gamma w \otimes \nabla_\Gamma w) = \|\nabla_\Gamma w\|^2$ and that $\Delta b = \text{tr } D^2b = 2H$. We obtain that the elastic energy of the system is given by

$$\begin{aligned} \mathcal{E}_p(\mathbf{e}) &= \frac{\lambda l}{2} \left| 2H w + \text{div}_\Gamma e_\Gamma + \frac{1}{2} \|\nabla_\Gamma w\|^2 \right|_{0,\Gamma}^2 \\ &\quad + \mu l \int_\Gamma \text{tr} \left[\left(\varepsilon_\Gamma(e_\Gamma) + D^2b w + V_\Gamma e_\Gamma + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right)^2 \right] \end{aligned}$$

$$+ \frac{\lambda l^3}{24} |\Delta_\Gamma w|_{0,\Gamma}^2 + \frac{\mu l^3}{12} \int_\Gamma \text{tr} \left[(S_\Gamma w + G_\Gamma w)^2 \right]. \tag{35}$$

4.3. *Weak formulation of the model*

Let us consider a final time $\tau > 0$. Among all kinematically admissible displacements, the actual motion of the shell will make stationary the Lagrangian

$$\mathcal{L}(\mathbf{e}) = \int_0^\tau \mathcal{E}_k(\mathbf{e}) - \mathcal{E}_p(\mathbf{e}) = \int_0^\tau \mathcal{E}_k(\mathbf{e}) - \mathcal{E}_p^M(\mathbf{e}) - \mathcal{E}_p^F(\mathbf{e}).$$

We consider the Gâteaux derivative in a direction $\hat{\mathbf{e}} = \hat{e}_\Gamma + \hat{w} \nabla b$; we have

$$\forall \hat{\mathbf{e}}, \left. \frac{\partial \mathcal{L}(\mathbf{e} + \theta \hat{\mathbf{e}})}{\partial \theta} \right|_{\theta=0} = 0. \tag{36}$$

Note that S symmetric implies $S \cdot \cdot M = S \cdot \cdot (*M + M)$ so the two last terms of $B'_M(\mathbf{e}, \hat{\mathbf{e}})$ will often be replaced by twice one of them.

$$\left. \frac{\partial \mathcal{E}_p^M(\mathbf{e} + \theta \hat{\mathbf{e}})}{\partial \theta} \right|_{\theta=0} = \int_\Gamma (CB_M(\mathbf{e})) \cdot \cdot B'_M(\mathbf{e}, \hat{\mathbf{e}}) \tag{37a}$$

$$\left. \frac{\partial \mathcal{E}_p^F(\mathbf{e} + \theta \hat{\mathbf{e}})}{\partial \theta} \right|_{\theta=0} = \gamma \int_\Gamma (CB_F(\mathbf{e})) \cdot \cdot B_F(\hat{\mathbf{e}}) \tag{37b}$$

and it follows that

$$\begin{aligned} \left. \frac{\partial \mathcal{E}_p(\mathbf{e} + \theta \hat{\mathbf{e}})}{\partial \theta} \right|_{\theta=0} &= \int_\Gamma (CB_M(\mathbf{e})) \cdot \cdot B'_M(\mathbf{e}, \hat{\mathbf{e}}) + \gamma \int_\Gamma (CB_F(\mathbf{e})) \cdot \cdot B_F(\hat{\mathbf{e}}) \\ &= \lambda l \int_\Gamma \left(2Hw + \text{div}_\Gamma e_\Gamma + \frac{1}{2} \|\nabla_\Gamma w\|^2 \right) (2H\hat{w} + \text{div}_\Gamma \hat{e}_\Gamma + \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle) \\ &\quad + 2\mu l \int_\Gamma \text{tr} \left(\left(\varepsilon_\Gamma e_\Gamma + D^2 b w + V_\Gamma e_\Gamma + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) \right. \\ &\quad \times \left. \left(\varepsilon_\Gamma \hat{e}_\Gamma + D^2 b \hat{w} + V_\Gamma \hat{e}_\Gamma + \frac{1}{2} \nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w} \right) \right) \\ &\quad + \frac{\lambda l^3}{12} \int_\Gamma \Delta_\Gamma w \Delta_\Gamma \hat{w} + \frac{\mu l^3}{6} \int_\Gamma \text{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma \hat{w} + G_\Gamma \hat{w})). \end{aligned}$$

Consequently, if \mathbf{e} is an admissible displacement of the mid-surface of the shell which is sufficiently regular then the following expression vanishes for any $\hat{\mathbf{e}} = \hat{e}_\Gamma + \hat{w} \nabla b \in V^1(\Gamma)$

$$\begin{aligned} &\int_0^\tau \int_\Gamma -2\rho \langle \partial_t e_\Gamma, \partial_t \hat{e}_\Gamma \rangle - 2\rho \partial_t w \partial_t \hat{w} + \gamma \int_0^\tau \int_\Gamma -2\rho (D^2 b \partial_t e_\Gamma, D^2 b \partial_t \hat{e}_\Gamma) - 2\rho \langle \nabla_\Gamma \partial_t w, \nabla_\Gamma \partial_t \hat{w} \rangle \\ &\quad + \rho \langle D^2 b \partial_t e_\Gamma, \nabla_\Gamma \partial_t \hat{w} \rangle + \rho \langle D^2 b \partial_t \hat{e}_\Gamma, \nabla_\Gamma \partial_t w \rangle \\ &\quad + \int_0^\tau \int_\Gamma 8\lambda H^2 w \hat{w} + 4\lambda H \text{div}_\Gamma e_\Gamma \hat{w} + \lambda H \|\nabla_\Gamma w\|^2 \hat{w} \\ &\quad + 4\lambda H w \text{div}_\Gamma \hat{e}_\Gamma + 2\lambda \text{div}_\Gamma e_\Gamma \text{div}_\Gamma \hat{e}_\Gamma + 2\lambda \|\nabla_\Gamma w\|^2 \text{div}_\Gamma \hat{e}_\Gamma \\ &\quad + 4\lambda H w \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle + 2\lambda \text{div}_\Gamma e_\Gamma \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle \\ &\quad + \lambda \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle \|\nabla_\Gamma w\|^2 + 4\mu \text{tr}((\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)(\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma)) \\ &\quad + 4\mu w \text{tr}((\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma) D^2 b) + 4\mu \hat{w} \text{tr}((\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma) D^2 b) + 8\mu (H^2 - 2K) w \hat{w} \\ &\quad + 2\mu \|\nabla_\Gamma w\|^2 \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle + 2\mu \text{tr}((\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma)(\nabla_\Gamma w \otimes \nabla_\Gamma w)) + 2\mu \hat{w} \text{tr}(D^2 b(\nabla_\Gamma w \otimes \nabla_\Gamma w)) \\ &\quad + 2\mu \text{tr}((\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w + \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w})(\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)) \end{aligned}$$

$$\begin{aligned}
 &+ 2\mu w \text{tr}((\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w + \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w}) D^2 b) \\
 &+ \gamma \int_0^\tau \int_\Gamma 2\lambda(\Delta_\Gamma w)(\Delta_\Gamma \hat{w}) + 4\mu \text{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma \hat{w} + G_\Gamma \hat{w})).
 \end{aligned}$$

Therefore, for all $\hat{e}_\Gamma \in H_0^1(\Gamma)$, we have

$$\begin{aligned}
 &\int_0^\tau \int_\Gamma -2\rho \langle \partial_t e_\Gamma, \partial_t \hat{e}_\Gamma \rangle + \gamma \int_0^\tau \int_\Gamma -2\rho \langle D^2 b \partial_t e_\Gamma, D^2 b \partial_t \hat{e}_\Gamma \rangle + \rho \langle D^2 b \partial_t \hat{e}_\Gamma, \nabla_\Gamma \partial_t w \rangle \\
 &\quad + 4\lambda H w \text{div}_\Gamma \hat{e}_\Gamma + 2\lambda \text{div}_\Gamma e_\Gamma \text{div}_\Gamma \hat{e}_\Gamma + \lambda \|\nabla_\Gamma w\|^2 \text{div}_\Gamma \hat{e}_\Gamma \\
 &\quad + 4\mu \text{tr}((\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)(\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma)) + 4\mu w \text{tr}((\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma) D^2 b) \\
 &\quad + 2\mu \text{tr}((\varepsilon_\Gamma(\hat{e}_\Gamma) + V_\Gamma \hat{e}_\Gamma)(\nabla_\Gamma w \otimes \nabla_\Gamma w)) = 0
 \end{aligned}$$

and for all $\hat{w} \in H_0^2(\Gamma) \cap H_0^1(\Gamma)$ we have

$$\begin{aligned}
 &\int_0^\tau \int_\Gamma -2\rho \partial_t w \partial_t \hat{w} + \gamma \int_0^\tau \int_\Gamma -2\rho \langle \nabla_\Gamma \partial_t w, \nabla_\Gamma \partial_t \hat{w} \rangle + \rho \langle D^2 b \partial_t e_\Gamma, \nabla_\Gamma \partial_t \hat{w} \rangle \\
 &\quad + \int_0^\tau \int_\Gamma 8\lambda H^2 w \hat{w} + 4\lambda H \text{div}_\Gamma e_\Gamma \hat{w} + 2\lambda H \|\nabla_\Gamma w\|^2 \hat{w} \\
 &\quad + 4\lambda H w \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle + 2\lambda \text{div}_\Gamma e_\Gamma \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle \\
 &\quad + \lambda \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle \|\nabla_\Gamma w\|^2 + 4\mu \hat{w} \text{tr}((\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma) D^2 b) \\
 &\quad + 8\mu(H^2 - 2K)w \hat{w} + 2\mu \|\nabla_\Gamma w\|^2 \langle \nabla_\Gamma w, \nabla_\Gamma \hat{w} \rangle + 2\mu \hat{w} \text{tr}(D^2 b(\nabla_\Gamma w \otimes \nabla_\Gamma w)) \\
 &\quad + 2\mu \text{tr}((\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w + \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w})(\varepsilon_\Gamma(e_\Gamma) + V_\Gamma e_\Gamma)) \\
 &\quad + 2\mu w \text{tr}((\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w + \nabla_\Gamma w \otimes \nabla_\Gamma \hat{w}) D^2 b) \\
 &\quad + \gamma \int_0^\tau \int_\Gamma 2\lambda(\Delta_\Gamma w)(\Delta_\Gamma \hat{w}) + 4\mu \text{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma \hat{w} + G_\Gamma \hat{w})) = 0.
 \end{aligned}$$

These equations are in line with [2, Section 7.3].

Lemma 1. *The following identities hold true*

$$\text{tr}(V_\Gamma u \nabla_\Gamma v \otimes \nabla_\Gamma v) = 0 \tag{38}$$

$$\int_\Gamma \text{tr}(\varepsilon_\Gamma u \nabla_\Gamma v \otimes \nabla_\Gamma v) = - \int_\Gamma \langle \text{div}_\Gamma(\nabla v \otimes \nabla v), u \rangle. \tag{39}$$

Proof. The term $\text{tr}(V_\Gamma u \nabla_\Gamma v \otimes \nabla_\Gamma v)$ can be written as

$$\begin{aligned}
 &\frac{1}{2} \text{tr}(((D^2 b u) \otimes \nabla b)(\nabla_\Gamma v \otimes \nabla_\Gamma v) + (\nabla b \otimes (D^2 b u))(\nabla_\Gamma v \otimes \nabla_\Gamma v)) \\
 &= \frac{1}{2} \text{tr}(\langle \nabla b, \nabla_\Gamma v \rangle (D^2 b u) \otimes \nabla_\Gamma v + \langle D^2 b u, \nabla_\Gamma v \rangle (\nabla b \otimes \nabla_\Gamma v)) \\
 &= \frac{1}{2} \langle \nabla b, \nabla_\Gamma v \rangle \langle D^2 b u, \nabla_\Gamma v \rangle + \frac{1}{2} \langle D^2 b u, \nabla_\Gamma v \rangle \langle \nabla b, \nabla_\Gamma v \rangle
 \end{aligned}$$

and since $\langle \nabla b, \nabla_\Gamma v \rangle = 0$ we obtain the first identity. The second one can be obtained by writing

$$\int_\Gamma \text{tr}(\varepsilon_\Gamma u \nabla_\Gamma v \otimes \nabla_\Gamma v) = \int_\Gamma \varepsilon_\Gamma u \cdot \cdot (\nabla v \otimes \nabla v) = - \int_\Gamma \langle \text{div}_\Gamma(\nabla v \otimes \nabla v), u \rangle. \quad \square$$

If A is a symmetric matrix then $\text{tr}(AB^*) = \text{tr}(AB)$. Using this equality, **Lemma 1** and the calculations above gives that if e is an admissible displacement of the mid-surface of the shell then it satisfies (16).

4.4. PDE formulation of the model

In this subsection we are interested in showing why (12) governs the displacement of the shell. We go back to Eq. (19). We have

$$(\mathcal{M}\partial_{tt}\mathbf{e}, \hat{\mathbf{e}})_{\Gamma} = (2\rho(I + \gamma(D^2b)^2)\partial_{tt}e_{\Gamma} - \rho\gamma D^2b\partial_{tt}w, \hat{e}_{\Gamma})_{\Gamma} + [2\rho\partial_{tt}w - 2\rho\gamma\Delta_{\Gamma}\partial_{tt}w + \gamma\rho\operatorname{div}_{\Gamma}(D^2b\partial_{tt}e_{\Gamma})]\hat{w} \tag{40}$$

$$\int_{\Gamma} \mathcal{C}B_M(\mathbf{e}) \cdot \cdot B'_M(\mathbf{e}, \hat{\mathbf{e}}) = (-\operatorname{div}_{\Gamma}(\mathcal{C}B_M(\mathbf{e})), \hat{e}_{\Gamma})_{\Gamma} + [\operatorname{tr}(\mathcal{C}(B_M(\mathbf{e}))D^2b) - \operatorname{div}_{\Gamma}(\mathcal{C}(B_M(\mathbf{e}))\nabla_{\Gamma}w)]\hat{w}. \tag{41}$$

Using [6, Lemma 2.3], we get

$$\int_{\Gamma} \mathcal{C}B_F(\mathbf{e}) \cdot \cdot B_F(\hat{\mathbf{e}}) = [(\lambda + 2\mu)\Delta_{\Gamma}^2w + 2\mu\operatorname{div}_{\Gamma}(K\nabla_{\Gamma}w) + 2\mu\operatorname{div}_{\Gamma}((D^2b)^2\nabla_{\Gamma}w)]\hat{w}. \tag{42}$$

Combining (40)–(42), we obtain

$$\begin{cases} \rho\partial_{tt}w - \rho\gamma\Delta_{\Gamma}\partial_{tt}w + (\lambda + 2\mu)\gamma\Delta_{\Gamma}^2w + \rho\frac{\gamma}{2}\operatorname{div}_{\Gamma}(D^2b\partial_{tt}e_{\Gamma}) + \operatorname{tr}(\mathcal{C}(B_M(\mathbf{e}))D^2b) \\ \quad - \operatorname{div}_{\Gamma}(\mathcal{C}(B_M(\mathbf{e}))\nabla_{\Gamma}w) + 2\mu\gamma\operatorname{div}_{\Gamma}(K\nabla_{\Gamma}w) + 2\mu\gamma\operatorname{div}_{\Gamma}((D^2b)^2\nabla_{\Gamma}w) = 0 \\ \rho(I + \gamma(D^2b)^2)\partial_{tt}e_{\Gamma} - \rho\frac{\gamma}{2}D^2b\partial_{tt}w - \operatorname{div}_{\Gamma}(\mathcal{C}B_M(\mathbf{e})) = 0. \end{cases} \tag{43}$$

The definition of B_M (Definition 2) gives the desired result (12).

4.5. Total energy of the system

The total energy of the system is given by the sum of the kinetic and potential (elastic) energy:

$$\begin{aligned} \mathcal{E}(\mathbf{e}) &= \mathcal{E}_k(\mathbf{e}) + \mathcal{E}_p(\mathbf{e}) \\ &= \frac{\rho l}{2} \int_{\Gamma} |\partial_t e_{\Gamma}|^2 + |\partial_t w|^2 + \frac{\rho l \gamma}{2} \int_{\Gamma} |D^2b \partial_t e_{\Gamma}|^2 + |\nabla_{\Gamma} \partial_t w|^2 + |D^2b \partial_t e_{\Gamma} - \nabla_{\Gamma} \partial_t w|^2 \\ &\quad + \frac{\lambda l}{2} \left| 2H w + \operatorname{div}_{\Gamma} e_{\Gamma} + \frac{1}{2} \|\nabla_{\Gamma} w\|^2 \right|_{0, \Gamma}^2 \\ &\quad + \mu l \int_{\Gamma} \operatorname{tr} \left[\left(\varepsilon_{\Gamma}(e_{\Gamma}) + D^2b w + V_{\Gamma} e_{\Gamma} + \frac{1}{2} \nabla_{\Gamma} w \otimes \nabla_{\Gamma} w \right)^2 \right] \\ &\quad + \frac{\lambda l^3}{24} |\Delta_{\Gamma} w|_{0, \Gamma}^2 + \frac{\mu l^3}{12} \int_{\Gamma} \operatorname{tr}[(S_{\Gamma} w + G_{\Gamma} w)^2]. \end{aligned} \tag{44}$$

Note that

$$\mathcal{E}(\mathbf{e}(t)) \geq C(\|e_{\Gamma}(t)\|_{H^1(\Gamma)}^2 + \|\partial_t e_{\Gamma}(t)\|_{L^2(\Gamma)}^2 + \|\partial_t w(t)\|_{H^1(\Gamma)}^2 + \|w(t)\|_{H^2(\Gamma)}^2). \tag{45}$$

The sum of the kinetic energy with the bending (flexural) energy will be of some interest in Section 8. It is defined as

$$\mathcal{E}^F(\mathbf{e}) = \mathcal{E}_k(\mathbf{e}) + \mathcal{E}_p^F(\mathbf{e}) \tag{46}$$

which can also be written as

$$\mathcal{E}^F(\mathbf{e}) = \mathcal{E}(\mathbf{e}) - \frac{l}{2} \int_{\Gamma} \mathcal{C}(B_M(\mathbf{e})) \cdot \cdot B_M(\mathbf{e}). \tag{47}$$

Note that the nonlinear terms appear only in B_M whose energy terms are in \mathcal{E}_p^M . As a consequence the energy terms in \mathcal{E}^F all come from linear terms of the equations.

Proposition 1. *Let \mathbf{e} be a regular solution to the small finite deflections shell problem; then*

$$\forall t \in [0, \tau[, \quad \mathcal{E}(\mathbf{e}(t)) = \mathcal{E}(\mathbf{e}(0)).$$

Proof. Since $\mathbf{e} \in C^0([0, T[, V^2(\Gamma)) \cap C^1([0, \tau[, V^1(\Gamma)) \cap C^2([0, \tau[, V^0(\Gamma))$ one can replace $\hat{\mathbf{e}}$ by $\partial_t \mathbf{e}$ in (16); this yields

$$m(\partial_t \mathbf{e}, \partial_t \mathbf{e}) + a(\mathbf{e}, \partial_t \mathbf{e}) - n(\mathbf{e}, \partial_t \mathbf{e}) = 0$$

and after some computations, it follows that

$$\partial_t [m(\partial_t \mathbf{e}, \mathbf{e})] + \int_{\Gamma} \mathcal{C}(B_M(\mathbf{e})) \cdot \cdot B'_M(\mathbf{e}, \partial_t \mathbf{e}) + \gamma \int_{\Gamma} \mathcal{C}(B_F(\mathbf{e})) \cdot \cdot B_F(\partial_t \mathbf{e}) = 0.$$

Eqs. (37) yield

$$\partial_t \mathcal{E}_k(t) + \partial_t \mathcal{E}_p^F(t) + \partial_t \mathcal{E}_p^F(t) = 0$$

and therefore $\partial_t \mathcal{E}(t) = 0$, which leads to the desired result. \square

For the sake of simplicity, we shall not introduce new notation for the energy as a function of time and we shall use $\mathcal{E}(t)$ (resp. $\mathcal{E}_p(t), \dots$) to refer to $\mathcal{E}(\mathbf{e}(t))$ (resp. $\mathcal{E}_p(\mathbf{e}(t)), \dots$).

5. Preliminary properties

The remainder of this paper is devoted to the proof of [Theorems 1–3](#). The properties established in this section will be used in the proofs of these theorems.

5.1. Properties of the operators

Lemma 2. *We have $\langle \mathbb{A}X, X \rangle_{\mathcal{H}} = 0$ for all $X \in D(A)$.*

Proof. We note that the inner product above is on the space \mathcal{H} which is defined in Eq. (17). Let $X = [\mathbf{e}^1 \mathbf{e}^2]$. Thus we have

$$\begin{aligned} \langle \mathbb{A}X, X \rangle_{\mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}^{1/2})} &= -\langle \mathbf{e}^2, \mathbf{e}^1 \rangle_{\mathcal{D}(\mathcal{A}^{1/2})} + \langle \mathcal{M}^{-1} \mathcal{A} \mathbf{e}^1, \mathbf{e}^2 \rangle_{\mathcal{D}(\mathcal{M}^{1/2})} \\ &= -\langle \mathcal{A}^{1/2} \mathbf{e}^2, \mathcal{A}^{1/2} \mathbf{e}^1 \rangle_{[L_2(\Gamma)]^3} + \langle \mathcal{M}^{1/2} \mathcal{M}^{-1} \mathcal{A} \mathbf{e}^1, \mathcal{M}^{1/2} \mathbf{e}^2 \rangle_{[L_2(\Gamma)]^3} \\ &= -\langle \mathbf{e}^2, \mathcal{A} \mathbf{e}^1 \rangle_{[L_2(\Gamma)]^3} + \langle \mathcal{M}^{1/2} \mathcal{M}^{1/2} \mathcal{M}^{-1} \mathcal{A} \mathbf{e}^1, \mathbf{e}^2 \rangle_{[L_2(\Gamma)]^3} \\ &= -\langle \mathbf{e}^2, \mathcal{A} \mathbf{e}^1 \rangle_{[L_2(\Gamma)]^3} + \langle \mathcal{A} \mathbf{e}^1, \mathbf{e}^2 \rangle_{[L_2(\Gamma)]^3} = 0. \quad \square \end{aligned}$$

Lemma 3. *The bilinear form $a(\mathbf{e}, \hat{\mathbf{e}})$ defined in (14) is elliptic on $V^1(\Gamma)$, that is, there exists a constant $C > 0$ such that*

$$C \|\mathbf{e}\|_{V^1(\Gamma)}^2 \leq a(\mathbf{e}, \mathbf{e}) \quad \forall \mathbf{e} \in V^1(\Gamma). \tag{48}$$

In addition the bilinear form $m(\mathbf{e}, \hat{\mathbf{e}})$ is elliptic on the space $V^0(\Gamma)$. This gives coercivity of the related linear operators \mathcal{A}, \mathcal{M} .

Proof. This inequality for $a(\mathbf{e}, \hat{\mathbf{e}})$ is established by Bernadou and Oden [3] (see also [2]) provided the shell is shallow enough (as in Hypothesis (ii), Section 1). Though this model does not exactly satisfy the lemma of rigid body motion, the proof relies on the fact that the shallowness assumption allows us to consider the equations above as some perturbation of the special case of the plate. The corresponding proof for $m(\mathbf{e}, \hat{\mathbf{e}})$ is given in [5]. \square

5.2. Norm estimates

Lemma 4. *Let $w \in H^2(\Gamma)$; then*

$$\|\nabla_\Gamma w \otimes \nabla_\Gamma w\|_{L^2(\Gamma)} \leq C \|w\|_{H^2(\Gamma)} \|w\|_{H^1(\Gamma)}. \tag{49}$$

Proof. We have

$$\|\nabla_\Gamma w \otimes \nabla_\Gamma w\|_{L^2(\Gamma)} \leq \|\nabla_\Gamma w\|_{L^4(\Gamma)}^2.$$

Using Sobolev’s embedding $H^{1/2}(\Gamma) \subset L^4(\Gamma)$, we obtain

$$\|\nabla_\Gamma w \otimes \nabla_\Gamma w\|_{L^2(\Gamma)} \leq C \|\nabla_\Gamma w\|_{H^{1/2}(\Gamma)}^2$$

and the classical interpolation inequality $\|\nabla_\Gamma w\|_{H^{1/2}(\Gamma)}^2 \leq \|\nabla_\Gamma w\|_{H^1(\Gamma)} \|\nabla_\Gamma w\|_{L^2(\Gamma)}$ yields (49). \square

Let us denote by $\{\varphi_i\}$ the family of the eigenvectors of the Laplace operator and λ_i the eigenvalues corresponding to φ_i (the eigenvalues λ_n should not to be confused with the Lamé coefficient λ). Let P_n be the orthogonal projection on the subspace spanned by the n eigenvectors $\varphi_0, \dots, \varphi_{n-1}$. If $f \in L^2$ and $g \in H^1$, then the estimate resulting from Sobolev’s embedding and the Holder inequality gives [30]:

$$\|(P_n f)g\|_{L^2} \leq C \sqrt{\ln(1 + \lambda_n)} \|f\|_{L^2} \|g\|_{H^1} \tag{50a}$$

$$\|(P_n g)f\|_{L^2} \leq C \sqrt{\ln(1 + \lambda_n)} \|f\|_{L^2} \|g\|_{H^1} \tag{50b}$$

$$\|P_n g\|_{L^\infty} \leq C \sqrt{\ln(1 + \lambda_n)} \|g\|_{H^1}. \tag{50c}$$

We define $Q_n = I - P_n$.

Remark 3. The sequence (λ_n) is increasing and $\lim \lambda_n = +\infty$.

Lemma 5. *Let $\epsilon > 0, \alpha > 0$ and $f \in H^{\epsilon+\alpha}$; then*

$$\|Q_n f\|_{H^\epsilon}^2 \leq \frac{1}{\lambda_n^\alpha} \|Q_n f\|_{H^{\epsilon+\alpha}}^2. \tag{51}$$

Proof. We have

$$Q_n f = \sum_{i=n}^{+\infty} \langle f, \varphi_i \rangle \varphi_i$$

and, for $\eta > 0$, we have

$$\|Q_n f\|_{H^\eta}^2 = \sum_{i=n}^{+\infty} \langle f, \varphi_i \rangle^2 \lambda_i^\eta. \tag{52}$$

Applying (52) with $\eta = \epsilon$ yields

$$\|Q_n f\|_{H^\epsilon}^2 = \sum_{i=n}^{+\infty} \langle f, \varphi_i \rangle^2 \lambda_i^\epsilon \leq \left(\sup_{i \geq n} \frac{1}{\lambda_i^\alpha} \right) \sum_{i=n}^{+\infty} \langle f, \varphi_i \rangle^2 \lambda_i^{\epsilon+\alpha}.$$

Using that (λ_i) is an increasing sequence, we obtain

$$\|Q_n f\|_{H^\epsilon}^2 \leq \frac{1}{\lambda_n^\alpha} \sum_{i=n}^{+\infty} \langle f, \varphi_i \rangle^2 \lambda_i^{\epsilon+\alpha}.$$

Applying (52) with $\eta = \epsilon + \alpha$ yields

$$\|Q_n f\|_{H^\epsilon}^2 \leq \frac{1}{\lambda_n^\alpha} \|Q_n f\|_{H^{\epsilon+\alpha}}^2. \quad \square$$

Lemma 6. *Let $\epsilon > 0, \alpha > 0, f \in H^{\epsilon+\alpha}$ and $g \in H^1$; then*

$$\|(Q_n f)g\|_{L^2}^2 \leq \frac{1}{\lambda_n^\alpha} \|f\|_{H^{\epsilon+\alpha}}^2 \|g\|_{H^1}^2. \tag{53}$$

Proof. We have

$$\|(Q_n f)g\|_{L^2}^2 \leq \|Q_n f\|_{H^\epsilon}^2 \|g\|_{H^1}^2.$$

Lemma 5 gives

$$\|(Q_n f)g\|_{L^2}^2 \leq \frac{1}{\lambda_n^\alpha} \|Q_n f\|_{H^{\epsilon+\alpha}}^2 \|g\|_{H^1}^2.$$

Using $\|Q_n f\|_{H^{\epsilon+\alpha}}^2 \leq \|f\|_{H^{\epsilon+\alpha}}^2$ we obtain (53). \square

Proposition 2. *Let $\epsilon > 0, \alpha > 0, f \in H^{\epsilon+\alpha}$ and $g \in H^1$; then*

$$\|fg\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|f\|_{L^2}^2 + \frac{1}{\lambda_n^\alpha} \|f\|_{H^{\epsilon+\alpha}}^2 \right) \|g\|_{H^1}^2. \tag{54}$$

Proof. We have

$$\|fg\|_{L^2}^2 = \|(P_n f)g + (Q_n f)g\|_{L^2}^2 \leq \|(P_n f)g\|_{L^2}^2 + \|(Q_n f)g\|_{L^2}^2$$

applying (50a) to $\|(P_n f)g\|_{L^2}$ and (53) to $\|(Q_n f)g\|_{L^2}$ gives (54). \square

Corollary 1. *Let $\epsilon > 0, \alpha > 0, v_1 \in (H^{\epsilon+\alpha})^3$ and $v_2 \in (H^1)^3$; then*

$$\|\langle v_1, v_2 \rangle\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|v_1\|_{L^2}^2 + \frac{1}{\lambda_n^\alpha} \|v_1\|_{H^{\epsilon+\alpha}}^2 \right) \|v_2\|_{H^1}^2 \tag{55}$$

$$\|v_1 \otimes v_2\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|v_1\|_{L^2}^2 + \frac{1}{\lambda_n^\alpha} \|v_1\|_{H^{\epsilon+\alpha}}^2 \right) \|v_2\|_{H^1}^2. \tag{56}$$

Proof. These inequalities follow from Proposition 2 for each term of $\langle v_1, v_2 \rangle$ or each component of $v_1 \otimes v_2$. \square

6. Proof of Theorem 1 (existence of regular solutions)

In this section we prove Theorem 1, the existence and uniqueness of regular solutions to the problem (16). We shall use the nonlinear Galerkin method, essentially following the proof in [25] for the full von Kármán plate equation.

6.1. Semidiscrete approximation

Let h be a non-negative real number and $V_h^2(\Gamma)$ be a finite dimensional subspace dense in $V^2(\Gamma)$ as defined in (11). We consider the following semidiscrete approximation of (19). Given $\mathbf{e}^{0,h} = (e_\Gamma^{h,0}, w^{h,0}) \in V_h^2(\Gamma)$ and $\mathbf{e}^{h,1} = (e_\Gamma^{h,1}, w^{h,1}) \in V_h^1(\Gamma)$, find $\mathbf{e}^h = (e_\Gamma^h, w^h) \in V_h^2$ such that

$$(\mathcal{M} \partial_t \mathbf{e}^h, \hat{\mathbf{e}})_\Gamma + \int_\Gamma (CB_M(\mathbf{e}^h)) \cdot \cdot B'_M(\mathbf{e}^h, \hat{\mathbf{e}}) + \gamma \int_\Gamma (CB_F(\mathbf{e}^h)) \cdot \cdot B_F(\hat{\mathbf{e}}) = 0, \quad \forall \hat{\mathbf{e}} \in V_h^2(\Gamma) \tag{57a}$$

$$\mathbf{e}^h(0) = \mathbf{e}^{h,0} \tag{57b}$$

$$\partial_t \mathbf{e}^h(0) = \mathbf{e}^{h,1}. \tag{57c}$$

Let us define $E_h(t) = \mathcal{E}(\mathbf{e}^h)$ where \mathcal{E} is defined in Eq. (44). Let $E_h(t)$ denote the energy computed along the semidiscrete solution \mathbf{e}^h . Since $\mathbf{e}^h(t)$ belongs to $V_h^2(\Gamma)$ it is a regular solution, so Proposition 1 applies.

$$E_h(t) \leq E_h(0). \tag{58}$$

Global existence and uniqueness of semidiscrete solutions follows from the fact that the nonlinear terms contained in B_M are locally Lipschitz on $V_h^2(\Gamma)$ together with the standard *a priori* bound (58). It follows that

$$\|\partial_t w_h(t)\|_{H^1}^2 + \|\partial_t e_\Gamma(t)\|_{L^2}^2 + \|w_h(t)\|_{H^2}^2 + \|\partial_t e_\Gamma(t)\|_{H^1}^2 \leq E_h(t) \leq E_h(0) \tag{59}$$

$$\|\partial_t w_h\|_{L^2} \leq \sqrt{E_h(0)} \tag{60}$$

for all $\tau > 0$ such that the solution $\mathbf{e}^h(t) \in C^\infty([0, \tau[, V_h^2)$.

6.2. Stability in higher norms

In the following equations, we shall use $\bar{u} = \partial_t u^h$. Let us differentiate (57a) with respect to t ; then the following equation holds true for all tests $\hat{\mathbf{e}} \in V_h^2$

$$(\mathcal{M}\partial_{tt}\bar{\mathbf{e}}, \hat{\mathbf{e}}) + \int_\Gamma (\mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}})) \cdot \cdot B'_M(\mathbf{e}^h, \hat{\mathbf{e}}) + \gamma \int_\Gamma (\mathcal{C}B_F(\bar{\mathbf{e}})) \cdot \cdot B_F(\hat{\mathbf{e}}) = 0.$$

Replacing the test function by $\partial_t \hat{\mathbf{e}}$ in (57a) yields a simplification in the equation above; it results in the following equation holding true for all tests $\hat{\mathbf{e}} \in V_h^2$

$$(\mathcal{M}\partial_{tt}\bar{\mathbf{e}}, \hat{\mathbf{e}}) + \int_\Gamma \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot B'_M(\mathbf{e}^h, \hat{\mathbf{e}}) + \int_\Gamma \mathcal{C}B_M(\mathbf{e}^h) \cdot \cdot (\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma \bar{w}) + \gamma \int_\Gamma (\mathcal{C}B_F(\bar{\mathbf{e}})) \cdot \cdot B_F(\hat{\mathbf{e}}) = 0.$$

We note that the differentiation of the nonlinear quantities results in extra terms in the equation. In order to see them we will write explicitly:

$$\begin{aligned} & \int_\Gamma \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot B'_M(\mathbf{e}^h, \hat{\mathbf{e}}) + \int_\Gamma \mathcal{C}B_M(\mathbf{e}^h) \cdot \cdot (\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma \bar{w}) \\ &= \int_\Gamma \text{tr} \left[\mathcal{C} \left(\varepsilon_\Gamma \bar{e}_\Gamma + V_\Gamma \bar{e}_\Gamma + \bar{w} D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma \bar{w} + \frac{1}{2} \nabla_\Gamma \bar{w} \otimes \nabla_\Gamma w^h \right) \right. \\ & \quad \times \left. \left(\varepsilon_\Gamma \hat{e}_\Gamma + V_\Gamma \hat{e}_\Gamma + \hat{w} D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma \hat{w} + \frac{1}{2} \nabla_\Gamma \hat{w} \otimes \nabla_\Gamma w^h \right) \right] \\ & \quad + \int_\Gamma \text{tr} \left[\mathcal{C} \left(\varepsilon_\Gamma (e_\Gamma)^h + V_\Gamma e_\Gamma^h + w^h D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma w^h \right) (\nabla_\Gamma \hat{w} \otimes \nabla_\Gamma \bar{w}) \right]. \end{aligned} \tag{61}$$

Taking $\hat{\mathbf{e}} = \partial_t \bar{\mathbf{e}} = \partial_{tt} \mathbf{e}^h$, we obtain

$$\begin{aligned} & (\mathcal{M}\partial_{tt}\bar{\mathbf{e}}, \partial_t \bar{\mathbf{e}}) + \gamma \int_\Gamma \text{tr} [\mathcal{C} (S_\gamma \bar{w} + G_\Gamma \bar{w}) (S_\Gamma \partial_t \bar{w} + G_\Gamma \partial_t \bar{w})] \\ & \quad + \int_\Gamma \text{tr} \left[\mathcal{C} \left(\varepsilon_\Gamma \bar{e}_\Gamma + V_\Gamma \bar{e}_\Gamma + \bar{w} D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma \bar{w} + \frac{1}{2} \nabla_\Gamma \bar{w} \otimes \nabla_\Gamma w^h \right) \right. \\ & \quad \times \left. \left(\varepsilon_\Gamma \partial_t \bar{e}_\Gamma + V_\Gamma \partial_t \bar{e}_\Gamma + \partial_t \bar{w} D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma \partial_t \bar{w} + \frac{1}{2} \nabla_\Gamma \partial_t \bar{w} \otimes \nabla_\Gamma w^h \right) \right] \\ & \quad + \int_\Gamma \text{tr} \left[\mathcal{C} \left(\varepsilon_\Gamma (e_\Gamma)^h + V_\Gamma e_\Gamma^h + w^h D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma w^h \right) (\nabla_\Gamma \partial_t \bar{w} \otimes \nabla_\Gamma \bar{w}) \right] = 0. \end{aligned} \tag{62}$$

Since

$$\begin{aligned} \partial_t B_M(\mathbf{e}^h) &= \partial_t \left(\varepsilon_\Gamma (e_\Gamma)^h + V_\Gamma e_\Gamma^h + w^h D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma w^h \right) \\ &= \varepsilon_\Gamma \bar{e}_\Gamma + V_\Gamma \bar{e}_\Gamma + \bar{w} D^2 b + \frac{1}{2} \nabla_\Gamma w^h \otimes \nabla_\Gamma \bar{w} + \frac{1}{2} \nabla_\Gamma \bar{w} \otimes \nabla_\Gamma w^h = B'_M(\bar{\mathbf{e}}, \mathbf{e}^h) \end{aligned} \tag{63}$$

and $B_F(\bar{\mathbf{e}}) = S_\Gamma \bar{w} + G_\Gamma \bar{w}$, it follows that we can rewrite (62) as

$$\begin{aligned}
 & (\mathcal{M}\partial_t\bar{\mathbf{e}}, \partial_t\bar{\mathbf{e}}) + \int_{\Gamma} (\mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}})) \cdot \cdot \partial_t B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) + \gamma \int_{\Gamma} (\mathcal{C}B_F(\bar{\mathbf{e}})) \cdot \cdot \partial_t B_F(\bar{\mathbf{e}}) \\
 &= \int_{\Gamma} \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) - \int_{\Gamma} \mathcal{C}B_M(\mathbf{e}^h) \cdot \cdot (\nabla_{\Gamma}\partial_t\bar{w} \otimes \nabla_{\Gamma}\bar{w}).
 \end{aligned} \tag{64}$$

Let us define

$$\bar{E}(t) = \|\mathcal{M}\partial_t\bar{\mathbf{e}}\|_{L^2(\Gamma)} + \int_{\Gamma} \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) + \gamma \int_{\Gamma} \mathcal{C}B_F(\bar{\mathbf{e}}) \cdot \cdot B_F(\bar{\mathbf{e}}). \tag{65}$$

We have

$$\|\mathcal{M}\partial_t\bar{\mathbf{e}}(t)\|_{L^2(\Gamma)} + \|\bar{w}(t)\|_{H^2} + \|\bar{e}_{\Gamma}(t)\|_{H^1} \leq \bar{E}(t). \tag{66}$$

It is straightforward to verify that $\bar{E}(t)$ is bounded above and below by a constant depending on $\|\bar{\mathbf{e}}\|_{V^1(\Gamma)}$ and $\|\partial_t\bar{\mathbf{e}}\|_{V^0(\Gamma)}$. This fact will be used frequently and without further mention. Eq. (64) can be rewritten as

$$\begin{aligned}
 \frac{1}{2}\partial_t\bar{E}(t) &= \int_{\Gamma} \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) - \int_{\Gamma} \mathcal{C}B_M(\mathbf{e}^h) \cdot \cdot (\nabla_{\Gamma}\partial_t\bar{w} \otimes \nabla_{\Gamma}\bar{w}) \\
 &= \int_{\Gamma} \text{tr} \left[\mathcal{C} \left(\varepsilon_{\Gamma}e_{\Gamma} + V_{\Gamma}e_{\Gamma} + \hat{w}D^2b + \frac{1}{2}\nabla_{\Gamma}w^h \otimes \nabla_{\Gamma}\bar{w} + \frac{1}{2}\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}w^h \right) (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) \right] \\
 &\quad - \int_{\Gamma} \text{tr} \left[\mathcal{C} \left(\varepsilon_{\Gamma}e_{\Gamma}^h + V_{\Gamma}e_{\Gamma}^h + w^hD^2b + \frac{1}{2}\nabla_{\Gamma}w^h \otimes \frac{1}{2}\nabla_{\Gamma}w^h \right) (\nabla_{\Gamma}\partial_t\bar{w} \otimes \nabla_{\Gamma}\bar{w}) \right].
 \end{aligned} \tag{67}$$

Integrating this equation between 0 and t yields

$$\frac{1}{2}(\bar{E}(t) - \bar{E}(0)) = \int_0^t \int_{\Gamma} \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) - \frac{1}{2} \int_0^t \int_{\Gamma} \mathcal{C}B_M(\mathbf{e}^h) \cdot \cdot \partial_t(\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}). \tag{68}$$

Integrating by parts the second term of the right hand side of (68) gives

$$\bar{E}(t) = \bar{E}(0) + 3 \int_0^t \int_{\Gamma} \mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}}) \cdot \cdot (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) + \left[\int_{\Gamma} (\mathcal{C}B_M(\mathbf{e}^h)) \cdot \cdot (\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}) \right]_0^t. \tag{69}$$

Therefore

$$\begin{aligned}
 \bar{E}(t) &\leq \bar{E}(0) + \frac{3}{2} \int_0^t \left(\|\mathcal{C}B'_M(\mathbf{e}^h, \bar{\mathbf{e}})\|_{L^2(\Gamma)}^2 + \|\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}\|_{L^2(\Gamma)}^2 \right) ds \\
 &\quad + \left[\frac{1}{4\epsilon} \|\mathcal{C}B_M(\mathbf{e}^h)\|_{L^2(\Gamma)}^2 + \epsilon \|\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}\|_{L^2(\Gamma)}^2 \right]_0^t
 \end{aligned} \tag{70}$$

where ϵ is an arbitrarily small non-negative number.

From Lemma 4 we have $\|\nabla_{\Gamma}\bar{w} \otimes \nabla_{\Gamma}\bar{w}\|_{L^2(\Gamma)} \leq C\|\bar{w}\|_{H^2(\Gamma)}\|\bar{w}\|_{H^1(\Gamma)}$. Using (45) with (58) gives $\|\bar{w}\|_{H^1(\Gamma)} \leq E_h(0)$, and then using (66), we obtain

$$\|\nabla\bar{w} \otimes \nabla\bar{w}\|_{L^2(\Gamma)}^2 \leq C E_h(0)\|\bar{w}\|_{H^2(\Gamma)}^2 \leq C E_h(0)\bar{E}(t) \tag{71}$$

where C is a constant. In addition, the following inequalities follow from the fact that tensor \mathcal{C} is coercive and the *a priori* bound on the energies:

$$\|\mathcal{C}(B_M(\mathbf{e}^h))\|_{L^2(\Gamma)}^2 \leq C E_h(t) \leq C E_h(0) \tag{72a}$$

$$\|\mathcal{C}(B'_M(\mathbf{e}^h, \bar{\mathbf{e}}))\|_{L^2(\Gamma)}^2 \leq C \bar{E}(t). \tag{72b}$$

Therefore

$$\bar{E}(t) \leq \bar{E}(0) + C(1 + E_h(0)) \int_0^t \bar{E}(s) ds + \frac{1}{4\epsilon} C E_h(0) + C \epsilon E_h(0) \bar{E}(t). \tag{73}$$

Let us now set $\epsilon = 1/(2CE_h(0))$; then

$$\bar{E}(t) \leq \bar{E}(0) + C(1 + E_h(0)) \int_0^t \bar{E}(s) \, ds + \frac{1}{2}(CE_h(0))^2 + \frac{1}{2}\bar{E}(t) \tag{74}$$

and therefore

$$\bar{E}(t) \leq C(\bar{E}(0) + (E_h(0))^2) + C(1 + E_h(0)) \int_0^t \bar{E}(s) \, ds. \tag{75}$$

Gronwall’s lemma yields

$$\forall t \in [0, \tau], \quad \bar{E}(t) \leq C(\bar{E}(0) + [E_h(0)]^2) \exp(C(1 + E_h(0))\tau) \tag{76}$$

where C does not depend on h .

6.3. A priori bounds for the discrete initial data

In order to provide an effective estimate (independent of h) of the right hand side of Eq. (76) we need to estimate $E_h(0)$ and $\bar{E}(0)$. This is the point of the next two sections.

6.3.1. A priori bounds for $E_h(0)$

We consider $\mathbf{e}^{0,h}$ and $\mathbf{e}^{1,h}$ “good” approximations of the initial data, that is

$$\lim_{h \rightarrow 0} \|\mathbf{e}^0 - \mathbf{e}^{h,0}\|_{V^2(\Gamma)} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|\mathbf{e}^1 - \mathbf{e}^{h,1}\|_{V^1(\Gamma)} = 0. \tag{77}$$

By the stability result of the estimates resulting from (77) and the regularity of the initial condition, we obtain $E_h(0) \leq C_2$.

6.3.2. A priori bounds for $\bar{E}(0)$

Eq. (57a) along with the coercivity of \mathcal{M} gives

$$(\partial_t \bar{\mathbf{e}}(0), \hat{\mathbf{e}})_{V^0(\Gamma)} \leq C \left| \int_{\Gamma} (B_M(\mathbf{e}^{h,0})) \cdot B'_M(\mathbf{e}^{h,0}, \hat{\mathbf{e}}) \right| + C \left| \int_{\Gamma} (B_F(\mathbf{e}^{h,0})) \cdot B_F(\hat{\mathbf{e}}) \right|. \tag{78}$$

It follows that

$$\begin{aligned} (\partial_t \bar{\mathbf{e}}(0), \hat{\mathbf{e}})_{V^0(\Gamma)} &\leq C \left| \int_{\Gamma} \operatorname{div}_{\Gamma} B_M(\mathbf{e}^{h,0}) \cdot \hat{e}_{\Gamma} \right| + C \left| \int_{\Gamma} B_M(\mathbf{e}^{h,0}) \cdot (V_{\Gamma} \hat{e}_{\Gamma} + \hat{w} D^2 b) \right| \\ &\quad + C \left| \int_{\Gamma} ((\varepsilon_{\Gamma} e_{\Gamma}^{h,0} + V_{\Gamma} e_{\Gamma}^{h,0} + w^{h,0} D^2 b) \nabla_{\Gamma} w^{h,0}, \nabla_{\Gamma} \hat{w}) \right| \\ &\quad + C \left| \int_{\Gamma} \langle (\nabla_{\Gamma} w^{h,0} \otimes \nabla_{\Gamma} w^{h,0}) \nabla_{\Gamma} w^{h,0}, \nabla_{\Gamma} \hat{w} \rangle \right| \\ &\quad + C \left| \int_{\Gamma} S_{\Gamma} w^{h,0} \cdot S_{\Gamma} \hat{w} \right| + C \left| \int_{\Gamma} G_{\Gamma} w^{h,0} \cdot S_{\Gamma} \hat{w} \right| \\ &\quad + C \left| \int_{\Gamma} (S_{\Gamma} w^{h,0} + G_{\Gamma} w^{h,0}) \cdot G_{\Gamma} \hat{w} \right| \end{aligned} \tag{79}$$

and therefore

$$\begin{aligned} (\partial_t \bar{\mathbf{e}}(0), \hat{\mathbf{e}})_{V^0(\Gamma)} &\leq C \|\operatorname{div}_{\Gamma} B_M(\mathbf{e}^{h,0})\|_{L^2(\Gamma)} \|\hat{e}_{\Gamma}\|_{L^2(\Gamma)} + C \|B_M(\mathbf{e}^{h,0})\|_{L^2(\Gamma)} \|V_{\Gamma} \hat{e}_{\Gamma} + \hat{w} D^2 b\|_{L^2(\Gamma)} \\ &\quad + C \|(\varepsilon_{\Gamma} e_{\Gamma}^{h,0} + V_{\Gamma} e_{\Gamma}^{h,0} + w^{h,0} D^2 b)\|_{L^4(\Gamma)} \|\nabla_{\Gamma} w^{h,0}\|_{L^4(\Gamma)} \|\hat{w}\|_{H^1(\Gamma)} \\ &\quad + C \|\nabla_{\Gamma} w^{h,0}\|_{L^6(\Gamma)}^3 \|\hat{w}\|_{H^1(\Gamma)} \\ &\quad + C \left| \int_{\Gamma} \langle \operatorname{div}_{\Gamma} S_{\Gamma} w^{h,0}, \nabla_{\Gamma} \hat{w} \rangle \right| + C \left| \int_{\Gamma} \langle \operatorname{div}_{\Gamma} G_{\Gamma} w^{h,0}, \nabla_{\Gamma} \hat{w} \rangle \right| \\ &\quad + C \|w^{h,0}\|_{H^2(\Gamma)} \|\hat{w}\|_{H^1(\Gamma)}. \end{aligned} \tag{80}$$

Using $H^1(\Gamma) \subset L^6(\Gamma) \subset L^4(\Gamma)$ as well as $\|\hat{w}\|_{H^1(\Gamma)} \leq \|\hat{\mathbf{e}}\|_{V^0(\Gamma)}$ and $\|\hat{e}_\Gamma\|_{L^2(\Gamma)} \leq \|\hat{\mathbf{e}}\|_{V^0(\Gamma)}$ we get

$$(\partial_t \bar{\mathbf{e}}(0), \hat{\mathbf{e}})_{V^0(\Gamma)} \leq C(\|\mathbf{e}^{h,0}\|_{V^0(\Gamma)} + \|\mathbf{e}^{h,0}\|_{V^1(\Gamma)} + \|\mathbf{e}^{h,0}\|_{V^2(\Gamma)})\|\hat{\mathbf{e}}\|_{V^0(\Gamma)} \tag{81a}$$

$$\leq C\|\mathbf{e}^{h,0}\|_{V^2(\Gamma)}\|\hat{\mathbf{e}}\|_{V^0(\Gamma)}. \tag{81b}$$

For h small enough, $\|\mathbf{e}^{h,0}\|_{V^2(\Gamma)} \leq 2\|\mathbf{e}^0\|_{V^2(\Gamma)}$, and it follows that

$$(\partial_t \bar{\mathbf{e}}(0), \hat{\mathbf{e}})_{V^0(\Gamma)} \leq C\|\mathbf{e}^0\|_{V^2(\Gamma)}\|\hat{\mathbf{e}}\|_{V^0(\Gamma)} \tag{82}$$

where C is a constant independent of h ; therefore

$$\|\partial_t \bar{\mathbf{e}}(0)\|_{V^0(\Gamma)} \leq C\|\mathbf{e}^0\|_{V^2(\Gamma)}. \tag{83}$$

Using $\|\bar{\mathbf{e}}(0)\|_{V^2(\Gamma)} \leq C$ and (83) we conclude that $\bar{E}(0)$ is bounded by a constant independent of h . Eq. (76) yields, for all $t < \tau$,

$$\bar{E}(t) \leq C. \tag{84}$$

Hence the *a priori* bounds

$$\|\mathbf{e}^h\|_{V^1(\Gamma)} + \|\partial_t \mathbf{e}^h\|_{V^1(\Gamma)} + \|\partial_{tt} \mathbf{e}^h\|_{V^0(\Gamma)} \leq C \tag{85}$$

hold uniformly in h , for all $t < \tau$ where τ is arbitrary.

6.4. Passage to the limit

We can now extract convergent subsequences, denoted by the same symbol, such that the following weak* convergence holds:

$$\mathbf{e}^h \rightharpoonup \mathbf{e} \quad \text{in } L^\infty(0, \tau; V^1(\Gamma)) \tag{86a}$$

$$\partial_t \mathbf{e}^h \rightharpoonup \partial_t \mathbf{e} \quad \text{in } L^\infty(0, \tau; V^1(\Gamma)) \tag{86b}$$

$$\partial_{tt} \mathbf{e}^h \rightharpoonup \partial_{tt} \mathbf{e} \quad \text{in } L^\infty(0, \tau; V^0(\Gamma)). \tag{86c}$$

Now, we can pass to the limit on the original semidiscrete form of Eq. (57a). Note that this is possible due to the weak continuity of $\nabla_\Gamma w \otimes \nabla_\Gamma w$. The clamped boundary condition also passes to the limit thanks to the strong convergence of the boundary traces. Therefore, we can conclude that \mathbf{e} satisfies the weak form of the original equation (16). Moreover, it has the following regularity

$$\mathbf{e} \in C(0, \tau, V^1(\Gamma)), \quad \partial_t \mathbf{e} \in C(0, \tau, V^1(\Gamma)), \quad \partial_{tt} \mathbf{e} \in C(0, \tau, V^0(\Gamma)). \tag{87}$$

In order to obtain existence of *weak* solutions, we use the *a priori* bound for the original energy function which implies weak convergence of

$$\mathbf{e}^h \rightharpoonup \mathbf{e} \quad \text{in } L^\infty(0, \tau; V^1(\Gamma)) \tag{88}$$

$$\partial_t \mathbf{e}^h \rightharpoonup \partial_t \mathbf{e} \quad \text{in } L^\infty(0, \tau; V^0(\Gamma)). \tag{89}$$

The above weak convergence and weak continuity of nonlinear terms allows passing to the limit. This part of the argument is straightforward. The limit equation defines a weak solution.

6.5. The improved spatial regularity

Using the second equation of (12) we have

$$\begin{aligned} & \operatorname{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)) \\ &= \rho(I + \gamma(D^2 b)^2)\partial_{tt} e_\Gamma - \rho \frac{\gamma}{2} D^2 b \partial_{tt} w - \operatorname{div}_\Gamma(\mathcal{C}(w D^2 b)) - \frac{1}{2} \operatorname{div}_\Gamma(\mathcal{C}(\nabla_\Gamma w \otimes \nabla_\Gamma w)). \end{aligned} \tag{90}$$

The regularity established in Section 6.4 gives the following regularities:

$$\partial_{tt}e_\Gamma \in [L^2(\Gamma)]^2, \quad \partial_{tt}e_\Gamma \in [L^2(\Gamma)]^2, \quad \operatorname{div}_\Gamma(\mathcal{C}(wD^2b)) \in [H^1(\Gamma)]^2.$$

In addition $\nabla_\Gamma w \otimes \nabla_\Gamma w$ is in $[H^{1-\epsilon}(\Gamma)]^2$ as the product of two functions in $[H^1(\Gamma)]^2$. It follows that

$$\operatorname{div}_\Gamma(\mathcal{C}(\nabla_\Gamma w \otimes \nabla_\Gamma w)) \in [H^{-\epsilon}(\Gamma)]^2$$

and consequently $\operatorname{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma)) \in [H^{-\epsilon}(\Gamma)]^2$ so

$$e_\Gamma \in C(0, \tau; [H^{2-\epsilon}(\Gamma)]^2). \tag{91}$$

For w , the first equation of (12) gives

$$\begin{aligned} (\lambda + 2\mu)\gamma \Delta_\Gamma^2 w &= -\rho \partial_{tt} w + \rho \gamma \Delta_\Gamma \partial_{tt} w - \rho \frac{\gamma}{2} \operatorname{div}_\Gamma(D^2b \partial_{tt} e_\Gamma) - \operatorname{tr}(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) D^2b) \\ &\quad - \operatorname{tr} \left(\mathcal{C} \left(w D^2b + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) D^2b \right) + \operatorname{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) \nabla_\Gamma w) \\ &\quad + \operatorname{div}_\Gamma \left(\mathcal{C} \left(w D^2b + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) \nabla_\Gamma w \right) - 2\mu\gamma \operatorname{div}_\Gamma(K \nabla_\Gamma w) - 2\mu\gamma \operatorname{div}_\Gamma((D^2b)^2 \nabla w). \end{aligned} \tag{92}$$

We now inspect the regularity of each term of (92). This regularity follows from Section 6.4 for all but the fifth term:

$$\begin{aligned} \partial_{tt} w &\in H^1(\Gamma), \quad \Delta_\Gamma \partial_{tt} w \in H^{-1}(\Gamma), \quad \operatorname{div}_\Gamma(D^2b \partial_{tt} e_\Gamma) \in H^{-1}(\Gamma) \\ \operatorname{tr}(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) D^2b) &\in L^2(\Gamma), \quad \operatorname{tr} \left(\mathcal{C} \left(w D^2b + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) D^2b \right) \in H^{1-\epsilon}(\Gamma) \\ \operatorname{div}_\Gamma \left(\mathcal{C} \left(w D^2b + \frac{1}{2} \nabla_\Gamma w \otimes \nabla_\Gamma w \right) \nabla_\Gamma w \right) &\in H^{-\epsilon}(\Gamma), \quad \operatorname{div}_\Gamma(K \nabla_\Gamma w) \in L^2(\Gamma), \\ \operatorname{div}_\Gamma((D^2b)^2 \nabla w) &\in L^2(\Gamma) \end{aligned}$$

where we have used that $\nabla w \in [H^1(\Gamma)]^2$ implies $\nabla_\Gamma w \otimes \nabla_\Gamma w \in [H^{1-\epsilon}(\Gamma)]^2$. The regularity of the sixth term of (92) follows from (91). We have $\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) \in H^{1-\epsilon}(\Gamma)$ and $\nabla_\Gamma w \in [H^1(\Gamma)]^2$ so the product is in $H^{1-\epsilon}(\Gamma)$,

$$\operatorname{div}_\Gamma(\mathcal{C}(\varepsilon_\Gamma e_\Gamma + V_\Gamma e_\Gamma) \nabla_\Gamma w) \in [H^{-\epsilon}(\Gamma)]^2$$

and consequently $\Delta_\Gamma^2 w$ belongs to $H^{-1}(\Gamma)$ so

$$w \in C(0, \tau; H^3(\Gamma)). \tag{93}$$

Going back to (90) we further improve the regularity with ϵ , obtaining $e_\Gamma \in C(0, \tau; [H^2(\Gamma)]^2)$. It follows that

$$\mathbf{e} \in C(0, \tau, V^2(\Gamma)) \tag{94}$$

as desired for the proof of the existence of regular solutions.

7. Proof of Theorem 2 (uniqueness property)

In order to prove Theorem 2 we use a method developed by V.I. Sedenko for Marguerre–Vlasov equations and then utilized for the von Kármán equations in [25]. Let \mathbf{e}^1 and \mathbf{e}^2 be two weak solutions to the small finite deflections shell problem, with the same initial condition. Let $\tilde{\mathbf{e}} = \mathbf{e}^2 - \mathbf{e}^1$. The goal of this section is to prove that $\tilde{\mathbf{e}} = 0$.

7.1. First step

Lemma 7. *The following inequality holds true*

$$\left\| \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} \right\|_{\mathcal{H}} \leq \int_0^t \left\| \mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}[N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix} \right\|_{\mathcal{H}} ds. \tag{95}$$

Proof. Eq. (20) yields

$$\partial_t \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} + \mathbb{A} \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} = \mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}[N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix} \tag{96}$$

and consequently

$$\begin{aligned} & \left(\partial_t \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix}, \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} \right)_{\mathcal{H}} + \left(\mathbb{A} \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix}, \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} \right)_{\mathcal{H}} \\ &= \left(\mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}_\gamma^{-1}[N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix}, \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} \right)_{\mathcal{H}}. \end{aligned}$$

Within this proof, we shall define

$$X(s) = \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}}(s) \\ \partial_t \tilde{\mathbf{e}}(s) \end{pmatrix} \quad \text{and} \quad F(s) = \mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}[N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix}.$$

Using Lemma 2 on (96) yields

$$(\partial_t X(s), X(s))_{\mathcal{H}} = (X(s), F(s))_{\mathcal{H}}$$

and therefore

$$\frac{1}{2} \int_0^t \partial_s \|X(s)\|_{\mathcal{H}}^2 ds = \int_0^t (X(s), F(s))_{\mathcal{H}} ds$$

whence it follows that

$$\frac{1}{2} \|X(t)\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|X(0)\|_{\mathcal{H}}^2 + \int_0^t \|X(s)\|_{\mathcal{H}} \|F(s)\|_{\mathcal{H}} ds.$$

Since $\tilde{\mathbf{e}}(0) = \partial_t \mathbf{e}(0) = 0$ and \mathbb{A}^{-1} is linear, we have $X(0) = 0$; it follows that

$$\frac{1}{2} \|X(t)\|_{\mathcal{H}}^2 \leq \left(\sup_{s \in [0, t]} \|X(s)\|_{\mathcal{H}} \right) \left(\int_0^t \|F(s)\|_{\mathcal{H}} ds \right)$$

and thus, for all $\bar{t} \leq t$, we have

$$\begin{cases} \frac{1}{2} \|X(\bar{t})\|_{\mathcal{H}}^2 \leq \left(\sup_{s \in [0, \bar{t}]} \|X(s)\|_{\mathcal{H}} \right) \left(\int_0^{\bar{t}} \|F(s)\|_{\mathcal{H}} ds \right) \\ \sup_{s \in [0, \bar{t}]} \|X(s)\|_{\mathcal{H}}^2 \leq \sup_{s \in [0, t]} \|X(s)\|_{\mathcal{H}}^2 \\ \int_0^{\bar{t}} \|F(s)\|_{\mathcal{H}} ds \leq \int_0^t \|F(s)\|_{\mathcal{H}} ds \end{cases}$$

whence it follows that

$$\frac{1}{2} \|X(\bar{t})\|_{\mathcal{H}}^2 \leq \left(\sup_{s \in [0, t]} \|X(s)\|_{\mathcal{H}} \right) \left(\int_0^t \|F(s)\|_{\mathcal{H}} ds \right)$$

and thus

$$\frac{1}{2} \sup_{\bar{t} \in [0, t]} \|X(\bar{t})\|_{\mathcal{H}}^2 \leq \left(\sup_{s \in [0, t]} \|X(s)\|_{\mathcal{H}} \right) \left(\int_0^t \|F(s)\|_{\mathcal{H}} ds \right)$$

and so consequently

$$\sup_{s \in [0, t]} \|X(s)\|_{\mathcal{H}} \leq \int_0^t \|F(s)\|_{\mathcal{H}} ds$$

and using $\|X(t)\|_{\mathcal{H}} \leq \sup_{s \in [0,t]} \|X(s)\|_{\mathcal{H}}$, we get

$$\|X(t)\|_{\mathcal{H}} \leq \int_0^t \|F(s)\|_{\mathcal{H}} ds,$$

that is (95). \square

Lemma 8. *The following inequality holds true*

$$\|\mathcal{M}_\gamma^{1/2} \tilde{\mathbf{e}}\|_{L^2} \leq \left\| \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} \right\|_{\mathcal{H}}. \tag{97}$$

Proof. We have

$$\mathbb{A}^{-1} \begin{pmatrix} \mathbf{e} \\ \partial_t \tilde{\mathbf{e}} \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{-1} \mathcal{M}_\gamma \partial_t \mathbf{e} \\ -\tilde{\mathbf{e}} \end{pmatrix}$$

and therefore

$$\left\| \mathbb{A}^{-1} \begin{pmatrix} \tilde{\mathbf{e}} \\ \partial_t \mathbf{e} \end{pmatrix} \right\|_{\mathcal{H}}^2 = \|\mathcal{A}^{-1/2} \mathcal{M}_\gamma \partial_t \tilde{\mathbf{e}}\|_{L^2}^2 + \|\mathcal{M}_\gamma^{1/2} \tilde{\mathbf{e}}\|_{L^2}^2.$$

Inequality (97) follows. \square

Proposition 3. *The following inequality holds true*

$$\|\mathcal{M}_\gamma^{1/2} \tilde{\mathbf{e}}\|_{L^2} \leq \int_0^t \|\mathcal{A}^{-1/2} [N(\mathbf{e}^2(s)) - N(\mathbf{e}^1)(s)]\|_{L^2} ds. \tag{98}$$

Proof. We have

$$\mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}_\gamma^{-1} [N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{-1} [N(\mathbf{e}^2) - N(\mathbf{e}^1)] \\ 0 \end{pmatrix}$$

and therefore

$$\left\| \mathbb{A}^{-1} \begin{pmatrix} 0 \\ \mathcal{M}_\gamma^{-1} [N(\mathbf{e}^2) - N(\mathbf{e}^1)] \end{pmatrix} \right\|_{\mathcal{H}} = \|\mathcal{A}^{-1/2} [N(\mathbf{e}^2) - N(\mathbf{e}^1)]\|_{L^2}.$$

From Lemma 7, we get

$$\left\| \mathbb{A}^{-1} \begin{pmatrix} \mathbf{e} \\ \partial_t \mathbf{e} \end{pmatrix} \right\|_{\mathcal{H}} \leq \int_0^t \|\mathcal{A}^{-1/2} [N(\mathbf{e}^2) - N(\mathbf{e}^1)]\|_{L^2} ds.$$

Lemma 8 yields

$$\|\mathcal{M}_\gamma^{1/2} \tilde{\mathbf{e}}\|_{L^2} \leq \int_0^t \|\mathcal{A}^{-1/2} [N(\mathbf{e}^2) - N(\mathbf{e}^1)]\|_{L^2} ds$$

which proves (98). \square

Corollary 2. *Following Proposition 3, we obtain*

$$\|\tilde{\mathbf{e}}_\Gamma\|_{L^2}^2 + \|\tilde{w}\|_{H^1}^2 \leq t \int_0^t (\|N_\Gamma(w^2) - N_\Gamma(w^1)\|_{H^{-1}}^2 + \|N_n(e_\Gamma^2, w^2) - N_n(e_\Gamma^1, w^1)\|_{H^{-2}}^2) ds. \tag{99}$$

Proof. The corollary follows from the use of $(\int_0^t f(t) dt)^2 \leq t \int_0^t f(t)^2 dt$ on (98). \square

7.2. Second step

Lemma 9. *We have*

$$N_\Gamma(w^2) - N_\Gamma(w^1) = \lambda \nabla_\Gamma(\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle) + 2\mu \operatorname{div}_\Gamma(\nabla_\Gamma \tilde{w} \otimes \nabla_\Gamma w^2 + \nabla_\Gamma w^1 \otimes \nabla_\Gamma \tilde{w}) \tag{100}$$

$$\begin{aligned} N_n(e_\Gamma^2, w^2) - N_n(e_\Gamma^1, w^1) &= -2H\lambda \langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle \\ &+ 4\lambda \operatorname{div}_\Gamma(Hw^2 \nabla_\Gamma \tilde{w} + H\tilde{w} \nabla_\Gamma w^1) + 4\mu \operatorname{div}_\Gamma(D^2b(w^2 \nabla_\Gamma \tilde{w} + \tilde{w} \nabla_\Gamma w^1)) \\ &+ (\lambda + 2\mu) \operatorname{div}_\Gamma(\|\nabla_\Gamma w^2\|^2 \nabla_\Gamma \tilde{w} + \nabla_\Gamma w^1 \langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle) \\ &- 2\mu \langle D^2b \nabla_\Gamma \tilde{w}, \nabla_\Gamma w^2 \rangle - 2\mu \langle D^2b \nabla_\Gamma w^1, \nabla_\Gamma \tilde{w} \rangle - 2\lambda \operatorname{div}_\Gamma(\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2 + \operatorname{div}_\Gamma e_\Gamma^1 \nabla_\Gamma \tilde{w}) \\ &- 4\mu \operatorname{div}_\Gamma((\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w} + (\varepsilon_\Gamma \tilde{e}_\Gamma + V_\Gamma \tilde{e}_\Gamma) \nabla_\Gamma w^1). \end{aligned} \tag{101}$$

Proposition 4. *Let $\epsilon > 0$ and $\alpha > 0$ be such that $\epsilon + \alpha \leq 1$. There exists a real C depending only on λ, μ, w^1 and w^2 such that*

$$\|N_\Gamma(w^2) - N_\Gamma(w^1)\|_{H^{-1}}^2 \leq C \ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + \frac{C}{\lambda_n^\alpha}. \tag{102}$$

Proof. Following Definition 7 gives

$$\begin{aligned} \|N_\Gamma(w^2) - N_\Gamma(w^1)\|_{H^{-1}}^2 &\leq \lambda \|\nabla_\Gamma \langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle\|_{H^{-1}}^2 \\ &+ 2\mu \|\operatorname{div}_\Gamma(\nabla_\Gamma \tilde{w} \otimes \nabla_\Gamma w^2 + \nabla_\Gamma w^1 \otimes \nabla_\Gamma \tilde{w})\|_{H^{-1}}^2. \end{aligned} \tag{103}$$

Since $\operatorname{div}_\Gamma$ is a bounded operator on H^{-1} , there exists a constant C such that

$$\begin{aligned} \|N_\Gamma(w^2) - N_\Gamma(w^1)\|_{H^{-1}}^2 &\leq \lambda^2 C \|\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle\|_{L^2}^2 \\ &+ 4\mu^2 C \|\nabla_\Gamma \tilde{w} \otimes \nabla_\Gamma w^2 + \nabla_\Gamma w^1 \otimes \nabla_\Gamma \tilde{w}\|_{L^2}^2. \end{aligned} \tag{104}$$

Applying (55) with $v_1 = \nabla_\Gamma \tilde{w}$ and $v_2 = \nabla_\Gamma(w^2 + w^1)$ yields

$$\|\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma(w^2 + w^1) \rangle\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + \frac{1}{\lambda_n^\alpha} \|w^2 - w^1\|_{H^{1+\epsilon+\alpha}}^2 \right) \|w^2 + w^1\|_{H^2}^2. \tag{105}$$

Applying (56) with $v_1 = \nabla_\Gamma \tilde{w}$ and $v_2 = \nabla_\Gamma w^2$ yields

$$\|\nabla_\Gamma \tilde{w} \otimes \nabla_\Gamma w^2\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + \frac{1}{\lambda_n^\alpha} \|w^2 - w^1\|_{H^{1+\epsilon+\alpha}}^2 \right) \|w^2\|_{H^2}^2. \tag{106}$$

Applying (56) with $v_1 = \nabla_\Gamma \tilde{w}$ and $v_2 = \nabla_\Gamma w^1$ yields

$$\|\nabla_\Gamma w^1 \otimes \nabla_\Gamma \tilde{w}\|_{L^2}^2 \leq C \left(\ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + \frac{1}{\lambda_n^\alpha} \|w^2 - w^1\|_{H^{1+\epsilon+\alpha}}^2 \right) \|w^1\|_{H^2}^2. \tag{107}$$

These inequalities, and w^1 and w^2 belonging to H^2 , give (102). \square

Proposition 5. *Let $\epsilon > 0$ and $\alpha > 0$ be such that $\epsilon + \alpha \leq 1$. There exists a real C depending only on $\lambda, \mu, w^1, w^2, e_\Gamma^1, e_\Gamma^2$, and b , such that*

$$\|N_n(e_\Gamma^2, w^2) - N_n(e_\Gamma^1, w^1)\|_{H^{-2}}^2 \leq C \ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + C \ln(1 + \lambda_n) \|\tilde{e}_\Gamma\|_{L^2}^2 + \frac{C}{\lambda_n^\alpha}. \tag{108}$$

Proof. Let us define

$$\begin{aligned} X^1(\tilde{w}, w^1, w^2) &= -2\lambda H \langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma (w^2 + w^1) \rangle + 4\lambda \operatorname{div}_\Gamma (H w^2 \nabla_\Gamma \tilde{w} + H \tilde{w} \nabla_\Gamma w^1) \\ &\quad + 4\mu \operatorname{div}_\Gamma (D^2 b (w^2 \nabla_\Gamma \tilde{w} + \tilde{w} \nabla_\Gamma w^1)) \\ &\quad - 2\mu \langle D^2 b \nabla_\Gamma \tilde{w}, \nabla_\Gamma w^2 \rangle - 2\mu \langle D^2 b \nabla_\Gamma w^1, \nabla_\Gamma \tilde{w} \rangle \\ &\quad + (\lambda + 2\mu) \operatorname{div}_\Gamma (\nabla_\Gamma \tilde{w} \|\nabla_\Gamma w^2\|^2 + \nabla_\Gamma w^1 \langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma (w^2 + w^1) \rangle) \\ X^2(\tilde{w}, \tilde{e}_\Gamma, w^1, e_\Gamma^1, w^2, e_\Gamma^2) &= -2\lambda \operatorname{div}_\Gamma (\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2 + \operatorname{div}_\Gamma e_\Gamma^1 \nabla_\Gamma \tilde{w}) \\ &\quad - 4\mu \operatorname{div}_\Gamma ((\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w} + (\varepsilon_\Gamma \tilde{e}_\Gamma + V_\Gamma \tilde{e}_\Gamma) \nabla_\Gamma w^1) \end{aligned}$$

so we have

$$N_n(e_\Gamma^2, w^2) - N_n(e_\Gamma^1, w^1) = X^1(\tilde{w}, w^1, w^2) + X^2(\tilde{w}, \tilde{e}_\Gamma, w^1, e_\Gamma^1, w^2, e_\Gamma^2) \tag{109}$$

and therefore

$$\|N_n(e_\Gamma^2, w^2) - N_n(e_\Gamma^1, w^1)\|_{H^{-2}} \leq \|X^1(\tilde{w}, w^1, w^2)\|_{H^{-2}} + \|X^2(\tilde{w}, \tilde{e}_\Gamma, w^1, e_\Gamma^1, w^2, e_\Gamma^2)\|_{H^{-2}}. \tag{110}$$

Estimation of the first term of the right hand side of (110)

$$\begin{aligned} \|X^1(\tilde{w}, w^1, w^2)\|_{H^{-2}} &\leq C(\|\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma (w^2 + w^1) \rangle\|_{H^{-2}} + \|w^2 \nabla_\Gamma \tilde{w}\|_{H^{-1}} \\ &\quad + \|\tilde{w} \nabla_\Gamma w^1\|_{H^{-1}} + \|\langle D^2 b \nabla_\Gamma \tilde{w}, \nabla_\Gamma w^2 \rangle\|_{H^{-2}} + \|\langle D^2 b \nabla_\Gamma w^1, \nabla_\Gamma \tilde{w} \rangle\|_{H^{-2}} \\ &\quad + \|\operatorname{div}_\Gamma ((\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w} + (\varepsilon_\Gamma \tilde{e}_\Gamma + V_\Gamma \tilde{e}_\Gamma) \nabla_\Gamma w^1)\|_{H^{-2}}). \end{aligned}$$

Bounding $\|X^1(\tilde{w}, w^1, w^2)\|_{H^{-2}}$ is an easy task, because all the terms are of lower order. For instance,

$$\begin{aligned} \|\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma (w^2 + w^1) \rangle\|_{H^{-2}} &\leq C \|\langle \nabla_\Gamma \tilde{w}, \nabla_\Gamma (w^2 + w^1) \rangle\|_{L^2} \\ &\leq C \|\nabla_\Gamma \tilde{w}\|_{L^2} \|\nabla_\Gamma (w^2 + w^1)\|_{L^2} \\ &\leq C \|\tilde{w}\|_{H^1} \|w^2 + w^1\|_{H^1} \\ &\leq C \|\tilde{w}\|_{H^1} \end{aligned}$$

where C depends only on w^1 and w^2 . An analogous argument is repeated for the four other terms. Eventually, we obtain that

$$\|X^1(\tilde{w}, w^1, w^2)\|_{H^{-2}} \leq C \|\tilde{w}\|_{H^1} \tag{111}$$

where C depends only on w^1, w^2 , the Lamé coefficients and b .

Estimation of the second term of the right hand side of (110)

$$\begin{aligned} \|X^2(\tilde{w}, \tilde{e}_\Gamma, w^1, e_\Gamma^1, w^2, e_\Gamma^2)\|_{H^{-2}} &\leq C(\|\operatorname{div}_\Gamma (\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2 + \operatorname{div}_\Gamma e_\Gamma^1 \nabla_\Gamma \tilde{w})\|_{H^{-2}} \\ &\quad + \|\operatorname{div}_\Gamma ((\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w} + (\varepsilon_\Gamma \tilde{e}_\Gamma + V_\Gamma \tilde{e}_\Gamma) \nabla_\Gamma w^1)\|_{H^{-2}}) \\ &\leq C(\|\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2\|_{H^{-1}} + \|\operatorname{div}_\Gamma e_\Gamma^1 \nabla_\Gamma \tilde{w}\|_{H^{-1}} \\ &\quad + \|(\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w}\|_{H^{-1}} + \|((\varepsilon_\Gamma \tilde{e}_\Gamma + V_\Gamma \tilde{e}_\Gamma) \nabla_\Gamma w^1)\|_{H^{-1}}). \end{aligned}$$

Let us look first at the term $\|\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2\|_{H^{-1}}$. Using Green’s formula for tangential derivatives (5) gives

$$\begin{aligned} \|\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2\|_{H^{-1}} &= \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \int_\Gamma (\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2, \varphi)_{H^{-1} \times H_0^1} \\ &= \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \int_\Gamma -\langle \tilde{e}_\Gamma, \nabla_\Gamma (\nabla_\Gamma w^2, \varphi)_{H^{-1} \times H_0^1} \rangle \\ &\quad + 2 \int_\Gamma H (\nabla_\Gamma w^2, \varphi)_{H^{-1} \times H_0^1} \langle \tilde{e}_\Gamma, \nabla b \rangle + \int_{\partial\Gamma} \langle \nabla_\Gamma (\nabla_\Gamma w^2, \varphi)_{H^{-1} \times H_0^1} \tilde{e}_\Gamma, \nu \rangle \\ &= - \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \int_\Gamma \langle \tilde{e}_\Gamma, D_\Gamma^2 w^2 \varphi \rangle + \langle \tilde{e}_\Gamma, \nabla_\Gamma w^2 D_\Gamma \varphi \rangle \end{aligned}$$

since the shell is clamped on the boundary and $\langle \tilde{e}_\Gamma, \nabla b \rangle = 0$. Using Cauchy–Schwarz and the inequality (56) we have

$$\begin{aligned} \left| \int_\Gamma \langle \tilde{e}_\Gamma, D_\Gamma^2 w^2 \varphi \rangle \right| &= \left| \int_\Gamma D_\Gamma^2 w^2 \cdot \cdot (\tilde{e}_\Gamma \otimes \varphi) \right| \\ &\leq \|D_\Gamma^2 w^2\|_{L_2} \|\tilde{e}_\Gamma \otimes \varphi\|_{L_2} \\ &\leq C |w^2|_{H^2} \left[\ln(1 + \lambda_n) \|\tilde{e}_\Gamma\|_{L_2} + \frac{1}{\lambda_n^\alpha} \|\tilde{e}_\Gamma\|_{H^{\epsilon+\alpha}} \right] \|\varphi\|_{H^1} \end{aligned}$$

and the second term can be bounded as well:

$$\begin{aligned} \left| \int_\Gamma \langle \tilde{e}_\Gamma, \nabla_\Gamma w^2 D_\Gamma \varphi \rangle \right| &= \left| \int_\Gamma D_\Gamma \varphi \cdot \cdot \tilde{e}_\Gamma \otimes \nabla_\Gamma w^2 \right| \\ &\leq \|D_\Gamma \varphi\|_{L_2} \|\tilde{e}_\Gamma \otimes \nabla_\Gamma w^2\|_{L_2} \\ &\leq C \|\varphi\|_{H^1} \left[\ln(1 + \lambda_n) \|\tilde{e}_\Gamma\|_{L_2} + \frac{1}{\lambda_n^\alpha} \|\tilde{e}_\Gamma\|_{H^{\epsilon+\alpha}} \right] \|w^2\|_{H^2} \end{aligned}$$

so that

$$\|\operatorname{div}_\Gamma \tilde{e}_\Gamma \nabla_\Gamma w^2\|_{H^{-1}} \leq C \left[\ln(1 + \lambda_n) \|\tilde{e}_\Gamma\|_{L_2} + \frac{1}{\lambda_n^\alpha} \|\tilde{e}_\Gamma\|_{H^{\epsilon+\alpha}} \right] \|w^2\|_{H^2}. \tag{112}$$

The second and third terms are more direct

$$\begin{aligned} \|\operatorname{div}_\Gamma e_\Gamma^1 \nabla_\Gamma \tilde{w}\|_{H^{-1}} &= \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \int_\Gamma \operatorname{div}_\Gamma e_\Gamma^1 \langle \nabla_\Gamma \tilde{w}, \varphi \rangle \\ &\leq \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \|\operatorname{div}_\Gamma e_\Gamma^1\|_{L_2} \|\langle \nabla_\Gamma \tilde{w}, \varphi \rangle\|_{L_2} \\ &\leq C \left[\ln(1 + \lambda_n) \|\tilde{w}\|_{H^1} + \frac{1}{\lambda_n^\alpha} \|\tilde{w}\|_{H^{1+\epsilon+\alpha}} \right] \|e_\Gamma^1\|_{H^1} \end{aligned} \tag{113}$$

and

$$\begin{aligned} \|(\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \nabla_\Gamma \tilde{w}\|_{H^{-1}} &= \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \int_\Gamma (\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2) \cdot \cdot (\nabla_\Gamma \tilde{w} \otimes \varphi) \\ &\leq \sup_{\{\varphi, \|\varphi\|_{H_0^1}=1\}} \|(\varepsilon_\Gamma e_\Gamma^2 + V_\Gamma e_\Gamma^2)\|_{L_2} \|\nabla_\Gamma \tilde{w} \otimes \varphi\|_{L_2} \\ &\leq C \left[\ln(1 + \lambda_n) \|\tilde{w}\|_{H^1} + \frac{1}{\lambda_n^\alpha} \|\tilde{w}\|_{H^{1+\epsilon+\alpha}} \right] \|e_\Gamma^2\|_{H^1}. \end{aligned} \tag{114}$$

The fourth term requires the following additional integrations using the Green’s formulas (7):

$$\begin{aligned} \int_\Gamma \varepsilon_\Gamma \tilde{e}_\Gamma \cdot \cdot (\nabla_\Gamma w^1 \otimes \varphi) &= \frac{1}{2} \int_\Gamma D_\Gamma \tilde{e}_\Gamma \cdot \cdot (\nabla_\Gamma w^1 \otimes \varphi) + D_\Gamma \tilde{e}_\Gamma \cdot \cdot (\varphi \otimes \nabla_\Gamma w^1) \\ &= -\frac{1}{2} \int_\Gamma \langle \tilde{e}_\Gamma, \operatorname{div}_\Gamma (\nabla_\Gamma w^1 \otimes \varphi) \rangle + \frac{1}{2} \int_\Gamma 2H \langle \tilde{e}_\Gamma, (\nabla_\Gamma w^1 \otimes \varphi) \nabla b \rangle \\ &\quad - \frac{1}{2} \int_\Gamma \langle \tilde{e}_\Gamma, \operatorname{div}_\Gamma (\varphi \otimes \nabla_\Gamma w^1) \rangle + \frac{1}{2} \int_\Gamma 2H \langle \tilde{e}_\Gamma, (\varphi \otimes \nabla_\Gamma w^1) \nabla b \rangle \\ &= -\frac{1}{2} \int_\Gamma \langle \tilde{e}_\Gamma, \operatorname{div}_\Gamma (\nabla_\Gamma w^1 \otimes \varphi) + \operatorname{div}_\Gamma (\varphi \otimes \nabla_\Gamma w^1) \rangle + \int_\Gamma H \langle \tilde{e}_\Gamma, \nabla_\Gamma w^1 \rangle \langle \varphi, \nabla b \rangle \end{aligned}$$

and

$$V_\Gamma \tilde{e}_\Gamma \cdot \cdot (\nabla_\Gamma w^1 \otimes \varphi) = \frac{1}{2} \langle D^2 b \tilde{e}_\Gamma, \nabla_\Gamma w^1 \rangle \langle \varphi, \nabla b \rangle$$

so that we have

$$\begin{aligned} \left| \int_{\Gamma} (\varepsilon_{\Gamma}(\tilde{e}_{\Gamma}) + V_{\Gamma}(\tilde{e}_{\Gamma})) \cdot (\nabla_{\Gamma} w^1 \otimes \varphi) \right| &\leq \frac{1}{2} \left| \int_{\Gamma} \langle e_{\Gamma}, \operatorname{div}_{\Gamma}(\nabla_{\Gamma} w \otimes \varphi) + \operatorname{div}_{\Gamma}(\varphi \otimes \nabla_{\Gamma} w) \rangle \right| \\ &\quad + \frac{1}{2} \left| \langle 2H\tilde{e}_{\Gamma} + D^2b\tilde{e}_{\Gamma}, \nabla_{\Gamma} w^1 \rangle \langle \varphi, \nabla b \rangle \right| \\ &\leq \frac{1}{2} \int_{\Gamma} |\langle \tilde{e}_{\Gamma}, D_{\Gamma}^2 w^1 \varphi \rangle + \langle \tilde{e}_{\Gamma}, \nabla_{\Gamma} w^1 \operatorname{div}_{\Gamma} \varphi \rangle| \\ &\quad + |\langle \tilde{e}_{\Gamma}, D_{\Gamma} \varphi \nabla_{\Gamma} w^1 \rangle + \langle \tilde{e}_{\Gamma}, \Delta_{\Gamma} w^1 \varphi \rangle| \\ &\quad + C \int_{\Gamma} \langle \tilde{e}_{\Gamma}, \nabla_{\Gamma} w^1 \rangle \langle \varphi, \nabla b \rangle \\ &\leq C(\|D_{\Gamma}^2 w^1\|_{L_2} \|\tilde{e}_{\Gamma} \otimes \varphi\|_{L_2} + \|\operatorname{div}_{\Gamma} \varphi\|_{L_2} \|\langle \tilde{e}_{\Gamma}, \nabla_{\Gamma} w^1 \rangle\|_{L_2} \\ &\quad + \|D_{\Gamma} \varphi\|_{L_2} \|\tilde{e}_{\Gamma} \otimes \nabla_{\Gamma} w^1\|_{L_2} + \|\Delta_{\Gamma} w^1\|_{L_2} \|\langle \tilde{e}_{\Gamma}, \varphi \rangle\|_{L_2} \\ &\quad + \|\langle \tilde{e}_{\Gamma}, \nabla_{\Gamma} w^1 \rangle\|_{L_2} \|\langle \varphi, \nabla b \rangle\|_{L_2}) \end{aligned}$$

and after application of Eqs. (55) and (56) we have

$$\|(\varepsilon_{\Gamma} \tilde{e}_{\Gamma}) + (V_{\Gamma} \tilde{e}_{\Gamma}) \nabla_{\Gamma} w^1\|_{H^{-1}} \leq C \left[\ln(1 + \lambda_n) \|\tilde{e}_{\Gamma}\|_{L_2} + \frac{1}{\lambda_n^{\alpha}} \|\tilde{e}_{\Gamma}\|_{H^{\epsilon+\alpha}} \right] \|w^1\|_{H^2}. \tag{115}$$

Finally, collecting equations (112) through (115) allows us to estimate the second term as

$$\begin{aligned} \|X^2(\tilde{w}, \tilde{e}_{\Gamma}, w^1, e_{\Gamma}^1, w^2, e_{\Gamma}^2)\|_{H^{-2}} &\leq C \ln(1 + \lambda_n) \|\tilde{w}\|_{H^1} + C \ln(1 + \lambda_n) \|\tilde{e}_{\Gamma}\|_{L_2} \\ &\quad + \frac{C}{\lambda_n^{\alpha}} \|\tilde{w}\|_{H^{1+\epsilon+\alpha}} + \frac{C}{\lambda_n^{\alpha}} \|\tilde{e}_{\Gamma}\|_{H^{\epsilon+\alpha}} \end{aligned} \tag{116}$$

where C depends only on w^1, w^2 , the Lamé coefficients and b .

Conclusion: From (110), (111) and (116), we derive

$$\|N_n(e_{\Gamma}^2, w^2) - N_{\Gamma}(e_{\Gamma}^1, w^1)\|_{H^{-2}}^2 \leq C \ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + C \ln(1 + \lambda_n) \|\tilde{e}_{\Gamma}\|_{L_2}^2 + \frac{C}{\lambda_n^{\alpha}}$$

which is the announced result. \square

7.3. Third step

From now on we will consider $\alpha = 1 - \epsilon$. This choice of α complies with the requirements of Proposition 4 and Proposition 5.

Lemma 10. *The following inequality holds true*

$$\|\tilde{e}_{\Gamma}\|_{L_2}^2 + \|\tilde{w}\|_{H^1}^2 \leq Ct^2 \lambda_n^{Ct^2-\alpha} \tag{117}$$

where C depends only on $\mathbf{e}^1, \mathbf{e}^2$, the Lamé coefficients and the geometric information embedded in b .

Proof. Propositions 3–5 give

$$\|\tilde{e}_{\Gamma}\|_{L_2}^2 + \|\tilde{w}\|_{H^1}^2 \leq t \int_0^t \left(C \ln(1 + \lambda_n) \|\tilde{w}\|_{H^1}^2 + C \ln(1 + \lambda_n) \|\tilde{e}_{\Gamma}\|_{L_2}^2 + \frac{C}{\lambda_n^{\alpha}} \right) ds \tag{118}$$

and therefore

$$\|\tilde{e}_{\Gamma}\|_{L_2}^2 + \|\tilde{w}\|_{H^1}^2 \leq \frac{Ct^2}{\lambda_n^{\alpha}} + Ct \ln(1 + \lambda_n) \int_0^t (\|\tilde{w}\|_{H^1}^2 + \|\tilde{e}_{\Gamma}\|_{L_2}^2) ds. \tag{119}$$

Gronwall’s inequality yields

$$\|\tilde{e}_\Gamma\|_{L^2}^2 + \|\tilde{w}\|_{H^1}^2 \leq \frac{Ct^2}{\lambda_n^\alpha} \exp\left(Ct^2 \ln(1 + \lambda_n)\right) \tag{120}$$

and consequently

$$\|\tilde{e}_\Gamma\|_{L^2}^2 + \|\tilde{w}\|_{H^1}^2 \leq \frac{Ct^2}{\lambda_n^\alpha} (1 + \lambda_n)^{Ct^2} \tag{121}$$

Eq. (117) follows. \square

Corollary 3. *We have $\tilde{\mathbf{e}} = 0$.*

Proof. The sequence (λ_n) tends toward $+\infty$. When $t \in \left[0, \sqrt{\frac{\alpha}{C}}\right]$, the right hand side of (117) tends to 0. Consequently $\tilde{\mathbf{e}} = 0$ on $\left[0, \sqrt{\frac{\alpha}{C}}\right]$. The bootstrap argument completes the proof of the corollary and the proof of Theorem 2. \square

Remark 4. The proof of uniqueness does not give the continuous dependence, which requires a separate argument. This is the point of the next section of this paper.

8. Proof of Theorem 3 (continuous dependence with respect to the initial data)

The main technical difficulty in the proof of Theorem 3 is the derivation of the energy identity for weak solutions. We denote as $B([0, \tau[; X)$ the space of X -valued functions which are bounded on the interval $[0, \tau[$. This space is endowed with the norm $\|x\|_{B([0, \tau[; X)} = \sup_{t \in [0, \tau[} |x(t)|_X$.

8.1. Energy identity result

Lemma 11. *We consider solutions to (16) with the following a priori regularity:*

$$\mathbf{e} \in B([0, \tau[; V^1(\Gamma)), \quad \partial_t \mathbf{e} \in B([0, \tau[; V^0(\Gamma)).$$

Then, the following energy identity holds:

$$\mathcal{E}(t) = \mathcal{E}(s), \quad 0 \leq s \leq t \leq \tau \tag{122}$$

where \mathcal{E} is defined in (44).

Note that Proposition 1 cannot be applied as is because of the lack of regularity of \mathbf{e} . In order to prove the energy identity we shall use a finite difference approximation of time derivatives (the same approximation as was used in [1, 18] for the von Kármán plate).

Let $h > 0$ be a small parameter designed to go to zero. Let $g \in B([0, \tau[; X)$ where X is a Hilbert space. We extend $g(t)$ to all $t \in \mathbb{R}$ by defining $g(t) = g(0)$ for $t \leq 0$ and $g(t) = g(\tau)$ for $t \geq \tau$. With these extensions we define three finite difference operators depending on the parameter h .

$$\begin{aligned} g_h^+(t) &\equiv g(t+h) - g(t) \\ g_h^-(t) &\equiv g(t) - g(t-h) \\ \partial_h g(t) &\equiv \frac{1}{2h} [g_h^+(t) + g_h^-(t)]. \end{aligned} \tag{123}$$

Proposition 6. *The following properties hold true:*

(i) *Let g be weakly continuous with values in X . Then*

$$\lim_{h \rightarrow 0} \int_0^\tau (g(t), \partial_h g(t))_X dt = \frac{1}{2} [\|g(\tau)\|_X^2 - \|g(0)\|_X^2].$$

(ii) Let $g \in H^1([0, \tau[; X)$. Then the following limits are well defined in $L_2([0, \tau[; X)$.

$$\lim_{h \rightarrow 0} \partial_h g = \partial_t g; \quad \lim_{h \rightarrow 0} \frac{1}{h} g_h^+ = \partial_t g; \quad \lim_{h \rightarrow 0} \frac{1}{h} g_h^- = \partial_t g.$$

Moreover if $\partial_t g$ is weakly continuous with the values in X , then for every $t \in (0, \tau)$, $\partial_h g(t) \rightarrow \partial_t g(t)$, weakly in X , and

$$\frac{1}{h} g_h^-(\tau) \rightarrow \partial_t g(\tau); \quad \frac{1}{h} g_h^+(0) \rightarrow \partial_t g(0); \quad \text{weakly in } X.$$

(iii) In addition to previous assumptions, let $V \subset X \subset V'$,

$$\partial_{tt} g \in L_2([0, \tau[, V'); \quad g \in L_2([0, \tau[, V),$$

then

$$\lim_{h \rightarrow 0} \int_0^\tau (\partial_{tt} g(t), \partial_h g(t))_X dt = \frac{1}{2} [\|\partial_t g(\tau)\|_X^2 - \|\partial_t g(0)\|_X^2].$$

Proof. The proof of Proposition 6 is elementary and the main steps are detailed in [18]. □

We return to the proof of Lemma 11.

Proof of Lemma 11. Eq. (47) gives

$$\mathcal{E}^F(t) \equiv \mathcal{E}(t) - \frac{l}{2} \int_\Gamma (CB_M \mathbf{e}, B_M \mathbf{e}). \tag{124}$$

Recall that the definition of \mathcal{E}^F includes the kinetic energy of the shell as well as the flexural terms from the potential energy \mathcal{E}_p (see (46) and (35) for a definition of these quantities).

$$\begin{aligned} \mathcal{E}^F(t) = & \frac{\rho l}{2} \int_\Gamma |\partial_t e_\Gamma|^2 + |\partial_t w|^2 + \frac{\rho l \gamma}{2} \int_\Gamma |D^2 b \partial_t e_\Gamma|^2 + |\nabla_\Gamma \partial_t w|^2 + |D^2 b \partial_t e_\Gamma - \nabla_\Gamma \partial_t w|^2 \\ & + \frac{\lambda \gamma l}{2} \langle \Delta_\Gamma w, \Delta_\Gamma \hat{w} \rangle_\Gamma + \mu \gamma l \int_\Gamma \text{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma \hat{w} + G_\Gamma \hat{w})). \end{aligned} \tag{125}$$

We will use the variational form (16) with the test functions $\partial_h \mathbf{e} = \partial_h \hat{e}_\Gamma + \partial_h \hat{w} \nabla b \in V^1(\Gamma)$. For reasons of readability we omit the details of the calculation, but this choice of test functions, dividing by 2, and the first three formulas of Proposition 6 easily give the following identity:

$$\frac{1}{2} \mathcal{E}^F(\tau) + \lim_{h \rightarrow 0} \int_0^\tau X_h dt = \frac{1}{2} \mathcal{E}^F(0) \tag{126}$$

where, in terms of the tensor B_M , we have

$$X_h \equiv \int_\Gamma CB_M(\mathbf{e}) \cdot \cdot (\varepsilon(\partial_h e_\Gamma) + V_\Gamma(\partial_h e_\Gamma) + D^2 b \partial_h w + \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w). \tag{127}$$

We note that

$$\int_\Gamma CB_M \mathbf{e} \cdot \cdot \partial_h B_M \mathbf{e} = \int_\Gamma CB_M(\mathbf{e}) \cdot \cdot (\varepsilon(\partial_h e_\Gamma) + V_\Gamma(\partial_h e_\Gamma) + D^2 b \partial_h w + \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w) \tag{128}$$

so that we can rewrite X_h as

$$X_h = \int_\Gamma CB_M \mathbf{e} \cdot \cdot \partial_h B_M \mathbf{e} - \int_\Gamma CB_M \mathbf{e} \cdot \cdot \left(\frac{1}{2} \partial_h (\nabla_\Gamma w \otimes \nabla_\Gamma w) - \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w \right). \tag{129}$$

Direct calculations give us that

$$\begin{aligned} \frac{1}{2} \partial_h (\nabla_\Gamma w(t) \otimes \nabla_\Gamma w(t)) &= \frac{1}{4h} [\nabla_\Gamma w(t+h) \otimes \nabla_\Gamma w(t+h) - \nabla_\Gamma w(t-h) \otimes \nabla_\Gamma w(t-h)] \\ &= \frac{1}{2} [\partial_h \nabla_\Gamma w(t) \otimes \nabla_\Gamma w(t+h) + \nabla_\Gamma w(t-h) \otimes \partial_h \nabla_\Gamma w(t)]. \end{aligned}$$

Since $\mathcal{C}(B_M \mathbf{e})$ is a symmetric tensor we have

$$\begin{aligned} & \mathcal{C}B_M(\mathbf{e}) \cdot \cdot \left(\frac{1}{2} \partial_h (\nabla_\Gamma w(t) \otimes \nabla_\Gamma w(t)) - \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w(t) \right) \\ &= \frac{1}{2} \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes \nabla_\Gamma w(t+h) - \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w(t)) \\ & \quad + \frac{1}{2} \mathcal{C}B_M \mathbf{e} \cdot \cdot (\nabla_\Gamma w(t-h) \otimes \partial_h \nabla_\Gamma w(t) - \nabla_\Gamma w \otimes \nabla_\Gamma \partial_h w(t)) \\ &= \frac{1}{2} \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes \nabla_\Gamma (w(t+h) - \nabla_\Gamma w(t))) \\ & \quad + \frac{1}{2} \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes \nabla_\Gamma (w(t-h) - \nabla_\Gamma w(t))) \\ &= \frac{1}{2} \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes (\nabla_\Gamma w_h^+(t) - \nabla_\Gamma w_h^-(t))). \end{aligned}$$

From these equations we see that

$$X_h = \int_\Gamma \mathcal{C}B_M \mathbf{e} \cdot \cdot \partial_h B_M \mathbf{e} - \frac{1}{2} \int_\Gamma \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes (\nabla_\Gamma w_h^+(t) - \nabla_\Gamma w_h^-(t))) \tag{130}$$

and applying the first identity of Proposition 6 together with weak continuity of $B_M \mathbf{e}$ we have

$$\lim_{h \rightarrow 0} \int_0^\tau X_h \, dt = \frac{1}{2} \|\mathcal{C}^{1/2} B_M \mathbf{e}(\tau)\|_{L_2(\Gamma)}^2 - \frac{1}{2} \|\mathcal{C}^{1/2} B_M \mathbf{e}(0)\|_{L_2(\Gamma)}^2 - \frac{1}{2} \lim_{h \rightarrow 0} \int_0^\tau Y_h \, dt \tag{131}$$

with

$$Y_h \equiv \int_\Gamma \mathcal{C}B_M \mathbf{e} \cdot \cdot (\partial_h \nabla_\Gamma w(t) \otimes (\nabla_\Gamma w_h^+(t) - \nabla_\Gamma w_h^-(t))). \tag{132}$$

Our goal is to show that

$$\lim_{h \rightarrow 0} \int_0^\tau Y_h = 0. \tag{133}$$

Once this is shown, the energy identity will follow from combining (126) and (131). Eqs. (131) and (133) give

$$\int_0^\tau X_h \, dt \rightarrow \frac{1}{2} \|\mathcal{C}^{1/2} B_M \mathbf{e}(\tau)\|_{L_2(\Gamma)}^2 - \frac{1}{2} \|\mathcal{C}^{1/2} B_M \mathbf{e}(0)\|_{L_2(\Gamma)}^2 \tag{134}$$

and thus from (126) we have

$$\mathcal{E}^F(\tau) + \|\mathcal{C}^{1/2} B_M \mathbf{e}(\tau)\|_{L_2(\Gamma)}^2 = \mathcal{E}^F(0) + \|\mathcal{C}^{1/2} B_M \mathbf{e}(0)\|_{L_2(\Gamma)}^2 \tag{135}$$

which gives the result of Lemma 11 for points $s = 0, t = \tau$. Other points in the interval are treated the same way, due to the fact that the argument is local.

It remains only to show that the Eq. (133) holds. The *a priori* regularity of weak solutions gives that $B_M \mathbf{e} \in B([0, \tau]; L_2(\Gamma))$ so that the assertion follows if

$$\int_0^\tau \|\partial_h \nabla_\Gamma w \otimes [\nabla_\Gamma w_h^+ - \nabla_\Gamma w_h^-]\|_{L_2(\Gamma)} \rightarrow 0 \quad \text{as } h \rightarrow 0. \tag{136}$$

Applying Cauchy–Schwarz lets us define

$$Z_h = \int_0^\tau h \|\partial_h \nabla_\Gamma w\|_{L_4(\Gamma)}^2 + h^{-1} (\|\nabla_\Gamma w_h^+\|_{L_4(\Gamma)}^2 + \|\nabla_\Gamma w_h^-\|_{L_4(\Gamma)}^2) \, dt. \tag{137}$$

The fact that $Z_h \rightarrow 0$ as $h \rightarrow 0$ follows by a density argument after we show that

$$Z_h \leq c(\|w\|_{L_2(0,\tau;H^2(\Gamma))} + \|\partial_t w\|_{L_2(0,\tau;H^1(\Gamma))}). \tag{138}$$

We apply Sobolev’s inequality $\|g\|_{L_4(\Gamma)} \leq C\|g\|_{W^{1,1}(\Gamma)}$ which is known to be true for the tangential derivatives on the shell mid-surface [11]. Letting $g = \|\partial_h \nabla_\Gamma w\|^2$ gives

$$\begin{aligned} h\|\partial_h \nabla_\Gamma w\|_{L_4}^2 &\leq Ch\|\partial_h \nabla_\Gamma w\|_{W^{1,1}}^2 \leq Ch\|\nabla_\Gamma |\partial_h \nabla_\Gamma w|^2\|_{L_1} + Ch\|\partial_h \nabla_\Gamma w\|_{L_2}^2 \\ &\leq Ch \int_\Gamma |\nabla_\Gamma \langle \partial_h \nabla_\Gamma w, \partial_h \nabla_\Gamma w \rangle| + Ch \int_\Gamma \|\partial_h \nabla_\Gamma w\|^2 \\ &\leq Ch\|\partial_h \nabla_\Gamma w\|_{L_2(\Gamma)} \|\partial_h \nabla_\Gamma w\|_{H^1(\Gamma)} + Ch \int_\Gamma \|\partial_h \nabla_\Gamma w\|^2. \end{aligned}$$

Using the fact that $h\partial_h$ is bounded then gives that

$$h\|\partial_h \nabla_\Gamma w\|_{L_4}^2 \leq 2C\|\nabla_\Gamma w\|_{H^1(\Gamma)} \|\partial_h \nabla_\Gamma w\|_{L_2(\Gamma)}. \tag{139}$$

The argument is similar for the other terms. \square

8.2. *Completion of the proof*

Let us define

$$\mathbf{x}(t) = (\mathbf{e}, \partial_t \mathbf{e}) = (e_\Gamma(t), w(t), \partial_t e_\Gamma(t), \partial_t w(t))$$

where $\mathbf{e} = e_\Gamma(t) + w(t)\nabla b$ is a weak solution to the small finite deflections shell problem at the time t due to some initial data $\mathbf{e}(0)$.

We start with initial data $\mathbf{x}(0) \in \mathcal{H}$ such that $\mathbf{x}^n(0) \rightarrow \mathbf{x}(0)$ in \mathcal{H} as $n \rightarrow \infty$. Our aim is to prove that

$$\mathbf{x}^n(0) \rightarrow \mathbf{x}(0) \quad \text{in } C([0, \tau], \mathcal{H}). \tag{140}$$

We will follow the road map presented in [18].

The potential energy defined by (35) only differs from the potential energy used in [5] by lower order terms. As a consequence, it is topologically equivalent to $V^1(\Gamma)$. In particular, using Sobolev’s embedding and Korn’s inequality (both of which are valid for the tangential calculus [11]) we have the inequalities

$$\begin{aligned} \|e_\Gamma\|_{H^1(\Gamma)} &\leq C[\|B_M \mathbf{e}\|_{L_2(\Gamma)} + \|\nabla_\Gamma w\|_{L_4(\Gamma)}^2 + \|e_\Gamma\|_{L_2(\Gamma)}] \\ &\leq C[\|B_M \mathbf{e}\|_{L_2(\Gamma)} + \|w\|_{H^2(\Gamma)}^2 + \|e_\Gamma\|_{L_2(\Gamma)}]. \end{aligned} \tag{141}$$

By Lemma 11 and the inequality (141), we have

$$\|\mathbf{x}\|_{\mathcal{H}} \leq C\mathcal{E}(\mathbf{x}(t)) \tag{142}$$

so that

$$\|\mathbf{e}^n(t)\|_{\mathcal{H}} \leq C(\|\mathbf{e}^n(0)\|_{\mathcal{H}}) \leq C(\|\mathbf{e}(0)\|_{\mathcal{H}}). \tag{143}$$

Hence, on a subsequence denoted by the same index we have

$$\mathbf{x}^n(t) \rightarrow \mathbf{x}^*(t) \quad \text{weakly}^* \text{ in } L^\infty(0, \tau; \mathcal{H}).$$

By using the variational equality together with weak continuity of nonlinear terms, we can show that $\mathbf{x}^*(t)$ coincides with a weak solution to (16) due to the initial data $\mathbf{x}(0)$. By uniqueness of the weak solution $\mathbf{x}^*(t) = \mathbf{x}(t)$. Hence we have

$$\mathbf{x}^n(t) \rightarrow \mathbf{x}(t) \quad \text{weakly}^* \text{ in } L^\infty(0, \tau; \mathcal{H}). \tag{144}$$

In view of this, to prove the theorem it is enough to show the norm convergence of

$$\|\mathbf{x}^n(t)\|_{\mathcal{H}} \rightarrow \|\mathbf{x}(t)\|_{\mathcal{H}} \quad \text{in } C(0, \tau).$$

We will use the equality in the energy relation from Lemma 11 to show this. We denote by $\mathcal{E}(\mathbf{x}(t))$ the energy corresponding to the solution $\mathbf{x}(t)$. From Lemma 11 we have

$$\mathcal{E}(\mathbf{x}^n(t)) = \mathcal{E}(\mathbf{x}^n(0)).$$

By the continuity of energy with respect to the strong topology in \mathcal{H} we have

$$\lim_{n \rightarrow \infty} \mathcal{E}(\mathbf{x}^n(0)) = \mathcal{E}(\mathbf{x}(0)).$$

On the other hand by applying the lemma again we have

$$\mathcal{E}(\mathbf{x}(0)) = \mathcal{E}(\mathbf{x}(t))$$

so that by uniqueness of weak solutions we have that

$$\lim_{n \rightarrow \infty} \mathcal{E}(\mathbf{x}^n(t)) = \mathcal{E}(\mathbf{x}(t))$$

where the limit is taken in $C(0, \tau)$. Combining weak convergence and norm convergence gives the convergence of the corresponding solutions in $C([0, \tau]; \mathcal{H})$.

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