



Boundary regularity for Maxwell's equations with applications to shape optimization

John Cagnol, Matthias Eller

► To cite this version:

John Cagnol, Matthias Eller. Boundary regularity for Maxwell's equations with applications to shape optimization. *Journal of Differential Equations*, 2011, 250 (2), pp.1114-1136. <10.1016/j.jde.2010.08.004>. <hal-01570315>

HAL Id: hal-01570315

<https://hal.science/hal-01570315v1>

Submitted on 26 Aug 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde



Boundary regularity for Maxwell's equations with applications to shape optimization

John Cagnol^a, Matthias Eller^{b,*},¹

^a Ecole Centrale Paris, Applied Mathematics and Systems Department, Grande Voie des Vignes, 92295 Châtenay-Malabry, France

^b Department of Mathematics, Georgetown University, Washington, DC 20057, United States

ARTICLE INFO

Article history:

Received 1 April 2010

Available online 21 August 2010

Keywords:

Maxwell's equations

Boundary value problems

Shape optimization

ABSTRACT

The dynamic Maxwell equations with a strictly dissipative boundary condition is considered. Sharp trace regularity for the electric and the magnetic field are established for both: weak and differentiable solutions. As an application a shape optimization problem for Maxwell's equations is considered. In order to characterize the shape derivative as a solution to a boundary value problem, the aforementioned sharp regularity of the boundary traces is critical.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, and connected set with a Lipschitz boundary $\Gamma = \partial\Omega$. The evolution of the electric field $e = e(t, x)$ and the magnetic field $h = h(t, x)$ in the space–time cylinder $Q = (0, T) \times \Omega$ is given by Maxwell's equations

$$\begin{aligned} \partial_t(\varepsilon e) - \nabla \times h &= f_1 \\ \partial_t(\mu h) + \nabla \times e &= f_2 \end{aligned} \quad \text{in } Q = (0, T) \times \Omega. \quad (1.1)$$

Here $\varepsilon = \varepsilon(t, x)$ is the electric permittivity, $\mu = \mu(t, x)$ is the magnetic permeability which are both positive definite Hermitian 3×3 matrices. The functions f_1, f_2 represent current densities, $\nabla \times$ denotes the curl operator, ∂_t is the differentiation with respect to t , and the final time T can be infinity. We add the boundary condition

* Corresponding author.

E-mail addresses: john.cagnol@ecp.fr (J. Cagnol), mme4@georgetown.edu (M. Eller).

¹ Research of M.E. supported by the National Science Foundation through grant DMS-0606118.

$$\nu \times e - \alpha h_\tau = g \quad \text{in } \Sigma = (0, T) \times \partial\Omega \quad (1.2)$$

and the initial conditions

$$e|_{t=0} = e^0 \quad \text{and} \quad h|_{t=0} = h^0 \quad \text{in } \Omega \quad (1.3)$$

where ν is the exterior unit normal vector along Γ , $\nu \times e$ is the cross product of the vectors ν and e , $\alpha = \alpha(t, x)$ is a positive function, and $h_\tau = (\nu \times h) \times \nu = h - (h \cdot \nu)\nu$ is the tangential component of the vector h on Γ . The boundary condition (1.3) is an *absorbing boundary* condition, see for example Section 7.12 in [7]. If $\alpha \equiv 1$ this is the Silver–Müller boundary condition. The quantity α represents the inverse of the surface conductivity on Γ and the function g is an external surface current density.

Maxwell's system is known to be symmetric hyperbolic. Indeed with

$$A^0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, \quad A^j \partial_j = \begin{bmatrix} 0 & -\nabla \times \\ \nabla \times & 0 \end{bmatrix}, \quad w = \begin{bmatrix} e \\ h \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad (1.4)$$

Maxwell's equations (1.1) can be written as

$$\partial_t(A^0 w) + A^j \partial_j w = f \quad (1.5)$$

where $\partial_j = \partial/\partial x_j$ and the summation convention is used. The boundary condition (1.3) is an example of a *strictly dissipative* boundary condition. Note that the boundary Σ is characteristic, i.e. $\det(A^j \nu_j) = 0$. Maxwell's equation are strictly hyperbolic only in the isotropic case, that is if $\varepsilon = \kappa \mu$ for some scalar function κ .

In the sequel the linear space of Lebesgue measurable functions on the open set $X \subset \mathbb{R}^n$ with absolutely integrable power p is denoted by $L_p(X)$. We write $L_p(X)^N$ for the linear space of vector-valued functions with N components with each component in $L_p(X)$. For the scalar product in $L_2(X)^N$ we will use the notation $(\cdot, \cdot)_X$, i.e.

$$(w_1, w_2)_X = \int_X w_1 \cdot \bar{w}_2 dX$$

and the corresponding norm is $\|w\|_X = \sqrt{(w, w)_X}$. The L_2 -based Sobolev spaces will be denoted by $H^s(X)$ for $s \in \mathbb{R}$ and the linear space of k times continuously differentiable functions on the closure of X is denoted by $C^k(\bar{X})$. By $C^k([0, T], Y)$ we denote the linear space of k times continuously differentiable functions with values in the linear space Y . Furthermore, let

$$\mathcal{H}(X) = \{(e, h) \in L_2(X)^6 : \nabla \cdot (\varepsilon e) = \nabla \cdot (\mu h) = 0\}$$

where $\nabla \cdot (\varepsilon e) = \partial_j (\varepsilon^{jk} e_k)$. If A is a Hermitian positive definite matrix we write $A > 0$. The transpose of an $m \times n$ matrix A is A^T and the Hermitian transpose A^H .

There is a wealth of results regarding the solutions to boundary value problems for Maxwell's system. However, there are only few results which discuss a non-homogeneous boundary condition and establish regularity of boundary traces. The existence, uniqueness and regularity of weak solutions are established in the following proposition.

Proposition 1.1. *Let $\varepsilon, \mu, \partial_t \varepsilon, \partial_t \mu \in L_\infty(Q)$, $\alpha \in L_\infty(\Sigma)$ be such that $\varepsilon, \mu \geq c > 0$ almost everywhere on Q and $\alpha \geq c > 0$ almost everywhere on Σ . Given $f \in L_2(Q)^6$, $g \in L_2(\Sigma)^3$ with $\nu \cdot g = 0$ and $(e^0, h^0) \in L_2(\Omega)^6$ there exists a unique weak solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ such that $(e_\tau, h_\tau)|_\Sigma \in L_2(\Sigma)^6$. Moreover, there exists a constant γ_0 such that*

$$\begin{aligned} & \|e^{-\gamma T}(e, h)(T)\|_{\Omega}^2 + \gamma \|e^{-\gamma t}(e, h)\|_Q^2 + \|e^{-\gamma t}(e_{\tau}, h_{\tau})\|_{\Sigma}^2 \\ & \lesssim \|(e^0, h^0)\|_{\Omega}^2 + \frac{1}{\gamma} \|e^{-\gamma t} f\|_Q^2 + \|e^{-\gamma t} g\|_{\Sigma}^2 \end{aligned}$$

for $\gamma \geq \gamma_0$.

Here and henceforth $a \lesssim b$ means $a \leq Cb$ for some constant C which depends only on Ω , ε , μ , and α . Note that we do not obtain regularity of the normal components of the fields e and h . In the isotropic case, this proposition is a corollary of Theorem 1.12 [9]. In the case of time-independent coefficients, this result can be found in the book by Lagnese and Leugering [8, Chapter 7].

Our first result shows how to obtain regularity results for the normal components of e and h on Σ . Define the functions

$$\begin{aligned} \rho_1(t, x) &= \int_0^t \nabla \cdot f_1(s, x) ds + \nabla \cdot (\varepsilon(t, x)e^0(x)), \\ \rho_2(t, x) &= \int_0^t \nabla \cdot f_2(s, x) ds + \nabla \cdot (\mu(t, x)h^0(x)), \end{aligned} \quad (1.6)$$

which are known as charge densities.

Theorem 1.2. Assume that Γ is of class C^2 and let $\varepsilon, \mu \in C^1(\overline{Q})$ and let $\alpha \in L_{\infty}(\Sigma)$ be such that $\alpha \geq c > 0$ almost everywhere on Σ . Furthermore, let $g \in L_2(\Sigma)^3$ with $\nu \cdot g = 0$, $f \in L_2(Q)^6$, $(e^0, h^0) \in L_2(\Omega)^6$.

If $\rho_1, \rho_2 \in L_2(Q)$, there exists a unique weak solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ to (1.1)–(1.3) such that $(e, h)|_{\Sigma} \in L_2(\Sigma)^6$. Moreover, there exists a positive constant γ_0 such that

$$\begin{aligned} & \|e^{-\gamma T}(e, h)(T)\|_{\Omega}^2 + \gamma \|e^{-\gamma t}(e, h)\|_Q^2 + \|e^{-\gamma t}(e, h)\|_{\Sigma}^2 \\ & \lesssim \|(e^0, h^0)\|_{\Omega}^2 + \frac{1}{\gamma} \|e^{-\gamma t}(f, \rho)\|_Q^2 + \|e^{-\gamma t} g\|_{\Sigma}^2 \end{aligned}$$

for $\gamma \geq \gamma_0$.

Our second result discusses differentiable solutions to the initial–boundary value problem (1.1)–(1.3). To our best knowledge, the only result concerning differentiable solutions for a symmetric hyperbolic system with characteristic boundary can be found in the paper by Majda and Osher [9, Theorem 3]. This result is only valid for the strictly hyperbolic case. Nevertheless, Theorem 3 in [9] shows that the step from the weak solution to a differentiable solution is a non-trivial matter in the presence of a characteristic boundary.

We will show that additional regularity assumptions on the functions ρ_1, ρ_2 will produce a differentiable solution. To formulate the related estimate, we need to introduce some weighted H^k -norms. Define

$$\begin{aligned} \|u\|_{1, \gamma, Q}^2 &= \gamma^2 \|e^{-\gamma t} u\|_Q^2 + \|e^{-\gamma t} \partial_t u\|_Q^2 + \|e^{-\gamma t} \nabla u\|_Q^2, \\ \|u(t)\|_{1, \gamma, \Omega}^2 &= \gamma^2 \|u(t)\|_{\Omega}^2 + \|e^{-\gamma t} \nabla u(t)\|_{\Omega}^2, \\ \|u\|_{1, \gamma, \Sigma}^2 &= \gamma^2 \|e^{-\gamma t} u\|_{\Sigma}^2 + \|e^{-\gamma t} \partial_t u\|_{\Sigma}^2 + \|e^{-\gamma t} \nabla_{\tau} u\|_{\Sigma}^2 \end{aligned} \quad (1.7)$$

for $\gamma > 0$. Here $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)$ is the gradient of u and $\nabla_\tau u_j = (\nu \times \nabla u_j) \times \nu$ is the tangential gradient of u_j along Γ .

Theorem 1.3. Suppose that Γ is of class C^2 and let $\varepsilon, \mu \in C^1(\overline{Q})$ and $\alpha \in C^1(\overline{\Sigma})$. Consider the initial-boundary value problem (1.1)–(1.3) with $g \in H^1(\Sigma)^3$ with $\nu \cdot g = 0$ on Σ , $f \in H^1(Q)^6$, $(e^0, h^0) \in H^1(\Omega)^6$ subject to the compatibility condition

$$g(0) = \nu \times e^0 - \alpha(0)h_\tau^0 \quad \text{in } L_2(\Omega).$$

If $\rho_1, \rho_2 \in H^1(Q)$, there exists a unique differentiable solution $(e, h) \in C([0, T], H^1(\Omega)^6) \cap C^1([0, T], L_2(\Omega)^6)$ such that $(e, h)|_\Sigma \in H^1(\Sigma)^6$ and $(\partial_\nu e, \partial_\nu h)|_\Sigma \in L_2(\Sigma)^6$. Furthermore, there exists a positive constant γ_0 such that

$$\begin{aligned} & \| (e, h)(T) \|_{1, \gamma, \Omega}^2 + \gamma \| (e, h) \|_{1, \gamma, Q}^2 + \| (e, h) \|_{1, \gamma, \Sigma}^2 + \| e^{-\gamma t} (\partial_\nu e, \partial_\nu h) \|_\Sigma^2 \\ & \lesssim \| (e^0, h^0) \|_{1, \gamma, Q}^2 + \frac{1}{\gamma} \| (f, \rho) \|_{1, \gamma, Q}^2 + \| g \|_{1, \gamma, \Sigma}^2 \end{aligned}$$

for $\gamma \geq \gamma_0$. Here $\partial_\nu e$ and $\partial_\nu h$ are the exterior normal derivatives of e and h , respectively.

Note that the boundary regularity results obtained in Proposition 1.1 and Theorems 1.2 and 1.3 cannot be inferred from the interior regularity of the solutions and the trace theorem in Sobolev spaces. Hence, these boundary regularity results for hyperbolic problems are referred to as “hidden regularity”.

Now we turn our attention to the following optimization problem. Minimize the functional

$$J(\Omega) = \frac{1}{2} \int_Q [(e - E)^H \varepsilon (e - E) + (h - H)^H \mu (h - H)] dt dx \quad (1.8)$$

over a collection of bounded, open and connected sets $\Omega \subset \mathbb{R}^3$ that are subsets of a fixed bounded, open and connected set $D \subset \mathbb{R}^3$. Here $(e, h) = (e, h)(\Omega)$ is a solution to the initial-boundary value problem (1.1)–(1.3), (E, H) is a desired target state and $(e - E)^H \varepsilon (e - E) = \varepsilon^{jk} (e - E)_j (\bar{e} - \bar{E})_k$.

In order to discuss the derivative of this functional with respect to Ω , perturbations of the set Ω in direction of a vector field V are introduced in [3,11]. Let \mathcal{O} be the collections of open, connected sets $\Omega \subset D$ whose boundary $\Gamma = \partial\Omega$ is of class C^2 . For some small positive number $S > 0$ we consider a vector field $V \in C([0, S]; C^2(D)^3)$ which is tangential on the boundary ∂D , i.e. $V \cdot \nu_{\partial D} = 0$ for all $x \in \partial D$. The flow $F_s(x)$ of this vector field is the solution of the ODE

$$\frac{\partial F_s(x)}{\partial s} = V(s, F_s(x)).$$

We summarize some results concerning the existence, uniqueness and regularity of F_s [11, Theorem 2.16]: For S sufficiently small this differential equation augmented by the initial condition $F_0(x) = x$ for all $x \in D$ has a unique solution $F_s(x)$ for all $s \in [0, S]$. Furthermore, the solution is of class $C^1([0, S]; C^2(D)^3)$ and is one-to-one and onto. The inverse mapping is also of class $C^1([0, S]; C^2(D)^3)$. In the sequel we will assume that $S > 0$ is chosen sufficiently small to guarantee these results.

Given $\Omega \in \mathcal{O}$ we obtain a family of perturbed domains $\Omega_s(V) = \{F_s(x) : x \in \Omega\}$ which are also in \mathcal{O} . The exterior unit normal vector of Ω_s along $\partial\Omega_s = \Gamma_s$ is denoted by ν_s . We denote by $Q_s = (0, T) \times \Omega_s$ the perturbed cylinder and by $\Sigma_s = (0, T) \times \Gamma_s$ the perturbed lateral boundary.

Now we can state our result regarding the differentiability of the shape functional above. For simplicity we focus on the case with constant coefficients, vanishing right-hand sides and divergence free initial data.

Theorem 1.4. Assume that ε, μ are constant, Hermitian positive definite matrices and let α be a positive constant. Furthermore, let $f \equiv 0$, $g \equiv 0$ and $(e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$ and $(E, H) \in H^1((0, T) \times D) \cap \mathcal{H}((0, T) \times D)$.

Then, the shape functional is Fréchet differentiable at $\Omega \in \mathcal{O}$ in direction of the vector field V with Fréchet derivative

$$dJ(\Omega, V) = \Re \int_{\Sigma} [q \cdot \{\partial_v \bar{e} \times v + \alpha [\partial_v \bar{h}]_{\tau}\} + \text{curl}_{\Gamma}(\bar{e}_v q) - \alpha \text{div}_{\Gamma}(\bar{h}_v q)] V_v dt d\Gamma + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon (e - E) + (h - H)^H \mu (h - H)] V_v dt d\Gamma \quad (1.9)$$

where $V_v = V(0) \cdot v$, $e_v = e \cdot v$, $h_v = h \cdot v$, div_{Γ} is the surface divergence, curl_{Γ} is the surface curl, and (p, q) is the solution to the backward adjoint initial-boundary value problem

$$\begin{aligned} \varepsilon \partial_t p - \nabla \times q &= \varepsilon (e - E), & \mu \partial_t q + \nabla \times q &= \mu (h - H) \quad \text{in } Q, \\ p|_{t=T} &= 0, & q|_{t=T} &= 0 \quad \text{in } \Omega, \\ v \times p + \alpha q_{\tau} &= 0 \quad \text{in } \Sigma. \end{aligned} \quad (1.10)$$

According to Theorem 1.3 the solution of the primal problem satisfies $(e, h) \in H^1(Q)^6$, $(e, h)|_{\Sigma} \in H^1(\Sigma)^6$, $(\partial_v e, \partial_v h) \in L_2(\Sigma)^6$ and the solution to the adjoint problem satisfies $q|_{\Sigma} \in H^1(\Sigma)^3$. Hence, the expression for the Fréchet derivative (1.9) is well defined. The function

$$G = \Re q \cdot \{\partial_v \bar{e} \times v + \alpha [\partial_v \bar{h}]_{\tau}\} + \text{curl}_{\Gamma}(\bar{e}_v q) - \alpha \text{div}_{\Gamma}(\bar{h}_v q) + \frac{1}{2} (e - E)^H \varepsilon (e - E) + \frac{1}{2} (h - H)^H \mu (h - H)$$

is the *shape gradient* of $J(\Omega)$ in direction of V . Theorem 1.3 implies $G \in L_1(\Sigma)$.

Theorem 1.2 and Theorem 1.3 will be proved in Section 2. Since the boundary Σ is characteristic for Maxwell's equation we will not work with Maxwell's system directly. A priori estimates will be established for a larger 8×8 symmetric hyperbolic system for which Σ is non-characteristic. The existence, uniqueness and regularity of weak and differentiable solutions follow from the a priori estimates as explained in the book by Chazarain and Piriou [2, Chapter 7]. After that, sharp trace regularity results are established through a multiplier identity.

The technique developed in Section 2 can be used to establish existence, uniqueness, and regularity of solutions with higher regularity. If $\Gamma \in C^{k+1}$, ε, μ, α , are of class C^k and $f, \rho, e^0, h^0, g \in H^k$ where k is a positive integer and corresponding compatibility conditions are met, then there exists a unique solution $(e, h) \in C^l([0, T]; H^{k-l}(\Omega)^6)$ and $(\partial_v^l e, \partial_v^l h)|_{\Sigma} \in H^{k-l}(\Sigma)^6$ for $0 \leq l \leq k$. The corresponding estimate is

$$\begin{aligned} \|(e, h)(T)\|_{k, \gamma, \Omega}^2 + \gamma \|(e, h)\|_{k, \gamma, Q}^2 + \sum_{j=0}^k \|(\partial_v^j e, \partial_v^j h)\|_{k-j, \gamma, \Sigma}^2 \\ \lesssim \|(e^0, h^0)\|_{k, \gamma, Q}^2 + \frac{1}{\gamma} \|(f, \rho)\|_{k, \gamma, Q}^2 + \|g\|_{k, \gamma, \Sigma}^2 \end{aligned}$$

where the norms are weighted norms in H^k defined as in (1.7), see [10].

Moreover, our results are valid in the case of a more general boundary condition of the form

$$\nu \times e - (I + S)^{-1}(I - S)h_\tau = g$$

where $S = S(t, x)$ is a 3×3 matrix with spectral norm uniformly less than one on Σ and with $S\nu = S^T\nu = 0$. In the case of an isotropic medium, i.e. if ε and μ are scalar functions, the restriction on S can be weakened from spectral norm uniformly less than one to spectral radius of S uniformly less than one [9, Section 2]. However it is not clear whether this boundary condition will work in the generic anisotropic case. We conjecture that in the case of an anisotropic medium the strictly dissipative boundary conditions coincide with the boundary conditions that satisfy the Kreiss–Sakamoto condition (uniform Lopatinskii condition).

Theorem 1.4 is proved in Section 3 which contains the shape sensitivity analysis of Maxwell's equation. We will show that the initial-boundary value problem (1.1)–(1.3) is shape differentiable. In order to characterize the shape derivative as a solution to an initial-boundary value problem the boundary regularity for the differentiable solution from Theorem 1.3 will be necessary. This situation is similar to the shape derivative of the Dirichlet problem for the wave equation, studied earlier by Cagnol and Zolésio [3]. On the other hand we like to point out that our analysis is valid for the anisotropic Maxwell equations which cannot be reduced to vector wave equations.

Finally, in Section 4 we discuss the case $\alpha \equiv 0$. From the viewpoint of applications this may be the most interesting case. It models the boundary Γ as a perfect conductor. We discuss the boundary value problem in this case and establish weaker regularity results for some boundary traces. Those results will suffice to show that the shape functional (1.8) is Fréchet differentiable even if $\alpha \equiv 0$. Here our work complements a recent paper by Zolésio where shape differentiability for a shape functional different from (1.8) in the case of the isotropic Maxwell system is established [12].

2. Proof of Theorem 1.2 and Theorem 1.3

We introduce the linear operator $L : H^1(Q)^8 \rightarrow L_2(Q)^8$ defined by

$$L(\partial_t, \partial_x)u = \begin{bmatrix} \partial_t(\varepsilon u_1) - \nabla \times u_3 - \varepsilon \nabla u_4 \\ \partial_t u_2 + \nabla \cdot (\mu u_3) \\ \partial_t(\mu u_3) + \nabla \times u_1 + \mu \nabla u_2 \\ \partial_t u_4 - \nabla \cdot (\varepsilon u_1) \end{bmatrix}. \quad (2.1)$$

Here $u = (u_1, u_2, u_3, u_4)^T$ where u_1, u_3 are vector-valued functions with three components each, u_2, u_4 are scalar-valued functions, and $\varepsilon = \varepsilon(t, x)$ and $\mu = \mu(t, x)$ are Hermitian positive definite matrix functions in Q . This operator is symmetric hyperbolic and the lateral boundary Σ is non-characteristic since $\det L(0, \nu) = (\nu^T \varepsilon \nu)^2 (\nu^T \mu \nu)^2 > 0$.

Let $\beta \in L_\infty(\Sigma)$ be such that $|\beta| \leq c < 1$ almost everywhere on Σ and set

$$Mu = (1 + \beta)\nu \times u_1 - (1 - \beta)u_{3,\tau} \quad (2.2)$$

where $u_{3,\tau}$ is the tangential component of u_3 . Note that this boundary operator can be used to express the boundary condition (1.2) if one sets $\beta = (1 - \alpha)/(1 + \alpha)$. Throughout this section we assume that Ω has a Lipschitz boundary Γ and that $\varepsilon, \mu, \partial_t \varepsilon, \partial_t \mu \in L_\infty(Q)$ such that $\varepsilon, \mu \geq c > 0$ almost everywhere in Q .

The first a priori estimate follows from the theory of symmetric hyperbolic systems with dissipative boundary conditions.

Proposition 2.1. *Let $u \in H^1(Q)^8$ be such that $u_2 = u_4 = 0$ on Σ . Then, there exists a $\gamma_0 > 0$ such that*

$$\begin{aligned} & \|e^{-\gamma T} u(T)\|_{\Omega}^2 + \gamma \|e^{-\gamma t} u\|_Q^2 + \|e^{-\gamma t} (u_{1,\tau}, u_{3,\tau})\|_{\Sigma}^2 \\ & \lesssim \frac{1}{\gamma} \|e^{-\gamma t} Lu\|_Q^2 + \|u(0)\|_{\Omega}^2 + \|e^{-\gamma t} Mu\|_{\Sigma}^2. \end{aligned} \quad (2.3)$$

Here $u_{j,\tau} = (v \times u_j) \times v$ is the tangential component of u_j on Σ for $j = 1$ or 3 .

Proof. We multiply the operator (2.1) by $e^{-\gamma t}$ and take the scalar product with $v := e^{-\gamma t} u$ in $L_2(Q)$

$$(e^{-\gamma t} Lu, v)_Q = (Lv, v)_Q + \gamma (Av, v)_Q$$

where A is the Hermitian, positive definite 8×8 matrix

$$A = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Taking the real part and performing integration by parts give

$$\begin{aligned} & \frac{1}{2} (Av(T), v(T))_{\Omega} + \gamma (Av, v)_Q + \frac{1}{2} (\partial_t Av, v)_Q + \Re \int_{\Sigma} (v \times v_1) \cdot \bar{v}_{3,\tau} d\Gamma dt \\ & + \Re \int_{\Sigma} v_2 v \cdot (\overline{\mu v_3}) d\Gamma dt - \Re \int_{\Sigma} v_4 v \cdot (\overline{\varepsilon v_1}) d\Gamma dt = \frac{1}{2} (Av(0), v(0))_{\Omega} + \Re (e^{-\gamma t} Lu, v)_Q \end{aligned}$$

and the positivity of A and the Cauchy–Schwarz inequality lead to the inequality

$$\begin{aligned} & \|v(T)\|_{\Omega}^2 + \gamma \|v\|_Q^2 + \Re \int_{\Sigma} (v \times v_1) \cdot \bar{v}_{3,\tau} d\Gamma dt + \Re \int_{\Sigma} v_2 v \cdot (\overline{\mu v_3}) d\Gamma dt - \Re \int_{\Sigma} v_4 v \cdot (\overline{\varepsilon v_1}) d\Gamma dt \\ & \lesssim \|v(0)\|_{\Omega}^2 + \frac{1}{\gamma} \|e^{-\gamma t} Lu\|_Q^2 \end{aligned} \quad (2.4)$$

for $\gamma \geq \gamma_0$ where γ_0 is a constant which depends only on ε and μ . The last two boundary integrals on the left-hand side vanish since $u_2 = u_4 = 0$ on Σ . Note that

$$\Re \int_{\Sigma} (v \times v_1) \cdot \bar{v}_{3,\tau} d\Gamma dt = \frac{1}{4} \|v \times v_1 + v_{3,\tau}\|_{\Sigma}^2 - \frac{1}{4} \|v \times v_1 - v_{3,\tau}\|_{\Sigma}^2$$

and that

$$|v_{1,\tau}|^2 + |v_{3,\tau}|^2 = \frac{1}{2} |v \times v_1 - v_{3,\tau}|^2 + \frac{1}{2} |v \times v_1 + v_{3,\tau}|^2.$$

Hence, the proposition is proved once we show that

$$|v \times v_1 - v_{3,\tau}|^2 + |v \times v_1 + v_{3,\tau}|^2 \lesssim |v \times v_1 + v_{3,\tau}|^2 - |v \times v_1 - v_{3,\tau}|^2 + |Mv|^2 \quad (2.5)$$

almost everywhere on Σ .

To prove this inequality we set $\lambda_1 = |\nu \times v_1 + v_{3,\tau}|$ and $\lambda_2 = |\nu \times v_1 - v_{3,\tau}|$ and observe that via (2.2)

$$\lambda_1^2 = \lambda_1^2 - \lambda_2^2 + \lambda_2^2 \leq \lambda_1^2 - \lambda_2^2 + (1 + \delta)|M\nu|^2 + \left(1 + \frac{1}{\delta}\right)\beta^2\lambda_1^2$$

for all $\delta > 0$. Choose $\delta = \frac{|\beta|}{1-|\beta|}$ and move the last term on the right-hand side into the left-hand side

$$(1 - |\beta|)\lambda_1^2 \leq \lambda_1^2 - \lambda_2^2 + \frac{1}{1 - |\beta|}|M\nu|^2.$$

Also, by (2.2)

$$\frac{1 - |\beta|}{2}\lambda_2^2 \leq (1 + \delta)\frac{1 - |\beta|}{2}|M\nu|^2 + \left(1 + \frac{1}{\delta}\right)\beta^2\frac{1 - |\beta|}{2}\lambda_1^2 \leq \frac{1}{2}|M\nu|^2 + |\beta|\frac{1 - |\beta|}{2}\lambda_1^2$$

where we used again $\delta = \frac{|\beta|}{1-|\beta|}$. Combining the last two inequalities gives

$$\frac{1 - |\beta|}{2}[\lambda_1^2 + \lambda_2^2] \leq \lambda_1^2 - \lambda_2^2 + \left[\frac{1}{1 - |\beta|} + \frac{1}{2}\right]|M\nu|^2$$

which is exactly (2.5). \square

Remark 2.1. Note that the proof remains true if β is replaced by a 3×3 matrix function $S = S(t, x)$ with entries in $L_\infty(\Sigma)$ satisfying $|Sz| \leq c|z|$ for some constant $c < 1$ almost everywhere on Σ as well as $S\nu = S^T\nu = 0$.

Corollary 2.2. If $F \in L_2(Q)^8$, $u^0 \in L_2(\Omega)^8$, and $g \in L_2(\Sigma)^3$ with $\nu \cdot g = 0$, then the initial-boundary value problem

$$Lu = F \quad \text{in } Q, \quad u|_{t=0} = u^0 \quad \text{in } \Omega, \quad Mu = g, \quad u_2 = u_4 = 0 \quad \text{in } \Sigma \quad (2.6)$$

has a unique weak solution $u \in C([0, T], L_2(\Omega)^8)$ with $(u_{1,\tau}, u_{3,\tau})|_\Sigma \in L_2(\Sigma)^6$. Moreover, estimate (2.3) is valid for $\gamma \geq \gamma_0$.

Note that the statement and the proof of Proposition 2.1 can be easily adjusted to the boundary value problem (1.1)–(1.3). This way the a priori estimate corresponding to Proposition 1.1 is proved.

Since the boundary Σ is non-characteristic one can establish the unique existence for more regular solutions. Following the approach by Rauch [10] we establish the estimate (2.3) at the H^1 -level.

Proposition 2.3. Suppose that Γ is of class C^2 , let $\varepsilon, \mu \in C^1(\overline{Q})$ be 3×3 , and let $u \in H^2(Q)^8$ be such that $u_2 = u_4 = 0$ on Σ . Furthermore, let $\beta \in C^1(\overline{\Sigma})$.

Then, there exists a $\gamma_0 > 0$ such that

$$\begin{aligned} & \|e^{-\gamma T}u(T)\|_{1,\gamma,\Omega}^2 + \gamma \|e^{-\gamma t}u\|_{1,\gamma,Q}^2 + \|e^{-\gamma t}(u_{1,\tau}, u_{3,\tau})\|_{1,\gamma,\Sigma}^2 \\ & \leq \frac{1}{\gamma} \|e^{-\gamma t}Lu\|_{1,\gamma,Q}^2 + \|u(0)\|_{1,\gamma,\Omega}^2 + \|e^{-\gamma t}Mu\|_{1,\gamma,\Sigma}^2 \end{aligned} \quad (2.7)$$

for $\gamma \geq \gamma_0$.

Proof. Assuming that Γ is of class C^2 the distance function $\phi(x) = \text{dist}(x, \Gamma)$ is of class C^2 in

$$W = \{x \in \bar{\Omega}: \phi(x) < r\}$$

where r is some positive number [4, Theorem 3.2]. Furthermore $\nu(y) = -\nabla\phi(y)$ for $y \in \Gamma$ and the exterior unit normal is extended to W by setting $\nu(x) = -\nabla\phi(x)$ for $x \in W$. Let

$$u_{j,\tau} = (\nu \times u_j) \times \nu, \quad u_{j,\nu} = u_j \cdot \nu, \quad (t, x) \in [0, T] \times W,$$

which extends the tangential and normal component of u from Σ to the collar $(0, T) \times W$.

At first establish (2.7) in the case that $u \equiv 0$ in $\{\phi(x) < r/2\}$. Then there will be no boundary terms in the estimate. Apply inequality (2.3) with u replaced by $\partial_j u$ for $j = 0, 1, 2, 3$ where $\partial_0 = \partial_t$. Since the coefficients of L are differentiable, the commutator of the operators L and ∂_j can be estimated

$$\|e^{-\gamma t} L \partial_j u\|_Q \lesssim \|e^{-\gamma t} \partial_j Lu\|_Q + \sum_{k=0}^3 \|e^{-\gamma t} \partial_k u\|_Q, \quad j = 0, 1, 2, 3.$$

Adding estimate (2.3) multiplied by γ^2 to the estimates for $\partial_j u$ gives

$$\|u(T)\|_{1,\gamma,\Omega}^2 + \gamma \|u\|_{1,\gamma,Q}^2 \lesssim \|u(0)\|_{1,\gamma,Q}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2 + \frac{1}{\gamma} \|u\|_{1,\gamma,Q}^2 + \|\partial_t u(0)\|_{\Omega}^2. \quad (2.8)$$

Using the definition of L one has $\|\partial_t u(0)\|_{\Omega}^2 \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \|Lu(0)\|_{\Omega}^2$ and using integration by parts

$$\begin{aligned} \|w(0)\|_{\Omega}^2 &= 2\gamma \int_Q e^{-2\gamma t} \left(1 - \frac{t}{T}\right) |w|^2 dt dx + \frac{1}{T} \int_Q e^{-2\gamma t} |w|^2 dt dx \\ &\quad - 2\Re \int_Q e^{-2\gamma t} \left(1 - \frac{t}{T}\right) w \cdot \partial_t \bar{w} dt dx \\ &\leq 4\gamma \|e^{-\gamma t} w\|_{\Omega}^2 + \frac{1}{\gamma} \|e^{-\gamma t} \partial_t w\|_{\Omega}^2 \leq \frac{4}{\gamma} \|w\|_{1,\gamma,Q}^2 \end{aligned} \quad (2.9)$$

for $w \in H^1(Q)^8$, provided $\gamma \geq 1/T$. Hence, with $w = Lu$ one obtains

$$\|\partial_t u(0)\|_{\Omega}^2 \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2. \quad (2.10)$$

The term next to the last term in the right-hand side of (2.8) can be moved into the left-hand side for γ sufficiently large. We obtain

$$\|u(T)\|_{1,\gamma,\Omega}^2 + \gamma \|u\|_{1,\gamma,Q}^2 \lesssim \|u(0)\|_{1,\gamma,Q}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2 \quad (2.11)$$

which is exactly (2.7) in this case.

Now we have to establish (2.7) for $u \in H^2(Q)^8$ where $u \equiv 0$ in $\{\phi(x) > r\}$. This part is more challenging since now the boundary terms will be active. Multiply the estimate (2.3) by γ^2 and add the estimate (2.3) applied to $\partial_t w$ and to $\nabla_{\tau} w$. This gives

$$\begin{aligned} & \gamma^2 \|e^{-\gamma T} u(T)\|_{\Omega}^2 + \|e^{-\gamma T} \nabla_{\tau} u(T)\|_{\Omega}^2 + \|e^{-\gamma T} \partial_t u(T)\|_{\Omega}^2 + \gamma^3 \|e^{-\gamma t} u\|_Q^2 + \gamma \|e^{-\gamma t} \nabla_{\tau} u\|_Q^2 \\ & + \gamma \|e^{-\gamma t} \partial_t u\|_Q^2 + \gamma^2 \|e^{-\gamma t} (u_{1,\tau}, u_{3,\tau})\|_{\Sigma}^2 + \|e^{-\gamma t} (\nabla_{\tau} u_{1,\tau}, \nabla_{\tau} u_{3,\tau})\|_{\Sigma}^2 \\ & + \|e^{-\gamma t} (\partial_t u_{1,\tau}, \partial_t u_{3,\tau})\|_{\Sigma}^2 \\ & \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \|\partial_t u(0)\|_{\Omega}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2 + \|Mu\|_{1,\gamma,\Sigma} + \frac{1}{\gamma} \|u\|_{1,\gamma,Q}^2 + \|e^{-\gamma t} u\|_{\Sigma}^2. \end{aligned}$$

The last two terms on the right-hand side are due to the commutator of the operator L with $\partial_t, \nabla_{\tau}$ and the commutator of M with $\partial_t, \nabla_{\tau}$, respectively. To estimate these commutators the regularity assumptions on the coefficients ε, μ, α and Γ are needed. The last term can be placed into the left-hand side for γ sufficiently large. Furthermore, the term $\|\partial_t u(0)\|_{\Omega}^2$ can be estimated as in (2.10). This gives

$$\begin{aligned} & \gamma^2 \|e^{-\gamma T} u(T)\|_{\Omega}^2 + \|e^{-\gamma T} \nabla_{\tau} u(T)\|_{\Omega}^2 + \|e^{-\gamma T} \partial_t u(T)\|_{\Omega}^2 + \gamma^3 \|e^{-\gamma t} u\|_Q^2 + \gamma \|e^{-\gamma t} \nabla_{\tau} u\|_Q^2 \\ & + \gamma \|e^{-\gamma t} \partial_t u\|_Q^2 + \|(u_{1,\tau}, u_{3,\tau})\|_{1,\gamma,\Sigma}^2 \\ & \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2 + \|Mu\|_{1,\gamma,\Sigma} + \frac{1}{\gamma} \|u\|_{1,\gamma,Q}^2. \end{aligned} \quad (2.12)$$

In order to obtain an estimate for the first derivative of u we estimate the normal derivative of u . Since Σ is non-characteristic for L ,

$$\partial_{\nu} u = B(x, \partial_t, \nabla_{\tau})u + D(x)Lu \quad (2.13)$$

where $B(x, \partial_t, \nabla_{\tau})$ is a matrix differential operator and $D(x)$ is an 8×8 matrix. This yields an estimate for the normal derivative of u in Q and in Ω at $t = T$,

$$\begin{aligned} & \|e^{-\gamma t} \partial_{\nu} u\|_Q^2 \lesssim \|e^{-\gamma t} \nabla_{\tau} u\|_Q^2 + \|e^{-\gamma t} \partial_t u\|_Q^2 + \|e^{-\gamma t} Lu\|_Q^2, \\ & \|e^{-\gamma T} \partial_{\nu} u(T)\|_{\Omega}^2 \lesssim \|e^{-\gamma T} \nabla_{\tau} u(T)\|_{\Omega}^2 + \|e^{-\gamma T} \partial_t u(T)\|_{\Omega}^2 + \|e^{-\gamma T} Lu(T)\|_{\Omega}^2. \end{aligned} \quad (2.14)$$

Hence, if we now combine estimate (2.12) with (2.14) we obtain

$$\begin{aligned} & \|u(T)\|_{1,\gamma,Q}^2 + \gamma \|u\|_{1,\gamma,Q}^2 + \|(u_{1,\tau}, u_{3,\tau})\|_{1,\gamma,\Sigma}^2 \\ & \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2 + \|Mu\|_{1,\gamma,\Sigma}^2 + \|e^{-\gamma T} Lu(T)\|_{\Omega}^2 + \frac{1}{\gamma} \|u\|_{1,\gamma,Q}^2. \end{aligned} \quad (2.15)$$

The last term can be moved into the left-hand side since γ is a large parameter. The term next to it is estimated using integration by parts

$$\|e^{-\gamma T} Lu(T)\|_{\Omega}^2 = \int_Q \partial_t [e^{-2\gamma t} |Lu|^2] dt dx + \|Lu(0)\|_{\Omega}^2 \lesssim \frac{1}{\gamma} \|Lu\|_{1,\gamma,Q}^2$$

where (2.9) was applied as well. Thus we have established

$$\|u(T)\|_{1,\gamma,Q}^2 + \gamma \|u\|_{1,\gamma,Q}^2 + \|(u_{1,\tau}, u_{3,\tau})\|_{1,\gamma,\Sigma}^2 \lesssim \|u(0)\|_{1,\gamma,\Omega}^2 + \|Lu\|_{1,\gamma,Q}^2 + \|Mu\|_{1,\gamma,\Sigma}^2$$

for $u \in H^2(Q)^8$ with $u \equiv 0$ for $\phi(x) > r$.

To obtain the a priori estimate for $u \in H^1(Q)^8$, choose $\chi \in C^2(\overline{\Omega})$ such that $\chi \equiv 1$ in $\{\phi(x) > r\}$ and $\chi \equiv 0$ in $\{\phi(x) < r/2\}$. Then $u = \chi u + (1 - \chi)u$ and (2.13) follows now from (2.11) applied to χu and the estimate above applied to $(1 - \chi)u$. \square

Observe that

$$\partial_\nu[u_\tau] = \partial_\nu[(v \times u) \times v] = (v \times \partial_\nu u) \times v + (\partial_\nu v \times u) \times v + (v \times u) \times \partial_\nu v = [\partial_\nu u]_\tau \quad (2.16)$$

since $\partial_\nu v = (\nabla \phi \cdot \nabla) \nabla \phi = \nabla |\nabla \phi|^2 / 2 = 0$.

Corollary 2.4. Suppose that Γ is of class C^2 . Let $\varepsilon, \mu \in C^1(\overline{Q})$ and let $\beta \in C^1(\overline{\Sigma})$. Moreover, let $F \in H^1(Q)^8$, $u^0 \in H^1(\Omega)^8$, $g \in H^1(\Sigma)^3$ with $v \cdot g = 0$ and assume that the compatibility condition $M(0)u^0 = g(0)$ in Ω is satisfied.

Then, there exists a unique solution $u \in C([0, T], H^1(Q)^8)$ to the initial-boundary value problem (2.6) with $(u_{1,\tau}, u_{3,\tau})|_\Sigma \in H^1(\Sigma)^6$. Furthermore, estimate (2.7) is valid.

Now we will show that the solutions to the initial-boundary value problem (2.6) can be used to obtain solutions to the initial-boundary value problem for Maxwell's equations (1.1)–(1.3)

Proposition 2.5.

- a) Let $(e^0, h^0) \in L_2(\Omega)^6$, $(f_1, f_2) \in L_2(Q)^6$, and $g \in L_2(\Sigma)^3$ be such that $v \cdot g = 0$ in Σ . If $\rho_1, \rho_2 \in L_2(Q)$, then there exists a unique weak solution $(e, h) \in C([0, T], L_2(\Omega)^6)$ to (1.1)–(1.3) with $(e_\tau, h_\tau)|_\Sigma \in L_2(\Sigma)^6$.
- b) Suppose that Γ is of class C^2 , $\varepsilon, \mu \in C^1(\overline{Q})$, and $\beta \in C^1(\overline{\Sigma})$. Furthermore, assume $(e^0, h^0) \in H^1(\Omega)^6$, $(f_1, f_2) \in H^1(Q)^6$, and $g \in H^1(\Sigma)$ such that $v \cdot g = 0$ in Σ and

$$g(0) = (1 + \beta(0))v \times e^0 - (1 - \beta(0))h_\tau^0 \quad \text{in } \Omega.$$

If $\rho_1, \rho_2 \in H^1(Q)$, then there exists a unique differentiable solution $(e, h) \in C([0, T], H^1(\Omega)^6)$ to (1.1)–(1.3) with $(e_\tau, h_\tau)|_\Sigma \in H^1(\Sigma)^6$.

Proof. Consider the unique weak solution to (2.6) with $u_2^0 = u_4^0 = 0$, $u_1^0 = e^0$, $u_3^0 = h^0$ and $F = (f_1, \rho_2, f_2, -\rho_1)$. Take the first (vector)-equation and the last equation

$$\partial_t(\varepsilon u_1) - \nabla \times u_3 - \varepsilon \nabla u_4 = f_1,$$

$$\partial_t u_4 - \nabla \cdot (\varepsilon u_1) = -\rho_1.$$

Applying the divergence to the first equation and the time derivative to the last equation, and adding the resulting equations together yield $\partial_t^2 u_4 - \nabla \cdot (\varepsilon \nabla u_4) = 0$ because of (1.6). Similarly one can show that $\partial_t^2 u_2 - \nabla \cdot (\mu \nabla u_2) = 0$.

These are both scalar wave equations. These two wave equations are complemented by the initial conditions

$$u_2|_{t=0} = u_4|_{t=0} = \partial_t u_2|_{t=0} = \partial_t u_4|_{t=0} = 0$$

where the vanishing initial velocities is a consequence of (1.6). Furthermore, we have the boundary condition $u_2 = u_4 = 0$ on Σ . Both u_2 and u_4 are solutions to a scalar wave equation with homogeneous initial and boundary data. Hence $u_2 \equiv u_4 \equiv 0$ and $e \equiv u_1$ and $h \equiv u_3$ are the solutions to the initial-boundary value problem (1.1)–(1.3). This proves part a). The proof of part b) is very similar and will be omitted. \square

In both cases a) and b) we have the estimates (2.3) for the weak solutions and (2.7) for differentiable solutions. Furthermore, the solutions satisfy the divergence equations

$$\nabla \cdot (\varepsilon e) = \rho_1, \quad \nabla \cdot (\mu h) = \rho_2 \quad \text{in } Q. \quad (2.17)$$

Now we will show that both: the weak solution and the differentiable solution to the Maxwell problem established in the last proposition have more regular traces than implied by the estimates (2.3) and (2.7), respectively. We start with the weak solution. In what follows we will abbreviate

$$\|w\|_{\gamma, X} = \|e^{-\gamma t} w\|_X \quad \text{and} \quad (w_1, w_2)_{\gamma, X} = (e^{-\gamma t} w_1, e^{-\gamma t} w_2)_X.$$

Proposition 2.6. Assume the boundary Γ is of class C^2 and let $\varepsilon, \mu \in C^1(\overline{Q})$, $u = (e, h) \in H^1(Q)^6$. Then, there exists a $\gamma_0 > 0$ such that

$$\begin{aligned} \|e_v\|_{\gamma, \Sigma}^2 + \|h_v\|_{\gamma, \Sigma}^2 &\lesssim \|e_\tau\|_{\gamma, \Sigma}^2 + \|h_\tau\|_{\gamma, \Sigma}^2 + \frac{1}{\gamma} \|(f, \rho)\|_{\gamma, Q}^2 + \gamma \|(e, h)\|_{\gamma, Q}^2 \\ &\quad + \|(e, h)(T)\|_{\gamma, \Omega}^2 + \|(e, h)(0)\|_{\Omega}^2 \end{aligned} \quad (2.18)$$

for $\gamma \geq \gamma_0$. Here f, ρ are computed using (1.1) and (2.17).

Proof. Let P, z be real C^1 -vector fields and let κ be a Hermitian matrix function. Note that

$$\begin{aligned} (\nabla \times z) \cdot (P \times \kappa z) &= \frac{1}{2} (P \cdot \nabla) (z^T \kappa z) - \frac{1}{2} z^T (P \cdot \nabla) \kappa z - (\kappa z \cdot \nabla) z \cdot P \\ &= \frac{1}{2} (P \cdot \nabla) (z^T \kappa z) - \frac{1}{2} z^T (P \cdot \nabla) \kappa z - \nabla (z \cdot P) \cdot (\kappa z) + (\kappa z \cdot \nabla) P \cdot z \end{aligned}$$

and hence,

$$\begin{aligned} (\nabla \times z) \cdot (P \times \kappa z) - \nabla \cdot (\kappa z) P \cdot z \\ = \frac{1}{2} (P \cdot \nabla) (z^T \kappa z) - \frac{1}{2} z^T (P \cdot \nabla) \kappa z - \nabla \cdot [(z \cdot P) \cdot (\kappa z)] + (\kappa z \cdot \nabla) P \cdot z. \end{aligned} \quad (2.19)$$

Furthermore,

$$\Re \{ \partial_t (\varepsilon e) \cdot (\overline{P \times \mu h}) - \partial_t (\mu h) \cdot (\overline{P \times \varepsilon e}) \} = -\Re \partial_t [\mu h \cdot (\overline{P \times \varepsilon e})].$$

Using now (2.19) with $z = e$, $\kappa = \varepsilon$ and $z = h$, $\kappa = \mu$ we establish the following identity

$$\begin{aligned} &\Re \{ f_1 \cdot (\overline{P \times \mu h}) - f_2 \cdot (\overline{P \times \varepsilon e}) + \rho_2 (\overline{P \cdot h}) + \rho_1 (\overline{P \cdot e}) \} \\ &= -\Re \partial_t [\mu h \cdot (\overline{P \times \varepsilon e})] - \frac{1}{2} (P \cdot \nabla) (e^H \varepsilon e + h^H \mu h) + \Re \{ \nabla \cdot [(e \cdot P) \cdot (\overline{\varepsilon e}) + (h \cdot P) \cdot (\overline{\mu h})] \} \\ &\quad + \frac{1}{2} [e^H (P \cdot \nabla) \varepsilon e + h^H (P \cdot \nabla) \mu h] - \Re [(\overline{\varepsilon e} \cdot \nabla) P \cdot e + (\overline{\mu h} \cdot \nabla) P \cdot h]. \end{aligned}$$

We multiply this identity by $e^{-2\gamma t}$ and integrate over Q and perform integration by parts in the first three terms on the right-hand side:

$$\begin{aligned}
 & \Re \{ (f_1, P \times \mu h)_{\gamma, Q} - (f_2, P \times \varepsilon e)_{\gamma, Q} + (\rho_2, P \cdot h)_{\gamma, Q} + (\rho_1, P \cdot e)_{\gamma, Q} \} \\
 &= -2\gamma \Re(\mu h, P \times \varepsilon e)_{\gamma, Q} - (\mu h(T), P \times \varepsilon e(T))_{\gamma, \Omega} + (\mu h(0), P \times \varepsilon e(0))_{\Omega} \\
 &\quad - \frac{1}{2} \int_{\Sigma} e^{-2\gamma t} (P \cdot \nu) (e^H \varepsilon e + h^H \mu h) dt d\Gamma + \Re \int_{\Sigma} e^{-2\gamma t} [(e \cdot P) \nu \cdot \bar{\varepsilon} e + (h \cdot P) \nu \cdot \bar{\mu} h] dt d\Gamma \\
 &\quad + \frac{1}{2} (e, (P \cdot \nabla) \varepsilon e)_{\gamma, Q} + \frac{1}{2} (h, (P \cdot \nabla) \mu h)_{\gamma, Q} - \Re((\varepsilon e \cdot \nabla) P, e)_{\gamma, Q} - \Re((\mu h \cdot \nabla) P, h)_{\gamma, Q}.
 \end{aligned} \tag{2.20}$$

Choose now P such that $P = \nu$ on Σ . Since $\Gamma \in C^2$ we know that $\nu \in C^1$. Then

$$\begin{aligned}
 & -\frac{1}{2} \int_{\Sigma} e^{-2\gamma t} (e^H \varepsilon e + h^H \mu h) dt d\Gamma + \Re \int_{\Sigma} e^{-2\gamma t} [(\nu \cdot e) \nu \cdot \bar{\varepsilon} e + (\nu \cdot h) \nu \cdot \bar{\mu} h] dt d\Gamma \\
 & \lesssim \frac{1}{\gamma} \|(f, \rho)\|_{\gamma, Q}^2 + \gamma \|(e, h)\|_{\gamma, Q}^2 + \|(e, h)(T)\|_{\gamma, \Omega}^2 + \|(e, h)(0)\|_{\Omega}^2.
 \end{aligned} \tag{2.21}$$

With $e = e_\tau + (\nu \cdot e)\nu$, $h = h_\tau + (\nu \cdot h)\nu$ the integrands of the integrals over Σ can be rewritten

$$\begin{aligned}
 & \frac{1}{2} (e^H \varepsilon e + h^H \mu h) - \Re(\nu \cdot \bar{e}) \nu \cdot \varepsilon e - \Re(\nu \cdot \bar{h}) \nu \cdot \mu h \\
 &= \frac{1}{2} \Re \{ e_\tau^H \varepsilon e - (\nu \cdot \bar{e})(\nu \cdot \varepsilon e) + h_\tau^H \mu h - (\nu \cdot \bar{h})(\nu \cdot \mu h) \} \\
 &= \frac{1}{2} \Re \{ [e_\tau - (\nu \cdot e)\nu]^H \varepsilon e + [h_\tau - (\nu \cdot h)\nu]^H \mu h \} \\
 &= \frac{1}{2} \Re \{ [e_\tau - (\nu \cdot e)\nu]^H \varepsilon [e_\tau + (\nu \cdot e)\nu] + [h_\tau - (\nu \cdot h)\nu]^H \mu [h_\tau + (\nu \cdot h)\nu] \} \\
 &= \frac{1}{2} \{ e_\tau^H \varepsilon e_\tau + h_\tau^H \mu h_\tau - |\nu \cdot e|^2 \nu^T \varepsilon \nu - |\nu \cdot h|^2 \nu^T \mu \nu \}.
 \end{aligned}$$

Using this identity in (2.21) finishes the proof. \square

By density, the identity (2.20) is also valid for weak solutions to Maxwell's equations. Theorem 1.2 follows now from Proposition 2.5, part a), and Proposition 2.6.

If we replace e and h by the tangential derivatives of e and h along Σ in the estimate (2.18) we obtain

$$\begin{aligned}
 & \|e_\nu\|_{1, \gamma, \Sigma}^2 + \|h_\nu\|_{1, \gamma, \Sigma}^2 \lesssim \|e_\tau\|_{1, \gamma, \Sigma}^2 + \|h_\tau\|_{1, \gamma, \Sigma}^2 + \frac{1}{\gamma} \|(f, \rho)\|_{1, \gamma, Q}^2 \\
 & \quad + \gamma \|(e, h)\|_{1, \gamma, Q}^2 + \|(e, h)(T)\|_{1, \gamma, \Omega}^2 + \|(e, h)(0)\|_{1, \gamma, \Omega}^2
 \end{aligned}$$

under the same regularity assumptions on the coefficients. This improves the boundary regularity of the differentiable solutions of Maxwell's system (part b) of Proposition 2.5) to $(e, h)|_\Sigma \in H^1(\Sigma)^6$ with the estimate

$$\begin{aligned}
 & \|(e, h)(T)\|_{1, \gamma, \Omega}^2 + \gamma \|(e, h)\|_{1, \gamma, Q}^2 + \|(e, h)\|_{1, \gamma, \Sigma}^2 \\
 & \lesssim \|(e^0, h^0)\|_{1, \gamma, Q}^2 + \frac{1}{\gamma} \|(f, \rho)\|_{1, \gamma, Q}^2 + \|g\|_{1, \gamma, \Sigma}^2.
 \end{aligned}$$

This proves Theorem 1.3 up to the regularity statement of the normal derivatives. The regularity of the normal derivatives follows from (2.13) which yields the estimate

$$\|e^{-\gamma t} \partial_\nu u\|_\Sigma^2 \lesssim \|e^{-\gamma t} \nabla_\tau u\|_\Sigma^2 + \|e^{-\gamma t} \partial_t u\|_\Sigma^2 + \|e^{-\gamma t} Lu\|_\Sigma^2.$$

The last term can be estimated by $\|Lu\|_{1,\gamma,Q}^2/\gamma$ since by the divergence theorem,

$$\begin{aligned} \|e^{-\gamma t} w\|_\Sigma^2 &= \int_Q e^{-2\gamma t} \nabla \cdot (|w|^2 \nu) dt dx = \int_Q e^{-2\gamma t} \nabla \cdot \nu |w|^2 dt dx + 2\Re \int_Q e^{-2\gamma t} (w \cdot \nabla) \bar{w} \cdot \nu dt dx \\ &\lesssim \|e^{-\gamma t} w\|_Q^2 + \gamma \|e^{-\gamma t} w\|_Q^2 + \frac{1}{\gamma} \|e^{-\gamma t} \nabla w\|_Q^2 \lesssim \frac{1}{\gamma} \|w\|_{1,\gamma,Q}^2 \end{aligned}$$

for all $w \in H^1(Q)$ ⁸.

3. Shape sensitivity analysis for Maxwell's system

Here we will prove Theorem 1.4. The Gâteaux derivative (Eulerian derivative) of the shape functional (1.8) in direction of V is

$$\begin{aligned} dJ(\Omega, V) &= \lim_{s \rightarrow 0} \frac{J(\Omega_s) - J(\Omega)}{s} \\ &= \Re \int_Q [(e - E)^H \varepsilon e' + (h - H)^H \mu h'] dt dx \\ &\quad + \frac{1}{2} \int_\Sigma [(e - E)^H \varepsilon (e - E) + (h - H)^H \mu (h - H)] V_\nu dt d\Gamma \end{aligned} \quad (3.1)$$

where (e', h') is the *shape derivative* of (1.1)–(1.3) and $V_\nu = V(0) \cdot \nu$ [11, Section 2.31]. In order to calculate this derivative we need to establish shape differentiability of Maxwell's equations and characterize the shape derivative.

Throughout this section we assume that ε, μ are constant Hermitian, positive definite matrices and that α is a positive constant. In contrast to Section 2 we write $u = (e, h)$. With initial data $u^0 = (e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$ we associate to each element $\Omega \in \mathcal{O}$ the solution $u(\Omega) = (e, h)(\Omega)$ of the initial-boundary value problem (1.1)–(1.3) with $f \equiv 0$, $g \equiv 0$. Note that $\rho_j \equiv 0$ for $j = 1, 2$, see (1.6).

Let Y be a space of functions defined on $(0, T) \times D$. The mapping $\Omega \rightarrow u(\Omega)$ is shape differentiable at $\Omega \in \mathcal{O}$ in direction V in Y if there exists a function $U \in C^1([0, S]; Y)$ such that $U(s, \cdot)|_{Q_s} = u(\Omega_s)$ and

$$\frac{U(s) - U(0)}{s} \longrightarrow \partial_s U(0) \quad \text{strongly in } Y, \text{ as } s \rightarrow 0.$$

The shape derivative is the function $u'(\Omega, V) = \partial_s U(0)|_Q$. If

$$\frac{U(s) - U(0)}{s} \rightharpoonup \partial_s U(0) \quad \text{weakly in } Y \text{ as } s \rightarrow 0,$$

then the mapping $\Omega \rightarrow u(\Omega)$ is weakly shape differentiable in Y .

Proposition 3.1. Let $(e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$, $f = 0$, $g = 0$ and let $(e, h) \in C([0, T], H^1(\Omega)^6) \cap C^1([0, T], L_2(\Omega)^6)$ be the solution to the initial-boundary value problem (1.1)–(1.3) guaranteed by Theorem 1.3.

This initial-boundary value problem is weakly shape differentiable with shape derivative $(e', h') \in C([0, T], L_2(\Omega)^6)$ and the shape derivative is a weak solution to the initial-boundary value problem

$$\varepsilon \partial_t e' - \nabla \times h' = \mu \partial_t h' + \nabla \times e' = 0 \quad \text{in } Q \quad (3.2)$$

with zero initial data

$$u'|_{t=0} = 0 \quad \text{in } \Omega \quad (3.3)$$

and the lateral boundary condition

$$\nu \times e' - \alpha h'_\tau = (V_\nu \partial_\nu e_\tau + e_\nu \nabla_\tau V_\nu) \times \nu + \alpha (V_\nu \partial_\nu h_\tau + h_\nu \nabla_\tau V_\nu) \quad \text{in } \Sigma. \quad (3.4)$$

To establish shape differentiability of the initial-boundary value problem (1.1)–(1.3) we will follow the approach by Cagnol and Zolésio for the wave equation with Dirichlet boundary conditions [3]. Certain aspects of the analysis will simplify since we work with an operator with constant coefficients and with zero right-hand sides. Following the approach given in [3] it should be possible to obtain the shape derivative for our Maxwell problem with non-zero f , ρ , and g and also with variable coefficients α , ε , μ , provided certain assumptions on shape differentiability are satisfied.

3.1. The material derivative

At first we will establish the material differentiability of the initial-boundary value problem (1.1)–(1.3) with $f = g = 0$ and divergence free initial data. We will show that this initial-boundary value problem is weakly material differentiable and that the material derivative is the weak solution to an initial-boundary value problem (1.1)–(1.3) with f and g depending on (e, h) and also the vector field $V(0)$.

For our purpose it will be useful to represent the hyperbolic system (1.5) in a different way. Denote the entries of the matrices A^l introduced in (1.4) by a^l_{jk} . Form the 6×3 matrices

$$\mathcal{A}_k = (a^l_{jk})_{\substack{1 \leq j \leq 6 \\ 1 \leq l \leq 3}}, \quad k = 1, 2, \dots, 6,$$

and the gradient of u_k is the column vector $\nabla u_k = (\partial_1 u_k, \partial_2 u_k, \partial_3 u_k)^T$ for $k = 1, 2, \dots, 6$. Then

$$Pu = A^0 \partial_t u + \sum_{k=1}^6 \mathcal{A}_k \nabla u_k$$

is the Maxwell operator corresponding to the differential equations (1.1). Let $u^0 = (e^0, h^0) \in \mathcal{H}(D) \cap H^1(D)^6$. Denote the unique solutions to the family of initial-boundary value problems

$$Pu_s = 0 \quad \text{in } Q_s, \quad \nu_s \times e_s - \alpha h_{s,\tau} = 0 \quad \text{in } \Sigma_s, \quad u_s|_{t=0} = u^0 \quad \text{in } \Omega_s \quad (3.5)$$

by u_s . By Theorem 1.3 we know $u_s = (e_s, h_s) \in C([0, T], H^1(\Omega_s)^6) \cap C^1([0, T], L_2(\Omega_s)^6)$ for $0 < s \leq S$. The solution to (3.5) corresponding to $s = 0$ will be denoted by u .

The pullback of u_s to Q is $u^s = u_s \circ F_s$ and the operator $P_s : H^1(Q)^6 \rightarrow L_2(Q)^6$ is defined by

$$P_s u = A^0 \partial_t u + \sum_{k=1}^6 \mathcal{A}_k (J_s^{-1})^T \nabla u_k$$

where $J_s = D_x F_s$ is the derivative of F_s (Jacobian matrix). By the chain rule one verifies that

$$P_s(v \circ F_s) = (Pv) \circ F_s \quad \text{for all } v \in H^1(\Omega_s)^6.$$

The shape difference quotient is $w^s = (u^s - u)/s$ for $0 < s \leq S$ and

$$Pw^s = P \left[\frac{u^s - u}{s} \right] = \frac{1}{s} (P - P_s) u^s$$

since $Pu = 0$ in Q and also $P_s u^s = P_s(u_s \circ F_s) = (Pu_s) \circ F_s = 0$ in Q . This shows that the shape difference quotient satisfies a non-homogeneous hyperbolic differential equation. More precisely, setting $\mathcal{A}_k^s = \mathcal{A}_k (J_s^{-1})^T$, we have

$$Pw^s = \frac{1}{s} (P - P_s) u^s = \frac{1}{s} \sum_{k=1}^6 (\mathcal{A}_k - \mathcal{A}_k^s) \nabla u_k^s. \quad (3.6)$$

Furthermore,

$$w^s|_{t=0} = \frac{u^s|_{t=0} - u|_{t=0}}{s} = \frac{(u_s|_{t=0} \circ F_s) - u|_{t=0}}{s} = \frac{(u^0 \circ F_s) - u^0}{s} \quad (3.7)$$

which provides the initial data of the shape difference quotient. Finally, we discuss the boundary condition. From Proposition 2.48 [11] we know that the pullback of the exterior unit normal ν_s along Γ_s to Ω is

$$\nu^s = \nu_s \circ F_s = \frac{(J_s^{-1})^T \nu}{|(J_s^{-1})^T \nu|}.$$

Hence, the pullback u^s satisfies the boundary condition

$$M_s u^s := \nu^s \times e^s - \alpha [h^s - (h^s \cdot \nu^s) \nu^s] = 0 \quad \text{on } \Sigma$$

for $0 \leq s < S$ and we write M in the case $s = 0$. Note that the definition of the boundary operator M is slightly different from (2.2). Hence

$$Mw^s = M \frac{u^s - u}{s} = \frac{1}{s} (M - M_s) u^s \quad \text{on } \Sigma$$

since also $Mu = 0$ on Σ . In particular,

$$Mw^s = \frac{\nu - \nu^s}{s} \times e^s + \alpha \left[(h^s \cdot \nu) \frac{\nu - \nu^s}{s} + \left(h^s \cdot \frac{\nu - \nu^s}{s} \right) \nu^s \right] \quad (3.8)$$

on Σ . Note that $w^s \in C([0, T], H^1(\Omega)^6)$ for $0 < s \leq S$. Then $\lim_{s \rightarrow 0} w^s$ is the *material derivative*, provided the limit exists. In order to establish the existence of the material derivative we will show that the shape difference quotient $w^s(t)$ at time t is uniformly bounded in $L_2(\Omega)$ with respect to s .

Proposition 3.2. *There exists a constant $C > 0$ such that*

$$\|w^s(t)\|_{\Omega} \leq C \quad \text{for all } s \in (0, S), t \in [0, T], \quad (3.9)$$

where C depends only on $\varepsilon, \mu, \alpha, \Omega$ and $\|u^0\|_{1,\gamma,\Omega}$.

Proof. By the chain rule the pullback u^s has the same regularity as the solution u_s . We know that w^s satisfies the initial-boundary value problem (3.6), (3.7), (3.8). In order to obtain a uniform bound for w^s we need to show that Pw^s , $w^s|_{t=0}$, Mw^s are uniformly bounded in L_2 . Hence we will show that the limits

$$\lim_{s \rightarrow 0} \|Pw^s\|_{(0,t) \times \Omega}, \quad \lim_{s \rightarrow 0} \|w^s|_{t=0}\|_{\Omega}, \quad \lim_{s \rightarrow 0} \|Mw^s\|_{(0,t) \times \Gamma} \quad (3.10)$$

exist for all $t \in [0, T]$. Once we have established these limits, formula (3.9) follows from Proposition 1.1.

By the chain rule, one has

$$\lim_{s \rightarrow 0} \frac{\mathcal{A}_k - \mathcal{A}_k^s}{s} = -\partial_s \mathcal{A}_k^s|_{s=0} = \mathcal{A}_k[\partial_s J_s^T](J_s^T)^{-2}$$

and using the formula $\partial_s J_s|_{s=0} = DV(0)$ (Lemma 2.31 [11]) one obtains $\lim_{s \rightarrow 0} (\mathcal{A}_k - \mathcal{A}_k^s)/s = \mathcal{A}_k[DV(0)]^T$ where $DV = D_x V$ is the derivative of V . Hence,

$$\lim_{s \rightarrow 0} Pw^s = \sum_{k=1}^6 \mathcal{A}_k[DV(0)]^T \nabla u_k \quad (3.11)$$

for every $t \in [0, T]$. From formula (3.7) we conclude with the aid of Proposition 2.32 [11]

$$\lim_{s \rightarrow 0} w^s|_{t=0} = [V(0) \cdot \nabla]u^0. \quad (3.12)$$

Finally we evaluate $\lim_{s \rightarrow 0} Mw^s$. Compute

$$\begin{aligned} \partial_s v_s &= \partial_s \frac{(J_s^{-1})^T v}{|(J_s^{-1})^T v|} = -\frac{1}{2} \frac{\partial_s((J_s^{-1})^T v, (J_s^{-1})^T v)}{|(J_s^{-1})^T v|^3} (J_s^{-1})^T v + \frac{(J_s^{-2})^T \partial_s J_s^T v}{|(J_s^{-1})^T v|} \\ &= -\frac{((J_s^{-2})^T \partial_s J_s^T v, (J_s^{-1})^T v)}{|(J_s^{-1})^T v|^3} (J_s^{-1})^T v + \frac{(J_s^{-2})^T \partial_s J_s^T v}{|(J_s^{-1})^T v|} \end{aligned}$$

where the notation (\cdot, \cdot) denotes the scalar product in \mathbb{R}^3 . Since $\lim_{s \rightarrow 0} J_s = I$ (the identity matrix) and $\partial_s J_s|_{s=0} = DV(0)$, this yields $-\partial_s v_s|_{s=0} = (DV(0)^T v, v)v - DV(0)^T v$ which is the tangential component of the vector $DV(0)^T v$. We will write $\dot{v} := \partial_s v_s|_{s=0} = -[DV(0)^T v]_{\tau}$ and thus

$$\lim_{s \rightarrow 0} Mw^s = -\dot{v} \times e - \alpha[(h \cdot v)\dot{v} + (h \cdot \dot{v})v]. \quad (3.13)$$

Formulas (3.11)–(3.13) establish the limits postulated in (3.10) since $u \in C([0, T], H^1(\Omega)^6)$. \square

Because of the uniform bound (3.9) there exists sequence $s_i \searrow 0$ such that w^{s_i} converges weakly in $L_2(\Omega)^6$ for every t . This weak limit is the material derivative \dot{u} , i.e.

$$w^{s_i}(t) \rightharpoonup \dot{u}(t) \quad \text{weakly in } L_2(\Omega)^6, \quad t \in [0, T].$$

Using (3.11)–(3.13) we see that the material derivative $\dot{u} = (\dot{e}, \dot{h})$ is the unique solution to the initial-boundary value problem:

$$\begin{aligned} P\dot{u} &= \sum_{k=1}^6 \mathcal{A}_k [DV(0)]^T \nabla u_k \quad \text{in } Q, \\ \dot{u}|_{t=0} &= [V(0) \cdot \nabla] u^0 \quad \text{in } \Omega, \\ v \times \dot{e} - \alpha \dot{h} &= -\dot{v} \times e - \alpha [(h \cdot v)\dot{v} + (h \cdot \dot{v})v] \quad \text{in } \Sigma. \end{aligned} \quad (3.14)$$

By Proposition 1.1, we have $\dot{u} \in C([0, T], L_2(\Omega)^6)$.

3.2. The shape derivative

The shape derivative can now be computed

$$u'(\Omega, V) = \dot{u} - DuV(0) = [V(0) \cdot \nabla] u \quad (3.15)$$

[3, Corollary 5], and since $u \in C([0, T], H^1(\Omega)^6)$ we know $u' \in C([0, T], L_2(\Omega)^6)$. Observe the homogeneous initial condition $u'|_{t=0} = 0$.

To prove Theorem 3.1 we need to show that the shape derivative satisfies the homogeneous Maxwell equations as well as the boundary condition (3.4). In what follows we will write V instead of $V(0)$. To establish (3.2) compute

$$\begin{aligned} Pu' &= P\dot{u} - P[(V \cdot \nabla)u] = \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k - A^0 \partial_t [(V \cdot \nabla)u] - \sum_{k=1}^6 \mathcal{A}_k \nabla [V \cdot \nabla u_k] \\ &= \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k - (V \cdot \nabla) [A^0 \partial_t u] - \sum_{k=1}^6 \mathcal{A}_k \{ [DV]^T \nabla u_k + (V \cdot \nabla) \nabla u_k \} \\ &= \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k - (V \cdot \nabla) [A^0 \partial_t u] - \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k - (V \cdot \nabla) \left(\sum_{k=1}^6 \mathcal{A}_k \nabla u_k \right) \\ &= \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k - (V \cdot \nabla) (Pu) - \sum_{k=1}^6 \mathcal{A}_k [DV]^T \nabla u_k = 0 \end{aligned}$$

since $Pu = 0$ in Q . For the boundary condition, observe that $V \cdot \nabla = V_\tau \cdot \nabla_\tau + V_\nu \partial_\nu$ where ∇_τ denotes the tangential gradient on Γ and $V_\tau = V - V_\nu \nu$. Since $v \times e - \alpha h_\tau = 0$ along Γ ,

$$\begin{aligned} 0 &= (V_\tau \cdot \nabla_\tau) [v \times e - \alpha (v \times h) \times v] = (V_\tau \cdot \nabla_\tau) v \times e + v \times (V_\tau \cdot \nabla_\tau) e \\ &\quad - \alpha \{ [(V_\tau \cdot \nabla_\tau) v \times h] \times v + [v \times (V_\tau \cdot \nabla_\tau) h] \times v + [v \times h] \times (V_\tau \cdot \nabla_\tau) v \} \end{aligned}$$

which gives

$$\begin{aligned} \nu \times (V \cdot \nabla) e - \alpha [\nu \times (V \cdot \nabla) h] \times \nu &= \nu \times (V_\nu \partial_\nu e) - \alpha [\nu \times (V_\nu \partial_\nu h)] \times \nu - (V_\tau \cdot \nabla_\tau) \nu \times e \\ &\quad + \alpha \{ [(V_\tau \cdot \nabla_\tau) \nu \times h] \times \nu + [\nu \times h] \times (V_\tau \cdot \nabla_\tau) \nu \} \\ &= \nu \times (V_\nu \partial_\nu e) - \alpha [\nu \times (V_\nu \partial_\nu h)] \times \nu - (V_\tau \cdot \nabla_\tau) \nu \times e \\ &\quad - \alpha \{ (V_\tau \cdot \nabla_\tau) \nu (h \cdot \nu) + \nu (h \cdot (V_\tau \cdot \nabla_\tau) \nu) \}. \end{aligned}$$

Hence the shape derivative satisfies the boundary condition

$$\begin{aligned} \nu \times e' - \alpha h'_\tau &= \nu \times \dot{e} - \alpha \dot{h}_\tau - \nu \times (V \cdot \nabla) e + \alpha [(V \cdot \nabla) h]_\tau \\ &= -\dot{\nu} \times e - \alpha \{ (h \cdot \nu) \dot{\nu} + (h \cdot \dot{\nu}) \nu \} - \nu \times (V \cdot \nabla) e + \alpha [(V \cdot \nabla) h]_\tau \\ &= -\nu \times (V_\nu \partial_\nu e) + \alpha [\nu \times (V_\nu \partial_\nu h)] \times \nu + [(V_\tau \cdot \nabla_\tau) \nu - \dot{\nu}] \times e \\ &\quad + \alpha \{ [(V_\tau \cdot \nabla_\tau) \nu - \dot{\nu}] (h \cdot \nu) + \nu (h \cdot [(V_\tau \cdot \nabla_\tau) \nu - \dot{\nu}]) \}. \end{aligned} \quad (3.16)$$

Since $\nabla_\tau V_\nu = \nabla_\tau (V \cdot \nu) = [(DV)^T \nu]_\tau + [(DV)^T V]_\tau$ one obtains

$$\begin{aligned} (V_\tau \cdot \nabla_\tau) \nu - \dot{\nu} &= (V_\tau \cdot \nabla_\tau) \nu + [(DV)^T \nu]_\tau = [(DV) V]_\tau + [(DV)^T \nu]_\tau \\ &= \nabla_\tau V_\nu + [(DV) V]_\tau - [(DV)^T V]_\tau. \end{aligned}$$

Recall from Section 2 that $\nu = -\nabla \phi$ where $\phi(x) = \text{dist}(\Gamma, x)$. Hence $D\nu = (DV)^T$ which yields $(V_\tau \cdot \nabla_\tau) \nu - \dot{\nu} = \nabla_\tau V_\nu$, and the boundary condition (3.16) simplifies to

$$\begin{aligned} \nu \times e' - \alpha h'_\tau &= -\nu \times (V_\nu \partial_\nu e) + \alpha [\nu \times (V_\nu \partial_\nu h)] \times \nu + \nabla_\tau V_\nu \times e \\ &\quad + \alpha \{ \nabla_\tau V_\nu (h \cdot \nu) + \nu (h \cdot \nabla_\tau V_\nu) \}. \end{aligned}$$

Finally, since $\nu \times e = \alpha h_\tau$ we have also $e_\tau = -\alpha(\nu \times h)$ and hence,

$$\begin{aligned} \nabla_\tau V_\nu \times e &= \nabla_\tau V_\nu \times (e_\tau + e_\nu \nu) = -\alpha \nabla_\tau V_\nu \times (\nu \times h) + \nabla_\tau V_\nu \times e_\nu \nu \\ &= -\alpha (h \cdot \nabla_\tau V_\nu) \nu + \nabla_\tau V_\nu \times e_\nu \nu. \end{aligned}$$

Combining the last two formulas completes the proof of Proposition 3.1 in view of (2.16).

3.3. Proof of Theorem 1.4

Use the solution to the adjoint problem (1.10) in formula (3.1) and compute the Gâteaux derivative

$$\begin{aligned} dJ(\Omega, V) &= \Re \int_Q (\varepsilon \partial_t p - \nabla \times q) \cdot \bar{e}' dt dx + \Re \int_Q (\mu \partial_t q + \nabla \times p) \cdot \bar{h}' dt dx \\ &\quad + \frac{1}{2} \int_\Sigma [(e - E)^H \varepsilon (e - E) + (h - H)^H \mu (h - H)] V_\nu dt d\Gamma \\ &= \Re \int_\Sigma [(\nu \times p) \cdot \bar{h}'_\tau + q \cdot (\nu \times \bar{e}')] dt d\Gamma \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon(e - E) + (h - H)^H \mu(h - H)] V_{\nu} dt d\Gamma \\
 & = \Re \int_{\Sigma} q \cdot [(V_{\nu} \partial_{\nu} \bar{e}_{\tau} + \bar{e}_{\nu} \nabla_{\tau} V_{\nu}) \times \nu + \alpha (V_{\nu} \partial_{\nu} \bar{h}_{\tau} + \bar{h}_{\nu} \nabla_{\tau} V_{\nu})] dt d\Gamma \\
 & + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon(e - E) + (h - H)^H \mu(h - H)] V_{\nu} dt d\Gamma
 \end{aligned}$$

where we used that (p, q) solves the adjoint problem (1.10) and that (e', h') is a weak solution to (3.2)–(3.4). To finish the proof we remove the derivatives of V_{ν} by integration by parts on Γ

$$\begin{aligned}
 dJ(\Omega, V) & = \Re \int_{\Sigma} [q \cdot V_{\nu} \{\partial_{\nu} \bar{e}_{\tau} \times \nu + \alpha \partial_{\nu} \bar{h}_{\tau}\} + \nabla_{\tau} V_{\nu} \cdot (\nu \times \bar{e}_{\nu} q) + \alpha \bar{h}_{\nu} q \nabla_{\tau} V_{\nu}] dt d\Gamma \\
 & + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon(e - E) + (h - H)^H \mu(h - H)] V_{\nu} dt d\Gamma \\
 & = \Re \int_{\Sigma} [q \cdot \{\partial_{\nu} \bar{e}_{\tau} \times \nu + \alpha \partial_{\nu} \bar{h}_{\tau}\} - \operatorname{div}_{\Gamma}(\nu \times \bar{e}_{\nu} q) - \alpha \operatorname{div}_{\Gamma}(\bar{h}_{\nu} q)] V_{\nu} dt d\Gamma \\
 & + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon(e - E) + (h - H)^H \mu(h - H)] V_{\nu} dt d\Gamma \\
 & = \Re \int_{\Sigma} [q \cdot \{\partial_{\nu} \bar{e}_{\tau} \times \nu + \alpha \partial_{\nu} \bar{h}_{\tau}\} + \operatorname{curl}_{\Gamma}(\bar{e}_{\nu} q) - \alpha \operatorname{div}_{\Gamma}(\bar{h}_{\nu} q)] V_{\nu} dt d\Gamma \\
 & + \frac{1}{2} \int_{\Sigma} [(e - E)^H \varepsilon(e - E) + (h - H)^H \mu(h - H)] V_{\nu} dt d\Gamma,
 \end{aligned}$$

see for example [1, Section 2.3]. Here $\operatorname{div}_{\Gamma}$ is the surface divergence and $\operatorname{curl}_{\Gamma}$ is the surface curl. Note that the Gâteaux derivative turns out to be a linear function of V_{ν} which proves that the shape functional is Fréchet-differentiable.

4. The case $\alpha \equiv 0$

Here we will show that the shape functional (1.8) is Fréchet differentiable in the case $\alpha \equiv 0$. For that we will investigate the dependence of the Fréchet derivative (1.9) on α . As in Section 2 we will work with $\beta = (1 - \alpha)/(1 + \alpha)$. We will show that $dJ(\Omega, V)$ is uniformly bounded with respect to β for $\beta \in [-1, 1]$.

4.1. The boundary value problem with dissipative boundary conditions

At first we will study the initial-boundary value problem for Maxwell's equations

$$\begin{aligned}
 \partial_t(\varepsilon e) - \nabla \times h & = f_1, & \partial_t(\mu h) + \nabla \times e & = f_2 \quad \text{in } Q, \\
 (e, h)|_{t=0} & = (e^0, h^0) \quad \text{in } \Omega, \\
 (1 + \beta)\nu \times e + (1 - \beta)h_{\tau} & = 0 \quad \text{in } \Sigma
 \end{aligned} \tag{4.1}$$

for $\beta \in L_\infty(\Sigma)$ satisfying $|\beta| \leq 1$ almost everywhere. In contrast to the previous sections this boundary condition is only dissipative but not necessarily strictly dissipative.

Consider the proof of Proposition 2.1 and let $u \in H^1(Q)^8$ be such that $u_2 = u_4 = 0$ and $Mu = 0$ on Σ and let $\beta \in L_\infty(\Sigma)$ with $|\beta| \leq 1$. Then

$$\Re \int_{\Sigma} (v \times v_1) \cdot \bar{v}_{3,\tau} d\Gamma dt = \Re \int_{\Sigma} \frac{1-\beta}{1+\beta} |v_{3,\tau}|^2 = \Re \int_{\Sigma} \frac{1+\beta}{1-\beta} |v_{1,\tau}|^2 \geq 0.$$

This implies the estimate

$$\|e^{-\gamma T} u(T)\|_{\Omega}^2 + \gamma \|e^{-\gamma t} u\|_Q^2 \lesssim \frac{1}{\gamma} \|e^{-\gamma t} Lu\|_Q^2 + \|u(0)\|_{\Omega}^2$$

where $a \lesssim b$ means now $a \leq Cb$ for some constant C depending on ε, μ, Ω but not on β . Following the results of Section 2 one can solve the initial-boundary value problem (4.1) with $|\beta| \leq 1$. Since we want to discuss the dependence of the solution on the coefficient function β of the boundary operator M we will denote the solution of the initial-boundary value problem (4.1) by $(e, h)_{\beta}$.

To be more precise, if $\varepsilon, \mu \in L_\infty(Q)$, $f \in L_2(Q)^6$, $(e^0, h^0) \in L_2(\Omega)^6$ one obtains a unique weak solution $(e, h)_{\beta} \in C([0, T], L_2(\Omega)^6)$ with the estimate

$$\|e^{-\gamma T} (e, h)_{\beta}(T)\|_{\Omega}^2 + \gamma \|e^{-\gamma t} (e, h)_{\beta}\|_Q^2 \lesssim \|(e^0, h^0)\|_{\Omega}^2 + \frac{1}{\gamma} \|e^{-\gamma t} f\|_Q^2 \quad (4.2)$$

for $\gamma \geq \gamma_0$. If $\Gamma \in C^2$, $\varepsilon, \mu \in C^1$, $f, \rho, e^0, h^0 \in H^1$ one obtains a unique differentiable solution $(e, h)_{\beta} \in C([0, T], H^1(\Omega)^6) \cap C^1([0, T], L_2(\Omega)^6)$ and

$$\|(e, h)(T)_{\beta}\|_{1,\gamma,\Omega}^2 + \gamma \|(e, h)_{\beta}\|_{1,\gamma,Q}^2 \lesssim \|(e^0, h^0)\|_{1,\gamma,\Omega}^2 + \frac{1}{\gamma} \|(f, \rho)\|_{1,\gamma,Q}^2.$$

Since a differentiable solution satisfies also $(e, h)_{\beta} \in H^1(Q)^6$, the trace theorem in Sobolev spaces give $(e, h)_{\beta}|_{\Sigma} \in H^{1/2}(\Sigma)^6$. Furthermore, one can show that $\partial_{\nu} e_{\tau}, \partial_{\nu} h_{\tau} \in H^{-1/2}(\Sigma)$.

Lemma 4.1. Let $(e^0, h^0) \in H^1(\Omega)$ and $f \in H^1(Q)^6$ be such that $\rho \in H^1(Q)^2$. Then the unique differentiable solution $(e, h)_{\beta} \in C([0, T], H^1(\Omega)^6) \cap C^1([0, T], L_2(\Omega)^6)$ satisfies

$$[\partial_j e_{\beta}]_{\tau}, [\partial_j h_{\beta}]_{\tau} \in H^{-1/2}(\Sigma)^3 \quad (4.3)$$

for $j = 1, 2, 3$, uniformly in β for $|\beta| \leq 1$.

Proof. Here we will only sketch the proof. The full proof requires a smoothing of the solution by Friedrichs's mollifiers. For brevity we will drop the subscript β . Let $(\underline{e}, \underline{h}) \in H^{1/2}(\Sigma)^3$. By the extension theorem in Sobolev spaces there exist $(\underline{E}, \underline{H}) \in H^1(Q)^3$ such that $\|(\underline{E}, \underline{H})\|_{H^1(Q)} \leq C\|(\underline{e}, \underline{h})\|_{H^{1/2}(\Sigma)}$. Integration by parts gives

$$\begin{aligned} & \int_{\Sigma} (v \times \underline{e}) \cdot \partial_j \bar{h} dt d\Gamma - \int_{\Sigma} (v \times \underline{h}) \cdot \partial_j \bar{e} dt d\Gamma \\ &= \int_Q \underline{E} \cdot [\partial_t (\varepsilon \partial_j \bar{e}) - \nabla \times \partial_j \bar{h}] dt dx + \int_Q \underline{H} \cdot [\partial_t (\mu \partial_j \bar{h}) + \nabla \times \partial_j \bar{e}] dt dx \end{aligned}$$

$$\begin{aligned}
 & + \int_Q (\varepsilon \partial_t \underline{E} - \nabla \times \underline{H}) \cdot \partial_j \bar{e} \, dt \, dx + \int_Q (\mu \partial_t \underline{H} + \nabla \times \underline{E}) \cdot \partial_j \bar{h} \, dt \, dx \\
 & - \int_\Omega [\partial_j e^H \varepsilon \underline{E} + \partial_j h^H \mu \underline{H}] \, dx \Big|_{t=0}^{t=T}.
 \end{aligned}$$

Choosing $\underline{h} = 0$ one obtains the estimate

$$\left| \int_\Sigma (\nu \times \underline{e}) \cdot \partial_j \bar{h} \, dt \, d\Gamma \right| \leq C \|\underline{e}\|_{H^{1/2}(\Sigma)} \|(e, h)\|_{H^1(Q)}$$

and by choosing $\underline{e} = 0$ one has

$$\left| \int_\Sigma (\nu \times \underline{h}) \cdot \partial_j \bar{e} \, dt \, d\Gamma \right| \leq C \|\underline{h}\|_{H^{1/2}(\Sigma)} \|(e, h)\|_{H^1(Q)}.$$

These estimates show that $[\partial_j e]_\tau$, $[\partial_j h]_\tau$ are linear functionals on $H^{1/2}(\Sigma)$. \square

These regularity results are weaker than the ones established in Theorem 1.3. However, they are also valid in the case that $\beta = 1$ or $\beta = -1$.

4.2. The derivative of the shape functional in the case $\alpha \equiv 0$

As in Section 3 let ε , μ be constant Hermitian, positive matrices and let α be a positive constant. Note that the Fréchet derivative of the shape functional (1.9) can be interpreted as a duality between the two Sobolev spaces $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$. The regularity results of the previous subsection guarantee that $\lim_{\alpha \rightarrow 0} J(\Omega)$ exists. Hence we have proved the following result.

Proposition 4.2. *Suppose that ε, μ are constant, Hermitian positive definite matrices and that $(E, H) \in H^1((0, T) \times D) \cap \mathcal{H}((0, T) \times D)$. Furthermore, let $f \equiv 0$, $g \equiv 0$ and $(e^0, h^0) \in H^1(D)^6 \cap \mathcal{H}(D)$ and let $(e, h) \in C([0, T], H^1(\Omega)^6)$ be the solution to (1.1)–(1.3) with $\alpha \equiv 0$.*

Then, the shape functional (1.8) is Fréchet differentiable at $\Omega \in \mathcal{O}$ in direction of the vector field V with Fréchet derivative

$$\begin{aligned}
 dJ(\Omega, V) &= \Re \langle q V_\nu, \partial_\nu e_\tau \times \nu \rangle + \Re \int_\Sigma \nabla_\tau V_\nu \cdot (\nu \times \bar{e}_\nu q) \, dt \, d\Gamma \\
 &+ \frac{1}{2} \int_\Sigma [(e - E)^H \varepsilon (e - E) + (h - H)^H \mu (h - H)] V_\nu \, dt \, d\Gamma
 \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ and (p, q) is the solution to the backward adjoint initial–boundary value problem

$$\begin{aligned}
 \varepsilon \partial_t p - \nabla \times q &= \varepsilon (e - E), & \mu \partial_t q + \nabla \times q \mu &= \mu (h - H) \quad \text{in } Q, \\
 p|_{t=T} &= 0, & q|_{t=T} &= 0 \quad \text{in } \Omega, \\
 \nu \times p &= 0 \quad \text{in } \Sigma.
 \end{aligned}$$

Remark 4.1. The initial–boundary value problem characterizing the material derivative (3.14) can be used to establish $\dot{u} \in C([0, T], L_2(\Omega)^6)$ even in the case $\alpha \equiv 0$ [5, Corollary 1.4]. The shape derivative has the same regularity because of (3.15).

Note that the boundary condition of the shape derivative in the case $\alpha \equiv 0$ simplifies to

$$\nu \times e' = (V_\nu \partial_\nu e_\tau + e_\nu \nabla_\tau V_\nu) \times \nu \quad \text{in } \Sigma.$$

In the case that ε and μ are scalar functions, it has been shown that the trace of the normal derivative of the electric field is square integrable, i.e. $\partial_\nu e|_\Sigma \in L_2(\Sigma)^3$ [6, Theorem 1.1], [12, Proposition 6.1]. However, this result is not sufficient to establish the regularity of the shape derivative from the boundary value problem (3.2)–(3.4).

We believe that our approach can be used for other shape optimization problems where the constraint is a hyperbolic boundary value problem with a conservative boundary condition. Examples include the scalar wave equation with Neumann boundary conditions and the elastic wave equations with a zero traction boundary condition.

References

- [1] Michel Cessenat, *Mathematical Methods in Electromagnetism. Linear Theory and Applications*, Ser. Adv. Math. Appl. Sci., vol. 41, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [2] Jacques Chazarain, Alain Piriou, *Introduction to the Theory of Linear Partial Differential Equations*, Stud. Math. Appl., vol. 14, North-Holland Publishing Co., Amsterdam, 1982, translated from French.
- [3] John Cagnol, Jean-Paul Zolésio, Shape derivative in the wave equation with Dirichlet boundary conditions, *J. Differential Equations* 158 (2) (1999) 175–210.
- [4] Michel C. Delfour, Jean-Paul Zolésio, Oriented distance function and its evolution equation for initial sets with thin boundary, *SIAM J. Control Optim.* 42 (6) (2004) 2286–2304 (electronic).
- [5] Matthias Eller, Symmetric hyperbolic systems with boundary conditions that do not satisfy the Kreiss–Sakamoto condition, *Appl. Math. (Warsaw)* 35 (3) (2008) 323–333.
- [6] Matthias Eller, On boundary regularity of solutions to Maxwell's equations with a homogeneous conservative boundary condition, *Discrete Contin. Dyn. Syst. Ser. S* 2 (3) (2009) 473–481.
- [7] Mauro Fabrizio, Angelo Morro, *Electromagnetism of Continuous Media. Mathematical Modelling and Applications*, Oxford Sci. Publ., Oxford University Press, Oxford, 2003.
- [8] John E. Lagnese, Günter Leugering, *Domain Decomposition Methods in Optimal Control of Partial Differential Equations*, Internat. Ser. Numer. Math., vol. 148, Birkhäuser Verlag, Basel, 2004.
- [9] Andrew Majda, Stanley Osher, Initial–boundary value problems for hyperbolic equations with uniformly characteristic boundary, *Comm. Pure Appl. Math.* 28 (5) (1975) 607–675.
- [10] Jeffrey Rauch, \mathcal{L}_2 is a continuable initial condition for Kreiss' mixed problems, *Comm. Pure Appl. Math.* 25 (1972) 265–285.
- [11] Jan Sokołowski, Jean-Paul Zolésio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Ser. Comput. Math., vol. 16, Springer-Verlag, Berlin, 1992.
- [12] Jean-Paul Zolésio, Hidden boundary shape derivative for the solution to maxwell equations and non-cylindrical wave equations, in: *Optimal Control of Coupled Systems of Partial Differential Equations*, in: Internat. Ser. Numer. Math., vol. 158, Birkhäuser, Basel, 2009, pp. 319–345.