Landau phonon-roton theory revisited for superfluid helium 4 and Fermi gases

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Liquid helium and spin-1/2 cold-atom Fermi gases both exhibit in their superfluid phase two distinct types of excitations, gapless phonons and gapped rotons or fermionic pair-breaking excitations. In the long wavelength limit, revising and extending Landau and Khalatnikov's theory initially developed for helium [ZhETF **19**, 637 (1949)], we obtain universal expressions for three- and four-body couplings among these two types of excitations. We calculate the corresponding phonon damping rates at low temperature and compare them to those of a pure phononic origin, paving the way to experimental observations.

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Introduction – A homogeneous superfluid of particles with short-range interactions exhibits, at sufficiently low temperature, phononic excitations as the only microscopic degrees of freedom. In this universal limit, all superfluids of this type reduce to a weakly interacting phonon gas, irrespective of the statistics of the underlying particles and of the strength of their interactions. Phonon damping then only depends on the phonon dispersion relation close to the zero wavenumber (namely, its slope and its third derivative) and on the phonon nonlinear coupling, deduced solely from the system equation of state through Landau and Khalatnikov's quantum hydrodynamic theory [1].

In the experimental reality, however, the temperature is not always low enough to make the dynamics purely phononic. Other elementary excitations can enrich the problem, such as rotons in liquid helium 4 and fermionic BCS-type pair-breaking excitations in spin-1/2 cold-atom Fermi gases. These excitations, denoted here as γ quasiparticles, exhibit in both cases an energy gap $\Delta > 0$. Remarkably, as shown by Landau and Khalatnikov, the phonon-roton coupling, and more generally the phonon coupling to all gapped excitations as we shall see, depend to leading order in temperature only on a few parameters of the dispersion relation of the γ quasiparticles, namely the value of the minimum Δ and its location k_0 in the wavenumber space, their derivatives with respect to density, and the effective mass m_* close to $k = k_0$. We have discovered however that the $\phi - \gamma$ coupling of Ref.[1] is not exact, a fact to our knowledge unnoticed in the literature. Our goal here is to complete the result of Landau and Khalatnikov, and to quantitatively obtain the phonon damping rate due to the $\phi - \gamma$ coupling as a function of temperature, an *a priori* nontrivial task in the considered strongly interacting systems. We restrict to the collisionless regime $\omega_{\mathbf{q}}\tau_\gamma \gg 1$ and $\omega_{\mathbf{q}}\tau_\phi \gg 1$, where $\omega_{\mathbf{q}}$ is the angular eigenfrequency of the considered phonon mode of wavevector $\mathbf{q}$, and τ_γ and τ_ϕ are respectively the typical collision times of a thermal γ quasiparticle and of a thermal phonon. An extension to the hydrodynamic regime $\omega_{\mathbf{q}}\tau_\gamma \lesssim 1$ or $\omega_{\mathbf{q}}\tau_\phi \lesssim 1$ may however be obtained from kinetic equations [2]. An experimental test of our results seems nowadays at hand, either in liquid helium 4, extending the recent work of Ref.[3], or in homogeneous cold Fermi gases, that the breakthrough of flat-bottom traps [4] now allows one to prepare in the lab [5].

Landau-Khalatnikov revisited – We briefly recall the reasoning of Ref.[1] to get the phonon-roton coupling in liquid helium 4, extending it to the phonon-fermionic quasiparticle coupling in an unpolarised spin-1/2 Fermi gas. We first treat in first quantized form the case of a single roton or fermionic excitation, considered as a quasiparticle γ of position $\mathbf{r}$, momentum $\mathbf{p}$ and spin $s = 0$ or $s = 1/2$. In a homogeneous superfluid of density ρ, its Hamiltonian is given by $\epsilon(\mathbf{p}, \rho)$, a rotationally invariant function of $\mathbf{p}$ such that $p \mapsto \epsilon(\mathbf{p}, \rho)$ is the γ quasiparticle dispersion relation. In presence of acoustic waves (phonons), the superfluid acquires a position-dependent density $\rho(\mathbf{r})$ and velocity $\mathbf{v}(\mathbf{r})$. For a phonon wavelength large enough as compared to the coherence length of the γ quasiparticle (its thermal de Broglie wavelength $\left(\frac{2\pi\hbar^2}{m_* k_B T}\right)^{1/2}$) and a phonon angular frequency small enough as compared to the γ quasiparticle "internal" energy Δ, we can write the γ quasiparticle Hamiltonian in the local density approximation:

$$\mathcal{H} = \epsilon(\mathbf{p}, \rho(\mathbf{r})) + \mathbf{p}\cdot\mathbf{v}(\mathbf{r}) \tag{1}$$

The last term is a Doppler effect reflecting the energy difference in the lab frame and in the frame moving with the superfluid [1]. For a weak phononic perturbation of the superfluid, we expand the Hamiltonian to second order in the density fluctuations $\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \rho$:

$$\mathcal{H} \simeq \epsilon(\mathbf{p}, \rho) + \partial_\rho\epsilon(\mathbf{p}, \rho)\delta\rho(\mathbf{r}) + \mathbf{p}\cdot\mathbf{v}(\mathbf{r}) + \frac{1}{2}\partial_\rho^2\epsilon(\mathbf{p}, \rho)\delta\rho^2(\mathbf{r}) \tag{2}$$

not paying attention yet to the noncommutation of $\mathbf{r}$ and $\mathbf{p}$. Phonons are bosonic quasiparticles connected to the expansion of $\delta\rho(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ on eigenmodes of the quantum hydrodynamic equations linearised around the homogeneous solution at rest:

$$\delta\rho(\mathbf{r}) = \frac{1}{\mathcal{V}^{1/2}}\sum_{\mathbf{q}\neq\mathbf{0}}\rho_q(\hat{b}_{\mathbf{q}}+\hat{b}^\dagger_{-\mathbf{q}})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \quad (3)$$

$$\mathbf{v}(\mathbf{r}) = \frac{1}{\mathcal{V}^{1/2}}\sum_{\mathbf{q}\neq\mathbf{0}}\mathbf{v}_{\mathbf{q}}(\hat{b}_{\mathbf{q}}-\hat{b}^\dagger_{-\mathbf{q}})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \quad (4)$$

The resulting annihilation and creation operators $\hat{b}_{\mathbf{q}}$ and $\hat{b}^\dagger_{\mathbf{q}}$ of a phonon of wavevector $\mathbf{q}$ and energy $\hbar\omega_{\mathbf{q}}$ obey the usual commutation relations $[\hat{b}_{\mathbf{q}},\hat{b}^\dagger_{\mathbf{q}'}]=\delta_{\mathbf{q},\mathbf{q}'}$. We have taken here periodic boundary conditions in a cubic box of volume $\mathcal{V}$, and we recall the expressions of the angular eigenfrequency $\omega_{\mathbf{q}}=cq$ and of the modal amplitudes $\rho_q=\left(\frac{\hbar\rho q}{2mc}\right)^{1/2}$ and $\mathbf{v}_{\mathbf{q}}=\left(\frac{\hbar c}{2m\rho q}\right)^{1/2}\mathbf{q}$, where m is the mass of a superfluid particle and c is the sound velocity.

For an arbitrary number of γ quasiparticles, we switch to second quantisation and rewrite Eq.(2) as

$$\hat{H}=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k}}\hat{\gamma}^\dagger_{\mathbf{k}\sigma}\hat{\gamma}_{\mathbf{k}\sigma}+\sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\sigma}\frac{\mathcal{A}_1(\mathbf{k},\mathbf{q};\mathbf{k}')}{\mathcal{V}^{1/2}}(\hat{\gamma}^\dagger_{\mathbf{k}'\sigma}\hat{\gamma}_{\mathbf{k}\sigma}\hat{b}_{\mathbf{q}}+\text{h.c.})\times\delta_{\mathbf{k}+\mathbf{q},\mathbf{k}'}+\sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}',\sigma}\frac{\mathcal{A}_2(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')}{\mathcal{V}}\hat{\gamma}^\dagger_{\mathbf{k}'\sigma}\hat{\gamma}_{\mathbf{k}\sigma}\delta_{\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q}'}\times[\hat{b}^\dagger_{\mathbf{q}'}\hat{b}_{\mathbf{q}}+\frac{1}{2}(\hat{b}_{-\mathbf{q}'}\hat{b}_{\mathbf{q}}+\text{h.c.})] \quad (5)$$

where $\hat{\gamma}_{\mathbf{k}\sigma}$ and $\hat{\gamma}^\dagger_{\mathbf{k}\sigma}$ are bosonic (rotons, $s=0$, $\sigma=0$) or fermionic ($s=1/2$, $\sigma=\uparrow,\downarrow$) annihilation and creation operators of a γ quasiparticle of wavevector $\mathbf{k}=\mathbf{p}/\hbar$ in spin component σ, obeying the usual commutation or anticommutation relations. The first sum in the right-hand side of Eq.(5) gives the energy of the γ quasiparticles in the unperturbed superfluid, with $\epsilon_{\mathbf{k}}\equiv\epsilon(\hbar\mathbf{k},\rho)$. The second sum, originating from the Doppler term and the term linear in $\delta\rho$ in Eq.(2), describes inelastic processes of absorption or emission of a phonon by a γ quasiparticle, characterised by the amplitude

$$\mathcal{A}_1(\mathbf{k},\mathbf{q};\mathbf{k}')=\rho_q\frac{\partial_\rho\epsilon_{\mathbf{k}}+\partial_\rho\epsilon_{\mathbf{k}'}}{2}+\mathbf{v}_{\mathbf{q}}\cdot\frac{\hbar\mathbf{k}+\hbar\mathbf{k}'}{2} \quad (6)$$

where $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{k}'$ are the wavevectors of the incoming phonon and the incoming and outgoing γ quasiparticles. The writing in Eq.(6) is invariant under the exchange of $\mathbf{k}$ and $\mathbf{k}'$. It results from the symmetrisation of the various terms, in the form $[f(\mathbf{p})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}+\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}f(\mathbf{p})]/2$ with $\mathbf{r}$ and $\mathbf{p}$ canonically conjugated operators, ensuring that the correct form of Eq.(2) is indeed hermitian. The third sum in Eq.(5), originating from the terms quadratic in $\delta\rho$ in Eq.(2), describes the direct elastic scattering of a phonon on a γ quasiparticle, with the symmetrised amplitude

$$\mathcal{A}_2(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')=\rho_q\rho_{q'}\frac{\partial^2_\rho\epsilon_{\mathbf{k}}+\partial^2_\rho\epsilon_{\mathbf{k}'}}{2} \quad (7)$$

where the primed wavevectors are the ones of the emerging quasiparticles. It also describes negligible two-phonon absorption and emission processes. The effective amplitude for the elastic $\phi-\gamma$ scattering is obtained by adding the contributions of the direct process, and of the absorption-emission or emission-absorption process treated to second order in perturbation theory [1]:

$$\mathcal{A}_2^{\text{eff}}(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')=\mathcal{A}_2(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')+\frac{\mathcal{A}_1(\mathbf{k},\mathbf{q};\mathbf{k}+\mathbf{q})\mathcal{A}_1(\mathbf{k}',\mathbf{q}';\mathbf{k}'+\mathbf{q}')}{\hbar\omega_{\mathbf{q}}+\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}}+\frac{\mathcal{A}_1(\mathbf{k}-\mathbf{q}',\mathbf{q}';\mathbf{k})\mathcal{A}_1(\mathbf{k}-\mathbf{q}',\mathbf{q};\mathbf{k}')}{\epsilon_{\mathbf{k}}-\hbar\omega_{\mathbf{q}'}-\epsilon_{\mathbf{k}-\mathbf{q}'}} \quad (8)$$

Up to this point this agrees with the theory of Landau and Khalatnikov, except that the first derivative of Δ with respect to ρ in Eq.(6), thought to be anomalously small in helium at low pressure, was neglected in Ref.[1]. Expression (8), issued from a local density approximation, holds only to leading order in a low-energy limit (it also holds only on the energy shell, that is for energy conserving scattering processes, which we used to rewrite the second denominator in (8), since quantum hydrodynamics is only an effective theory [6]). We then take the $T\to 0$ limit with the scaling laws

$$q\approx T, k-k_0\approx T^{1/2} \quad (9)$$

reflecting the fact that the thermal energy of a phonon is $\hbar cq\approx k_BT$ and the effective kinetic energy $\epsilon_{\mathbf{k}}-\Delta$ of a quasiparticle γ is $\approx k_BT$, as seen from the expansion

$$\epsilon_{\mathbf{k}}\underset{k\to k_0}{=}\Delta+\frac{\hbar^2(k-k_0)^2}{2m_*}+O(k-k_0)^3 \quad (10)$$

The coupling amplitudes $\mathcal{A}_1$ and the energy denominators in Eq.(8) must be expanded up to relative corrections of order T, that is to order $T^{3/2}$ for $\mathcal{A}_1$ and T^2 for the energy denominators, q' being deduced from $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{q}'/q'$ by energy conservation $[q-q'=\frac{\hbar(k-k_0)q(u-u')}{m_*c}[1+\frac{\hbar(k-k_0)u'}{m_*c}]+\frac{\hbar q^2(u-u')^2}{2m_*c}+O(T^{5/2})$ where u and u' are defined below Eq.(11)]. On the contrary, it suffices to expand $\mathcal{A}_2$ to leading order T in temperature. We hence get our main result, the effective coupling amplitude of the elastic $\phi-\gamma$ process to leading order in temperature:

$$\mathcal{A}_2^{\text{eff}}(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')\underset{T\to 0}{\sim}\frac{\hbar q}{mc\rho}\Bigg\{\frac{1}{2}\rho^2\Delta''+\frac{(\hbar\rho k_0')^2}{2m_*}+\frac{\hbar^2k_0^2}{2m_*}\times\Bigg\{\left(\frac{\rho\Delta'}{\hbar ck_0}\right)^2uu'+\frac{\rho\Delta'}{\hbar ck_0}\left[(u+u')\left(uu'-\frac{\rho k_0'}{k_0}\right)+\frac{2m_*c}{\hbar k_0}w\right]+\frac{m_*c}{\hbar k_0}(u+u')w+u^2u'^2-\frac{\rho k_0'}{k_0}(u^2+u'^2)\Bigg\}\Bigg\} \quad (11)$$

Here Δ', k_0', Δ'' are the first and second derivatives of Δ and k_0 with respect to ρ; $u = \frac{\mathbf{q}\cdot\mathbf{k}}{qk}$, $u' = \frac{\mathbf{q}'\cdot\mathbf{k}}{q'k}$, $w = \frac{\mathbf{q}\cdot\mathbf{q}'}{qq'}$ are cosines of the angles between $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{q}'$; our results hold in the case $k_0 = 0$ provided the limit $k_0 \to 0$ is taken in Eq.(11). In Ref.[1], the Δ' terms were neglected as said, but the last term in (11), with the factor $\rho k_0'/k_0$, was forgotten.

Damping rates – A straightforward application of Eq.(11) is a Fermi golden rule calculation of the damping rate $\Gamma^{\rm el}_{\mathbf{q}}$ of phonons $\mathbf{q}$ due to elastic scattering on γ quasiparticles. The γ quasiparticles are in thermal equilibrium with Bose or Fermi mean occupation numbers $\bar{n}_{\gamma,\mathbf{k}} = [\exp(\epsilon_{\mathbf{k}}/k_BT) - (-1)^{2s}]^{-1}$. So are the phonons in the modes $\mathbf{q}' \neq \mathbf{q}$, with Bose occupation numbers $\bar{n}_{b,\mathbf{q}'} = [\exp(\hbar\omega_{q'}/k_BT) - 1]^{-1}$; the mode $\mathbf{q}$ is initially excited with an arbitrary number $n_{b,\mathbf{q}}$ of phonons. By including both the loss $\mathbf{q}+\mathbf{k} \to \mathbf{q}'+\mathbf{k}'$ and the inverse gain $\mathbf{q}'+\mathbf{k}' \to \mathbf{q}+\mathbf{k}$ process, with the corresponding bosonic amplification factors, and after summation over $\mathbf{k}$, $\mathbf{q}'$, σ, passage to the thermodynamic limit, use of energy conservation and of the relations $1+(-1)^{2s}\bar{n} = e^{\epsilon/k_BT}\bar{n}$, one finds that $\frac{\mathrm{d}}{\mathrm{d}t}n_{b,\mathbf{q}} = -\Gamma^{\rm el}_{\mathbf{q}}(n_{b,\mathbf{q}} - \bar{n}_{b,\mathbf{q}})$ with

$$\Gamma^{\rm el}_{\mathbf{q}} = \frac{2\pi}{\hbar}(2s+1)\int \frac{\mathrm{d}^3k\,\mathrm{d}^3q'}{(2\pi)^6}\left[\mathcal{A}_2^{\rm eff}(\mathbf{k},\mathbf{q};\mathbf{k}',\mathbf{q}')\right]^2 \\ \times\,\delta(\epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}} - \epsilon_{\mathbf{k}'} - \hbar\omega_{\mathbf{q}'})\frac{\bar{n}_{b,\mathbf{q}'}\bar{n}_{\gamma,\mathbf{k}'}[1+(-1)^{2s}\bar{n}_{\gamma,\mathbf{k}}]}{\bar{n}_{b,\mathbf{q}}} \tag{12}$$

and $\mathbf{k}' = \mathbf{k}+\mathbf{q}-\mathbf{q}'$. As our low energy theory only holds for $k_BT \ll \Delta$, the gas of γ quasiparticles is non degenerate, and $\bar{n}_{\gamma,\mathbf{k}} \simeq \exp(-\epsilon_{\mathbf{k}}/k_BT) \ll 1$ in the numerator of Eq.(12). By taking the $T \to 0$ limit at fixed $\hbar cq/k_BT$ and setting $\mathcal{A}_2^{\rm eff} = \frac{\hbar\omega_q}{\rho}f$, where the dimensionless quantity f only depends on angle cosines, we obtain the equivalent

$$\hbar\Gamma^{\rm el}_{\mathbf{q}} \underset{T\to 0}{\sim} (2s+1)\frac{\mathrm{e}^{-\Delta/k_BT}}{(2\pi)^{9/2}}\frac{k_0^2q^4c}{\rho^2}(m_*k_BT)^{1/2}I \tag{13}$$

where the integral $I = \int \mathrm{d}^2\Omega_{\mathbf{k}}\int \mathrm{d}^2\Omega_{\mathbf{q}'}f^2(u,u',w)$ over solid angles of direction $\mathbf{k}$ and $\mathbf{q}'$ is independent of $\frac{\mathbf{q}}{q}$ [we find $I/(4\pi)^2 = (\frac{\hbar^2k_0^2}{2m_*mc^2})^2[\frac{1}{25} - \frac{4\alpha}{15} + \frac{28}{45}\alpha^2 + \frac{2\beta^2}{9} + A(\frac{2}{9} - \frac{4\alpha}{3}) + A^2 + 4\beta B(\frac{1}{15} - \frac{\alpha}{9}) + B^2(\frac{2}{15} - \frac{4\alpha}{9} + \frac{2\alpha^2}{3} + \frac{4\beta^2}{3}) + \frac{4\beta}{9}B^3 + \frac{B^4}{9}]$, with $\alpha = \frac{\rho k_0'}{k_0}$, $\beta = \frac{m_*c}{\hbar k_0}$, $A = \frac{m_*\rho^2\Delta''}{(\hbar k_0)^2} + \alpha^2$, $B = \frac{\rho\Delta'}{\hbar ck_0}$].

One proceeds similarly for the calculation of the damping rate $\Gamma^{\rm inel}_{\mathbf{q}}$ of phonons $\mathbf{q}$ due to the inelastic processes of absorption $\mathbf{q}+\mathbf{k} \to \mathbf{k}'$ or emission $\mathbf{k}' \to \mathbf{q}+\mathbf{k}$ by the thermal equilibrium γ quasiparticles. We obtain

$$\Gamma^{\rm inel}_{\mathbf{q}} = \frac{2\pi}{\hbar}(2s+1)\int \frac{\mathrm{d}^3k}{(2\pi)^3}[\mathcal{A}_1(\mathbf{k},\mathbf{q};\mathbf{k}')]^2 \\ \times\,\delta(\hbar\omega_{\mathbf{q}} + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})(\bar{n}_{\gamma,\mathbf{k}} - \bar{n}_{\gamma,\mathbf{k}'}) \tag{14}$$

with $\mathbf{k}' = \mathbf{k}+\mathbf{q}$. The low degeneracy of the γ quasiparticles and energy conservation allow us to write $\bar{n}_{\gamma,\mathbf{k}} - \bar{n}_{\gamma,\mathbf{k}'} \simeq \exp(-\epsilon_{\mathbf{k}}/k_BT)/(1+\bar{n}_{b,\mathbf{q}})$. Energy conservation leads here to a scaling law on k that differs from Eq.(9) as it forces the wavevector k to be at a nonzero distance from k_0, even in the low phonon energy limit: When $q \to 0$ at fixed $\mathbf{k}$, the Dirac delta in Eq.(14) becomes

$$\delta(\hbar\omega_{\mathbf{q}} + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \underset{q\to 0}{\sim} (\hbar cq)^{-1}\delta\left(1 - u\frac{\frac{\mathrm{d}\epsilon_k}{\mathrm{d}k}}{\hbar c}\right) \tag{15}$$

and imposes that the group velocity $\frac{1}{\hbar}\frac{\mathrm{d}\epsilon_k}{\mathrm{d}k}$ of the incoming γ quasiparticle is larger in absolute value than that, c, of the phonons. This condition, that reminds us of the Landau criterion, restricts the wavenumber k to a subset D of $\mathbb{R}^+$. In the low q limit, $q \ll k\,\forall k \in D$ significantly contributing to Eq.(14), but with no constraint on the ratio $\hbar cq/k_BT$, we write $\mathcal{A}_1$ in Eq.(6) to the leading order $q^{1/2}$ in q, and integrate over the direction of $\mathbf{k}$, to obtain

$$\Gamma^{\rm inel}_{\mathbf{q}} \simeq \frac{(2s+1)\rho}{4\pi mc}\int_D \frac{\mathrm{d}k\,k^2}{|\frac{\mathrm{d}\epsilon_k}{\mathrm{d}k}|}\frac{\mathrm{e}^{-\epsilon_{\mathbf{k}}/k_BT}}{1+\bar{n}_{b,\mathbf{q}}}\left|\partial_\rho\epsilon_k + \frac{\hbar^2c^2k}{\rho\frac{\mathrm{d}\epsilon_k}{\mathrm{d}k}}\right|^2 \tag{16}$$

$$\underset{T\to 0}{\sim} \frac{(2s+1)\rho k_*^2}{4\pi\hbar^2mc^3}\left|\partial_\rho\epsilon_{k_*} + \frac{\hbar ck_*}{\rho\eta_*}\right|^2\frac{k_BT\mathrm{e}^{-\epsilon_{k_*}/k_BT}}{1+\bar{n}_{b,\mathbf{q}}} \tag{17}$$

In the second line we give an equivalent when $T \to 0$ at fixed $\hbar cq/k_BT$, k_* being the element of the border of D ($\frac{\mathrm{d}\epsilon_k}{\mathrm{d}k}|_{k=k_*} = \eta_*\hbar c$, $\eta_* = \pm$) with minimal energy ϵ_k (when more than one of such k_* exists, one has to sum their contributions). As $\epsilon_{k_*} > \Delta$, the elastic damping rate dominates the inelastic one in the mathematical limit $T \to 0$; we shall see however that this is not always the case for typical temperatures in current experiments.

To be complete, we give a low temperature equivalent of the damping rate of the γ quasiparticles $\mathbf{k}$ due to their interaction with thermal phonons. With $k - k_0 = O(T^{1/2})$ as in Eq.(9):

$$\hbar\Gamma^{\gamma\phi}_{\mathbf{k}} \underset{T\to 0}{\sim} \frac{\pi}{42}\frac{(k_BT)^7}{(\hbar c\rho^{1/3})^6}I \tag{18}$$

where the factor $2s+1$ is gone but I is the same angular integral as in Eq.(13). This results from the elastic scattering processes: At low enough temperature, k is close to k_0, emission processes $\mathbf{k} \leftrightarrow \mathbf{q}+\mathbf{k}'$ are forbidden by energy conservation, and absorption processes $\mathbf{k}+\mathbf{q} \leftrightarrow \mathbf{k}'$, that conserve energy only for $q \geq q_* \simeq 2m_*c/\hbar$, are suppressed by an exponential factor $\mathrm{e}^{-\hbar\omega_{q_*}/k_BT}$. Eq.(18) can be used to check that the figures below are in the collisionless regime $\omega_{\mathbf{q}}\tau_\gamma \gg 1$. Indeed, at low temperature, the density of γ quasiparticles is $O(\mathrm{e}^{-\Delta/k_BT})$, their interactions become negligible, and their thermal collision time reduces to $\tau_\gamma \simeq 1/\Gamma^{\gamma\phi}_{\mathbf{k}}$. Similarly, we checked that $\omega_{\mathbf{q}}\tau_\phi \gg 1$ on the figures.

Application to helium – Precise measurements of the equation of state (relating ρ to the pressure) and of

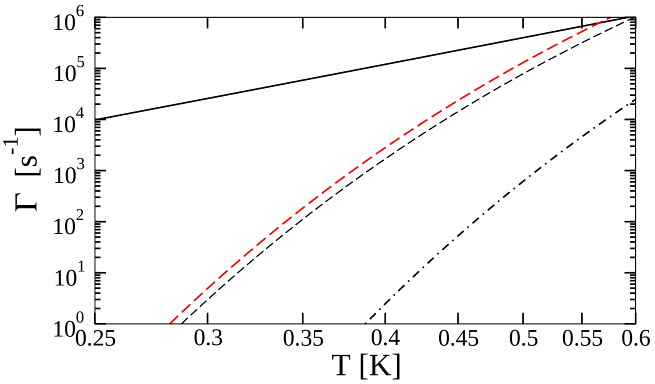

FIG. 1: Phonon damping rates at the angular frequency $\omega_{\mathbf{q}} = 2\pi \times 165$ GHz ($q = 0.3\text{Å}^{-1}$) in liquid ^{4}He at pressure $P = 20$ bar as a function of temperature. Solid line: purely phononic damping $\Gamma_{\phi\phi}$ due to the Landau-Khalatnikov four-phonon process [1, 6, 15]; it depends on the curvature parameter γ defined as $\omega_{\mathbf{q}} = cq[1 + \frac{\gamma}{8}(\frac{\hbar q}{mc})^2 + O(q^4)]$. Interpolating the measurements of $P \mapsto \gamma(P)$ in Refs.[16, 17] gives $\gamma = -6.9$. Dashed black line/dash-dotted black line: damping due to elastic/inelastic processes, see Eq.(13)/(16). Red dashed line: original formula of Ref.[1] for the elastic phonon-roton damping rate. The roton parameters are extracted from the measurement of their dispersion relation $k \mapsto \epsilon_{\mathbf{k}}$ at various pressures [18]: $\Delta/k_B = 7.44$K, $k_0 = 2.05\text{Å}^{-1}$, $m_*/m = 0.11$, $\rho k_0'/k_0 = 0.39$, $\rho\Delta'/\Delta = -1.64$, $\rho^2\Delta''/\Delta = -8.03$, $\rho m_*'/m_* = -4.7$. In Eq.(16), the parabolic approximation (10) has been used (hence $\epsilon_{k_*}/\Delta \simeq 1.43$). The speed of sound $c = 346.6\,$m/s and the Grüneisen parameter $u = \frac{\mathrm{d}\ln c}{\mathrm{d}\ln\rho} = 2.274$ which enters in $\Gamma_{\phi\phi}$, are taken from the equation of state (A1) of Ref.[19]. The very low values of $\frac{\hbar q}{mc} = 0.13$ and of $\frac{k_B T}{mc^2} < 10^{-2}$ justify our use of quantum hydrodynamics.

the roton dispersion relation for various pressures have been performed in liquid ^{4}He at very low temperature ($k_B T \ll mc^2, \Delta$). They give access to the parameters k_0, Δ, their derivatives and m_*. The measured speed of sound is in excellent agreement with the thermodynamic relation $mc^2 = \rho\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$, where μ is the zero temperature chemical potential of the liquid. With this we are able to plot in Fig. 1 the phonon damping rates as functions of temperature, for a fixed value of the angular frequency $\omega_{\mathbf{q}}$. At the chosen high pressure, the phonon dispersion relation is concave at low q, therefore the Beliaev-Landau [6–11] three-phonon process $\phi \leftrightarrow \phi\phi$ is energetically forbidden at low temperature and the Landau-Khalatnikov [1] process $\phi\phi \leftrightarrow \phi\phi$ is dominant. Our high yet experimentally accessible [12, 13] value of $\omega_{\mathbf{q}}$ leads to not too weak damping rates $\Gamma_{\mathbf{q}}$, thus to attenuation lengths $2c/\Gamma_{\mathbf{q}}$ short enough to be measured in centimetric cells. As visible on Fig. 1, the damping of sound is in fact dominated by the four-phonon Landau-Khalatnikov processes up to a temperature $T \simeq 0.6$ K. In this regime one would directly observe this phonon-phonon damping mechanism, which would be a premiere. The sound attenuation measurements of Ref.[14] in helium at 23 bars and $\omega_{\mathbf{q}} = 2\pi \times 1.1$ GHz are indeed limited to $T > 0.8$ K where the damping is still dominated by the rotons.

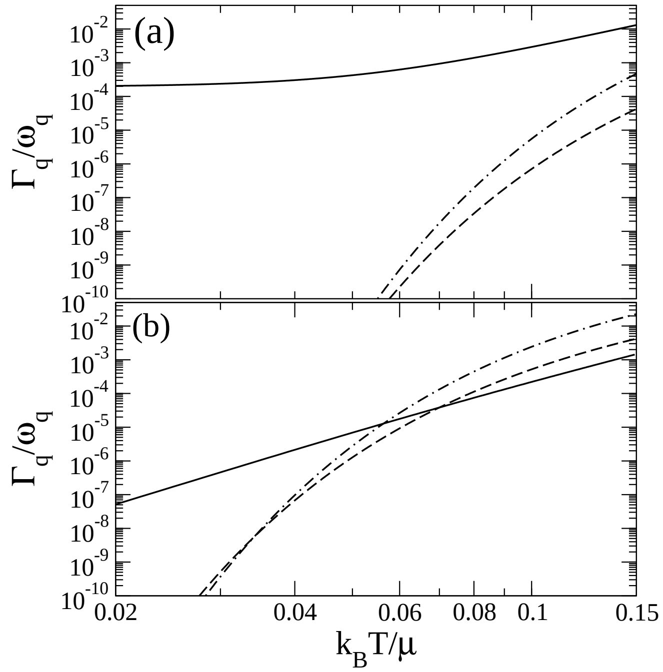

FIG. 2: Phonon damping rates at the wavenumber $q = mc/2\hbar$ in an unpolarized homogeneous [5] cold-atom Fermi gas in the thermodynamic limit as a function of temperature. (a) At unitarity $a^{-1} = 0$, where most parameters of the phonons and the fermionic quasiparticles have been measured (see text). (b) On the BCS side $1/k_{\mathrm{F}}a = -0.389$, these parameters are estimated in the BCS theory ($\mu/\epsilon_{\mathrm{F}} \simeq 0.809$, $\Delta/\mu \simeq 0.566$, $\frac{m_*}{m} = \frac{\Delta}{2\mu}$, $\rho\mu'/\mu \simeq 0.602$, $\rho\Delta'/\Delta \simeq 0.815$, $\rho^2\Delta''/\Delta \simeq -0.209$, $u = \frac{\mathrm{d}\ln c}{\mathrm{d}\ln\rho} \simeq 0.303$). In both cases the curvature parameter γ defined in the caption of Fig. 1 is estimated in the RPA [33]. Solid line: phonon-phonon (a) Beliaev-Landau damping $\phi \leftrightarrow \phi\phi$ (for $\gamma > 0$) as in Eqs. (121) and (122) of Ref.[6] (independent of $|\gamma|$) and (b) Landau-Khalatnikov damping $\phi\phi \leftrightarrow \phi\phi$ (for $\gamma \simeq -0.30 < 0$) [6, 15]. Dashed line/dash-dotted line: elastic/inelastic phonon-fermionic quasiparticle processes, as in Eq.(13)/(16). In Eq.(16), we took for ϵ_k (a) the form proposed in Ref.[30] (hence $\epsilon_{k_*}/\Delta \simeq 1.12$) and (b) the BCS form (hence $\epsilon_{k_*}/\Delta \simeq 1.14$). μ is the chemical potential of the gas at $T = 0$, and the plotted quantities are in fact inverse quality factors.

Application to fermions – In cold-atom Fermi gases, interactions occur in s-wave between opposite-spin atoms. Of negligible range, they are characterized by the scattering length a tunable by Feshbach resonance [20–25].

Precise measurements of the parameters k_0 and Δ of the fermionic excitations have been performed at unitarity $a^{-1} = 0$ [26]. Due to scale invariance of the unitary gas [27–29], k_0 is proportional to the Fermi wavenumber $k_{\mathrm{F}} = (3\pi^2\rho)^{1/3}$, $k_0 \simeq 0.92k_{\mathrm{F}}$ [26], and Δ is proportional to the Fermi energy $\epsilon_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2m}$, $\Delta \simeq 0.44\epsilon_{\mathrm{F}}$ [26]. This also determines their derivatives with respect to ρ. Similarly, the equation of state measured at $T=0$ is simply $\mu=\xi\epsilon_{\rm F}$, where $\xi\simeq 0.376$ [25], and the critical temperature is $T_c\simeq 0.167\epsilon_{\rm F}/k_{\rm B}$ [25]. For the effective mass of the fermionic excitations and their dispersion relation at non vanishing $k-k_0$, we must rely on the results of a dimensional $\epsilon=4-d$ expansion, $m_*/m\simeq 0.56$ and $\epsilon_k\simeq\Delta+\frac{\hbar^2(k^2-k_0^2)^2}{8m_*k_0^2}$ [30]. We also trust the prediction of Anderson's RPA [31, 32] that the third derivative of the phononic dispersion relation is positive at $q=0$ [33]. The damping rates of phonons with wavenumber $q=mc/2\hbar$ are plotted in Fig. 2a. The contribution of the three-phonon Landau-Beliaev processes $\phi\leftrightarrow\phi\phi$, here energetically allowed, is the dominant one; it is computed in the quantum hydrodynamic approximation where it is independent of the value of the third derivative.

The phononic excitation branch becomes concave in the BCS limit $k_{\rm F}a\to 0^-$ [34]. As visible on Fig. 2b, the phonon-phonon damping (now governed by the Landau-Khalatnikov processes mentioned earlier) is much weaker, and dominates the $\phi-\gamma$ damping only at very low temperatures. At the commonly reached temperatures $T>0.05\epsilon_{\rm F}/k_B$ [35], the damping is in fact dominated by the inelastic $\phi-\gamma$ processes which (unlike in helium) prevail over the elastic ones because of the small value of ϵ_{k_*}/Δ. Although the associated quality factors $\omega_{\mathbf{q}}/\Gamma_{\mathbf{q}}$ may seem impressive, the lifetimes $\Gamma_{\mathbf{q}}^{-1}$ of the modes do not exceed one second in a gas of ^{6}Li with a typical Fermi temperature $T_{\rm F}=1\mu$K, which is shorter than what has been observed in a BEC [36]. Our predictions, less quantitative than on Fig. 2a, are based on the BCS approximation for both the equation of state and the dispersion relation of the fermionic excitations $\epsilon_k\simeq\epsilon_k^{\rm BCS}=[(\frac{\hbar^2k^2}{2m}-\mu)^2+\Delta_{\rm BCS}^2]^{1/2}$ and on the RPA for the third derivative of ω_q in $q=0$ (whose precise value matters here). A cutting remark on Ref. [37]: even in the BCS approximation to which it is restricted, we disagree with its expression of $\Gamma_{\mathbf{q}}^{\rm inel}$.

Conclusion – By complementing the Landau approach [1] with a systematic low temperature expansion, we derived the definitive leading order expression of the phonon-roton coupling in liquid helium and we generalized it to the phonon-pair-breaking excitation coupling in Fermi gases. The ever-improving experimental technics in these systems give access to the microscopic parameters determining the coupling and allow for a verification in the near future. Our result also clarifies the regime of temperature and interaction strength in which the purely phononic $\phi\phi\leftrightarrow\phi\phi$ Landau-Khalatnikov sound damping in a superfluid, unobserved to this day, is the dominant process.

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