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# Estimating functions for SDE driven by stable Lévy processes

Emmanuelle Clément\*      Arnaud Gloter†

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## Abstract

This paper is concerned with parametric inference for a stochastic differential equation driven by a pure-jump Lévy process, based on high frequency observations on a fixed time period. Assuming that the Lévy measure of the driving process behaves like that of an  $\alpha$ -stable process around zero, we propose an estimating functions based method which leads to asymptotically efficient estimators for any value of  $\alpha \in (0, 2)$  and does not require any integrability assumptions on the process. The main limit theorems are derived thanks to a control in total variation distance between the law of the normalized process, in small time, and the  $\alpha$ -stable distribution. This method is an alternative to the non Gaussian quasi-likelihood estimation method proposed by Masuda [20] where the Blumenthal-Gettoor index  $\alpha$  is restricted to belong to the interval  $[1, 2)$ .

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**Key words:** Lévy process, Stable process, Stochastic Differential Equation, Parametric inference, Estimating functions, Malliavin Calculus.

## 1 Introduction

Pure-jump processes are widely used and appropriate in several fields such as traffic modeling, energy market modeling and estimation of such processes is a currently active topic. In this paper we are interested in parametric estimation of the drift and scale coefficients for a one-dimensional stochastic differential equation given by

$$dX_t = b(X_t, \theta)dt + a(X_{t-}, \sigma)dL_t,$$

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\*Laboratoire MICS, Fédération de Mathématiques FR 3487, Université Paris-Saclay, CentraleSupélec, 91190 Gif-sur-Yvette, France.

†LaMME, Univ Evry, CNRS, Université Paris-Saclay, 91025, Evry, France.

where  $L$  is a locally stable pure-jump Lévy process with Bumenthal-Gettoor index  $\alpha \in (0, 2)$ . This choice of  $L$  encompasses stable processes and also many other interesting processes.

We study estimation of  $(\theta, \sigma)$  from discrete equidistant observations of  $X$  on a fixed time interval with time grid mesh shrinking to zero. In this high-frequency observation context it is known that the estimation of  $\theta$  is impossible if the driving process  $L$  contains a Brownian part (which corresponds to the case  $\alpha = 2$ ). However when  $\alpha < 2$ , both parameters  $\theta$  and  $\sigma$  can be estimated. This problem has been studied first when  $X$  is a Lévy process (which corresponds to constant coefficients  $a$  and  $b$ ) in several papers, see for example Aït-Sahalia and Jacod [1] [2], Kawai and Masuda [14] [15], Masuda [19], Ivanenko, Kulik and Masuda [11]. In all these papers, the increments of the observed process  $X$  are independent with an explicit characteristic function. However, contrarily to the jump-diffusion case, for a general pure-jump driven SDE the literature is much smaller. It has been established recently in Clément and Gloter [7], Clément, Gloter and Nguyen [9] that the Local Asymptotic Mixed Normality property holds when the scale coefficient  $a$  is assumed to be constant and  $L$  is a truncated  $\alpha$ -stable process. This result permits to identify the Fisher information matrix for the parameters  $(\theta, \sigma)$  and that the rate of convergence are respectively  $n^{1/\alpha-1/2}$  for the estimation of  $\theta$  and  $n^{1/2}$  for the estimation of  $\sigma$ . It is important to remark that the estimation rate for  $\theta$  is slower than the usual rate  $n^{1/2}$  when  $\alpha > 1$  and faster when  $\alpha < 1$ . Concerning the estimation problem, this has been addressed by Masuda [20] assuming that  $\alpha \in [1, 2)$  using a quasi-likelihood estimation method. Indeed, in that case, the drift contribution is negligible compared to the jump part and the law of the normalized increment  $h^{-1/\alpha}(X_{t+h} - X_t - hb(X_t, \theta))/a(X_t, \sigma)$  is close to the  $\alpha$ -stable distribution as  $h$  goes to zero. However, this method does not seem to be extended to the case  $\alpha \in (0, 1)$  mainly because of the great contribution of the drift.

In this paper, we propose an estimating functions based method which applies to any value of  $\alpha \in (0, 2)$ . Estimating equation methods are useful alternative methods in situations where the likelihood function is not known in a tractable form and have been widely used in estimating diffusion processes from discrete time observations (see for example Bibby and Sørensen [5], Kessler and Sørensen [16]). We also refer to the papers by Barndorff-Nielsen and Sørensen [4], Sørensen [22] for general asymptotic results on estimating equation methods. In this work, we consider estimating equations derived by approximating the score function and by changing the above normalized increment by  $h^{-1/\alpha}(X_{t+h} - \xi_h^{X_t}(\theta))/a(X_t, \sigma)$ , where  $(\xi_t^x(\theta))_t$  is solution to the ordinary differential equation

$$\xi_t^x(\theta) = x + \int_0^t b(\xi_s^x(\theta), \theta) ds.$$

The introduction of this ordinary differential equation is convenient for dealing with any value of  $\alpha$  but we can also replace it by a numerical approximation scheme whose order has to be high when  $\alpha$  is small. Conditionally on  $X_t$  the density of  $h^{-1/\alpha}(X_{t+h} - \xi_h^{X_t}(\theta))/a(X_t, \sigma)$  converges when  $h$  tends to zero to the density of the  $\alpha$ -stable distribution. This has been established by Kulik [17] when  $L$  is an  $\alpha$ -stable process (and also in [8] when  $L$  is a truncated  $\alpha$ -stable process assuming that  $a$  is constant). However to prove consistency and asymptotic mixed normality of our estimators, the convergence of the densities is not sufficient and we also need a rate of convergence. This is the most technical part of the paper. For  $L$  a locally  $\alpha$ -stable process, we prove that the total variation distance between the conditional law of  $h^{-1/\alpha}(X_{t+h} - \xi_h^{X_t}(\theta))/a(X_t, \sigma)$  and the  $\alpha$ -stable distribution is bounded by  $\varepsilon_h$  such that  $h^{-1/2}\varepsilon_h$  tends to zero. This result is the key ingredient to derive some limit theorems (Law of Large Numbers, Central Limit Theorem) for functionals of normalized discrete time observations of the process  $X$ .

At last, it should be noted that the estimation method proposed in this paper requires that the Blumenthal-Gettoor index  $\alpha$  is known. This is also the case in Masuda [20]. A large literature is devoted to the estimation of the jump activity of jump-diffusion processes from high frequency observations, based on truncated power variation or on empirical characteristic function. We mention among others the papers by Ait-Sahalia and Jacod [3], Todorov [23], Todorov and Tauchen [24].

This paper is organized as follows. Section 2 introduces the notations and assumptions. Section 3 is devoted to the estimating function method and states consistency and asymptotic mixed normality of our estimators. These results are proved in Section 4 after establishing appropriate limit theorems. Finally, Section 5 is dedicated to some critical total variation distance estimates.

## 2 Notation and setup

We consider the process  $(X_t)_{t \in [0,1]}$  solution to the stochastic differential equation :

$$X_t = x_0 + \int_0^t b(X_s, \theta) ds + \int_0^t a(X_{s-}, \sigma) dL_s, \quad t \in [0, 1], \quad (1)$$

where  $a$  and  $b$  are known functions from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$  and  $(\theta, \sigma)$  are real parameters. We assume that  $(L_t)_{t \in [0,1]}$  is a pure-jump Lévy process defined on a filtered space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ .

We introduce also the solution to the ordinary differential equation

$$\xi_t^{x_0}(\theta) = x_0 + \int_0^t b(\xi_s^{x_0}(\theta), \theta) ds, \quad t \in [0, 1]. \quad (2)$$

We observe the process on discrete times  $t_i = i/n$  for  $i = 1, \dots, n$  for the value  $(\theta_0, \sigma_0)$  of the parameter and based on these observations our aim is to estimate  $(\theta_0, \sigma_0)$ .

We make the following assumptions.

**H1(Regularity)** : (a) Let  $V_{\theta_0} \times V_{\sigma_0}$  be a neighborhood of  $(\theta_0, \sigma_0)$ . We assume that  $a$ , respectively  $b$ , are  $\mathcal{C}^2$  on  $\mathbb{R} \times V_{\sigma_0}$ , respectively on  $\mathbb{R} \times V_{\theta_0}$ , with bounded derivatives with respect to  $x$  :

$$\sup_{\theta \in V_{\theta_0}} \sup_x \max_{1 \leq l \leq 2} (|\partial_x^l b(x, \theta)| + |\partial_x^l a(x, \sigma_0)|) \leq C,$$

$a$  is lower bounded : there exists  $\underline{a} > 0$  such that

$$\forall x \in \mathbb{R}, \forall \sigma \in V_{\sigma_0} \quad a(x, \sigma) \geq \underline{a}.$$

(b)  $\forall x \in \mathbb{R}$ ,  $\theta \mapsto b(x, \theta)$  and  $\sigma \mapsto a(x, \sigma)$  are  $\mathcal{C}^3$  and such that

$$\sup_{(\theta, \sigma) \in V_{\theta_0} \times V_{\sigma_0}} \max_{1 \leq l \leq 3} (|\partial_\theta^l b(x, \theta)| + |\partial_\sigma^l a(x, \sigma)|) \leq C(1 + |x|^p), \quad \text{for some } p > 0,$$

$$\sup_{\theta \in V_{\theta_0}} |\partial_x \partial_\theta b(x, \theta)| \leq C.$$

**H2 (Lévy measure)** : (a) The Lévy measure of  $(L_t)$  satisfies  $\nu(dz) = \frac{g(z)}{|z|^{\alpha+1}} 1_{\mathbb{R}^*}(z) dz$ , where  $\alpha \in (0, 2)$  and  $g : \mathbb{R} \mapsto \mathbb{R}$  is a continuous symmetric non negative bounded function with  $g(0) = 1$

(b)  $g$  is derivable on  $\{0 < |z| \leq \eta\}$  for some  $\eta > 0$  with continuous derivative and such that  $\left| \frac{g'}{g} \right|$  is bounded.

This assumption is satisfied by a large class of processes :  $\alpha$ -stable process ( $g = 1$ ), truncated  $\alpha$ -stable process ( $g = \tau$  a truncation function), tempered stable process ( $g(z) = e^{-\lambda|z|}$ ,  $\lambda > 0$ ).

**H3 (Non degeneracy)** : Almost surely,  $\exists t_1, t_2 \in (0, 1)$ , such that  $\partial_\sigma a(X_{t_1}, \sigma_0) \neq 0$ ,  $\partial_\theta b(X_{t_2}, \theta_0) \neq 0$ , where  $(X_t)_{t \in [0, 1]}$  is the solution to (1) for the value  $(\theta_0, \sigma_0)$  of the parameter.

We will use the following notation. We note  $\|\cdot\|$  a vector norm or a matrix norm and  $A^T$  the transpose of a matrix  $A$ . For a bounded function  $f : \mathbb{R} \mapsto \mathbb{R}$ ,  $\|f\|_\infty = \sup_x |f(x)|$ . In the sequel, we note  $a'$ ,  $b'$  the derivative of  $a$ ,  $b$  with respect to  $x$  and  $\dot{a}$ ,  $\dot{b}$  the derivative with respect to the parameter. We will also note  $\xi_i(\theta) = \xi_{\frac{1}{n}}^{X_{i/n}}(\theta)$ , for  $i = 0$  to  $n$ .

Throughout the paper,  $C$  or  $C_p$  denote some constants whose value does not depend on  $n$  and may change from line to line.

### 3 Estimating functions

To estimate  $(\theta, \sigma)$ , we will use the estimating function method (see [22]). To this end, we consider the functions

$$G_n(\theta, \sigma) = \begin{pmatrix} G_n^1(\theta, \sigma) \\ G_n^2(\theta, \sigma) \end{pmatrix},$$

with for  $k = 1, 2$

$$G_n^k(\theta, \sigma) = \sum_{i=1}^n g^k \left( X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma \right),$$

where  $g^k$  will be specified below.

If we note  $p_{1/n}(x, y)$  the transition density of the Markov chain  $(X_{i/n})_i$  solution to (1), we can prove the convergence as  $n$  goes to infinity

$$\frac{a(x, \sigma)}{n^{1/\alpha}} p_{1/n}(x, \frac{a(x, \sigma)}{n^{1/\alpha}} y + \xi_{1/n}^x(\theta)) \longrightarrow \varphi_\alpha(y),$$

where  $\varphi_\alpha$  is the density of  $L_1^\alpha$ , a stable random variable with characteristic function  $e^{-C(\alpha)|u|^\alpha}$  (see [8] assuming that  $a$  is constant and  $(L_t)$  is a truncated  $\alpha$ -stable process or [17] for more general assumptions on the coefficients assuming that  $(L_t)$  is an  $\alpha$ -stable process). From this observation, we can approximate  $p_{1/n}(x, y)$  by  $\frac{n^{1/\alpha}}{a(x, \sigma)} \varphi_\alpha \left( n^{1/\alpha} \frac{(y - \xi_{1/n}^x(\theta))}{a(x, \sigma)} \right)$  and approximating the score function, a natural choice of estimating functions is

$$\begin{aligned} g^1(x, y, \theta, \sigma) &= \frac{n^{1/\alpha}}{n} \frac{\dot{b}(x, \theta)}{a(x, \sigma)} \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)), \\ g^2(x, y, \theta, \sigma) &= \frac{\dot{a}(x, \sigma)}{a(x, \sigma)} \left( 1 + z_n(x, y, \theta, \sigma) \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) \right), \end{aligned}$$

where

$$z_n(x, y, \theta, \sigma) = n^{1/\alpha} \frac{(y - \xi_{1/n}^x(\theta))}{a(x, \sigma)}. \quad (3)$$

For this choice of estimating functions, we can prove the following theorem.

**Theorem 1** *Under the assumptions H1, H2 and H3, there exists an estimator  $(\hat{\theta}_n, \hat{\sigma}_n)$  solution to the equation  $G_n(\hat{\theta}_n, \hat{\sigma}_n) = 0$  with probability tending to 1 that converges in probability to  $(\theta_0, \sigma_0)$ . Moreover we have the convergence in law*

$$\begin{pmatrix} n^{1/\alpha-1/2}(\hat{\theta}_n - \theta_0) \\ n^{1/2}(\hat{\sigma}_n - \sigma_0) \end{pmatrix} \Longrightarrow I(\theta_0, \sigma_0)^{-1/2} \mathcal{N},$$

where  $\mathcal{N}$  is a standard Gaussian variable independent of  $I(\theta_0, \sigma_0)$  and

$$I(\theta, \sigma) = \begin{pmatrix} \int_0^1 \frac{b(X_s, \theta)^2}{a(X_s, \sigma)^2} ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du & 0 \\ 0 & \int_0^1 \frac{\dot{a}(X_s, \sigma)^2}{a(X_s, \sigma)^2} ds \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du \end{pmatrix}.$$

**Remark 1** The LAMN property with information  $I(\theta, \sigma)$  has been established in [9] for this experiment, assuming that  $a$  is constant and  $g$  is a truncation function. This result permits to deduce that our estimator is efficient.

**Remark 2** If the solution to the ordinary differential equation (2) is not explicit, we can replace it in the expression of  $z_n$  by any numerical scheme  $\bar{\xi}_{1/n}^x(\theta)$  with order  $k$  such that  $\sup_{\theta \in V_{\theta_0}} |\xi_{1/n}^x(\theta) - \bar{\xi}_{1/n}^x(\theta)| \leq C(1 + |x|^p)/n^k$  and  $k > 1/\alpha + 1/2$ . This gives estimating functions sufficiently close to those studied here to preserve the sufficient conditions C1 and C2 stated below to prove the convergence and asymptotic mixed normality of the estimator. In the case  $\alpha > 2/3$ , the choice  $\bar{\xi}_{1/n}^x(\theta) = x + b(x, \theta)/n$  is convenient since we can check from H1 that  $\sup_{\theta \in V_{\theta_0}} |\xi_{1/n}^x(\theta) - x - b(x, \theta)/n| \leq C(1 + |x|)/n^2$ .

The proof of Theorem 1 is based on the results established in Sørensen [22] to obtain the consistency and asymptotic normality of estimators constructed from estimating functions. We first remark that from H3, we have  $I(\theta_0, \sigma_0) > 0$  almost surely.

We define the matrices  $u_n$  and  $J_n((\theta_1, \sigma_1), (\theta_2, \sigma_2))$  by

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha-1/2}} & 0 \\ 0 & \frac{1}{n^{1/2}} \end{pmatrix}$$

$$J_n((\theta_1, \sigma_1), (\theta_2, \sigma_2)) = \sum_{i=1}^n \begin{pmatrix} \partial_\theta g^1(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_1, \sigma_1) & \partial_\sigma g^1(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_1, \sigma_1) \\ \partial_\theta g^2(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_2, \sigma_2) & \partial_\sigma g^2(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_2, \sigma_2) \end{pmatrix}.$$

We also consider the neighborhood of  $(\theta_0, \sigma_0)$ , for  $\eta > 0$ :

$$V_n^{(\eta)}(\theta_0, \sigma_0) = \{(\theta, \sigma); \|u_n^{-1}(\theta - \theta_0, \sigma - \sigma_0)^T\| \leq \eta\}.$$

With these notations, the result of Theorem 1 is a consequence of the two following sufficient conditions:

**C1** :  $\forall \eta > 0$ , we have the convergence in probability :

$$\sup_{(\theta_1, \sigma_1), (\theta_2, \sigma_2) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \|u_n J_n((\theta_1, \sigma_1), (\theta_2, \sigma_2)) u_n - I(\theta_0, \sigma_0)\| \rightarrow 0.$$

**C2** :  $(u_n G_n(\theta_0, \sigma_0))_n$  stably converges in law to  $I(\theta_0, \sigma_0)^{1/2} \mathcal{N}$ , where  $\mathcal{N}$  is a standard Gaussian variable independent of  $I(\theta_0, \sigma_0)$  and the convergence is stable with respect to the  $\sigma$ -field  $\sigma(L_s, s \leq 1)$ .

These conditions imply the ones given in [22] which are sufficient to prove consistency in the case of a random information matrix. In [22] the Central Limit Theorem is established for a deterministic information only, however the proof can be easily extended to the random case enhancing the convergence in law by the stable convergence in law.

The proof of **C1** and **C2** follows from limit theorems for functionals of  $(X_{i/n})_{0 \leq i \leq n}$  and is postponed to the next section.

## 4 Limit theorems

### 4.1 Law of Large Numbers

In this section, we first establish an uniform Law of Large Numbers (for  $(\theta, \sigma)$  in a neighborhood of  $(\theta_0, \sigma_0)$ ) for the normalized sums

$$\frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)),$$

where  $z_n$  is defined by (3).

**Proposition 1** *We assume that H1 and H2(a)-(b) hold. Let  $h : \mathbb{R} \mapsto \mathbb{R}$  be a continuous bounded function with bounded derivative and  $f : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}$  a continuous function with continuous partial derivative with respect to the second variable such that*

$$\sup_{(\theta, \sigma) \in V_{\theta_0} \times V_{\sigma_0}} (|f(x, \theta, \sigma)| + |\partial_{\theta} f(x, \theta, \sigma)| + |\partial_{\sigma} f(x, \theta, \sigma)|) \leq C(1 + |x|^p), \quad \text{for some } p > 0,$$

then we have the convergence in probability:

i)

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - \int_0^1 f(X_s, \theta_0, \sigma_0) ds \mathbb{E}h(L_1^{\alpha}) \right| \rightarrow 0,$$

ii) Moreover if  $\mathbb{E}h(L_1^{\alpha}) = 0$

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) \right| \rightarrow 0 \quad \text{in probability .}$$

Obviously ii) is a consequence of i) in the case  $\alpha \leq 1$  and only the case  $\alpha > 1$  requires a proof.

Before proceeding to the proof of this proposition, we first recall the following useful result to prove convergence in probability of triangular arrays (see [13]).



Let  $(\zeta_i^n)$  be a triangular array such that  $\zeta_i^n$  is  $\mathcal{F}_{\frac{i}{n}}$ -measurable then the two following conditions imply the convergence in probability  $\sum_{i=1}^n \zeta_i^n \rightarrow 0$  :

$$\sum_{i=1}^n |\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} \zeta_i^n| \rightarrow 0 \quad \text{in probability,} \quad (4)$$

$$\sum_{i=1}^n \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} |\zeta_i^n|^2 \rightarrow 0 \quad \text{in probability.} \quad (5)$$

To prove Proposition 1, the idea is to replace  $z_n$  defined by (3), for  $(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)$ , by the normalized increment  $n^{1/\alpha} \Delta L_i = n^{1/\alpha} (L_{\frac{i}{n}} - L_{\frac{i-1}{n}})$ . This approximation is justify by the next lemma which is an extension of Lemma 3 given in Section 5.1.

**Lemma 1** *Assuming H1 and H2(a), there exists  $p, q > 0$  such that*

$$\forall \varepsilon > 0, \mathbb{P}_{|\mathcal{F}_{\frac{i-1}{n}}} \left( \sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| > \varepsilon \right) \leq C(\varepsilon)(1 + |X_{\frac{i-1}{n}}|^p)/n^q, \quad (6)$$

where  $C(\varepsilon)$  is a positive constant and  $\Delta L_i = L_{\frac{i}{n}} - L_{\frac{i-1}{n}}$ .

**Proof** We have the decomposition

$$\begin{aligned} z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) &= \frac{a(X_{\frac{i-1}{n}}, \sigma_0)(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i)}{a(X_{\frac{i-1}{n}}, \sigma)} + \frac{a(X_{\frac{i-1}{n}}, \sigma_0)}{a(X_{\frac{i-1}{n}}, \sigma)} n^{1/\alpha} \Delta L_i \\ &\quad + \frac{n^{1/\alpha}}{a(X_{\frac{i-1}{n}}, \sigma)} (\xi_{i-1}(\theta_0) - \xi_{i-1}(\theta)). \end{aligned}$$

From H1 we have for  $t \in [0, 1/n]$  and  $\theta \in V(\theta_0)$

$$|\xi_t^x(\theta) - \xi_t^x(\theta_0)| \leq \|b'\|_\infty \int_0^t |\xi_s^x(\theta) - \xi_s^x(\theta_0)| ds + |\theta - \theta_0| (1 + \sup_{t \in [0, 1/n]} |\xi_t^x(\theta_0)|^p)/n,$$

moreover from Gronwall's Lemma we check easily that

$$\sup_{t \in [0, 1/n]} |\xi_t^x(\theta_0)| \leq C(1 + |x|).$$

This leads to the bound (using once again Gronwall's Lemma)

$$|\xi_{i-1}(\theta_0) - \xi_{i-1}(\theta)| \leq \frac{C}{n} |\theta - \theta_0| (1 + |X_{\frac{i-1}{n}}|^p).$$

Now for  $(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)$ , we have  $|\theta - \theta_0| \leq \eta/n^{1/\alpha-1/2}$  and  $|\sigma - \sigma_0| \leq \eta/n^{1/2}$  so from H1 we deduce  $|1/a(X_{\frac{i-1}{n}}, \sigma)| \leq C$  and we get that

$$\left| \frac{n^{1/\alpha}}{a(X_{\frac{i-1}{n}}, \sigma)} (\xi_{i-1}(\theta_0) - \xi_{i-1}(\theta)) \right| \leq C(1 + |X_{\frac{i-1}{n}}|^p)/n^{1/2}.$$

Using once again H1 and a Taylor expansion, we have for  $(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)$ ,

$$\left| \frac{a(X_{\frac{i-1}{n}}, \sigma_0)}{a(X_{\frac{i-1}{n}}, \sigma)} - 1 \right| \leq C(1 + |X_{\frac{i-1}{n}}|^p)/n^{1/2}.$$

This gives the bound

$$\begin{aligned} \sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| &\leq C|a(X_{\frac{i-1}{n}}, \sigma_0)(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i)| \\ &+ C(1 + |X_{\frac{i-1}{n}}|^p) \frac{n^{1/\alpha} |\Delta L_i|}{n^{1/2}} + C(1 + |X_{\frac{i-1}{n}}|^p) \frac{1}{n^{1/2}}. \end{aligned}$$

The Markov property and the result of Lemma 3 give for some  $p, q > 0$

$$\mathbb{P}_{|\mathcal{F}_{\frac{i-1}{n}}} (|a(X_{\frac{i-1}{n}}, \sigma_0)(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i)| > \varepsilon) \leq C(\varepsilon)(1 + |X_{\frac{i-1}{n}}|^p)/n^q.$$

Moreover from Theorem 2 in Luschgy-Pagès [18], we have  $\mathbb{E}|n^{1/\alpha} \Delta L_i|^q \leq C$ , for  $q < \alpha$  and we deduce that

$$\mathbb{P}_{|\mathcal{F}_{\frac{i-1}{n}}} ((1 + |X_{\frac{i-1}{n}}|^p) \frac{n^{1/\alpha} |\Delta L_i|}{n^{1/2}} > \varepsilon) \leq C(\varepsilon)(1 + |X_{\frac{i-1}{n}}|^{pq})/n^{q/2}.$$

This finally leads to the bound (6). □

We can now proceed to the proof of Proposition 1.

### Proof of Proposition 1

i) From the triangle inequality, it is sufficient to prove the four following convergences in probability:

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n} \sum_{i=1}^n (f(X_{\frac{i-1}{n}}, \theta, \sigma) - f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0)) h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) \right| \rightarrow 0, \quad (7)$$

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - h(n^{1/\alpha} \Delta L_i)) \right| \rightarrow 0, \quad (8)$$

$$\left| \frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) (h(n^{1/\alpha} \Delta L_i) - \mathbb{E}h(L_1^\alpha)) \right| \rightarrow 0 \quad (9)$$

$$\left| \frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) - \int_0^1 f(X_s, \theta_0, \sigma_0) ds \right| \rightarrow 0 \quad (10)$$

The last point (10) is the convergence of a Riemann sum since  $t \mapsto X_t$  is càdlàg and  $f$  is continuous.

Since  $h$  is bounded and  $\sup_{(\theta, \sigma) \in V_{\theta_0} \times V_{\sigma_0}} (|\partial_\theta f(x, \theta, \sigma)| + |\partial_\sigma f(x, \theta, \sigma)|) \leq C(1 + |x|^p)$ , we get easily (7) from a Taylor expansion and using  $|\theta - \theta_0| \leq \eta/n^{1/\alpha - 1/2}$  and  $|\sigma - \sigma_0| \leq \eta/n^{1/2}$ , for  $(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)$ .

To prove (8), we introduce the truncation  $1_{\{\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| \leq \varepsilon\}}$  for  $\varepsilon > 0$ , and since  $h$  and  $h'$  are bounded we get

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - h(n^{1/\alpha} \Delta L_i)| \leq \varepsilon \|h'\|_\infty + 2\|h\|_\infty 1_{\{\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| > \varepsilon\}}.$$

This yields

$$\begin{aligned} \sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - h(n^{1/\alpha} \Delta L_i)) \right| &\leq \varepsilon C(1 + \sup_{s \in [0, 1]} |X_s|^p) \|h'\|_\infty \\ &+ C \|h\|_\infty \frac{1}{n} \sum_{i=1}^n (1 + |X_{\frac{i-1}{n}}|^p) 1_{\{\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| > \varepsilon\}}. \end{aligned}$$

For any  $\varepsilon > 0$ ,  $\frac{1}{n} \sum_{i=1}^n (1 + |X_{\frac{i-1}{n}}|^p) 1_{\{\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - n^{1/\alpha} \Delta L_i| > \varepsilon\}}$  goes to zero in probability (we check easily (4) and (5) from Lemma 1) and we deduce (8) letting  $\varepsilon$  go to zero.

The proof of (9) is established by checking (4) and (5) with

$$\zeta_i^n = \frac{1}{n} f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) (h(n^{1/\alpha} \Delta L_i) - \mathbb{E}h(L_1^\alpha)).$$

From Proposition 4 we get (4) and the boundedness of  $h$  implies immediately (5).

ii) As mentioned before, only the case  $\alpha > 1$  requires a proof.

We first remark that we just have to prove the convergence in probability, for any  $K > 0$  fixed

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) \right| \rightarrow 0. \quad (11)$$

Indeed,  $1_{\{|X_{\frac{i-1}{n}}| > K\}} \leq 1_{\{\sup_{t \in [0, 1]} |X_t| > K\}}$  and since  $\mathbb{P}(\sup_{t \in [0, 1]} |X_t| > K)$  goes to zero as  $K$  goes to infinity, we deduce ii) from (11) letting successively  $n$  and  $K$  go to infinity.

The proof of (11) is obtained by establishing the three following convergences in probability :

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0))) \right| \rightarrow 0, \quad (12)$$

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma_0)) - \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0))) \right| \rightarrow 0, \quad (13)$$

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} \left| \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) \right| \rightarrow 0. \quad (14)$$

Considering first (12), we have

$$z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) = \frac{a(X_{\frac{i-1}{n}}, \sigma_0)}{a(X_{\frac{i-1}{n}}, \sigma)} z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) + \frac{n^{1/\alpha}}{a(X_{\frac{i-1}{n}}, \sigma)} (\xi_{i-1}(\theta_0) - \xi_{i-1}(\theta)).$$

Now, we have the bounds (this has been established in the proof of Lemma 1)

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} \left| \frac{n^{1/\alpha}}{a(X_{\frac{i-1}{n}}, \sigma)} (\xi_{i-1}(\theta_0) - \xi_{i-1}(\theta)) \right| \leq C(1 + |X_{\frac{i-1}{n}}|^p)/n^{1/2},$$

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} \left| \frac{a(X_{\frac{i-1}{n}}, \sigma_0)}{a(X_{\frac{i-1}{n}}, \sigma)} - 1 \right| \leq C(1 + |X_{\frac{i-1}{n}}|^p)/n^{1/2}.$$

This leads to

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)| \leq C(1 + |X_{\frac{i-1}{n}}|^p) \left( \frac{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)|}{n^{1/2}} + \frac{1}{n^{1/2}} \right),$$

and finally adding and subtracting  $n^{1/\alpha} \Delta L_i$

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} |z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) - z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)|$$

$$\leq C(1 + |X_{\frac{i-1}{n}}|^p) \frac{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i| + n^{1/\alpha} |\Delta L_i| + 1}{n^{1/2}}.$$

Introducing the truncation  $1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i| \leq \varepsilon\}}$  for  $\varepsilon > 0$ , we deduce that

$$\sup_{(\theta, \sigma) \in V_n^{(n)}(\theta_0, \sigma_0)} |f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)))| \leq$$

$$C_K \left( \|h'\|_\infty \frac{(\varepsilon + 1 + n^{1/\alpha} |\Delta L_i|)}{n^{1/2}} + \|h\|_\infty 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i| > \varepsilon\}} \right).$$

Observing that  $1/\alpha + 1/2 > 1$ , to prove (12) it remains to check the two convergences in probability

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n \frac{n^{1/\alpha} |\Delta L_i|}{n^{1/2}} \rightarrow 0, \quad \frac{1}{n^{1/\alpha}} \sum_{i=1}^n 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i| > \varepsilon\}} \rightarrow 0.$$

Since  $\alpha > 1$ , Theorem 2 in [18] gives  $\mathbb{E} n^{1/\alpha} |\Delta L_i| \leq C$  and we deduce  $\mathbb{E} [\frac{1}{n^{1/\alpha}} \sum_{i=1}^n \frac{n^{1/\alpha} |\Delta L_i|}{n^{1/2}}] \rightarrow 0$ .

Moreover using the result of Lemma 3, we deduce  $\mathbb{E} 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) - n^{1/\alpha} \Delta L_i| > \varepsilon\}} \leq C_K C(\varepsilon)/n^{1-\delta}$  for all  $\delta \in (0, 1)$  and this permits to obtain the second convergence. This achieves the proof of (12).

Turning to (14), since  $\mathbb{E} h(L_1^\alpha) = 0$ , we deduce from Proposition 4

$$|\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(n^{1/\alpha} \Delta L_i)| \leq C \varepsilon_n,$$

moreover from Proposition 3, we get

$$|\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - h(n^{1/\alpha} \Delta L_i))| \leq C(1 + |X_{\frac{i-1}{n}}|) \varepsilon_n,$$

with  $\sqrt{n} \varepsilon_n \rightarrow 0$ . This permits to conclude that (14) holds.

It remains to prove the uniform convergence of the martingale part (13). For any  $(\theta, \sigma) \in V_{\theta_0} \times V_{\sigma_0}$  the convergence in probability

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0))) \rightarrow 0,$$

is immediate (we check easily (5) since  $2/\alpha > 1$ ). To prove the uniform convergence we use a tightness criteria (see for example the Appendix of [10]). Denoting

$$M_n(\theta, \sigma) = \frac{1}{n^{1/\alpha}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0))),$$

it is sufficient to check, for  $(\theta_i, \sigma_i) \in V_{\theta_0} \times V_{\sigma_0}$ ,  $i = 1, 2$ ,

$$\sup_n \mathbb{E} |M_n(\theta_1, \sigma_1) - M_n(\theta_2, \sigma_2)|^4 \leq C \|(\theta_1, \sigma_1)^T - (\theta_2, \sigma_2)^T\|^4.$$

From Burkholder inequality for discrete martingale (see [21]) :

$$\mathbb{E} |M_n(\theta_1, \sigma_1) - M_n(\theta_2, \sigma_2)|^4 \leq \frac{C}{n^{4/\alpha}} \mathbb{E} \left( \sum_{i=1}^n |m_i(\theta_1, \sigma_1) - m_i(\theta_2, \sigma_2)|^2 \right)^2,$$

where  $m_i(\theta, \sigma) = f(X_{\frac{i-1}{n}}, \theta, \sigma) 1_{\{|X_{\frac{i-1}{n}}| \leq K\}} (h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)))$ , and from Cauchy-Schwarz inequality

$$\mathbb{E} |M_n(\theta_1, \sigma_1) - M_n(\theta_2, \sigma_2)|^4 \leq \frac{C}{n^{4/\alpha}} n \mathbb{E} \sum_{i=1}^n |m_i(\theta_1, \sigma_1) - m_i(\theta_2, \sigma_2)|^4.$$

This gives from a first order Taylor expansion of  $f$

$$\mathbb{E} |M_n(\theta_1, \sigma_1) - M_n(\theta_2, \sigma_2)|^4 \leq \frac{C_K}{n^{4/\alpha}} n^2 \|h\|_\infty \|(\theta_1, \sigma_1)^T - (\theta_2, \sigma_2)^T\|^4.$$

Since  $n^2/n^{4/\alpha} \rightarrow 0$ , the result is established. □

## 4.2 Central Limit Theorem

The asymptotic mixed normality of the estimators proposed in Section 3 is a consequence of the stable convergence in law of functionals of the form

$$\frac{1}{n^{1/2}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)).$$

Thanks to the control in total variation distance given in Section 5 between  $z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)$  and  $n^{1/\alpha} \Delta L_i$ , this can be reduced to the stable convergence in law for

$$\frac{1}{n^{1/2}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) h(n^{1/\alpha} \Delta L_i).$$

This is established in the next proposition.

**Proposition 2** *We assume that H1 and H2(a)-(b) hold. Let  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  be bounded functions and  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions. We assume that  $\mathbb{E}h_1(L_1^\alpha) = \mathbb{E}h_2(L_1^\alpha) = \mathbb{E}h_1(L_1^\alpha)h_2(L_1^\alpha) = 0$ . Then we have the stable convergence in law with respect to  $\sigma(L_s, s \leq 1)$ :*

$$\frac{1}{n^{1/2}} \sum_{i=1}^n \begin{pmatrix} f_1(X_{\frac{i-1}{n}}) h_1(n^{1/\alpha} \Delta L_i) \\ f_2(X_{\frac{i-1}{n}}) h_2(n^{1/\alpha} \Delta L_i) \end{pmatrix} \Longrightarrow \Sigma^{1/2} \mathcal{N},$$

where  $\mathcal{N}$  is a standard Gaussian variable independent of  $\Sigma$  and

$$\Sigma = \begin{pmatrix} \int_0^1 f_1^2(X_s) ds \mathbb{E}h_1^2(L_1^\alpha) & 0 \\ 0 & \int_0^1 f_2^2(X_s) ds \mathbb{E}h_2^2(L_1^\alpha) \end{pmatrix}.$$

From this proposition we deduce the following corollary.

**Corollary 1** *We assume that H1 and H2(a)-(b) hold. Let  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  be bounded functions with bounded derivative and  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions. We assume that  $\mathbb{E}h_1(L_1^\alpha) = \mathbb{E}h_2(L_1^\alpha) = \mathbb{E}h_1(L_1^\alpha)h_2(L_1^\alpha) = 0$ . Then we have the stable convergence in law with respect to  $\sigma(L_s, s \leq 1)$ :*

$$\frac{1}{n^{1/2}} \sum_{i=1}^n \begin{pmatrix} f_1(X_{\frac{i-1}{n}}) h_1(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) \\ f_2(X_{\frac{i-1}{n}}) h_2(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) \end{pmatrix} \Longrightarrow \Sigma^{1/2} \mathcal{N},$$

where  $\mathcal{N}$  is a standard Gaussian variable independent of  $\Sigma$  and

$$\Sigma = \begin{pmatrix} \int_0^1 f_1^2(X_s) ds \mathbb{E}h_1^2(L_1^\alpha) & 0 \\ 0 & \int_0^1 f_2^2(X_s) ds \mathbb{E}h_2^2(L_1^\alpha) \end{pmatrix}.$$

**Proof** From Proposition 2, it is sufficient to prove that for  $f : \mathbb{R} \mapsto \mathbb{R}$  a continuous function and  $h : \mathbb{R} \mapsto \mathbb{R}$  a bounded function with bounded derivative

$$\frac{1}{n^{1/2}} \sum_{i=1}^n f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) \left( h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - h(n^{1/\alpha} \Delta L_i) \right) \rightarrow 0, \quad \text{in probability.}$$

For this we check the conditions (4) and (5) with

$$\zeta_i^n = \frac{1}{n^{1/2}} f(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) \left( h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - h(n^{1/\alpha} \Delta L_i) \right).$$

From Proposition 3

$$|\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} h(n^{1/\alpha} \Delta L_i)| \leq C(1 + |X_{\frac{i-1}{n}}|) \varepsilon_n \|h\|_\infty,$$

where  $n^{1/2} \varepsilon_n \rightarrow 0$  and (4) is immediate. Turning to (5) and using that  $h$  and  $h'$  are bounded, we have for all  $\varepsilon > 0$

$$\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} |h(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - h(n^{1/\alpha} \Delta L_i)|^2 \leq C\varepsilon^2 + C \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - n^{1/\alpha} \Delta L_i| > \varepsilon\}}.$$

From Lemma 3,  $\forall \varepsilon > 0$ ,  $\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - n^{1/\alpha} \Delta L_i| > \varepsilon\}} \leq C(\varepsilon)(1 + |X_{\frac{i-1}{n}}|^p) \frac{1}{n^q}$  for  $p, q > 0$  and we deduce

$$\limsup_n \frac{1}{n} \sum_{i=1}^n f^2(X_{\frac{i-1}{n}}, \theta_0, \sigma_0) \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} 1_{\{|z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) - n^{1/\alpha} \Delta L_i| > \varepsilon\}} = 0 \quad a.s.$$

This yields

$$\limsup_n \sum_{i=1}^n \mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}} |\zeta_i^n|^2 \leq C\varepsilon^2, \quad a.s.$$

and we get (5) letting  $\varepsilon$  go to zero. □

### Proof of Proposition 2

We will prove the stable convergence in law with respect to  $\sigma(L_s, s \leq 1)$  of the process

$$\Gamma_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{[nt]} \begin{pmatrix} f_1(X_{\frac{i-1}{n}}) h_1(n^{1/\alpha} \Delta L_i) \\ f_2(X_{\frac{i-1}{n}}) h_2(n^{1/\alpha} \Delta L_i) \end{pmatrix}, \quad t \in [0, 1],$$

in  $\mathbb{D}([0, 1], \mathbb{R})$  equipped with the Skorokhod topology. To this end we introduce the processes

$$\bar{L}_t^n = \sum_{i=1}^{[nt]} \Delta L_i, \quad t \in [0, 1],$$

$$\Gamma_t^m = \frac{1}{n^{1/2}} \sum_{i=1}^{[nt]} \begin{pmatrix} h_1(n^{1/\alpha} \Delta L_i) \\ h_2(n^{1/\alpha} \Delta L_i) \end{pmatrix}, \quad t \in [0, 1].$$

The process  $(\bar{L}_t^n)_t$  converges in probability to  $(L_t)_t$  for the Skorokhod topology and according to Lemma 2.8 in [12], if  $(\bar{L}_1^n, \Gamma_1^m)$  converges in law to  $(L_1, \gamma')$  where  $\gamma'$  is a Gaussian variable independent of  $L_1$  with variance

$$\Sigma' = \begin{pmatrix} \mathbb{E}h_1^2(L_1^\alpha) & 0 \\ 0 & \mathbb{E}h_2^2(L_1^\alpha) \end{pmatrix}, \quad (15)$$

then there exists a two-dimensional standard Brownian motion  $(B_t) = (B_t^1, B_t^2)$  independent of  $(L_t)$  such that the processes  $(\bar{L}^n, \Gamma^n, \Gamma^m)$  converge in law to  $(L, \Gamma, (\Sigma')^{1/2}B)$ , where

$$\Gamma_t = \int_0^t \begin{pmatrix} f_1(X_s) & 0 \\ 0 & f_2(X_s) \end{pmatrix} (\Sigma')^{1/2} dB_s.$$

This result implies the stable convergence stated in Proposition 2.

To study the convergence in law of  $(\bar{L}_1^n, \Gamma_1^m)$ , we denote by  $\Phi_n$  the characteristic function of  $(\bar{L}_1^n, \Gamma_1^m)$ , and by  $\phi_n$  the characteristic function of the  $(L_{1/n}, \frac{1}{n^{1/2}}h_1(n^{1/\alpha}L_{1/n}), \frac{1}{n^{1/2}}h_2(n^{1/\alpha}L_{1/n}))$ . Then we have

$$\log \Phi_n = n \log \phi_n,$$

and we just have to study the asymptotic behavior of  $\phi_n$ . By definition

$$\phi_n(u, v, w) = \mathbb{E}e^{iuL_{1/n} + i\frac{v}{n^{1/2}}h_1(n^{1/\alpha}L_{1/n}) + i\frac{w}{n^{1/2}}h_2(n^{1/\alpha}L_{1/n})}.$$

A Taylor expansion of the exponential function gives :

$$\begin{aligned} e^{i\frac{v}{n^{1/2}}h_1(n^{1/\alpha}L_{1/n}) + i\frac{w}{n^{1/2}}h_2(n^{1/\alpha}L_{1/n})} &= 1 + i\frac{v}{n^{1/2}}h_1(n^{1/\alpha}L_{1/n}) + i\frac{w}{n^{1/2}}h_2(n^{1/\alpha}L_{1/n}) \\ &\quad - \frac{v^2}{2n}h_1^2(n^{1/\alpha}L_{1/n}) - \frac{w^2}{2n}h_2^2(n^{1/\alpha}L_{1/n}) - \frac{vw}{n}(h_1h_2)(n^{1/\alpha}L_{1/n}) + o(1/n), \end{aligned}$$

where for any  $p \geq 0$ ,  $o(1/n^p)$  is a bounded term such that  $n^p o(1/n^p) \rightarrow 0$  as  $n$  goes to infinity.

Consequently we obtain

$$\begin{aligned} \phi_n(u, v, w) &= \mathbb{E}e^{iuL_{1/n}} + i\frac{v}{n^{1/2}}\mathbb{E}e^{iuL_{1/n}}h_1(n^{1/\alpha}L_{1/n}) + i\frac{w}{n^{1/2}}\mathbb{E}e^{iuL_{1/n}}h_2(n^{1/\alpha}L_{1/n}) \\ &\quad - \frac{v^2}{2n}\mathbb{E}e^{iuL_{1/n}}h_1^2(n^{1/\alpha}L_{1/n}) - \frac{w^2}{2n}\mathbb{E}e^{iuL_{1/n}}h_2^2(n^{1/\alpha}L_{1/n}) - \frac{vw}{n}\mathbb{E}e^{iuL_{1/n}}(h_1h_2)(n^{1/\alpha}L_{1/n}) + o(1/n). \end{aligned}$$

From this expansion, we have to study the convergence of



1.  $\mathbb{E}e^{iuL_{1/n}}$ ,
2.  $\mathbb{E}e^{iuL_{1/n}}h(n^{1/\alpha}L_{1/n})$  for  $h$  a bounded function such that  $\mathbb{E}h(L_1^\alpha) = 0$ ,
3.  $\mathbb{E}e^{iuL_{1/n}}h^2(n^{1/\alpha}L_{1/n})$  for  $h$  a bounded function,
4.  $\mathbb{E}e^{iuL_{1/n}}(h_1h_2)(n^{1/\alpha}L_{1/n})$  for  $h_1, h_2$  bounded and such that  $\mathbb{E}(h_1h_2)(L_1^\alpha) = 0$

The terms 1., 3., and 4. are easy to study and only 2. requires some more work.

Term 1. We have  $\mathbb{E}e^{iuL_{1/n}} = 1 + \psi(u)/n + o(1/n)$ , where  $\psi$  is the Lévy-Khintchine exponent of  $L_1$ .

Term 3. We deduce from Proposition 4 that  $\mathbb{E}e^{iuL_{1/n}}h^2(n^{1/\alpha}L_{1/n}) = \mathbb{E}e^{iuL_1^\alpha/n^{1/\alpha}}h^2(L_1^\alpha) + o(1/n^{1/2})$ .

Moreover from dominated convergence Theorem, we have

$$\mathbb{E}e^{iuL_1^\alpha/n^{1/\alpha}}h^2(L_1^\alpha) = \mathbb{E}h^2(L_1^\alpha) + o(1),$$

we conclude that

$$\mathbb{E}e^{iuL_{1/n}}h^2(n^{1/\alpha}L_{1/n}) = \mathbb{E}h^2(L_1^\alpha) + o(1).$$

Term 4. In the same way, we have

$$\mathbb{E}e^{iuL_{1/n}}(h_1h_2)(n^{1/\alpha}L_{1/n}) = o(1).$$

Term 2. Proposition 4 yields

$$\mathbb{E}e^{iuL_{1/n}}h(n^{1/\alpha}L_{1/n}) = \mathbb{E}e^{iuL_1^\alpha/n^{1/\alpha}}h(L_1^\alpha) + o(1/n^{1/2}).$$

Now we observe that  $\mathbb{E}e^{iuL_1^\alpha/n^{1/\alpha}}h(L_1^\alpha) = \mathbb{E}(e^{iuL_1^\alpha/n^{1/\alpha}} - 1)h(L_1^\alpha)$ . To control this term, we consider separately the cases  $\alpha > 1$  and  $\alpha \leq 1$ .

- $\alpha > 1$ . Since  $\mathbb{E}|L_1^\alpha| < \infty$  and  $h$  is bounded, we immediately obtain

$$|\mathbb{E}(e^{iuL_1^\alpha/n^{1/\alpha}} - 1)h(L_1^\alpha)| \leq C|u|/n^{1/\alpha} = o(1/n^{1/2}).$$

- $\alpha \leq 1$ . From the Lévy-Itô representation, we have

$$L_1^\alpha = L_1^{\alpha,1} + L_1^{\alpha,2},$$

where

$$L_1^{\alpha,1} = \int_0^1 \int_{\mathbb{R}^*} z 1_{\{|z| \leq 1\}} \tilde{\mu}^\alpha(dt, dz), \quad L_1^{\alpha,2} = \int_0^1 \int_{\mathbb{R}^*} z 1_{\{|z| > 1\}} \mu^\alpha(dt, dz),$$

$\mu^\alpha$  is a Poisson random measure with compensator  $dt \frac{1}{|z|^{\alpha+1}} dz$  and  $\tilde{\mu}^\alpha$  is the compensated measure.

With these notations, we set

$$A_n = \{\mu^\alpha([0, 1] \times \{|z| > n^{1/\alpha}\}) = 0\}.$$

Since  $\mu^\alpha([0, 1] \times \{|z| > n^{1/\alpha}\})$  has a Poisson distribution with parameter bounded by  $C/n$ , we deduce that

$$\mathbb{P}(A_n^c) \leq C/n.$$

Using the truncation  $1_{A_n}$ , this permits to get the bound

$$|\mathbb{E}(e^{iuL_1^\alpha/n^{1/\alpha}} - 1)h(L_1^\alpha)| \leq C\mathbb{E}|e^{iuL_1^\alpha/n^{1/\alpha}} - 1|1_{A_n} + C/n \leq C\mathbb{E}(|L_1^\alpha|1_{A_n})/n^{1/\alpha} + C/n.$$

Obviously we have  $\mathbb{E}|L_1^{\alpha,1}| \leq C$  and for the big jumps component we get

$$\mathbb{E}|L_1^{\alpha,2}|1_{A_n} \leq \int_{\{1 < |z| \leq n^{1/\alpha}\}} \frac{|z|}{|z|^{\alpha+1}} dz \leq \begin{cases} C \log n, & \text{if } \alpha = 1 \\ Cn^{1/\alpha}/n & \text{if } \alpha < 1. \end{cases}$$

This leads to

$$|\mathbb{E}(e^{iuL_1^\alpha/n^{1/\alpha}} - 1)h(L_1^\alpha)| \leq \begin{cases} C \log n/n, & \text{if } \alpha = 1 \\ C/n & \text{if } \alpha < 1. \end{cases}$$

In both cases, we conclude for the term 2. that

$$|\mathbb{E}e^{iuL_{1/n}}h(n^{1/\alpha}L_{1/n})| \leq o(1/n^{1/2}).$$

Putting all these results together, we finally obtain the convergence

$$\log \Phi_n(u, v, w) = n \log \phi_n(u, v, w) \rightarrow \psi(u) - \frac{v^2}{2}\mathbb{E}h_1^2(L_1^\alpha) - \frac{w^2}{2}\mathbb{E}h_2^2(L_1^\alpha),$$

and we get the convergence in law of the vector  $(\bar{L}_1^n, \Gamma_1^n)$  to  $(L_1, \gamma')$  where  $\gamma'$  is a Gaussian variable independent of  $L_1$  with variance  $\Sigma'$  defined by (15).

This achieves the proof of Proposition 2. □

### 4.3 Proof of Theorem 1

We will check the conditions **C1** and **C2**. To this end, we first recall that

$$u_n J_n((\theta_1, \sigma_1), (\theta_2, \sigma_2)) u_n = \sum_{i=1}^n \begin{pmatrix} \frac{n}{n^{2/\alpha}} \partial_\theta g^1(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_1, \sigma_1) & \frac{1}{n^{1/\alpha}} \partial_\sigma g^1(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_1, \sigma_1) \\ \frac{1}{n^{1/\alpha}} \partial_\theta g^2(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_2, \sigma_2) & \frac{1}{n} \partial_\sigma g^2(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_2, \sigma_2) \end{pmatrix}.$$

We compute explicitly the partial derivatives appearing in this matrix. Observing that  $\partial_\theta z_n(x, y, \theta, \sigma) = -\frac{n^{1/\alpha}}{a(x, \sigma)} \dot{\xi}_{1/n}^x(\theta)$  and  $\partial_\sigma z_n(x, y, \theta, \sigma) = -\frac{\dot{a}(x, \sigma)}{a(x, \sigma)} z_n(x, y, \theta, \sigma)$  we get

$$\begin{aligned}\partial_\theta g^1(x, y, \theta, \sigma) &= \frac{n^{1/\alpha}}{n} \frac{\ddot{b}(x, \theta)}{a(x, \sigma)} \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) - \frac{n^{2/\alpha}}{n} \frac{\dot{b}(x, \theta)}{a(x, \sigma)^2} \dot{\xi}_{1/n}^x(\theta) \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)'(z_n(x, y, \theta, \sigma)), \\ \partial_\sigma g^1(x, y, \theta, \sigma) &= -\frac{n^{1/\alpha}}{n} \frac{\dot{a}(x, \sigma)}{a(x, \sigma)^2} \dot{b}(x, \theta) \left[ \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) + z_n(x, y, \theta, \sigma) \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)'(z_n(x, y, \theta, \sigma)) \right], \\ \partial_\theta g^2(x, y, \theta, \sigma) &= -n^{1/\alpha} \frac{\dot{a}(x, \sigma)}{a(x, \sigma)^2} \dot{\xi}_{1/n}^x(\theta) \left[ \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) + z_n(x, y, \theta, \sigma) \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)'(z_n(x, y, \theta, \sigma)) \right], \\ \partial_\sigma g^2(x, y, \theta, \sigma) &= \left( \frac{\ddot{a}a - \dot{a}^2}{a^2} \right) (x, \sigma) \left[ 1 + z_n(x, y, \theta, \sigma) \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) \right] \\ &\quad - \left( \frac{\dot{a}^2}{a^2} \right) (x, \sigma) z_n(x, y, \theta, \sigma) \left[ \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(x, y, \theta, \sigma)) + z_n(x, y, \theta, \sigma) \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)'(z_n(x, y, \theta, \sigma)) \right].\end{aligned}$$

### Proof of C1.

We will use intensively the result of Proposition 1.

We first remark that the condition **C1** reduces to the uniform convergence of  $u_n J_n((\theta, \sigma), (\theta, \sigma)) u_n$ . In the sequel we note  $h_\alpha = \frac{\varphi'_\alpha}{\varphi_\alpha}$  and  $g_\alpha(z) = h_\alpha(z) + zh'_\alpha(z)$ . The functions  $h_\alpha$  and  $g_\alpha$  are bounded with bounded derivative (see [2]). We also set :

$$u_n J_n((\theta, \sigma), (\theta, \sigma)) u_n = \begin{pmatrix} I_n^{1,1} & I_n^{1,2} \\ I_n^{2,1} & I_n^{2,2} \end{pmatrix}.$$

With these notations we have using the above calculus

$$\begin{aligned}I_n^{1,1} &= \frac{1}{n^{1/\alpha}} \sum_{i=1}^n \frac{\ddot{b}(X_{\frac{i-1}{n}}, \theta)}{a(X_{\frac{i-1}{n}}, \sigma)} h_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) - \sum_{i=1}^n \frac{\dot{b}(X_{\frac{i-1}{n}}, \theta)}{a(X_{\frac{i-1}{n}}, \sigma)^2} \dot{\xi}_{1/n}^{X_{\frac{i-1}{n}}}(\theta) h'_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)), \\ I_n^{1,2} &= -\frac{1}{n} \sum_{i=1}^n \frac{\dot{a}(X_{\frac{i-1}{n}}, \sigma)}{a(X_{\frac{i-1}{n}}, \sigma)^2} \dot{b}(X_{\frac{i-1}{n}}, \theta) g_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)), \\ I_n^{2,1} &= -\sum_{i=1}^n \frac{\dot{a}(X_{\frac{i-1}{n}}, \sigma)}{a(X_{\frac{i-1}{n}}, \sigma)^2} \dot{\xi}_{1/n}^{X_{\frac{i-1}{n}}}(\theta) g_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)), \\ I_n^{2,2} &= \frac{1}{n} \sum_{i=1}^n \left( \frac{\ddot{a}a - \dot{a}^2}{a^2} \right) (X_{\frac{i-1}{n}}, \sigma) \left[ 1 + z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) h_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)) \right] \\ &\quad - \frac{1}{n} \sum_{i=1}^n \left( \frac{\dot{a}^2}{a^2} \right) (X_{\frac{i-1}{n}}, \sigma) z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma) g_\alpha(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta, \sigma)).\end{aligned}$$

From H1, obviously the functions  $\dot{b}/a$ ,  $\dot{a}/a$ ,  $\ddot{b}/a$ ,  $\ddot{a}/a$  satisfy the assumptions of Proposition 1.

Since  $\mathbb{E}h_\alpha(L_1^\alpha) = 0$ , the first term in the expression of  $I_n^{1,1}$  goes to zero (Proposition 1 ii).

For the second term we observe that  $(\dot{\xi}_t^x(\theta))_t$  is solution to

$$\dot{\xi}_t^x(\theta) = \int_0^t b'(\xi_s^x(\theta), \theta) \dot{\xi}_s^x(\theta) ds + \int_0^t \dot{b}(\xi_s^x(\theta), \theta) ds,$$

and from H1 and Gronwall's Lemma we can deduce (we omit the details of this standard proof)

$$\sup_{\theta \in V_{\theta_0}} |\dot{\xi}_{1/n}^x(\theta) - \frac{1}{n} \dot{b}(x, \theta)| \leq C(1 + |x|^p)/n^2, \quad (16)$$

so from Proposition 1 i) we deduce the convergence of the second term of  $I_n^{1,1}$  to  $-\int_0^1 \frac{\dot{b}(X_s, \theta_0)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E} \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)' (L_1^\alpha)$ .

Since  $\int \varphi''_\alpha(x) dx = 0$ , we deduce  $\mathbb{E} \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)' (L_1^\alpha) = -\mathbb{E} \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)^2 (L_1^\alpha)$  and finally

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |I_n^{1,1} - \int_0^1 \frac{\dot{b}(X_s, \theta_0)^2}{a(X_s, \sigma_0)^2} ds \mathbb{E} \left( \frac{\varphi'_\alpha}{\varphi_\alpha} \right)^2 (L_1^\alpha)| \rightarrow 0.$$

Using the symmetry of the function  $\varphi_\alpha$  we have  $\mathbb{E}g_\alpha(L_1^\alpha) = 0$  and we deduce easily from Proposition 1 i) and (16) that

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |I_n^{1,2}| \rightarrow 0 \quad \text{and} \quad \sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |I_n^{2,1}| \rightarrow 0.$$

Turning to  $I_n^{2,2}$ , we have by integrating by part  $\mathbb{E}(1 + L_1^\alpha h_\alpha(L_1^\alpha)) = 0$  and consequently

$$\sup_{(\theta, \sigma) \in V_n^{(\eta)}(\theta_0, \sigma_0)} |I_n^{2,2} + \int_0^1 \left( \frac{\dot{a}^2}{a^2} \right) (X_s, \sigma_0) ds \mathbb{E} L_1^\alpha g_\alpha(L_1^\alpha)| \rightarrow 0,$$

and it remains to check  $\mathbb{E} L_1^\alpha g_\alpha(L_1^\alpha) = -\mathbb{E} \frac{(\varphi_\alpha(L_1^\alpha) + L_1^\alpha \varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2}$ . This is done by integrating by parts.

This achieves the proof of **C1**.

### Proof of C2.

We recall that

$$u_n G_n(\theta_0, \sigma_0) = \frac{1}{n^{1/2}} \sum_{i=1}^n \left( \begin{array}{c} \frac{\dot{b}(X_{\frac{i-1}{n}}, \theta_0)}{a(X_{\frac{i-1}{n}}, \sigma_0)} \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0)) \\ \frac{\dot{a}(X_{\frac{i-1}{n}}, \sigma_0)}{a(X_{\frac{i-1}{n}}, \sigma_0)} [1 + z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0) \frac{\varphi'_\alpha}{\varphi_\alpha}(z_n(X_{\frac{i-1}{n}}, X_{\frac{i}{n}}, \theta_0, \sigma_0))] \end{array} \right).$$

Applying Corollary 1 with  $f_1 = \dot{b}/a$ ,  $f_2 = \dot{a}/a$ ,  $h_1 = \varphi'_\alpha/\varphi_\alpha$ ,  $h_2(z) = 1 + zh_1(z)$  ( $h_1$  and  $h_2$  are bounded functions with bounded derivative, see for example [2]), we deduce immediately **C2**.

## 5 Total variation distance estimates

This section is the most technical part of the paper and contains some crucial estimates to derive the asymptotic properties of the estimating functions considered in the previous section. We consider here the process  $(X_t)_{t \in [0,1]}$  solution of (1) for the value  $(\theta_0, \sigma_0)$  of the parameter and to simplify the notation we omit the dependence on  $(\theta_0, \sigma_0)$  in the expressions of the functions  $a$ ,  $b$  and  $\xi^{x_0}$ .

We will prove that we can approximate  $n^{1/\alpha}(X_{\frac{1}{n}} - \xi_{\frac{1}{n}}^{x_0})$  by  $n^{1/\alpha}a(x_0)L_{\frac{1}{n}}$  and control this approximation. This is done by estimating the total variation distance between  $n^{1/\alpha}(X_{\frac{1}{n}} - \xi_{\frac{1}{n}}^{x_0})$  and  $n^{1/\alpha}a(x_0)L_{\frac{1}{n}}$ . We also give a weak rate of convergence of the rescaled Lévy process  $n^{1/\alpha}L_{1/n}$  to the  $\alpha$ -stable process  $L_1^\alpha$  which is estimated by the total variation distance between  $n^{1/\alpha}L_{1/n}$  and  $L_1^\alpha$ .

**Proposition 3** *We assume H1(a) and H2(a)-(b). There exists a constant  $C$  such that for all measurable bounded function  $h$ , we have :*

$$|\mathbb{E}h(n^{1/\alpha}(X_{\frac{1}{n}} - \xi_{\frac{1}{n}}^{x_0})) - \mathbb{E}h(n^{1/\alpha}a(x_0)L_{\frac{1}{n}})| \leq C(1 + |x_0|)\varepsilon_n \|h\|_\infty,$$

where

- if  $\alpha \leq 1$ ,  $\forall \varepsilon \in (0, 1)$ ,  $\varepsilon_n = \frac{1}{n^{1-\varepsilon}}$
- if  $\alpha > 1$ ,  $\forall \varepsilon \in (0, 1/\alpha)$ ,  $\varepsilon_n = \frac{1}{n^{1/\alpha-\varepsilon}}$ .

In particular, in both cases  $n^{1/2}\varepsilon_n \rightarrow 0$ .

**Proposition 4** *Under H2(a)-(b), we have for all bounded function  $h$  :*

$$|\mathbb{E}h(n^{1/\alpha}L_{1/n}) - \mathbb{E}h(L_1^\alpha)| \leq C\|h\|_\infty \varepsilon_n,$$

where  $\varepsilon_n$  is as in Proposition 3.

To prove these results, it is convenient to introduce an adequate truncation function and to consider a rescaled process. This is explained in the next subsections. Moreover, the proof of Propositions 3 and 4 requires some Malliavin calculus and we recall in that follows all the technical tools to make easier the understanding of the paper.

**Remark 3** *In the statement of Proposition 3, we give a control for the distance in total variation between the laws of the processes  $X$  and  $L$ , under a short time asymptotic  $1/n \rightarrow 0$ . If we assume that  $h$  admits a bounded derivative, it is possible to get some related control from the study of the strong*

error  $|n^{1/\alpha}(X_{\frac{1}{n}} - \xi_{\frac{1}{n}}^{x_0}) - n^{1/\alpha}a(x_0)L_{\frac{1}{n}}|$ . In Lemma 3 below we state an upper bound, in probability, for this error. In the case  $\alpha > 1$ , using the controls in  $L^p$ -norm given in the proof of Lemma 3, we can show

$$|\mathbb{E}h(n^{1/\alpha}(X_{\frac{1}{n}} - \xi_{\frac{1}{n}}^{x_0})) - \mathbb{E}h(n^{1/\alpha}a(x_0)L_{\frac{1}{n}})| \leq C(1 + |x_0|)(\|h\|_\infty + \|h'\|_\infty)/n^{1/\alpha-\varepsilon}.$$

Unfortunately, this proof does not work in the case  $\alpha \leq 1$  and we have not been able to give a simple proof of the above result in that case. The Malliavin calculus, especially the integration by part formula and the Malliavin weights, permit to compensate the lack of integrability of the process  $(L_t)$  and additionally to weaken the assumptions on the function  $h$ .

## 5.1 Localization and rescaling

We first introduce a truncation function in order to suppress the big jumps of  $(L_t)$ . Let  $\tau : \mathbb{R} \mapsto [0, 1]$  be a symmetric function, continuous with continuous derivative, such that  $\tau = 1$  on  $\{|z| \leq K(a)/2\}$ ,  $\tau = 0$  on  $\{|z| \geq K(a)\}$  where

$$K(a) = \frac{1}{2}(\eta \wedge \frac{1}{\|a'\|_\infty}), \quad (17)$$

for  $\eta$  defined in H2(b).

On the same probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ , we consider the Lévy process  $(L_t)_{t \in [0,1]}$  with Lévy measure  $\nu$  and the truncated Lévy process  $(L_t^\tau)_{t \in [0,1]}$  with Lévy measure  $\nu^\tau$  given by  $\nu^\tau(dz) = \frac{g(z)}{|z|^{\alpha+1}}\tau(z)1_{\mathbb{R}^*}(z)dz$ . This can be done by setting  $L_t = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}(ds, dz)$ , respectively  $L_t^\tau = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}^\tau(ds, dz)$ , where  $\tilde{\mu}$ , respectively  $\tilde{\mu}^\tau$ , are the compensated Poisson random measures associated respectively to

$$\mu(A) = \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t, z) \mu^g(dt, dz, du), \quad A \subset [0, 1] \times \mathbb{R}$$

$$\mu^\tau(A) = \int_{[0,1]} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t, z) 1_{\{u \leq \tau(z)\}} \mu^g(dt, dz, du), \quad A \subset [0, 1] \times \mathbb{R},$$

for  $\mu^g$  a Poisson random measure on  $[0, 1] \times \mathbb{R} \times [0, 1]$  with compensator  $\bar{\mu}^g(dt, dz, du) = dt \frac{g(z)}{|z|^{\alpha+1}} 1_{\mathbb{R}^*}(z) dz du$ .

By construction, the restrictions of the measures  $\mu$  and  $\mu^\tau$  to  $[0, 1/n] \times \mathbb{R}$  coincide on the event

$$\Omega_n = \{\omega \in \Omega; \mu^g([0, 1/n] \times \{z \in \mathbb{R}; |z| \geq K(a)/2\} \times [0, 1]) = 0\}. \quad (18)$$

Since  $\mu^g([0, 1/n] \times \{z \in \mathbb{R}; |z| \geq K(a)/2\} \times [0, 1])$  has a Poisson distribution with parameter

$$\lambda_n = \frac{1}{n} \int_{|z| \geq K(a)/2} g(z)/|z|^{\alpha+1} dz \leq C/n,$$

we deduce that

$$\mathbb{P}(\Omega_n^c) \leq C/n. \quad (19)$$

We consider now the truncated process solution to the equation

$$X_t^\tau = x_0 + \int_0^t b(X_s^\tau) ds + \int_0^t a(X_{s-}^\tau) dL_s^\tau, \quad t \in [0, 1]. \quad (20)$$

Obviously  $(X_t, L_t)_{t \in [0, 1/n]} = (X_t^\tau, L_t^\tau)_{t \in [0, 1/n]}$  on  $\Omega_n$  and consequently since  $\mathbb{P}(\Omega_n^c) \leq C/n$ , the result of Proposition 3 consists in proving

$$|\mathbb{E}h(n^{1/\alpha}(X_{1/n}^\tau - \xi_{1/n}^{x_0})) - \mathbb{E}h(n^{1/\alpha}a(x_0)L_{1/n}^\tau)| \leq C(1 + |x_0|)\varepsilon_n \|h\|_\infty.$$

To clarify the proofs, it will be useful to rescale the truncated process  $(X_t^\tau)_{t \in [0, 1/n]}$ . To this end we introduce an auxiliary Lévy process  $(L_t^n)_{t \in [0, 1]}$  defined possibly on an other filtered space  $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{\mathbb{P}})$  and admitting the decomposition

$$L_t^n = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}^n(dt, dz), \quad t \in [0, 1], \quad (21)$$

where  $\tilde{\mu}^n$  is a compensated Poisson random measure,  $\tilde{\mu}^n = \mu^n - \bar{\mu}^n$ , with compensator  $\bar{\mu}^n(dt, dz) = dt \frac{g(z/n^{1/\alpha})}{|z|^{\alpha+1}} \tau(z/n^{1/\alpha}) 1_{\mathbb{R}^*}(z) dz$ . By construction, the process  $(L_t^n)_{t \in [0, 1]}$  is equal in law to the rescaled truncated process  $(n^{1/\alpha}L_{t/n}^\tau)_{t \in [0, 1]}$ . We now consider the solution  $(Y_t^n)_{t \in [0, 1]}$  to the following equation

$$Y_t^n = x_0 + \frac{1}{n} \int_0^t b(Y_s^n) ds + \frac{1}{n^{1/\alpha}} \int_0^t a(Y_{s-}^n) dL_s^n, \quad t \in [0, 1], \quad (22)$$

and the solution  $(\xi_t^{n, x_0})_t$  to the ordinary differential equation

$$\xi_t^{n, x_0} = x_0 + \frac{1}{n} \int_0^t b(\xi_s^{n, x_0}) ds, \quad t \in [0, 1]. \quad (23)$$

The equality in law

$$(Y_t^n - \xi_t^{n, x_0}, L_t^n)_{t \in [0, 1]} \stackrel{\text{law}}{=} (X_{t/n}^\tau - \xi_{t/n}^{x_0}, n^{1/\alpha}L_{t/n}^\tau)_{t \in [0, 1]} \quad (24)$$

is straightforward and consequently with these notations the result of Proposition 3 follows from

$$|\mathbb{E}h(n^{1/\alpha}(Y_1^n - \xi_1^{n, x_0})) - \mathbb{E}h(a(x_0)L_1^n)| \leq C(1 + |x_0|)\varepsilon_n \|h\|_\infty.$$

It is worth to note that the jumps of  $(L_t^n)$  are bounded by  $n^{1/\alpha}K(a)$ , and then the processes  $(L_t^n)$  and  $(Y_t^n)$  admit moments of all orders. More precisely, we have the following result.

**Lemma 2** *Assuming  $H1(a)$  and  $H2(a)$ , we have*

$$\forall p \geq 1, \quad \sup_n \mathbb{E} \left( \sup_{t \in [0,1]} |Y_t^n|^p \right) \leq C_p (1 + |x_0|^p), \quad (25)$$

$$\forall p \geq 1 \text{ and } p > \alpha, \quad \mathbb{E} \sup_{t \in [0,1]} |Y_t^n - x_0|^p \leq C_p (1 + |x_0|^p) \frac{1}{n}. \quad (26)$$

**Proof** The proof of both inequalities is based on Burkholder type inequalities (see 2.1.36 and 2.1.37 in Lemma 2.1.5 of [13]) for purely discontinuous martingales and standard arguments (convexity inequality, Lipschitz assumption on the coefficients and Gronwall's lemma). We only prove (26).

Let  $p \geq 1$ , using a convexity inequality and the Lipschitz assumption on  $b$ , we have

$$\mathbb{E} \sup_{t \in [0,1]} |Y_t^n - x_0|^p \leq C_p \left( \frac{1}{n^p} (1 + \mathbb{E} \sup_{s \in [0,1]} |Y_s^n|^p) + \frac{1}{n^{p/\alpha}} \mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t a(Y_{s-}^n) dL_s^n \right|^p \right).$$

If  $p \geq 2$ , inequality 2.1.37 in [13] gives

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t a(Y_{s-}^n) dL_s^n \right|^p \leq C_p (1 + \mathbb{E} \sup_{t \in [0,1]} |Y_t^n|^p) \left( \int_0^{K(a)n^{1/\alpha}} \frac{|z|^p}{|z|^{\alpha+1}} dz + \left( \int_0^{K(a)n^{1/\alpha}} \frac{|z|^2}{|z|^{\alpha+1}} dz \right)^{p/2} \right)$$

where we used the boundedness of  $g$  and the definition of  $\tau$ . From (25), we deduce

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t a(Y_{s-}^n) dL_s^n \right|^p \leq C_p (1 + |x_0|^p) \left( \frac{n^{p/\alpha}}{n} + \frac{n^{p/\alpha}}{n^{p/2}} \right),$$

and (26) follows in that case.

Assuming now that  $p \in [1, 2)$  and  $p > \alpha$ , from inequality 2.1.36 in [13], we obtain

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t a(Y_{s-}^n) dL_s^n \right|^p \leq C_p (1 + \mathbb{E} \sup_{t \in [0,1]} |Y_t^n|^p) \int_0^{K(a)n^{1/\alpha}} \frac{|z|^p}{|z|^{\alpha+1}} dz \leq C_p (1 + |x_0|^p) \frac{n^{p/\alpha}}{n}.$$

This achieves the proof of (26). □

We end this subsection with a control of  $\sup_{t \leq 1/n} |n^{1/\alpha}(X_t - \xi_t^{x_0}(\theta_0)) - n^{1/\alpha}a(x_0)L_t|$  which can be established using both the truncation and rescaling procedure.

**Lemma 3** *Assuming  $H1(a)$  and  $H2(a)$ , there exists  $p > 0$  such that*

$$\forall \varepsilon > 0, \quad \mathbb{P} \left( \sup_{t \leq 1/n} |n^{1/\alpha}(X_t - \xi_t^{x_0}(\theta_0)) - n^{1/\alpha}a(x_0)L_t| > \varepsilon \right) \leq \begin{cases} C(\varepsilon)(1 + |x_0|^p) \frac{\log n}{n^\alpha} & \text{if } \alpha < 1, \\ C(\varepsilon)(1 + |x_0|^p) \frac{1}{n^{1-\delta}}, \quad \forall \delta \in (0, 1) & \text{if } \alpha \geq 1. \end{cases}$$

where  $C(\varepsilon)$  is a positive constant.



**Proof**

Recalling that  $(X_t, L_t)_{t \in [0, 1/n]} = (X_t^\tau, L_t^\tau)_{t \in [0, 1/n]}$  on  $\Omega_n$  and that  $\mathbb{P}(\Omega_n^c) \leq 1/n$ , it is sufficient to study the convergence in probability of  $\sup_{t \leq 1/n} |n^{1/\alpha}(X_t^\tau - \xi_t^{x_0}(\theta_0)) - n^{1/\alpha}a(x_0)L_t^\tau|$ . Now using the rescaled process  $(L_t^n)$  this is equivalent to study the convergence in probability of  $\sup_{t \in [0, 1]} |n^{1/\alpha}(Y_t^n - \xi_t^{n, x_0}) - a(x_0)L_t^n|$ . We have the inequality for  $t \in [0, 1]$

$$\begin{aligned} \sup_{s \leq t} |n^{1/\alpha}(Y_s^n - \xi_s^{n, x_0}) - a(x_0)L_s^n| &\leq \frac{\|b'\|_\infty}{n} \int_0^t \sup_{u \leq s} |n^{1/\alpha}(Y_u^n - \xi_u^{n, x_0}) - a(x_0)L_u^n| ds \\ &\quad + \frac{\|b'\|_\infty |a(x_0)|}{n} \sup_{t \in [0, 1]} |L_t^n| + \sup_{t \in [0, 1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^n \right|, \end{aligned}$$

and from Gronwall's inequality we deduce

$$\sup_{t \in [0, 1]} |n^{1/\alpha}(Y_t^n - \xi_t^{n, x_0}) - a(x_0)L_t^n| \leq C \left( \frac{1}{n} \sup_{t \in [0, 1]} |L_t^n| + \sup_{t \in [0, 1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^n \right| \right). \quad (27)$$

In that follows, we will distinguish between the small jumps and the big jumps of  $L_t^n$ . To this end we have

$$L_t^n = \int_0^t \int_{\{0 < |z| \leq 1\}} z \tilde{\mu}^n(dt, dz) + \int_0^t \int_{\{|z| > 1\}} z \tilde{\mu}^n(dt, dz) := L_t^{n,1} + L_t^{n,2}. \quad (28)$$

Control of the small jumps part  $L^{n,1}$

Since  $\mathbb{E} \sup_{t \in [0, 1]} |L_t^{n,1}|^2 \leq C$  we deduce  $\mathbb{P}(\frac{1}{n} \sup_{t \in [0, 1]} |L_t^{n,1}| > \varepsilon) \leq C/(n^2 \varepsilon^2)$ . Turning to the other term  $\sup_{t \in [0, 1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right|$ , we have using Lemma 2

$$\mathbb{E} \left( \sup_{t \in [0, 1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right|^2 \right) \leq C \mathbb{E} \left( \sup_{t \in [0, 1]} |Y_t^n - x_0|^2 \right) \leq C(1 + |x_0|^2)/n,$$

and we get  $\mathbb{P}(\sup_{t \in [0, 1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right| > \varepsilon) \leq C(\varepsilon)(1 + |x_0|^2)/n$ .

Control of the big jumps part  $L^{n,2}$

We distinguish between the cases  $\alpha \geq 1$  and  $\alpha < 1$ .

- $\alpha \geq 1$ .

Using inequality 2.1.36 in [13] with  $\alpha < p < 2$ , the boundedness of  $g$  and the definition of  $\tau$ , we obtain :

$$\mathbb{E} \left( \sup_{t \in [0, 1]} |L_t^{n,2}|^p \right) \leq C \int_{\{1 < |z| \leq K(a)n^{1/\alpha}\}} \frac{|z|^p}{|z|^{\alpha+1}} dz \leq Cn^{p/\alpha}/n,$$

and then from Markov inequality

$$\mathbb{P} \left( \frac{1}{n} \sup_{t \in [0, 1]} |L_t^{n,2}| > \varepsilon \right) \leq C(\varepsilon)n^{p/\alpha}/n^{p+1} \leq C(\varepsilon)/n.$$

Similarly we get using Lemma 2

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{2,n} \right|^p \leq C \mathbb{E} \left( \sup_{t \in [0,1]} |Y_t^n - x_0|^p \right) n^{p/\alpha} / n \leq C(1 + |x_0|^p) \frac{n^{p/\alpha}}{n^2},$$

this gives choosing  $p = \alpha(1 + \delta)$

$$\mathbb{P} \left( \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{2,n} \right| > \varepsilon \right) \leq C(\varepsilon)(1 + |x_0|^p) / n^{1-\delta}.$$

- $\alpha < 1$ .

Thanks to the symmetry of the compensator  $\bar{\mu}^n$ , we have  $L_t^{n,2} = \int_0^t \int_{|z|>1} z \mu^n(ds, dz)$ . Then we can write

$$|L_t^{n,2}|^\alpha = \left| \int_0^t \int_{\{|z|>1\}} z \mu^n(dt, dz) \right|^\alpha \leq \int_0^1 \int_{\{|z|>1\}} |z|^\alpha \mu^n(dt, dz),$$

consequently  $\mathbb{E}(\sup_{t \in [0,1]} |L_t^{n,2}|^\alpha) \leq C \log n$ , and then

$$\mathbb{P} \left( \frac{1}{n} \sup_{t \in [0,1]} |L_t^{n,2}| > \varepsilon \right) \leq C(\varepsilon) \frac{\log n}{n^\alpha}.$$

Similarly, we have

$$\left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,2} \right|^\alpha \leq \int_0^1 \int_{\{|z|>1\}} |a(Y_{s-}^n) - a(x_0)|^\alpha |z|^\alpha \mu^n(ds, dz).$$

Taking the expectation and using the lipschitz assumption on  $a$ , this yields

$$\mathbb{E} \left( \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,2} \right|^\alpha \right) \leq C \mathbb{E} \left( \sup_{t \in [0,1]} |Y_t^n - x_0|^\alpha \right) \log n \leq C \left( \mathbb{E} \sup_{t \in [0,1]} |Y_t^n - x_0| \right)^\alpha \log n,$$

where for the second inequality we used Hölder's inequality with  $p' = 1/\alpha > 1$ . From Lemma 2, we deduce

$$\mathbb{E} \left( \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,2} \right|^p \right) \leq C(1 + |x_0|^p) \log n / n^\alpha.$$

and we obtain  $\mathbb{P}(\sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^n \right| > \varepsilon) \leq C(\varepsilon)(1 + |x_0|^p) \log n / n^\alpha$ .

Putting all these results together, Lemma 3 is proved. □

## 5.2 Malliavin Calculus

In this section, we recall some results on Malliavin calculus for jump processes. We refer to [6] for a complete presentation and to [7] for the adaptation to our framework. We will work on the Poisson

space associated to the measure  $\mu^n$  defining the process  $(L_t^n)$  of Section 5.1, assuming that  $n$  is fixed. By construction, the support of  $\mu^n$  is contained in  $[0, 1] \times E_n$ , where

$$E_n = \{z \in \mathbb{R}; |z| < K(a)n^{1/\alpha}\}, \quad (29)$$

and  $K(a)$  is defined by (17). We recall that the measure  $\mu^n$  has compensator

$$\bar{\mu}^n(dt, dz) = dt \frac{g(z/n^{1/\alpha})}{|z|^{\alpha+1}} \tau(z/n^{1/\alpha}) 1_{\mathbb{R}^*} dz := dt F_n(z) dz. \quad (30)$$

In this section we assume that the truncation function  $\tau$  satisfies the additional assumption

$$\int_{\mathbb{R}} \left| \frac{\tau'(z)}{\tau(z)} \right|^p \tau(z) dz < \infty, \quad \forall p \geq 1. \quad (31)$$

We define the Malliavin operators  $L$  and  $\Gamma$  (we omit the dependence in  $n$ ) and their basic properties (see Bichteler, Gravereaux, Jacod, [6] Chapter IV, sections 8-9-10). For a test function  $f : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$  ( $f$  is measurable,  $\mathcal{C}^2$  with respect to the second variable, with bounded derivatives, and  $f \in \cap_{p \geq 1} \mathbf{L}^p(F_n(z) dz)$ ), we set  $\mu^n(f) = \int_0^1 \int_{\mathbb{R}} f(t, z) \mu^n(dt, dz)$ . As auxiliary function, we consider  $\rho : \mathbb{R} \mapsto [0, \infty)$  such that  $\rho$  is symmetric, two times differentiable and such that  $\rho(z) = z^4$  if  $z \in [0, 1/2]$  and  $\rho(z) = z^2$  if  $z \geq 1$ . Note that thanks to the truncation  $\tau$ , we do not need that  $\rho$  vanishes at infinity. Assuming H2(b), we check that  $\rho, \rho'$  and  $\rho \frac{F'_n}{F_n}$  belong to  $\cap_{p \geq 1} \mathbf{L}^p(F_n(z) dz)$ . With these notations, we define the Malliavin operator  $L$ , on a simple functional  $\mu^n(f)$  as follows

$$L(\mu^n(f)) = \frac{1}{2} \mu^n \left( \rho' f' + \rho \frac{F'_n}{F_n} f' + \rho f'' \right), \quad (32)$$

where  $f'$  and  $f''$  are the derivatives with respect to the second variable. This definition permits to construct a linear operator on a space  $D \subset \cap_{p \geq 1} \mathbf{L}^p$  which is self-adjoint :

$$\forall \Phi, \Psi \in D, \quad \mathbb{E} \Phi L \Psi = \mathbb{E} L \Phi \Psi.$$

We associate to  $L$ , the symmetric bilinear operator  $\Gamma$  :

$$\Gamma(\Phi, \Psi) = L(\Phi \Psi) - \Phi L \Psi - \Psi L \Phi. \quad (33)$$

If  $f$  and  $h$  are two test functions, we have :

$$\Gamma(\mu^n(f), \mu^n(h)) = \mu^n(\rho f' h'), \quad (34)$$

The operators  $L$  and  $\Gamma$  satisfy the chain rule property :

$$L F(\Phi) = F'(\Phi) L \Phi + \frac{1}{2} F''(\Phi) \Gamma(\Phi, \Phi), \quad (35)$$

$$\Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi). \quad (36)$$

These operators permit to establish the following integration by parts formula (see [6] Proposition 8-10 p.103).

**Proposition 5** *Let  $\Phi$  and  $\Psi$  be random variables in  $D$ , and  $f$  be a bounded function with bounded derivatives up to order two. If  $\Gamma(\Phi, \Phi)$  is invertible and  $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p \geq 1} \mathbf{L}^p$ , we have*

$$\mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi), \quad (37)$$

with

$$\mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)). \quad (38)$$

In the next section, we will apply this Malliavin calculus to the random variables  $L_1^n$  and  $Y_1^n$ , which belong to the domain of the operators  $L$  and  $\Gamma$ . From the preceding definitions, we compute easily  $L(L_1^n)$ ,  $\Gamma(L_1^n, L_1^n)$  and  $\mathcal{H}_{L_1^n}(1)$ . We have

$$\begin{aligned} L(L_1^n) &= \frac{1}{2}\mu^n \left( \rho' + \rho \frac{F'_n}{F_n} \right), \\ \Gamma(L_1^n, L_1^n) &= \mu^n(\rho), \\ \mathcal{H}_{L_1^n}(1) &= \frac{\Gamma(L_1^n, \Gamma(L_1^n, L_1^n))}{\Gamma(L_1^n, L_1^n)2} - 2 \frac{L(L_1^n)}{\Gamma(L_1^n, L_1^n)} = \frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2} - \frac{\mu^n(\rho' + \rho \frac{F'_n}{F_n})}{\mu^n(\rho)}. \end{aligned} \quad (39)$$

This leads to the expression

$$\mathcal{H}_{L_1^n}(1) = \frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2} - \frac{\mu^n(\rho' - (\alpha + 1)\frac{1}{z}\rho)}{\mu^n(\rho)} + \frac{1}{n^{1/\alpha}}R_n, \quad (40)$$

where  $R_n$  is given by

$$R_n = \frac{\mu^n(\rho \frac{g_n^1}{g_n})}{\mu^n(\rho)} + \frac{\mu^n(\rho \frac{\tau_n^1}{\tau_n})}{\mu^n(\rho)} := R_n^1 + R_n^2,$$

with the additional notations  $g_n(z) = g(z/n^{1/\alpha})$ ,  $g_n^1(z) = g'(z/n^{1/\alpha})$ ,  $\tau_n(z) = \tau(z/n^{1/\alpha})$ ,  $\tau_n^1(z) = \tau'(z/n^{1/\alpha})$ .

From the choice of  $\rho$  we can prove that

$$\mathbb{E} \frac{1}{\mu^n(\rho)^p} \leq C_p, \quad \forall p \geq 1. \quad (41)$$

This is obtain remarking that (see [7] p.2324 )

$$\mathbb{E} \frac{1}{\mu^n(\rho)^p} \leq \mathbb{E} \frac{1}{\mu^n(\rho \mathbf{1}_{\{|z| \leq 1/2\}})^p} = C_p \int_0^\infty u^{p-1} \mathbb{E}(e^{-u\mu^n(\rho \mathbf{1}_{\{|z| \leq 1/2\}})}) du.$$

From the classical exponential formula for Poisson measures, we have

$$\mathbb{E}e^{-u\mu^n(\rho 1_{\{|z|\leq 1/2\}})} = e^{-\int_{\{|z|\leq 1/2\}}(1-e^{-u\rho(z)})\frac{g_n(z)}{|z|^{\alpha+1}}\tau_n(z)dz} \leq e^{-\int_{\{|z|\leq 1/2\}}(1-e^{-uz^4})\frac{C}{|z|^{\alpha+1}}dz},$$

where we used that  $g$  is lower bounded by  $C > 0$  in a neighborhood of zero (recall that  $g$  is continuous and  $g(0) = 1$ ),  $\tau = 1$  and  $\rho(z) = z^4$  on  $\{|z| \leq 1/2\}$ . We conclude observing that  $\liminf_{u \rightarrow \infty} \frac{1}{\ln u} \int_{\{|z|\leq 1/2\}} 1_{\{z^4 \geq 1/u\}} \frac{C}{|z|^{\alpha+1}} dz = +\infty$ .

Moreover on  $E_n$  we observe that  $|z|/n^{1/\alpha} \leq \eta$  and then assuming H2(b) we have the bound

$$|R_n^1| \leq C. \quad (42)$$

Turning to  $R_n^2$ , and using the definition of  $\tau$  we have

$$|R_n^2| \leq \frac{\mu_n(\rho|\frac{\tau_n^1}{\tau_n}|1_{\{K(a)n^{1/\alpha}/2 < |z| < K(a)n^{1/\alpha}\}})}{\mu_n(\rho 1_{\{K(a)n^{1/\alpha}/2 < |z| < K(a)n^{1/\alpha}\}})} \leq \mu_n(|\frac{\tau_n^1}{\tau_n}|1_{\{K(a)n^{1/\alpha}/2 < |z| < K(a)n^{1/\alpha}\}}).$$

Since  $\mu_n = \tilde{\mu}_n + \bar{\mu}_n$ , we deduce from inequalities 2.1.36 and 2.1.37 in [13], the change of variable  $u = z/n^{1/\alpha}$  and assumption (31) that

$$\mathbb{E}|R_n^2|^p \leq C_p/n, \quad \forall p \geq 1. \quad (43)$$

This permits to deduce the following useful inequalities.

**Lemma 4** *We have*

$$\sup_n \mathbb{E}|\mathcal{H}_{L_1^n}(1)|^p \leq C_p, \quad \forall p \geq 1, \quad (44)$$

$$\sup_n \mathbb{E} \int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \mathcal{H}_{L_1^n}(1)^p \leq C_p, \quad \forall p \geq 1, \quad (45)$$

$$\sup_n \mathbb{E} \sup_{t \in [0,1]} |L_t^n| |\mathcal{H}_{L_1^n}(1)| \leq C. \quad (46)$$

**Proof** Obviously (46) is a consequence of (44) and (45). From (42) and (43), to prove (44) we just have to consider the first two terms in the right-hand side of (40). Distinguishing between the small jumps and the big jumps of the Poisson measure we have for the first term

$$\frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2} \leq \frac{\mu^n(\rho\rho'1_{\{|z|<1\}})}{\mu^n(\rho)^2} + \frac{\mu^n(\rho\rho'1_{\{|z|\geq 1\}})}{\mu^n(\rho 1_{\{|z|\geq 1\}})^2}.$$

We conclude immediately using (41) that  $\forall p \geq 1$ ,  $\sup_n \mathbb{E}|\frac{\mu^n(\rho\rho'1_{\{|z|<1\}})}{\mu^n(\rho)^2}|^p \leq C_p$ . Moreover, recalling that  $\rho(z) = z^2$  for  $|z| \geq 1$  we deduce that  $\frac{\mu^n(\rho\rho'1_{\{|z|\geq 1\}})}{\mu^n(\rho 1_{\{|z|\geq 1\}})^2} \leq 2$  and this yields  $\forall p \geq 1$ ,  $\sup_n \mathbb{E}|\frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2}|^p \leq C_p$ .

We proceed similarly for  $\frac{\mu^n(\rho' - (\alpha+1)\frac{1}{z}\rho)}{\mu^n(\rho)}$  and this achieves the proof of (44).

It remains to prove (45). We check immediately from (42) and (43) that

$$\mathbb{E} \left| \int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \frac{1}{n^{1/\alpha}} R_n \right|^p \leq C_p.$$

Turning to  $\int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2}$ , from the Cauchy-Schwarz inequality (and using  $\rho(z) = z^2$  if  $|z| \geq 1$ ) we get  $\int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \leq \mu_n(1_{\{|z|\geq 1\}})^{1/2} \mu_n(\rho 1_{\{|z|\geq 1\}})^{1/2}$  and we deduce the bound

$$\int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \frac{\mu^n(|\rho\rho'|)}{\mu^n(\rho)^2} \leq \mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho\rho'|)}{\mu^n(\rho)^{3/2}}.$$

Remarking that  $\mu_n(1_{\{|z|\geq 1\}})$  has a Poisson distribution with some parameter  $\lambda_\alpha^n$  bounded by  $\lambda_\alpha$  independent of  $n$  we get that

$$\sup_n \mathbb{E} \left| \mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho\rho'| 1_{\{|z|<1\}})}{\mu^n(\rho)^{3/2}} \right|^p \leq C_p, \quad \forall p \geq 1.$$

Considering the large jumps part, we have

$$\mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho\rho'| 1_{\{|z|\geq 1\}})}{\mu^n(\rho)^{3/2}} \leq \mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho\rho'| 1_{\{|z|\geq 1\}})}{\mu^n(\rho 1_{\{|z|\geq 1\}})^{3/2}} \leq 2\mu_n(1_{\{|z|\geq 1\}})^{1/2},$$

and this permits to conclude that

$$\sup_n \mathbb{E} \left| \int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \frac{\mu^n(\rho\rho')}{\mu^n(\rho)^2} \right|^p \leq C_p, \quad \forall p \geq 1.$$

In the same way, we have for the last term

$$\int_0^1 \int_{\{|z|>1\}} |z| \mu^n(ds, dz) \left| \frac{\mu^n(\rho' - (\alpha + 1)\frac{1}{z}\rho)}{\mu^n(\rho)} \right| \leq C \mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho' + \rho/z|)}{\mu^n(\rho)^{1/2}}.$$

We conclude as previously remarking that for the large jumps part we have, using once again the Cauchy-Schwarz inequality,

$$\mu_n(1_{\{|z|\geq 1\}})^{1/2} \frac{\mu^n(|\rho' + \rho/z| 1_{\{|z|\geq 1\}})}{\mu^n(\rho)^{1/2}} \leq 3\mu_n(1_{\{|z|\geq 1\}}).$$

This ends the proof of (45). □

With this background, we can proceed to the proof of Propositions 3 and 4.

### 5.3 Proof of Proposition 3

From the the localization and rescaling procedure, we just have to prove

$$|\mathbb{E}h(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0})) - \mathbb{E}h(a(x_0)L_1^n)| \leq C(1 + |x_0|)\varepsilon_n \|h\|_\infty.$$

Now, considering a regularizing sequence  $(h_p)$  converging to  $h$  in  $L^1$ -norm, such that  $\forall p$ ,  $h_p$  admits a bounded derivative and  $\|h_p\|_\infty \leq \|h\|_\infty$ , we may assume that  $h$  admits a bounded derivative.

Using the integration by part formula (37) and denoting by  $H$  any primitive function of  $h$

$$\mathbb{E}h(a(x_0)L_1^n) = \mathbb{E}H(a(x_0)L_1^n)\mathcal{H}_{a(x_0)L_1^n}(1),$$

and then from the triangle inequality, we have to bound the two following terms :

$$T_1 := |\mathbb{E}h(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}) - \mathbb{E}H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}))\mathcal{H}_{a(x_0)L_1^n}(1)| \quad (47)$$

$$T_2 := |\mathbb{E}H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}))\mathcal{H}_{a(x_0)L_1^n}(1) - \mathbb{E}H(a(x_0)L_1^n)\mathcal{H}_{a(x_0)L_1^n}(1)|. \quad (48)$$

### Bound for $T_2$

We have

$$T_2 \leq \|h\|_\infty \mathbb{E}|n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}) - a(x_0)L_1^n| |\mathcal{H}_{a(x_0)L_1^n}(1)|,$$

and from (27) we get

$$T_2 \leq \|h\|_\infty \frac{1}{|a(x_0)|} \left( \frac{1}{n} \mathbb{E} \sup_{t \in [0,1]} |L_t^n| |\mathcal{H}_{L_1^n}(1)| + \mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^n \right| |\mathcal{H}_{L_1^n}(1)| \right),$$

where as a consequence of the linearity property of the operators  $\Gamma$  and  $L$ , we have used  $\mathcal{H}_{a(x_0)L_1^n}(1) = \frac{1}{a(x_0)} \mathcal{H}_{L_1^n}(1)$ .

From (46), the first term in the right-hand side of the above inequality is bound by  $C/n$ .

Turning to the second term  $\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^n \right| |\mathcal{H}_{L_1^n}(1)|$ , we use once again the decomposition (28). For the small jumps part, Hölder's inequality and inequality 2.1.36 in [13] with  $1 < p < 2$ ,  $p > \alpha$  and  $q$  such that  $1/p + 1/q = 1$ , lead to

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right| |\mathcal{H}_{L_1^n}(1)| \leq C_p \mathbb{E} \sup_{t \in [0,1]} |a(Y_{t-}^n) - a(x_0)|^p \left( \mathbb{E} |\mathcal{H}_{L_1^n}(1)|^q \right)^{1/q}.$$

From the Lipschitz assumption on  $a$ , the result of Lemma 2 and (44), we conclude

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right| |\mathcal{H}_{L_1^n}(1)| \leq C_p (1 + |x_0|) \frac{1}{n^{1/p}}.$$

This gives the bound

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right| |\mathcal{H}_{L_1^n}(1)| \leq C_p (1 + |x_0|) \varepsilon_n,$$

with  $\varepsilon_n$  given as in Proposition 3.

For the big jumps part, we remark first that

$$\int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,2} = \int_0^t \int_{\{|z|>1\}} (a(Y_{s-}^n) - a(x_0)) z \mu^n(dt, dz),$$

and then

$$\mathbb{E} \sup_{t \in [0,1]} \left| \int_0^t (a(Y_{s-}^n) - a(x_0)) dL_s^{n,1} \right| \mathcal{H}_{L_1^n}(1) \leq \mathbb{E} \left[ \sup_{t \in [0,1]} |a(Y_{t-}^n) - a(x_0)| \int_0^t \int_{\{|z|>1\}} |z| \mu^n(dt, dz) \mathcal{H}_{L_1^n}(1) \right].$$

As for the small jumps part, we conclude applying successively Hölder's inequality, Lemma 2 and (45) and this shows finally

$$T_2 \leq C(1 + |x_0|)\varepsilon_n.$$

### Bound for $T_1$

It remains to consider (47).

From (38) and (33), we remark that

$$\mathcal{H}_{a(x_0)L_1^n}(1) = L \left( \frac{1}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} \right) a(x_0)L_1^n - \frac{L(a(x_0)L_1^n)}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} - L \left( \frac{a(x_0)L_1^n}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} \right),$$

and using the self-adjoint property of the operator  $L$  for the first and third terms we obtain

$$\begin{aligned} & \mathbb{E} H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0})) \mathcal{H}_{a(x_0)L_1^n}(1) = \\ & \mathbb{E} \left( \frac{L(H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}))a(x_0)L_1^n) - H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}))L(a(x_0)L_1^n) - L(H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0})))a(x_0)L_1^n}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} \right). \end{aligned}$$

Using again (33) and the chain rule we get

$$\mathbb{E} H(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0})) \mathcal{H}_{a(x_0)L_1^n}(1) = \mathbb{E} h(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0})) \frac{\Gamma(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}), a(x_0)L_1^n)}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)}$$

and so

$$T_1 \leq \|h\|_\infty \mathbb{E} \left| \frac{\Gamma(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}), a(x_0)L_1^n)}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} - 1 \right|.$$

From the linearity property of  $\Gamma$  and since  $\xi^{n,x_0}$  is deterministic, we have  $\Gamma(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}), a(x_0)L_1^n) = n^{1/\alpha}a(x_0)\Gamma(Y_1^n, L_1^n)$  and  $\Gamma(a(x_0)L_1^n, a(x_0)L_1^n) = a(x_0)^2\Gamma(L_1^n, L_1^n)$ .

Setting  $U_t^n = n^{1/\alpha}\Gamma(Y_t^n, L_t^n)$ ,  $t \in [0, 1]$ , this leads to the simplification

$$\left| \frac{\Gamma(n^{1/\alpha}(Y_1^n - \xi_1^{n,x_0}), a(x_0)L_1^n)}{\Gamma(a(x_0)L_1^n, a(x_0)L_1^n)} - 1 \right| = \frac{1}{|a(x_0)|} \frac{|U_1^n - a(x_0) \int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)},$$

and

$$T_1 \leq \|h\|_\infty \frac{1}{|a(x_0)|} \mathbb{E} \left( \frac{|U_1^n - a(x_0) \int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right). \quad (49)$$



From Theorem 10-3 p.130 in [6], we can prove that the process  $(U_t^n)$  satisfies the equation

$$U_t^n = \frac{1}{n} \int_0^t b'(Y_s^n) U_s^n ds + \frac{1}{n^{1/\alpha}} \int_0^t \int_{E_n} a'(Y_{s-}^n) U_{s-}^n z \tilde{\mu}^n(ds, dz) + \int_0^t \int_{E_n} a(Y_{s-}^n) \rho(z) \mu^n(ds, dz), \quad t \in [0, 1]. \quad (50)$$

Introducing the solution to the linear equation

$$Z_t^n = 1 + \frac{1}{n} \int_0^t b'(Y_s^n) Z_s^n ds + \frac{1}{n^{1/\alpha}} \int_0^t \int_{E_n} a'(Y_{s-}^n) Z_{s-}^n z \tilde{\mu}^n(ds, dz), \quad t \in [0, 1], \quad (51)$$

and using Itô's formula, we show that  $Z_t^n$  admits an inverse, denoted by  $(Z_t^n)^{-1}$ , and solution to the equation

$$\begin{aligned} (Z_t^n)^{-1} = 1 - \frac{1}{n} \int_0^t b'(Y_s^n) (Z_s^n)^{-1} ds - \frac{1}{n^{1/\alpha}} \int_0^t \int_{E_n} \frac{a'(Y_{s-}^n) z}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} (Z_{s-}^n)^{-1} \tilde{\mu}^n(ds, dz) \\ + \frac{1}{n^{2/\alpha}} \int_0^t \int_{E_n} \frac{(a'(Y_{s-}^n) z)^2}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} (Z_{s-}^n)^{-1} \bar{\mu}^n(ds, dz), \quad t \in [0, 1]. \end{aligned} \quad (52)$$

Note that on  $E_n$ ,  $0 < \frac{1}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} \leq 2$  and the above integrals are well defined.

With these processes we can solve (50) and we obtain the explicit expression

$$U_t^n = Z_t^n \int_0^t \int_{E_n} \frac{a(Y_{s-}^n) \rho(z)}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} (Z_{s-}^n)^{-1} \mu^n(ds, dz), \quad t \in [0, 1]. \quad (53)$$

Moreover we can prove the following bounds for the processes  $(Z^n)$  and  $(Z^n)^{-1}$ .

**Lemma 5** *Let  $p \geq 1$  and  $p > \alpha$ , then*

$$\mathbb{E} \sup_{t \in [0, 1]} |Z_t^n - 1|^p \leq C_p / n, \quad (54)$$

$$\mathbb{E} \sup_{t \in [0, 1]} |(Z_t^n)^{-1} - 1|^p \leq C_p / n, \quad (55)$$

The result of Lemma 5 follows from convexity inequality, inequality 2.1.36 (or 2.1.37 if  $p \geq 2$ ) in [13] and Gronwall's Lemma. We omit its standard proof.

Plugging (53) into (49), we split the right-hand side of (49) into four parts :

$$\begin{aligned} T_{1,1} &= \mathbb{E} \left( \frac{|Z_1^n \int_0^1 \int_{E_n} \frac{(a(Y_{s-}^n) - a(x_0)) \rho(z)}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} (Z_{s-}^n)^{-1} \mu^n(ds, dz)|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right), \\ T_{1,2} &= \mathbb{E} \left( \frac{|Z_1^n - 1| \int_0^1 \int_{E_n} \frac{a(x_0) \rho(z)}{1 + a'(Y_{s-}^n) z / n^{1/\alpha}} (Z_{s-}^n)^{-1} \mu^n(ds, dz)|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right), \end{aligned}$$

$$T_{1,3} = \mathbb{E} \left( \frac{\left| \int_0^1 \int_{E_n} \frac{((Z_{s-}^n)^{-1} - 1)}{1 + a'(Y_{s-}^n)z/n^{1/\alpha}} a(x_0) \rho(z) \mu^n(ds, dz) \right|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right),$$

$$T_{1,4} = \mathbb{E} \left( \frac{\left| \int_0^1 \int_{E_n} \frac{a'(Y_{s-}^n)z/n^{1/\alpha}}{1 + a'(Y_{s-}^n)z/n^{1/\alpha}} a(x_0) \rho(z) \mu^n(ds, dz) \right|}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right).$$

Considering first  $T_{1,1}$ , we have

$$T_{1,1} \leq 2\mathbb{E}(|Z_1^n| \sup_{t \in [0,1]} |(Z_t^n)^{-1}| \sup_{t \in [0,1]} |a(Y_{t-}^n) - a(x_0)|),$$

and from Hölder's inequality with conjugated  $p$  and  $q$  such that  $1 < p < 2$  and  $p > \alpha$  we obtain from Lemma 2 (with  $p$ ) and Lemma 5 (with  $q$ )

$$T_{1,1} \leq C_p(1 + |x_0|)\varepsilon_n,$$

where  $\varepsilon_n$  is defined in Proposition 3. Turning to  $T_{1,2}$ , we have

$$T_{1,2} \leq 2|a(x_0)|\mathbb{E}(|Z_1^n - 1| \sup_{t \in [0,1]} |(Z_t^n)^{-1}|),$$

and as previously using Lemma 5, we deduce

$$T_{1,2} \leq C_p(1 + |x_0|)\varepsilon_n.$$

The third term satisfies  $T_{1,3} \leq 2|a(x_0)|\mathbb{E}(\sup_{t \in [0,1]} |(Z_t^n)^{-1} - 1|)$  and so

$$T_{1,3} \leq C_p(1 + |x_0|)\varepsilon_n.$$

Finally for the last term, we observe that

$$T_{1,4} \leq 2|a(x_0)|\|a'\|_\infty \mathbb{E} \left( \frac{\int_0^1 \int_{E_n} (|z|/n^{1/\alpha}) \rho(z) \mu^n(ds, dz)}{\int_0^1 \int_{E_n} \rho(z) \mu^n(ds, dz)} \right),$$

$$\leq C(1 + |x_0|) \left( \frac{1}{n^{1/\alpha}} + \mathbb{E} \left( \frac{\int_0^1 \int_{E_n \cap \{|z| > 1\}} (|z|/n^{1/\alpha}) \rho(z) \mu^n(ds, dz)}{\int_0^1 \int_{E_n \cap \{|z| > 1\}} \rho(z) \mu^n(ds, dz)} \right) \right).$$

But remarking that

$$\frac{\int_0^1 \int_{E_n \cap \{|z| > 1\}} (|z|/n^{1/\alpha}) \rho(z) \mu^n(ds, dz)}{\int_0^1 \int_{E_n \cap \{|z| > 1\}} \rho(z) \mu^n(ds, dz)} \leq \int_0^1 \int_{E_n \cap \{|z| > 1\}} (|z|/n^{1/\alpha}) \mu^n(ds, dz)$$

and taking the expectation we deduce that if  $\alpha \neq 1$

$$\mathbb{E} \frac{\int_0^1 \int_{E_n \cap \{|z| > 1\}} (|z|/n^{1/\alpha}) \rho(z) \mu^n(ds, dz)}{\int_0^1 \int_{E_n \cap \{|z| > 1\}} \rho(z) \mu^n(ds, dz)} \leq C/n,$$

and if  $\alpha = 1$

$$\mathbb{E} \frac{\int_0^1 \int_{E_n \cap \{|z| > 1\}} (|z|/n^{1/\alpha}) \rho(z) \mu^n(ds, dz)}{\int_0^1 \int_{E_n \cap \{|z| > 1\}} \rho(z) \mu^n(ds, dz)} \leq C \log n/n.$$

This yields to

$$T_{1,4} \leq C \varepsilon_n.$$

Combining all these results we obtain

$$T_1 \leq C(1 + |x_0|) \varepsilon_n,$$

and the proof of Proposition 3 is finished.

## 5.4 Proof of Proposition 4

As in the proof of Proposition 3 we will use Malliavin calculus integration by part formula. The first step is to construct on the same probability space two random variables whose laws are close to the laws of  $n^{1/\alpha} L_{1/n}$  and  $L_1^\alpha$ . We recall briefly the notations of Section 5.1 :  $\mu^n$  is a Poisson random measure with compensator  $\bar{\mu}^n(dt, dz) = dt \frac{g(z/n^{1/\alpha})}{|z|^{\alpha+1}} \tau(z/n^{1/\alpha}) 1_{\mathbb{R}^*}(z) dz$ , where  $\tau$  is a truncation function, and the process  $(L_t^n)_t$  defined by  $L_t^n = \int_0^t \int_{\mathbb{R}} z \tilde{\mu}^n(ds, dz)$ , with  $\tilde{\mu}^n = \mu^n - \bar{\mu}^n$  is such that (see (19), (24))

$$\left| \mathbb{E}[h(n^{1/\alpha} L_{1/n})] - \mathbb{E}[h(L_1^n)] \right| \leq C \frac{\|h\|_\infty}{n}. \quad (56)$$

We now construct a variable approximating the law of  $L_1^\alpha$ , and based on the Poisson measure  $\mu^n$ .

For  $x > 0$  we define

$$G(x) = \int_x^\infty \frac{g(z)\tau(z)}{z^{1+\alpha}} dz, \quad \text{and, } H(x) = \int_x^\infty \frac{\tau(z)}{z^{1+\alpha}} dz. \quad (57)$$

Recall that  $\tau$  is a truncation function equal to 1 on  $[-K(a)/2, K(a)/2]$  and equal to 0 on  $[-K(a), -K(a)]^c$ . We assume for the sequel of the proof that  $\tau(z) > 0$ ,  $g(z) > 0$ , for  $|z| < K(a)$ . Indeed,  $\tau$  is a truncation function that can be chosen non vanishing on  $(K(a), K(a))$ , and up to reducing the value of  $\eta$  in the Assumption H2(b) we can assume that  $g$  does not vanish on  $[-K(a), -K(a)]$ . Then, it is immediate to check that  $G$  and  $H$  are non increasing, one to one, functions from  $(0, K(a)]$  to  $(\infty, 0]$ . We define  $s_n(z) = n^{1/\alpha} G^{-1}(H(n^{-1/\alpha} z))$  for  $z \in (0, n^{1/\alpha} K(a)]$ ,  $s_n(0) = 0$ , and  $s_n(z) = -s_n(-z)$  for  $z \in [-n^{1/\alpha} K(a), 0)$ . The function  $s_n$  is increasing, odd and one to one, from the interval  $[-n^{1/\alpha} K(a), n^{1/\alpha} K(a)]$  on itself and we let  $h_n = s_n^{-1} : [-n^{1/\alpha} K(a), n^{1/\alpha} K(a)] \rightarrow [-n^{1/\alpha} K(a), n^{1/\alpha} K(a)]$  be its inverse function. Let us admit temporarily the next lemma about the behaviour of the functions  $h_n$  as  $n \rightarrow \infty$ .

**Lemma 6** 1) *There exists  $\varepsilon > 0$  such that, for all  $|z| \leq \varepsilon n^{1/\alpha}$ ,*

$$\begin{aligned} |h_n(z) - z| &\leq C \frac{z^2}{n^{1/\alpha}} + C \frac{|z|^{\alpha+1}}{n}, \quad \text{if } \alpha \neq 1, \\ |h_n(z) - z| &\leq C \frac{z^2}{n} \left| \log\left(\frac{|z|}{n}\right) \right|, \quad \text{if } \alpha = 1. \end{aligned}$$

2) *The function  $h_n$  is  $\mathcal{C}^1$  on  $(-\varepsilon n^{1/\alpha}, \varepsilon n^{1/\alpha})$  and for all  $|z| < \varepsilon n^{1/\alpha}$ ,*

$$\begin{aligned} |h'_n(z) - 1| &\leq C \frac{|z|}{n^{1/\alpha}} + C \frac{|z|^\alpha}{n}, \quad \text{if } \alpha \neq 1, \\ |h'_n(z) - 1| &\leq C \frac{|z|}{n} \left| \log\left(\frac{|z|}{n}\right) \right|, \quad \text{if } \alpha = 1. \end{aligned}$$

Using the previous lemma, we can define a process  $(L_t^{\alpha,n})_{t \in [0,1]}$  by setting

$$L_t^{\alpha,n} = \int_0^t \int_{\{|z| \leq n^{1/\alpha} K(a)\}} h_n(z) \tilde{\mu}^n(ds, dz). \quad (58)$$

We can compute the characteristic function of the random variable  $L_1^{\alpha,n}$ . Indeed, using the exponential formula for Poisson measure,

$$\begin{aligned} \mathbb{E}[e^{iuL_1^{\alpha,n}}] &= \exp \left( \int_{\{|z| \leq K(a)n^{1/\alpha}\}} \left( e^{iuh_n(z)} - 1 - iuh_n(z) \right) \frac{g(zn^{-1/\alpha})\tau(zn^{-1/\alpha})}{|z|^{1+\alpha}} dz \right), \\ &= \exp \left( \int_{\{|w| \leq K(a)n^{1/\alpha}\}} \left( e^{iuw} - 1 - iuw \right) \frac{\tau(wn^{-1/\alpha})}{|w|^{1+\alpha}} dw \right), \end{aligned} \quad (59)$$

where in the second line we have used the change of variable  $w = h_n(z)$  and the relation

$$\frac{g(s_n(w)n^{-1/\alpha})\tau(s_n(w)n^{-1/\alpha})s'_n(w)}{|s_n(w)|^{1+\alpha}} = \frac{\tau(wn^{-1/\alpha})}{|w|^{1+\alpha}}, \quad (60)$$

that can be derived for  $w > 0$  from the differentiation of the relation  $G(n^{-1/\alpha}s_n(w)) = H(n^{-1/\alpha}w)$ , and is extended to  $w < 0$  by symmetry of  $g$  and  $\tau$ . From (59) we see that  $L_1^{\alpha,n}$  has the law of an  $\alpha$ -stable process whose jumps are truncated with the function  $\tau$ . Similarly to (56) (in the situation  $g = 1$ ), we deduce

$$|\mathbb{E}[h(L_1^\alpha)] - \mathbb{E}[h(L_1^{\alpha,n})]| \leq C \frac{\|h\|_\infty}{n}. \quad (61)$$

Proposition 4 is a consequence of (56), (61) and of the following lemma.

**Lemma 7** *Let  $h$  be as in Proposition 4, then we have,*

$$|\mathbb{E}[h(L_1^{\alpha,n})] - \mathbb{E}[h(L_1^n)]| \leq C\varepsilon_n \|h\|_\infty,$$

where  $\varepsilon_n$  is defined in the statement of Proposition 3.

**Proof** The scheme of the proof follows the same lines as the proof of Proposition 3, and is based on the comparison of the representation of the random variables (21) and (58). Since, in Lemma 6, the difference  $h_n(z) - z$  is only controlled for  $|z| \leq \varepsilon n^{1/\alpha}$  with some  $\varepsilon > 0$ , we need to introduce an additional localization procedure consisting in regularizing  $1_{\{\mu^n([0,1] \times \{z \in \mathbb{R}; |z| > \varepsilon n^{1/\alpha}\}) = 0\}}$ . Let  $\mathcal{I}$  be a smooth function defined on  $\mathbb{R}$ , and with values in  $[0, 1]$ , such that  $\mathcal{I}(x) = 1$  for  $x \leq 1/2$ , and  $\mathcal{I}(x) = 0$  for  $x \geq 1$ . We note  $\xi$  a smooth function on  $\mathbb{R}$ , with values in  $[0, 1]$  and such that  $\xi(z) = 0$  for  $|z| \leq 1/2$  and  $\xi(z) = 1$  for  $|z| \geq 1$ , and we set

$$V^n = \int_0^1 \int_{\mathbb{R}} \xi\left(\frac{z}{\varepsilon n^{1/\alpha}}\right) \mu^n(ds, dz) = \int_0^1 \int_{\{\frac{1}{2}\varepsilon n^{1/\alpha} \leq |z| \leq \varepsilon n^{1/\alpha}\}} \xi\left(\frac{z}{\varepsilon n^{1/\alpha}}\right) \mu^n(ds, dz) + \int_0^1 \int_{\{|z| > \varepsilon n^{1/\alpha}\}} \mu^n(ds, dz), \quad (62)$$

$$W^n = \mathcal{I}(V^n). \quad (63)$$

From the construction,  $W^n$  is a Malliavin differentiable random variable such that  $W^n \neq 0$  implies  $\mu^n([0, 1] \times \{z \in \mathbb{R}; |z| > \varepsilon n^{1/\alpha}\}) = 0$ , and one can show that  $P(W^n \neq 1) = O(n^{-1})$ . From the latter, it is clear that the proof of the lemma reduces in proving the upper bound

$$|\mathbb{E}[h(L_1^{\alpha,n})W^n] - \mathbb{E}[h(L_1^n)W^n]| \leq C\varepsilon_n \|h\|_\infty.$$

Using a regularizing sequence as in the proof of Proposition 3, we can assume that  $h$  is  $\mathcal{C}^1$  with bounded derivative. Then by the integration by part formula (37), we can write  $\mathbb{E}[h(L_1^n)W^n] = \mathbb{E}[H(L_1^n)\mathcal{H}_{L_1^n}(W^n)]$  where  $H$  is some primitive function of  $h$  and the Malliavin weight can be written, using (38) and the chain rule property of the operator  $\Gamma$ ,

$$\mathcal{H}_{L_1^n}(W^n) = W^n \mathcal{H}_{L_1^n}(1) - \frac{\Gamma(W^n, L_1^n)}{\Gamma(L_1^n, L_1^n)}. \quad (64)$$

Using the triangle inequality, we are now left to find upper bounds for the two following terms

$$\tilde{T}_1 := |\mathbb{E}[h(L_1^{\alpha,n})W^n] - \mathbb{E}[H(L_1^{\alpha,n})\mathcal{H}_{L_1^n}(W^n)]|, \quad (65)$$

$$\tilde{T}_2 := |\mathbb{E}[H(L_1^{\alpha,n})\mathcal{H}_{L_1^n}(W^n)] - \mathbb{E}[H(L_1^n)\mathcal{H}_{L_1^n}(W^n)]|. \quad (66)$$

### Bound for $\tilde{T}_2$

Using (64) and the Lipschitz property of the function  $H$ , we have

$$\tilde{T}_2 \leq \|h\|_\infty \mathbb{E}[|L_1^{\alpha,n} - L_1^n| |\mathcal{H}_{L_1^n}(1)| W^n] + \|h\|_\infty \mathbb{E}\left[|L_1^{\alpha,n} - L_1^n| \left| \frac{\Gamma(W^n, L_1^n)}{\Gamma(L_1^n, L_1^n)} \right|\right]. \quad (67)$$

We focus on the first expectation appearing in the right-hand side of (67). Using (21) and (58), we have

$$\begin{aligned} \mathbb{E} [ |L_1^{\alpha,n} - L_1^n| | \mathcal{H}_{L_1^n}(1) | W^n ] &= \mathbb{E} \left[ \left| \int_0^1 \int_{\mathbb{R}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) \right| | \mathcal{H}_{L_1^n}(1) | W^n \right] \\ &\leq \mathbb{E} \left[ \left| \int_0^1 \int_{\{|z| \leq 1\}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) \right| | \mathcal{H}_{L_1^n}(1) | W^n \right] \\ &\quad + \mathbb{E} \left[ \left| \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} (h_n(z) - z) \mu^n(ds, dz) \right| | \mathcal{H}_{L_1^n}(1) | W^n \right], \end{aligned} \quad (68)$$

where we have used that  $h_n$  is an odd function with the symmetry of the compensator  $\bar{\mu}^n$ , and the fact that on  $W^n \neq 0$  we have  $\mu^n([0, 1] \times \{z \in \mathbb{R}; |z| > \varepsilon n^{1/\alpha}\}) = 0$ . The two terms in the right-hand side of (68) are controlled using Lemma 6 1). For the sake of shortness, we only give the details of the proof in the case  $\alpha \neq 1$ . In the case  $\alpha = 1$ , one needs to modify this control with an additional logarithmic term. For the small jumps term, from inequality 2.1.37 in [13] and Lemma 6 1), we deduce  $\mathbb{E} | \int_0^1 \int_{\{|z| \leq 1\}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) |^p < C_p (n^{-1/\alpha} + n^{-1})^p$ , for all  $p \geq 2$  and using  $0 \leq W^n \leq 1$  and (44) we get from Hölder's inequality

$$\mathbb{E} \left[ \left| \int_0^1 \int_{\{|z| \leq 1\}} (h_n(z) - z) \tilde{\mu}^n(s, dz) \right| | \mathcal{H}_{L_1^n}(1) | W^n \right] \leq C n^{-1/\alpha} + C n^{-1}.$$

The large jumps term of (68) is upper bounded by

$$\mathbb{E} \left[ \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} \left( \frac{|z|^\alpha}{n} + \frac{|z|}{n^{1/\alpha}} \right) \mu^n(ds, dz) \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z| \mu^n(ds, dz) | \mathcal{H}_{L_1^n}(1) | W^n \right],$$

where we have used Lemma 6 1), and the basic inequality,

$$\int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z|^\beta \mu^n(ds, dz) \leq \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z|^{\beta-1} \mu^n(ds, dz) \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z| \mu^n(ds, dz)$$

for  $\beta \geq 1$ . From  $\mu^n = \tilde{\mu}^n + \bar{\mu}^n$  and inequality 2.1.36 in [13], one can easily show that for  $q \in (1, 2)$ ,

$$\begin{aligned} \mathbb{E} \left[ \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z| \mu^n(ds, dz) \right]^q &\leq C(1 + n^{q/\alpha-1}), \\ \mathbb{E} \left[ \int_0^1 \int_{\{1 < |z| \leq \varepsilon n^{1/\alpha}\}} |z|^\alpha \mu^n(ds, dz) \right]^q &\leq C n^{q-1}. \end{aligned}$$

By Hölder's inequality and (45), we deduce that the large jumps term of (68) is eventually smaller than  $C(n^{-1/\alpha} + n^{-1/q})$  for any  $q \in (1, 2)$ , and in turn

$$\mathbb{E} [ |L_1^{\alpha,n} - L_1^n| | \mathcal{H}_{L_1^n}(1) | W^n ] \leq C \varepsilon_n.$$

Let us now study the second expectation in the right-hand side of (67), which can be rewritten, using (63) and the chain rule property of the operator  $\Gamma$

$$\mathbb{E} \left[ |L_1^{\alpha,n} - L_1^n| |\mathcal{I}'(V^n)| \left| \frac{\Gamma(V^n, L_1^n)}{\Gamma(L_1^n, L_1^n)} \right| \right]. \quad (69)$$

Using (62), we get the explicit expression for  $\Gamma(V^n, L_1^n) = \int_0^1 \int_{\mathbb{R}} \rho(z) \xi'(\frac{z}{\varepsilon n^{1/\alpha}}) \mu^n(ds, dz) n^{-1/\alpha} \varepsilon^{-1}$ , from which we deduce  $\left| \frac{\Gamma(V^n, L_1^n)}{\Gamma(L_1^n, L_1^n)} \right| \leq C \|\xi'\|_{\infty} n^{-1/\alpha}$ . Hence, the term (69) is smaller than

$$C n^{-1/\alpha} \mathbb{E} \left[ \left| \int_0^1 \int_{\{|z| \leq K(a)n^{1/\alpha}\}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) \right| |\mathcal{I}'(V_n)| \right].$$

Remarking that  $\mathcal{I}'(x) = 0$  for  $x \geq 1$ , we deduce that  $|\mathcal{I}'(V^n)| \neq 0$  implies  $\mu^n([0, 1] \times \{z \in \mathbb{R}; |z| > \varepsilon n^{-1/\alpha}\}) = 0$ . Consequently, (69) is upper bounded by

$$C \|\mathcal{I}'\|_{\infty} n^{-1/\alpha} \mathbb{E} \left[ \left| \int_0^1 \int_{\{|z| \leq \varepsilon n^{1/\alpha}\}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) \right| \right],$$

where we used the symmetry of the compensator  $\bar{\mu}^n$ . Using Lemma 5 1), one can show that

$$\mathbb{E} \left[ \left| \int_0^1 \int_{\{|z| \leq \varepsilon n^{1/\alpha}\}} (h_n(z) - z) \tilde{\mu}^n(ds, dz) \right| \right] \leq C n^{1/\alpha-1},$$

and deduce that (69) is smaller than  $C n^{-1}$ . This finishes the proof that  $\tilde{T}_2 \leq C \|h\|_{\infty} \varepsilon_n$ .

### Bound for $\tilde{T}_1$

Using (33) and (38) we can write

$$\mathcal{H}_{L_1^n}(W^n) = \frac{-W^n L(L_1^n)}{\Gamma(L_1^n, L_1^n)} + L \left( \frac{W^n}{\Gamma(L_1^n, L_1^n)} \right) L_1^n - L \left( \frac{L_1^n W^n}{\Gamma(L_1^n, L_1^n)} \right),$$

and with computations using that  $L$  is a self-adjoint operator, as in the proof of Proposition 3, we get that

$$\begin{aligned} \tilde{T}_1 &= \left| \mathbb{E} [h(L_1^{\alpha,n}) W^n] - \mathbb{E} \left[ h(L_1^{\alpha,n}) \frac{\Gamma(L_1^{\alpha,n}, L_1^n)}{\Gamma(L_1^n, L_1^n)} W^n \right] \right| \\ &\leq \|h\|_{\infty} \mathbb{E} \left[ \left| \frac{\Gamma(L_1^n - L_1^{\alpha,n}, L_1^n)}{\Gamma(L_1^n, L_1^n)} \right| W^n \right] \end{aligned}$$

But  $\Gamma(L_1^n - L_1^{\alpha,n}, L_1^n) = \int_0^1 \int_{\{|z| \leq K(a)n^{1/\alpha}\}} \rho(z) (1 - h'_n(z)) \mu^n(ds, dz)$ . Using Lemma 6 2), we deduce that on the event  $W^n \neq 0$ ,

$$\begin{aligned} |\Gamma(L_1^n - L_1^{\alpha,n}, L_1^n)| &\leq C \int_0^1 \int_{\{|z| \leq \varepsilon n^{1/\alpha}\}} \rho(z) \left( \frac{|z|}{n^{1/\alpha}} + \frac{|z|^\alpha}{n} \right) \mu^n(ds, dz), \\ &\leq C \int_0^1 \int_{\{|z| \leq 1\}} \rho(z) \left( \frac{|z|}{n^{1/\alpha}} + \frac{|z|^\alpha}{n} \right) \mu^n(ds, dz) + \\ &\quad C \int_0^1 \int_{\{1 \leq |z| \leq \varepsilon n^{1/\alpha}\}} \rho(z) \mu^n(ds, dz) \int_0^1 \int_{\{1 \leq |z| \leq \varepsilon n^{1/\alpha}\}} \left( \frac{|z|}{n^{1/\alpha}} + \frac{|z|^\alpha}{n} \right) \mu^n(ds, dz). \end{aligned}$$

From this equation, we deduce that

$$\begin{aligned}\tilde{T}_1 &\leq C\|h\|_\infty(n^{-1/\alpha} + n^{-1}) + C\|h\|_\infty \mathbb{E} \int_0^1 \int_{\{1 \leq |z| \leq \varepsilon n^{1/\alpha}\}} \left(\frac{|z|}{n^{1/\alpha}} + \frac{|z|^\alpha}{n}\right) \mu^n(ds, dz), \\ &\leq C\|h\|_\infty(n^{-1/\alpha} + \log(n)n^{-1}).\end{aligned}$$

This is the required upper bound for  $\tilde{T}_1$ .  $\square$

**Proof of Lemma 6** As  $x \mapsto h_n(x)$  is an odd function, it is sufficient to study this function on  $[0, \infty)$ . Recall that for  $0 < x < K(a)n^{1/\alpha}$ ,  $h_n(x) = s_n^{-1}(x)$  and  $s_n(x) = n^{1/\alpha}G^{-1}(H(n^{-1/\alpha}x))$ , where  $G$  and  $H$  are defined in (57). As  $\tau(x) = 1$  for  $|x| \leq K(a)/2$ , we have  $H(x) = \alpha^{-1}x^{-\alpha} + \kappa_1$  for  $0 < x < K(a)/2$ , and where  $\kappa_1$  is some constant. Using that  $g(x) = 1 + O(x)$  as  $x \rightarrow 0$ , we get

$$G(x) = \int_x^{K(a)} \frac{g(z)\tau(z)}{z^{1+\alpha}} dz = \begin{cases} \alpha^{-1}x^{-\alpha} + \kappa_2 + O(x^{1-\alpha}), & \text{if } \alpha \neq 1 \\ \alpha^{-1}x^{-\alpha} + O(|\log(x)|), & \text{if } \alpha = 1 \end{cases}.$$

where  $\kappa_2$  is some constant. Then, by elementary computations we deduce that if  $u \in (0, \infty)$  is large enough,

$$\left|G^{-1}(u) - (\alpha u)^{-1/\alpha}\right| \leq \begin{cases} Cu^{-2/\alpha} + Cu^{-1-1/\alpha}, & \text{if } \alpha \neq 1 \\ Cu^{-2} |\log(u)|, & \text{if } \alpha = 1 \end{cases}. \quad (70)$$

From the expression  $H(x) = \alpha^{-1}x^{-\alpha} + \kappa_1$  and (70), it comes, for  $x/n^{1/\alpha}$  small enough,

$$|s_n(x) - x| = \left|n^{1/\alpha}G^{-1}(H(n^{-1/\alpha}x)) - x\right| \leq \begin{cases} C\frac{x^2}{n^{1/\alpha}} + C\frac{x^{1+\alpha}}{n}, & \text{if } \alpha \neq 1 \\ C\frac{x^2}{n} |\log(\frac{x}{n})| & \text{if } \alpha = 1 \end{cases}. \quad (71)$$

Now, the first part of the lemma follows from  $h_n = s_n^{-1}$ .

For the second part we use (60) to get, if  $x/n^{1/\alpha}$  is small enough,

$$s'_n(x) = \left(\frac{s_n(x)}{x}\right)^{1+\alpha} \frac{1}{g(n^{-1/\alpha}s_n(x))}.$$

From (71) and  $\frac{1}{g(n^{-1/\alpha}s_n(x))} = 1 + O(s_n(x)/n^{1/\alpha}) = 1 + O(x/n^{1/\alpha})$ , we deduce the second part of the lemma.  $\square$

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