Controllability of low Reynolds numbers swimmers of ciliate type

Jérôme Lohéac, Takéo Takahashi

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Abstract

We study the locomotion of a ciliated microorganism in a viscous incompressible fluid. We use the Blake ciliated model: the swimmer is a rigid body with tangential displacements at its boundary that allow it to propel in a Stokes fluid. This can be seen as a control problem: using periodical displacements, is it possible to reach a given position and a given orientation? We are interested in the minimal dimension d of the space of controls that allows the microorganism to swim. Our main result states the exact controllability with $d = 3$ generically with respect to the shape of the swimmer and with respect to the vector fields generating the tangential displacements. The proof is based on analyticity results and on the study of the particular case of spheroidal swimmer.

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1 Introduction

The aim of this article is to analyse the controllability of a system associated to a model of micro-swimmers. The swimmers considered here are ciliated microorganisms immersed in a viscous incompressible fluid. In the model considered here, the shape of the swimmer is fixed and we use the Blake ciliated model [5, 6] consisting in replacing the propelling mechanism of the cilia by time periodic tangential displacements. Due to the micro-scale of the swimmer (very low Reynolds numbers), the inertial forces are neglected and in particular, the fluid motion is governed by the steady-state Stokes system. For more details about this model, we refer the reader to [5, 6, 15, 26, 27, 35]. An important property of the corresponding system is that it can be rewritten as a finite dimensional nonlinear control problem and this permits the use of the geometric controllability theory. Such an approach is classical and comes back to [3, 32]. In the case of very high Reynolds numbers, one can assume that the fluid is potential and this leads also to a finite dimensional nonlinear controlled system that can again be studied with the geometric controllability theory: see [8] for one of the first results in that direction.

The study done here follows the works of J. San Martín, T. Takahashi and M. Tucsnak [32], M. Sigalotti and J.-C. Vivalda [34], where a similar model is considered. In this first model, the swimming mechanism is modeled by a tangential velocity which is unrelated to a tangential displacement. If we impose that this tangential velocity comes from a boundary displacement, the problem is more complicated and was only tackled in J. San Martín, T. Takahashi and M. Tucsnak [33]. In this last work, only axi-symmetric swimmers were considered and the control problem was to move the swimmer along the axis of symmetry.

Let us mention several other classes of swimmers which have been tackled in the literature. Apart ciliated swimmers, let us mention, among them, the three link swimmer introduced in [31], the three sphere swimmer introduced in [30] and for which the controllability has been shown in [3] (its extension, the n-sphere swimmer has been first studied in [2]). Another swimming mechanism consists in small deformations at its boundary. Such a model was considered in [24, 25]. Let us also mention some other related works: the case where the fluid is inviscid and potential leads to a very close theory see [29, 13, 14, 12].

In this paper, we deal with swimmers of arbitrary shape and our aim is to control all the rigid motions of the swimmer, i.e. its position and orientation. In order to explain our main result, let us briefly explain how the boundary displacement of the swimmer is built. First of all, the shape of the swimmer is defined as the image of the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ by some diffeomorphism $\mathrm{Id}+\Psi_0$ of $\mathbb{R}^3$. The displacement on the boundary is obtained from d vector fields $\delta_1, \cdots, \delta_d$ of $\mathbb{S}^2$ from which we build the map $\mathcal{X}_\delta(s) : y \in \mathbb{S}^2 \mapsto \exp_y\left(\sum_{i=1}^d s_i\delta_i(y)\right) \in \mathbb{S}^2$. For $s \in \mathbb{R}^d$ small enough, $\mathcal{X}_\delta(s)$ is a diffeomorphism of $\mathbb{S}^2$. The definition of the exponential map on manifolds can be found, for instance, in [22, 23, 28]. The main result (Theorem 2.5) states that for $d \geqslant 3$ and generically with respect to Ψ_0 and δ, the swimmer is controllable, i.e. any rigid position of the swimmer can be tracked and reached. Let us emphasize that here, we only need $d = 3$ elementary deformations. This is a novelty compared to other controllability results, see for instance [24, 2], where four elementary deformations are required to fully control the rigid position of the swimmer. Let us also point out the works [4, 19] where only three elementary deformations are required. Nevertheless, in these works, the fluid is only present on half of the space $\mathbb{R}^3$ and the controllability with less than four controls is obtained by using the boundary effects. Finally, let us also quote that due to the scallop Theorem [31], it is known that at least two elementary deformations are required to control the swimmer. We believe that our result holds true for $d = 2$ (see Remark 4.6) but the proof should follow a different method. Indeed, we use here explicit formulas for the Stokes system in the exterior of a ball. Unfortunately because of symmetry properties of the sphere, it seems that for such a shape, we need $d \geqslant 3$. In order to reach $d = 2$, we would need to remove such a symmetry, but in that case the difficulty would be to compute the solution of the Stokes system.

The other novelty comes from the fact that the swimming mechanism here is coming from the periodic tangential displacements of points located at the boundary of the swimmer. In particular, we present in the next section the model described above that allows us to write this control in a simple way and in particular to write the corresponding system in a suitable controlled system.

This article is organized as follows. In Section 2, we introduce the model corresponding to the ciliate locomotion. We introduce in particular the velocity fields $\delta = (\delta_1, \ldots, \delta_d)$ that generates the tangential displacement. The shape of the microorganism is parametrized by a transformation of the unit sphere of $\mathbb{R}^3$ through a diffeomorphism $\mathrm{Id}_{\mathbb{R}^3} + \Psi_0$. The corresponding fluid-structure system is written in (2.7). We also give the main result (Theorem 2.5), that is the exact controllability for $d \geqslant 3$ and generically with respect to Ψ_0 and δ. We rewrite the fluid-structure system in Section 3 so that we can apply general results from the geometric controllability theory and in particular the Rashevsky-Chow theorem. In order to use such a theorem, we need to compute the Lie brackets associated to the controls. These Lie brackets involve in particular several Stokes systems with

non-homogeneous Dirichlet boundary conditions. Consequently, to obtain explicit formulas we particularize the problem in Section 4 by considering the case where the shape of the swimmer is a ball. In that case, using the classical work of Brenner [7], we can consider particular cases for $\delta = (\delta_1, \dots, \delta_d)$ and show that for $d = 3$, there exists a choice such that the system is controllable. Using analytical properties of the system, we can then prove the main result.

2 The model and the main results

2.1 The swimmer mechanism

We denote by B_0 be the unit ball of $\mathbb{R}^3$ and by $\mathbb{S}^2$ the unit sphere of $\mathbb{R}^3$. For any $k \in \mathbb{N}^* \cup \{\infty\}$ and any $\Theta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)$, we can consider the mapping

$$\mathcal{X} : \mathbb{S}^2 \to \mathbb{S}^2, \quad y \mapsto \cos(|\Theta(y)|)y + \mathrm{sinc}(|\Theta(y)|)\Theta(y). \tag{2.1}$$

Here we recall that sinc is the cardinal sine function. If $\Theta \equiv 0$, then $\mathcal{X} = \mathrm{Id}_{\mathbb{S}^2}$. Formula (2.1) comes from the exponential formula $\mathcal{X} = \exp(\Theta)$ in the case of $\mathbb{S}^2$ (see for instance [22, 23, 28]) Expanding the sine and cosine functions, one can see using the above expression and [36] that for every $k \in \mathbb{N}^*$, $\Theta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2) \mapsto \mathcal{X} \in C^k(\mathbb{S}^2)$ is analytic. In fact, we have,

$$\begin{aligned}\mathcal{X} = \cos|\Theta|\ \mathrm{Id}_{\mathbb{S}^2} + \sin|\Theta|\ \frac{\Theta}{|\Theta|} &= \sum_{n=0}^{\infty}(-1)^n \frac{|\Theta|^{2n}}{(2n)!}\ \mathrm{Id}_{\mathbb{S}^2} + \sum_{n=0}^{\infty}(-1)^n \frac{|\Theta|^{2n}}{(2n+1)!}\,\Theta \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left((2n+1)\,\mathrm{Id}_{\mathbb{S}^2} + \Theta\right)|\Theta|^{2n}. \end{aligned} \tag{2.2}$$

Let us now consider for $k \in \mathbb{N}^* \cup \{\infty\}$ and $d \in \mathbb{N}^*$, $\delta = (\delta_1, \dots, \delta_d) \in C^k\left(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2\right)^d$. For any $s = (s_1, \dots, s_d) \in \mathbb{R}^d$, we write

$$\Theta_\delta(s) := \sum_{j=1}^{d} s_j \delta_j \tag{2.3}$$

and we consider the mapping $\mathcal{X}_\delta(s)$ obtained from (2.1) with $\Theta = \Theta_\delta(s)$. We denote by $\tilde{\mathcal{J}}(\delta)$ the set of $s \in \mathbb{R}^d$ such that $\mathcal{X}_\delta(s)$ is a diffeomorphism of $\mathbb{S}^2$ and by $\mathcal{J}(\delta)$ the connected component of $\tilde{\mathcal{J}}(\delta)$ containing 0.

Lemma 2.1. *Given $k \in \mathbb{N}^* \cup \{\infty\}$, $d \in \mathbb{N}^*$ and $\delta = (\delta_1, \cdots, \delta_d) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, $\tilde{\mathcal{J}}(\delta)$ is a nonempty open subset of $\mathbb{R}^d$.*

Remark 2.2. We deduce in particular the existence of $\varepsilon = \varepsilon(\delta) > 0$ such that for every $s \in \mathbb{R}^d$, with $|s| < \varepsilon$, $\mathcal{X}_\delta(s)$ is a C^k-diffeomorphism of $\mathbb{S}^2$.

Proof. In order to prove that $\tilde{\mathcal{J}}(\delta)$ is a nonempty open set of $\mathbb{R}^d$, we first recall (see [9, Proposition 2 p. 287] and [10, p. 1]) that the set of C^k-diffeomorphisms is a nonempty open set of $C^k(\mathbb{S}^2)$. In addition, the above comments ensure that the map $(\delta, s) \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2) \times \mathbb{R}^d \mapsto \mathcal{X}_\delta(s) \in C^k(\mathbb{S}^2)$ is analytic. This yields the result. □

We also consider $\Psi_0 \in \mathcal{D}_0^k$ (in particular $\mathrm{Id}_{\mathbb{R}^3} + \Psi_0$ is a C^k-diffeomorphism of $\mathbb{R}^3$, see Appendix A for the definition of $\mathcal{D}_0^k$). For every $d \in \mathbb{N}^*$ and every $k \in \mathbb{N}^* \cup \{\infty\}$, we denote by

$$\mathcal{SC}^k(d) = \mathcal{D}_0^k \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$$

the set of the *swimmer configurations*, $c = (\Psi_0, \delta)$. This space, is a subset of the Banach space $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$ which is endowed with the norm

$$\|c\| = \|\Psi_0\|_{C_0^k(\mathbb{R}^3)^3} + \|\delta\|_{C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d} \qquad (c = (\Psi_0, \delta)).$$

Finally, we define the global deformation of the swimmer X_c by

$$X_c(s) = (\mathrm{Id}_{\mathbb{R}^3} + \Psi_0) \circ \mathcal{X}_\delta(s) \qquad (s \in \mathcal{J}(\delta)). \tag{2.4}$$

For every $s \in \mathcal{J}(\delta)$, $X_c(s)$ is a C^k-diffeomorphism from $\mathbb{S}^2$ onto

$$\mathcal{S}_c := (\mathrm{Id}_{\mathbb{R}^3} + \Psi_0)\,(\mathbb{S}^2).$$

$\mathcal{S}_c$ is the shape of our swimmer and its swimming mechanism, which consists in periodical displacements at its boundary, is represented by $\mathcal{X}_\delta(s)$. More precisely, we are interested in the following problem: using a function $s : \mathbb{R}^+ \to \mathcal{J}(\delta)$ is it possible for the swimmer to control its position. In what follows, we also add the constraint that s should have the same value at the initial time and at the final time.

Let us define for every $d \in \mathbb{N}^*$ and every $k \in \mathbb{N}^* \cup \{\infty\}$, the set

$$\mathcal{A}^k(d) = \left\{((\Psi_0, \delta), s) \in \mathcal{SC}^k(d) \times \mathbb{R}^d \mid s \in \mathcal{J}(\delta)\right\}. \tag{2.5}$$

Proposition 2.3. *For every $d \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, $\mathcal{A}^k(d)$ is a connected open set of $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$. Furthermore, $\mathcal{A}^\infty(d)$ is dense in $\mathcal{A}^k(d)$.*

We recall that $C_0^k(\mathbb{R}^3)$ is defined in Appendix A.

Proof. In order to prove the connectivity of $\mathcal{A}^k(d)$, let us consider $f : \mathcal{A}^k(d) \to \{0, 1\}$ a continuous function. For every $\Psi_0 \in \mathcal{D}_0^k$, and every $\delta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, by construction, we have $\mathcal{J}(\delta)$ is connected. Consequently, $f(\Psi_0, \delta, s) = c(\Psi_0, \delta) \in \{0, 1\}$ for every $s \in \mathcal{J}(\delta)$. In particular, we have for every $\delta \in C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, $0 \in \mathcal{J}(\delta)$ and by continuity of $\delta \mapsto f(\Psi_0, \delta, 0)$ and by the connectivity of $C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, we conclude that $c(\Psi_0, \delta) \in \{0, 1\}$ is independent of δ. Similarly, by continuity of $\Psi_0 \mapsto f(\Psi_0, 0, 0)$ and by connectivity of $\mathcal{D}_0^k$, we also prove that $c(\Psi_0, \delta) \in \{0, 1\}$ is independent of Ψ_0. All in all, we have proven that f is a constant function, showing the connectivity of $\mathcal{A}^k$.

Proving that $\mathcal{A}^k(d)$ is open in $C_0^k(\mathbb{R}^3)^3 \times C^k(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d \times \mathbb{R}^d$ is similar as proving Lemma 2.1.
The density of $\mathcal{A}^\infty(d)$ in $\mathcal{A}^k(d)$ follows from the density of C^∞ functions in C^k functions and the open character of the set $\mathcal{A}^k(d)$. □

Remark 2.4. By analyticity of $\Theta \mapsto \mathcal{X}$, it is also obvious that $(c, s) \in \mathcal{A}^k(d) \mapsto X_c(s) \in C^k(\mathbb{S}^2, \mathbb{R}^3)$ is analytic.

2.2 Fluid-structure interactions and motion of the swimmer

Immersed into a viscous incompressible fluid, the swimmer described in the previous section can translate and rotate. We write for $Q \in \mathcal{SO}(3)$ and $h \in \mathbb{R}^3$,

$$X^\dagger(h, Q, s)(y) := QX_c(s)(y) + h \quad \text{and} \quad \mathcal{S}^\dagger(h, Q) := Q\mathcal{S}_c + h\,.$$

We also set $\mathcal{F}_c \subset \mathbb{R}^3$ (respectively $\mathcal{F}^\dagger(h, Q)$) the unbounded connected component of $\mathbb{R}^3 \setminus \mathcal{S}_c$ (respectively $\mathbb{R}^3 \setminus \mathcal{S}^\dagger(h, Q)$). These correspond to fluid domains.

A point on the surface of the swimmer can be parametrized as follows

$$x = X^\dagger(h, Q, s)(y) \qquad (y \in \mathbb{S}^2).$$

Assume that (h, Q, s) is a C^1 function with respect to the time. Then the velocity of the above point x is:

$$v^\dagger(t, x) = \dot{Q}Q^\top(x - h) + \dot{h} + Q\frac{\mathrm{d}\, X_c(s)}{\mathrm{d}t}\left(X_c(s)^{-1}\left(Q^\top(x - h)\right)\right).$$

Here and in what follows, $\cdot^\top$ denotes the matrix transposition and the dot above a function means its time derivative.

The system describing the motion of the swimmer is given by:

$$-\Delta u^\dagger + \nabla p^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(h, Q), \tag{2.6a}$$

$$\operatorname{div} u^\dagger = 0 \qquad \text{in } \mathcal{F}^\dagger(h, Q), \tag{2.6b}$$

$$u^\dagger(t, x) = v^\dagger(t, x) \qquad \text{on } \mathcal{S}^\dagger(h, Q), \tag{2.6c}$$

$$\lim_{|x| \to \infty} u^\dagger(t, x) = 0\,, \tag{2.6d}$$

$$\int_{\mathcal{S}^\dagger(h,Q)} \sigma(u^\dagger, p^\dagger)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.6e}$$

$$\int_{\mathcal{S}^\dagger(h,Q)} (x - h) \times \sigma(u^\dagger, p^\dagger)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.6f}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{F}^\dagger(h,Q)$ and where we have used the notation

$$\sigma(u^\dagger,p^\dagger) := 2\mathrm{D}(u^\dagger) - p^\dagger \mathrm{I}_3\,, \qquad \mathrm{D}(u^\dagger) := \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right).$$

The functions $u^\dagger$ and $p^\dagger$ are respectively the velocity and the pressure of the fluid. Equations (2.6a) and (2.6b) are the Stokes system, (2.6c) corresponds to the no-slip boundary condition. Finally, (2.6e) and (2.6f) are the Newton laws with the hypotheses that the inertial effects can be neglected.

We then perform a change of variable to work in a referential attached to the swimmer: we set

$$u^\dagger(t,x) = Q(t)u\left(t, Q(t)^\top(x-h(t))\right), \qquad p^\dagger(t,x) = p\left(t, Q(t)^\top(x-h(t))\right),$$

$$\ell(t) = Q(t)^\top \dot h(t)\,, \qquad Q(t)^\top \dot Q(t) = \mathbb{A}(\omega(t))\,,$$

where for any $\omega\in\mathbb{R}^3$,

$$\mathbb{A}(\omega) := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)\,.$$

After some calculation, we obtain the following system:

$$-\Delta u + \nabla p = 0 \qquad \text{in } \mathcal{F}_c, \tag{2.7a}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathcal{F}_c, \tag{2.7b}$$

$$u(t,x) = \ell(t) + \omega(t)\times x + \frac{\mathrm{d}\,X_c(s)}{\mathrm{d}t}\left(X_c(s)^{-1}(x)\right) \qquad \text{on } \mathcal{S}_c, \tag{2.7c}$$

$$\lim_{|x|\to\infty} u(t,x) = 0\,, \tag{2.7d}$$

$$\int_{\mathcal{S}_c} \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.7e}$$

$$\int_{\mathcal{S}_c} x\times\sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 0\,, \tag{2.7f}$$

$$\dot h = Q\ell\,, \tag{2.7g}$$

$$\dot Q = Q\mathbb{A}(\omega)\,. \tag{2.7h}$$

Let us remark that the above system can be written as a dynamical system with state (h,Q,s) and control input $\dot s$, see Section 3, eqs. (3.5) and (3.8).

2.3 Main results

We are now in position to state the main result of this paper:

Theorem 2.5. *Given $d\geqslant 3$, $\varepsilon,\eta>0$, $\overline{c}=(\overline{\Psi}_0,\overline{\delta})\in\mathcal{SC}^2(d)$, $T>0$ and $(\overline{h},\overline{Q},\overline{s})\in C^0\big([0,T],\mathbb{R}^3\times\mathcal{SO}(3)\times\mathcal{J}(\overline{\delta})\big)$.*

There exists $c=(\Psi_0,\delta)\in\mathcal{SC}^\infty(d)$ such that

$$\|c-\overline{c}\| < \varepsilon\,,$$

and there exists $s\in C^\infty([0,T],\mathbb{R}^d)$, with

$$s(t)\in\mathcal{J}(\delta)\,, \qquad s(0)=\overline{s}(0)\,, \qquad s(T)=\overline{s}(T) \quad \textit{and} \quad |s(t)-\overline{s}(t)|\leqslant\eta \qquad (t\in[0,T]),$$

such that the corresponding solution (h,Q) of (2.7) with initial conditions

$$h(0)=\overline{h}(0)\,, \quad Q(0)=\overline{Q}(0)$$

satisfies

$$\sup_{t\in[0,T]}\big(|h(t)-\overline{h}(t)|+|Q(t)-\overline{Q}(t)|\big)<\eta$$

together with

$$h(T)=\overline{h}(T)\,, \quad Q(T)=\overline{Q}(T)\,.$$

The proof of this theorem in given in Section 4.5.

Remark 2.6. In particular the set of pairs $c = (\Psi_0, \delta)$ such that the system (2.7) is controllable is an open dense set of $\mathcal{SC}^2(d)$.

Remark 2.7. In the above statement, we can in particular choose $\overline{s}$, so that $\overline{s}(0) = \overline{s}(T)$ and we have a periodic control.

Based on this result, we can also derive the existence of an optimal control we also refer to [12] for similar optimal control problems.

Theorem 2.8. *Given $d \geqslant 3$ and $c = (\Psi_0, \delta) \in \mathcal{SC}^2(d)$ such that the system (2.7) is controllable and set Λ a compact of $\mathbb{R}^d$ containing 0 in its interior and K a compact set of $\mathcal{J}(\delta)$ which is connected by C^1-arcs and has a nonempty interior. Let $g \in C^0(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that g is convex with respect to the fifth variable.*

Given $(h^0, Q^0), (h^1, Q^1) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ and s^0, $s^1 \in K$, we have:

1. *there exists $T_* > 0$ such that for every $T > T_*$, the optimal control problem*

$$\left|\begin{array}{ll} \min & \displaystyle\int_0^T g\big(t, h(t), Q(t), s(t), \dot{s}(t)\big)\,\mathrm{d}t \\ & s \in W^{1,\infty}(0,T)^d, \\ & s(t) \in K \qquad (t \in [0,T]), \\ & \dot{s}(t) \in \Lambda \qquad (t \in (0,T) \ \textit{a.e.}), \\ & (h,Q) \ \textit{solution of (2.7)}, \\ & h(0) = h^0, \quad Q(0) = Q^0, \quad s(0) = s^0, \\ & h(T) = h^1, \quad Q(T) = Q^1, \quad s(T) = s^1. \end{array}\right. \tag{2.8}$$

   admits a solution;

2. *the time optimal control problem*

$$\left|\begin{array}{ll} \min & T \\ & T > 0\,, \\ & s \in W^{1,\infty}(0,T)^d, \\ & s(t) \in K \qquad (t \in [0,T]), \\ & \dot{s}(t) \in \Lambda \qquad (t \in (0,T) \ \textit{a.e.}), \\ & (h,Q) \ \textit{solution of (2.7)}, \\ & h(0) = h^0, \quad Q(0) = Q^0, \quad s(0) = s^0, \\ & h(T) = h^1, \quad Q(T) = Q^1, \quad s(T) = s^1. \end{array}\right. \tag{2.9}$$

   admits a solution.

Proof. Let us scratch the proof for the first optimal control problem, that is (2.8).

We apply the Filippov theorem [11, Theorem 9.3.i p. 314] and its extension, see [11, § 9.5 p. 318]. One can check that conditions [11, (a), (b), (c) p. 317] are fulfilled with the above hypotheses.

The main difficulty is to check the existence of an admissible control, i.e. that there exists a triplet (h, Q, s) satisfying the constraints of (2.8). To this end, we are going to construct a trajectory on the time interval $[0, 1]$ satisfying the constraint on s. First of all, since K is connected by C^1-arcs and since the interior of K is nonempty, there exist a point $\overline{s}$ in the interior of K and two C^1-arcs $\widehat{s}_0 : [0, 1/3] \to K$ and $\widehat{s}_1 : [2/3, 1] \to K$ such that

$$\widehat{s}_0(0) = s^0, \quad \widehat{s}_0(1/3) = \overline{s}, \quad \widehat{s}_1(2/3) = \overline{s}, \quad \widehat{s}_1(2/3) = s^1.$$

Let us then define $(\widehat{h}^0, \widehat{Q}^0) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ the final value of the solution of (2.7) in $[0, 1/3]$ with initial condition (h^0, Q^0) and control $\widehat{s}_0$. Similarly, we define $(\widehat{h}^1, \widehat{Q}^1) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ the initial condition such that the solution of (2.7) in $[2/3, 1]$ with initial condition (at 2/3) $(\widehat{h}^1, \widehat{Q}^1)$ and control $\widehat{s}_1$ reaches (h^1, Q^1) at the final time (such a construction can be obtained by time reversion).
We conclude, using Theorem 2.5, together with the fact that $\overline{s}$ is in the interior of K, that there exists a control $\widehat{s}_{1/2} \in C^1([1/3, 2/3], \mathbb{R}^d)$ steering $(\widehat{h}^0, \widehat{Q}^0)$ to $(\widehat{h}^1, \widehat{Q}^1)$ and such that $\widehat{s}_{1/2}(t) \in K$ for every $t \in [1/3, 2/3]$.
All in all, by concatenation of $\widehat{s}_0$, $\widehat{s}_{1/2}$ and $\widehat{s}_1$, we have build a control $\widehat{s} \in W^{1,\infty}([0, 1], \mathbb{R}^d)$ steering (h^0, Q^0) to (h^1, Q^1) and such that $\widehat{s}(t) \in K$ for every $t \in [0, 1]$.
Nevertheless, the property $\mathrm{d}\widehat{s}(t)/\mathrm{d}t \in \Lambda$ may not hold. For $T > 0$, we take the control $s(t) = \widehat{s}(t/T)$ and we see that $s(t) \in K$ for every $t \in [0, T]$ and this control steers (h^0, Q^0) to (h^1, Q^1) in time T. Furthermore, we have

$\sup_{[0,T]} |\dot{s}| = \frac{1}{T} \sup_{[0,1]} |\mathrm{d}\widehat{s}/\mathrm{d}t|$. Since $\widehat{s} \in W^{1,\infty}([0,1], \mathbb{R}^d)$ and since 0 is an interior point of Λ, we conclude that for T larger than some T_* (depending on $\widehat{s}$ and Λ), s is an admissible control.

For the time optimal control problem, namely (2.9), the proof is similar and relies on [11, Theorem 9.2.i p. 311] and its extensions. □

3 Rewriting the system

This section is devoted to rewrite system (2.7) as a nonlinear finite-dimensional control problem (system (3.8)) and to compute Lie brackets that will be useful to apply the Rashevsky-Chow Theorem.

From now on, we assume $k \geqslant 2$. It is used in the regularity of the solution of the Stokes system.

3.1 Decomposition of the system

Let us rewrite the velocity corresponding to $X_c(s)$: we first define for every $i \in \{1, \cdots, d\}$,

$$D_c^i(s) = \partial_{s_i} X_c(s) \circ X_c(s)^{-1}\,, \tag{3.1}$$

and

$$\lambda = \dot{s}\,.$$

Then (2.7c) writes

$$u(t,x) = \sum_{i=1}^{3} \ell_i \boldsymbol{e}_i + \sum_{i=1}^{3} \omega_i \left(\boldsymbol{e}_i \times x\right) + \sum_{i=1}^{d} \lambda_i D_c^i(s)(x) \qquad (x \in \mathcal{S}_c),$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$. This leads to consider the following Stokes systems

$$\begin{aligned} -\Delta u_c^i + \nabla p_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} u_c^i &= 0 && \text{in } \mathcal{F}_c,\\ u_c^i &= \boldsymbol{e}_i && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} u_c^i(x) &= 0, \end{aligned} \qquad (i \in \{1,2,3\}), \tag{3.2a}$$

$$\begin{aligned} -\Delta u_c^i + \nabla p_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} u_c^i &= 0 && \text{in } \mathcal{F}_c,\\ u_c^i &= \boldsymbol{e}_{i-3} \times x && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} u_c^i(x) &= 0, \end{aligned} \qquad (i \in \{4,5,6\}), \tag{3.2b}$$

$$\begin{aligned} -\Delta v_c^i + \nabla q_c^i &= 0 && \text{in } \mathcal{F}_c,\\ \operatorname{div} v_c^i &= 0 && \text{in } \mathcal{F}_c,\\ v_c^i &= D_c^i(s) && \text{on } \mathcal{S}_c,\\ \lim_{|x|\to\infty} v_c^i(x) &= 0, \end{aligned} \qquad (i \in \{1, \cdots, d\}). \tag{3.2c}$$

Notice that v_c^i and q_c^i are also functions of s. In (3.2), the pairs (u_c^i, p_c^i) and (v_c^i, q_c^i) are well-defined in $\left(D^{1,2}(\mathcal{F}_c)^3 \cap H^2_{loc}(\overline{\mathcal{F}_c})^3\right) \times \left(L^2(\mathcal{F}_c) \cap H^1_{loc}(\overline{\mathcal{F}_c})\right)$, where $D^{1,2}(\mathcal{F}_c) = \left\{f \in L^2_{loc}(\mathcal{F}_c)\,,\ \nabla f \in L^2(\mathcal{F}_c)^3\right\}$. We refer to [18, Lemma V.1.1 p. 305, Theorem V.1.1 p. 306] for the well-posedness of the exterior Stokes problem.

Then

$$u = u(\ell, \omega, \lambda, s) := \sum_{i=1}^{3} \ell_i u_c^i + \sum_{i=1}^{3} \omega_i u_c^{i+3} + \sum_{i=1}^{d} \lambda_i v_c^i(s)$$

satisfies (2.7a)–(2.7c). In that case, (2.7e) and (2.7f) can also be rewritten. Indeed, after an integration by parts and using [18, Theorem V.3.2 p. 314], we find

$$\int_{\mathcal{S}_c} \boldsymbol{e}_i \cdot \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^i)\,\mathrm{d}x$$

and

$$\int_{\mathcal{S}_c} (\boldsymbol{e}_i \times x) \cdot \sigma(u,p)\mathbf{n}\,\mathrm{d}\Gamma = 2\int_{\mathcal{F}_c} \mathrm{D}(u) : \mathrm{D}(u_c^{i+3})\,\mathrm{d}x\,,$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathcal{F}_c$.

We define the matrices $K_c \in \mathcal{M}_6(\mathbb{R})$ and $N_c(s) \in \mathcal{M}_{6,d}(\mathbb{R})$ by:

$$K_c = 2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i) : \mathrm{D}(u_c^j)\,\mathrm{d}x\right)_{i,j\in\{1,\dots,6\}} \qquad \text{and} \qquad N_c(s) = -2\left(\int_{\mathcal{F}_c} \mathrm{D}(u_c^i) : \mathrm{D}(v_c^j)\,\mathrm{d}x\right)_{\substack{i\in\{1,\dots,6\}\\ j\in\{1,\dots,d\}}}, \tag{3.3}$$

so that relations (2.7e) and (2.7f) are equivalent to

$$K_c \begin{pmatrix} \ell \\ \omega \end{pmatrix} = N_c(s)\lambda\,. \tag{3.4}$$

The following result holds (see [24]):

Lemma 3.1. *Given $k \geqslant 2$, the mapping $(c,s) \in \mathcal{A}^k(d) \mapsto (K_c, N_c(s)) \in \mathcal{M}_6(\mathbb{R}) \times \mathcal{M}_{6,d}(\mathbb{R})$ is analytic and for every c, K_c is positive definite.*

We recall that $\mathcal{A}^k(d)$ is defined by (2.5). We refer to [36] for the definitions and properties of analytic functions in Banach spaces.

Finally, (2.7) writes

$$\dot{h} = Q\ell\,, \tag{3.5a}$$

$$\dot{Q} = Q\mathbb{A}(\omega)\,, \tag{3.5b}$$

$$\dot{s} = \lambda\,, \tag{3.5c}$$

$$\begin{pmatrix} \ell \\ \omega \end{pmatrix} = K_c^{-1} N_c(s)\lambda\,. \tag{3.5d}$$

This shows that (2.7) is a finite dimensional nonlinear dynamical system with control s. Since we also want to impose some conditions on s, we put s in the state of the system and the control of this extended system is λ.

3.2 Formulation of the system in a Lie group

Let us define:

$$P(h,Q,s) = \begin{pmatrix} \begin{pmatrix} Q & h \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} \mathrm{I}_d & s \\ 0 & 1 \end{matrix} \end{pmatrix} \in \mathcal{M}_{d+5}(\mathbb{R}) \qquad ((h,Q,s) \in \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R}) \times \mathbb{R}^d)$$

and

$$E(3,d) = \left\{ P(h,Q,s)\,,\ (h,Q,s) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \right\} \subset \mathcal{GL}_{d+5}(\mathbb{R})\,.$$

Notice that the map $P : \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d \to E(3,d)$ is a bijection. In addition, endowed with the matrix product, $E(3,d)$ is a Lie group whose Lie algebra is:

$$\mathfrak{e}(3,d) = \left\{ \mathfrak{p}(\ell,\omega,\lambda)\,,\ (\ell,\omega,\lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d \right\},$$

with,

$$\mathfrak{p}(\ell,\omega,\lambda) = \mathfrak{q}(\ell,\mathbb{A}(\omega),\lambda)$$

$$\text{and with} \quad \mathfrak{q}(\ell,M,s) = \begin{pmatrix} \begin{pmatrix} M & \ell \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & \lambda \\ 0 & 0 \end{matrix} \end{pmatrix} \in \mathcal{M}_{d+5}(\mathbb{R}) \qquad ((\ell,\omega,\lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d\,,\ M \in \mathcal{M}_3(\mathbb{R}))\,.$$

Clearly, $\mathfrak{p} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^d \to \mathfrak{e}(3,d)$ is a bijection.

Let us finally define:

$$I(Q) = P(0, Q, 0) \in E(3, d) \qquad (Q \in \mathcal{SO}(3)),$$

so that we have:

$$\mathrm{T}_{(h,Q,s)}\left(\mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^d\right) = \mathfrak{q}^{-1}\big(I(Q)\mathfrak{e}(3, d)\big)\,.$$

Let us define for every $j \in \{1, \dots, d\}$,

$$V_c^j(s) = \begin{pmatrix} \ell_c^j(s) \\ \omega_c^j(s) \end{pmatrix} = K_c^{-1} N_c(s) \boldsymbol{e}_j\,, \tag{3.6}$$

the j^{th} column of $K_c^{-1}N_c(s)$, with $\{\boldsymbol{e}_j\}_{j\in\{1,\dots,d\}}$ the canonical basis of $\mathbb{R}^d$. With such a notation, (3.5d) becomes,

$$\begin{pmatrix} \ell \\ \omega \end{pmatrix} = \sum_{j=1}^{d} \lambda_j \begin{pmatrix} \ell_c^j \\ \omega_c^j \end{pmatrix}.$$

Let us also define:

$$\mathfrak{f}_c^j(s) = \mathfrak{p}\left(\ell_c^j(s), \omega_c^j(s), \boldsymbol{e}_j\right) \in \mathfrak{e}(3, d) \qquad \text{and} \qquad f_c^j(h, Q, s) = I(Q)\mathfrak{f}_c^j(s)\,.$$

Based on Lemma 3.1 we obtain the following lemma.

Lemma 3.2. *Given $k \geqslant 2$ and $d \in \mathbb{N}^*$, for every $j \in \{1, \cdots, d\}$, the map $((c, s), h, Q) \in \mathcal{A}^k(d) \times \mathbb{R}^3 \times \mathcal{SO}(3) \mapsto f_c^j(h, Q, s) \in \mathcal{M}_{d+5}(\mathbb{R})$ is analytic.*

Relation (3.5) now reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}P(h, Q, s) = \begin{pmatrix} \begin{pmatrix} \dot{Q} & \dot{h} \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & \dot{s} \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} Q\mathbb{A}(\omega) & Q\ell \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & \lambda \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \end{pmatrix} = I(Q)\mathfrak{p}(\ell, \omega, \lambda) \\ &= I(Q)\sum_{j=1}^{d} \mathfrak{p}(\ell_c^j, \omega_c^j, \boldsymbol{e}_j)\lambda_j = \sum_{j=1}^{d} I(Q)\mathfrak{f}_c^j(s)\lambda_j = \sum_{j=1}^{d} f_c^j(h, Q, s)\lambda_j \end{aligned} \tag{3.7}$$

and can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} h \\ Q \\ s \end{pmatrix} = \sum_{j=1}^{d} \mathfrak{q}^{-1}\left(f_c^j(h, Q, s)\right)\lambda_j\,. \tag{3.8}$$

From [24, Proposition 1.6] (see also [17]), we deduce

Proposition 3.3. *Let $k \in \mathbb{N} \cup \{\infty\}$ with $k \geqslant 2$, $d \in \mathbb{N}^*$, $(\psi_0, \delta) \in \mathcal{SC}^k(d)$, $T > 0$ and $\lambda \in L^1_{loc}(\mathbb{R}_+)^d$ (respectively $\lambda \in C^{p-1}(\mathbb{R}_+)^d$, $p \in \mathbb{N}^*$).*

Then for every $(h_0, Q_0, s_0) \in \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\delta)$, the system (3.8) endowed with the initial condition $(h, Q, s)(0) = (h_0, Q_0, s_0)$ and control λ admits a unique maximal solution (h, Q, s) which is absolutely continuous (respectively of class C^p).

Furthermore, if for every $t \in [0, T]$, $s(t) = s_0 + \displaystyle\int_0^t \lambda(\tau)\,\mathrm{d}\tau$ belongs to $\mathcal{J}(\delta)$, then the solution (h, Q, s) of (3.8) endowed with the initial condition $(h, Q, s)(0) = (h_0, Q_0, s_0)$ is well-defined on $[0, T]$.

3.3 Lie brackets computations

Let us now compute the Lie brackets of the system (3.7). We have

$$\partial_h f_c^i(h, Q, s) \cdot \tilde{h} = 0\,, \quad \partial_s f_c^i(h, Q, s) \cdot \boldsymbol{e}_j = I(Q)\partial_{s_j}\mathfrak{f}_i(s) \quad \text{and} \quad \partial_Q f_c^i(h, Q, s) \cdot (Q\mathbb{A}(\omega)) = I(Q)\mathfrak{p}(0, \omega, 0)\mathfrak{f}_i(s)\,.$$

In order to make relations shorter, we set $\partial_{i_1,\dots,i_n}$ for $\partial_{s_{i_1}} \cdots \partial_{s_{i_n}}$. This notation will be used all along the article.

For $i,j\in\{1,\cdots,d\}$, we have

$$\begin{aligned}
[f_c^j,f_c^i] &= \partial_{(h,Q,s)}f_c^i\cdot f_c^j-\partial_{(h,Q,s)}f_c^j\cdot f_c^i\\
&= I(Q)\left(\partial_j\mathfrak{f}_i-\partial_i\mathfrak{f}_j+\mathfrak{p}(0,\omega_c^j,0)\mathfrak{f}_i-\mathfrak{p}(0,\omega_c^i,0)\mathfrak{f}_j\right)\\
&= I(Q)\left(\mathfrak{p}\left(\partial_j\ell_c^i-\partial_i\ell_c^j,\partial_j\omega_c^i-\partial_i\omega_c^j,0\right)+\begin{pmatrix}\mathbb{A}(\omega_c^j)\mathbb{A}(\omega_c^i)-\mathbb{A}(\omega_c^i)\mathbb{A}(\omega_c^j) & \mathbb{A}(\omega_c^j)\ell_c^i-\mathbb{A}(\omega_c^i)\ell_c^j & 0\\ 0 & 0 & 0\end{pmatrix}\right)\\
&= I(Q)\left(\mathfrak{p}\left(\partial_j\ell_c^i-\partial_i\ell_c^j,\partial_j\omega_c^i-\partial_i\omega_c^j,0\right)+\mathfrak{p}\left(\omega_c^j\times\ell_c^i-\omega_c^i\times\ell_c^j,\omega_c^j\times\omega_c^i,0\right)\right)\\
&= I(Q)\,\tilde{\mathfrak{p}}\left(\partial_jV_c^i-\partial_iV_c^j+V_c^j\wedge V_c^i\right),
\end{aligned}\tag{3.9}$$

reminding that $V_c^i=\begin{pmatrix}\ell_c^i\\ \omega_c^i\end{pmatrix}$ and where we have defined

$$\begin{pmatrix}\ell_c^j\\ \omega_c^j\end{pmatrix}\wedge\begin{pmatrix}\ell_c^i\\ \omega_c^i\end{pmatrix}=\begin{pmatrix}\omega_c^j\times\ell_c^i-\omega_c^i\times\ell_c^j\\ \omega_c^j\times\omega_c^i\end{pmatrix}\in\mathbb{R}^3\times\mathbb{R}^3\qquad(\ell_c^i,\ell_c^j,\omega_c^i,\omega_c^j\in\mathbb{R}^3)$$

$$\text{and}\qquad\tilde{\mathfrak{p}}\left(\begin{pmatrix}l\\ \omega\end{pmatrix}\right)=\mathfrak{p}(l,\omega,0)\qquad(l,\omega\in\mathbb{R}^3).\tag{3.10}$$

Let us also express the third order Lie brackets which will be useful in the following. With a similar computation to the one done in (3.9), we have, for every $i,j,k\in\{1,\cdots,d\}$,

$$[f_c^k,[f_c^j,f_c^i]]=I(Q)\,\tilde{\mathfrak{p}}\left(\partial_k\left(\partial_jV_c^i-\partial_iV_c^j\right)+\partial_kV_c^j\wedge V_c^i+V_c^j\wedge\partial_kV_c^i+V_c^k\wedge\left(\partial_jV_c^i-\partial_iV_c^j+V_c^j\wedge V_c^i\right)\right).\tag{3.11}$$

By induction, the following result can be easily proved.

Lemma 3.4. *Let $d\geqslant 2$, $(h,Q)\in\mathbb{R}^3\times\mathcal{SO}(3)$ and $(c,s)\in\mathcal{A}^\infty(d)$ then we have*

$$\dim\mathrm{Lie}_{(h,Q,s)}\left\{f_c^1,\cdots,f_c^d\right\}=\dim\mathrm{Lie}_{(0,\mathrm{I}_3,s)}\left\{f_c^1,\cdots,f_c^d\right\}\qquad((h,Q)\in\mathbb{R}^3\times\mathcal{SO}(3)).$$

Furthermore,

$$\begin{aligned}
d+6\geqslant\dim\mathrm{Lie}_{(0,\mathrm{I}_3,s)}&\left\{f_c^1,\cdots,f_c^d\right\}\\
&\geqslant d+\dim\Big(\mathrm{Span}\left\{\tilde{\mathfrak{p}}^{-1}([f_c^j,f_c^i](0,\mathrm{I}_3,s)),\ i,j\in\{1,\cdots,d\}\right\}\\
&\qquad+\mathrm{Span}\left\{\tilde{\mathfrak{p}}^{-1}([f_c^k,[f_c^j,f_c^i]](0,\mathrm{I}_3,s)),\ i,j,k\in\{1,\cdots,d\}\right\}\Big)\\
&\geqslant d+\dim\mathrm{Span}\left\{\tilde{\mathfrak{p}}^{-1}([f_c^j,f_c^i](0,\mathrm{I}_3,s)),\ i,j\in\{1,\cdots,d\}\right\}.
\end{aligned}$$

In order to compute these Lie brackets, one has to compute the derivatives of $s\mapsto V_c^i(s)$, where V_c^i is defined by (3.6). That is to say that we have to compute:

$$\begin{aligned}
\partial_s^\alpha N(s)\boldsymbol{e}_j&=-2\left(\partial_s^\alpha\left(\int_{\mathcal{F}_c}\mathrm{D}(u_c^i):\mathrm{D}\big(v_c^j(s)\big)\,\mathrm{d}x\right)\right)_{i\in\{1,\dots,6\}}\\
&=-2\left(\int_{\mathcal{F}_c}\mathrm{D}(u_c^i):\mathrm{D}\left(\partial_s^\alpha v_c^j(s)\right)\,\mathrm{d}x\right)_{i\in\{1,\dots,6\}}=-\begin{pmatrix}\displaystyle\int_{\mathcal{S}_c}\sigma\left(\partial_s^\alpha v_c^j(s),\partial_s^\alpha q_c^j(s)\right)\mathbf{n}\,\mathrm{d}x\\ \displaystyle\int_{\mathcal{S}_c}x\times\sigma\left(\partial_s^\alpha v_c^j(s),\partial_s^\alpha q_c^j(s)\right)\mathbf{n}\,\mathrm{d}x\end{pmatrix}
\end{aligned}\tag{3.12}$$

for $j\in\{1,\cdots,d\}$ and for $\alpha\in\mathbb{N}^d$.
In the above expression, $v_c^j(s)$ and $q_c^j(s)$ are the solutions of (3.2c). In particular, $(\partial_s^\alpha v_c^j(s),\partial_s^\alpha q_c^j(s))$ is solution of the following system

$$\begin{aligned}
-\Delta(\partial_s^\alpha v_j)+\nabla(\partial_s^\alpha q_j)&=0 &&\text{in }\mathcal{F}_c,\\
\mathrm{div}(\partial_s^\alpha v_j)&=0 &&\text{in }\mathcal{F}_c,\\
\partial_s^\alpha v_j(s)&=\partial_s^\alpha D_c^j(s) &&\text{on }\mathcal{S}_c,
\end{aligned}\tag{3.13}$$

with D_c^i defined by (3.1). In general, it is very difficult to an explicit formula for $\partial_s^\alpha N(s)\boldsymbol{e}_j$, but this can be done in the case of the sphere and for particular boundary conditions.

4 The case of the unit sphere

In this section we consider the situation where $\mathcal{S}_0 = \mathbb{S}^2$ and namely the case where $\Psi_0 = 0$.

4.1 Derivation of boundary conditions

In this paragraph, we compute the expressions of D_c^i given by (3.1) for $\Psi_0 = 0$ at $s = 0$. In that case, we have:

Proposition 4.1. *Let $d \geqslant 1$, $i, j, k \in \{1, \cdots, d\}$, $\delta = (\delta_1, \cdots, \delta_d) \in C^2(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$ and $c = (0, \delta)$. We have*

$$\begin{aligned}
D_c^i(0) &= \delta_i \,, \\
\partial_j D_c^i(0) &= -\,\mathrm{G}_\Gamma\, \delta_i \cdot \delta_j \,, \\
\partial_{k,j} D_c^i(0) &= \frac{1}{6}\big(-2\,\langle \delta_j, \delta_k\rangle\, \delta_i + \langle \delta_i, \delta_k\rangle\, \delta_j + \langle \delta_i, \delta_j\rangle\, \delta_k\big) \\
&\qquad + \frac{1}{2}\big(\mathrm{G}_\Gamma\,(\mathrm{G}_\Gamma\,\delta_i \cdot \delta_k) \cdot \delta_j + \mathrm{G}_\Gamma\,(\mathrm{G}_\Gamma\,\delta_i \cdot \delta_j) \cdot \delta_k + \mathrm{G}_\Gamma\,\delta_i \cdot (\mathrm{G}_\Gamma\,\delta_j \cdot \delta_k + \mathrm{G}_\Gamma\,\delta_k \cdot \delta_j)\big) \,.
\end{aligned}$$

In the above relations, the differential operator G_Γ is defined by (B.3)

This result in obtained by combining Lemmas 4.2 and 4.3.

Lemma 4.2. *For $c = (0, \delta) \in \mathcal{SC}^2(d)$, we have at $s = 0$, for $i, j, k \in \{1, \cdots, d\}$,*

$$\begin{aligned}
D_c^i &= \partial_i \mathcal{X}_\delta \,, \\
\partial_j D_c^i &= \partial_{j,i} \mathcal{X}_\delta - \nabla D_c^i \cdot D_c^j \,, \\
\partial_{k,j} D_c^i &= \partial_{k,j,i} \mathcal{X}_\delta - \Big(\nabla \partial_k D_c^i \cdot D_c^j + \nabla D_c^i \cdot \partial_k D_c^j + \nabla \partial_{i,j} \mathcal{X}_\delta \cdot D_c^k\Big) \,.
\end{aligned}$$

Proof. Let us first notice that for $\Psi_0 = 0$, we have $D_c^i = \partial_i \mathcal{X}_\delta \circ \mathcal{X}_\delta^{-1}$ or equivalently,

$$\partial_i \mathcal{X}_\delta = D_c^i \circ \mathcal{X}_\delta$$

and hence, at $s = 0$ ($\mathcal{X}_\delta(0) = \mathrm{Id}_{\mathbb{S}^2}$),

$$D_c^i = \partial_i \mathcal{X}_\delta \,;$$

- 1^{st} derivative:

$$\partial_{j,i} \mathcal{X}_\delta = \partial_j D_c^i \circ \mathcal{X}_\delta + \nabla D_c^i \circ \mathcal{X}_\delta \cdot \partial_j \mathcal{X}_\delta = \Big(\partial_j D_c^i + \nabla D_c^i \cdot D_c^j\Big) \circ \mathcal{X}_\delta$$

  and hence, at $s = 0$,

$$\partial_j D_c^i = \partial_{j,i} \mathcal{X}_\delta - \nabla D_c^i \cdot D_c^j \,;$$

- 2^{nd} derivative:

$$\partial_{k,j,i} \mathcal{X}_\delta = \Big(\partial_{k,j} D_c^i + \nabla \partial_k D_c^i \cdot D_c^j + \nabla D_c^i \cdot \partial_k D_c^j + \nabla \Big(\partial_j D_c^i + \nabla D_c^i \cdot D_c^j\Big) \cdot D_c^k\Big) \circ \mathcal{X}_\delta$$

  and hence, at $s = 0$,

$$\partial_{k,j} D_c^i = \partial_{k,j,i} \mathcal{X}_\delta - \Big(\nabla \partial_k D_c^i \cdot D_c^j + \nabla D_c^i \cdot \partial_k D_c^j + \nabla \partial_{i,j} \mathcal{X}_\delta \cdot D_c^k\Big) \,.$$

□

Let us now compute the derivatives of $\mathcal{X}_\delta$.

Lemma 4.3. *For $\delta = (\delta_1, \cdots, \delta_d) \in C^2(\mathbb{S}^2, \mathrm{T}\mathbb{S}^2)^d$, we have at $s = 0$, for $i, j, k \in \{1, \cdots, d\}$,*

$$\begin{aligned}
\mathcal{X}_\delta(s)|_{s=0} &= \mathrm{Id}_{\mathbb{S}^2} \,, \\
\partial_i \mathcal{X}_\delta(s)|_{s=0} &= \delta_i \,, \\
\partial_{j,i} \mathcal{X}_\delta(s)|_{s=0} &= -\,\langle \delta_i, \delta_j\rangle\, \mathrm{Id}_{\mathbb{S}^2} \,, \\
\partial_{k,j,i} \mathcal{X}_\delta(s)|_{s=0} &= \frac{-1}{3}\big(\langle \delta_j, \delta_k\rangle\, \delta_i + \langle \delta_i, \delta_k\rangle\, \delta_j + \langle \delta_i, \delta_j\rangle\, \delta_k\big) \,.
\end{aligned}$$

Proof. To simplify the notation, we set $\mathcal{X} = \mathcal{X}_\delta$ and $\Theta = \Theta_\delta(s) = \sum_{i=1}^d s_i\delta_i$.

For every $n \in \mathbb{N}^*$, set $A_n = ((2n+1)\,\mathrm{Id}_{\mathbb{S}^2} + \Theta)$, so that, according to (2.2), we have $\mathcal{X} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, A_n|\Theta|^{2n}$.

Then, for every $n \in \mathbb{N}^*$, we have:

- 1^{st} derivative:
$$\partial_i \left(A_n|\Theta|^{2n}\right) = \delta_i|\Theta|^{2n} + 2nA_n\langle\delta_i,\Theta\rangle|\Theta|^{2n-2}$$
and hence,
$$\partial_i \left(A_n|\Theta|^{2n}\right)\big|_{s=0} = \begin{cases}\delta_i & \text{if } n=0\,,\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \partial_i\mathcal{X}|_{s=0} = \delta_i\,;$$

- 2^{nd} derivative:
$$\begin{aligned}\partial_{j,i} \left(A_n|\Theta|^{2n}\right) &= 2n\delta_i\,\langle\delta_j,\Theta\rangle\,|\Theta|^{2n-2} + 2n\delta_j\,\langle\delta_i,\Theta\rangle\,|\Theta|^{2n-2}\\ &\qquad + 2nA_n\,\langle\delta_i,\delta_j\rangle\,|\Theta|^{2n-2} + 2n(2n-2)A_n\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\,|\Theta|^{2n-4}\\ &= 2n\left(\left(\delta_i\,\langle\delta_j,\Theta\rangle + \delta_j\,\langle\delta_i,\Theta\rangle + A_n\,\langle\delta_i,\delta_j\rangle\right)|\Theta|^{2n-2} + (2n-2)A_n\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\,|\Theta|^{2n-4}\right)\end{aligned}$$
and hence,
$$\partial_{j,i} \left(A_n|\Theta|^{2n}\right)\big|_{s=0} = \begin{cases}6\,\langle\delta_i,\delta_j\rangle\,\mathrm{Id}_{\mathbb{S}^2} & \text{if } n=1\,,\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \partial_{j,i}\mathcal{X}|_{s=0} = -\,\langle\delta_i,\delta_j\rangle\,\mathrm{Id}_{\mathbb{S}^2}\,;$$

- 3^{rd} derivative:
$$\begin{aligned}\partial_{k,j,i} \left(A_n|\Theta|^{2n}\right) &= 2n\left(\left(\delta_i\,\langle\delta_j,\delta_k\rangle + \delta_j\,\langle\delta_i,\delta_k\rangle + \delta_k\,\langle\delta_i,\delta_j\rangle\right)|\Theta|^{2n-2}\right.\\ &\qquad +(2n-2)\left(\delta_i\,\langle\delta_j,\Theta\rangle + \delta_j\,\langle\delta_i,\Theta\rangle + A_n\,\langle\delta_i,\delta_j\rangle\right)\langle\delta_k,\Theta\rangle\,|\Theta|^{2n-4}\\ &\qquad +(2n-2)\delta_k\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\,|\Theta|^{2n-4}\\ &\qquad +(2n-2)A_n\,\langle\delta_i,\delta_k\rangle\,\langle\delta_j,\Theta\rangle\,|\Theta|^{2n-4} + (2n-2)A_n\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\delta_k\rangle\,|\Theta|^{2n-4}\\ &\qquad \left.+(2n-2)(2n-4)A_n\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\,\langle\delta_k,\Theta\rangle\,|\Theta|^{2n-6}\right)\\ &= 2n\left(\left(\delta_i\,\langle\delta_j,\delta_k\rangle + \delta_j\,\langle\delta_i,\delta_k\rangle + \delta_k\,\langle\delta_i,\delta_j\rangle\right)|\Theta|^{2n-2}\right.\\ &\qquad +(2n-2)\left(\delta_i\,\langle\delta_j,\Theta\rangle\,\langle\delta_k,\Theta\rangle + \delta_j\,\langle\delta_i,\Theta\rangle\,\langle\delta_k,\Theta\rangle + \delta_k\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\right.\\ &\qquad\qquad \left.+A_n\left(\langle\delta_i,\delta_j\rangle\,\langle\delta_k,\Theta\rangle + \langle\delta_i,\delta_k\rangle\,\langle\delta_j,\Theta\rangle + \langle\delta_i,\Theta\rangle\,\langle\delta_j,\delta_k\rangle\right)\right)|\Theta|^{2n-4}\\ &\qquad \left.+(2n-2)(2n-4)A_n\,\langle\delta_i,\Theta\rangle\,\langle\delta_j,\Theta\rangle\,\langle\delta_k,\Theta\rangle\,|\Theta|^{2n-6}\right)\end{aligned}$$
and hence,
$$\partial_{k,j,i} \left(A_n|\Theta|^{2n}\right)\big|_{s=0} = \begin{cases}2\left(\delta_i\,\langle\delta_j,\delta_k\rangle + \delta_j\,\langle\delta_i,\delta_k\rangle + \delta_k\,\langle\delta_i,\delta_j\rangle\right) & \text{if } n=1\,,\\ 0 & \text{otherwise}\end{cases}$$
$$\text{and} \qquad \partial_{k,j,i}\mathcal{X}|_{s=0} = \frac{-1}{3}\left(\delta_i\,\langle\delta_j,\delta_k\rangle + \delta_j\,\langle\delta_i,\delta_k\rangle + \delta_k\,\langle\delta_i,\delta_j\rangle\right).$$

□

We are now in position to give the proof of Proposition 4.1.

Proof of Proposition 4.1. According to Lemmas 4.2 and 4.3 and (B.2), it is obvious that
$$D_c^i(0) = \delta_i \quad \text{and} \quad \partial_j D_c^i(0) = -\,\mathrm{G}_\Gamma\,\delta_i\cdot\delta_j\,.$$

We also have

$$\begin{aligned}
\partial_{k,j} D_c^i(0) &= \frac{-1}{3}\big(\langle\delta_j,\delta_k\rangle\,\delta_i + \langle\delta_i,\delta_k\rangle\,\delta_j + \langle\delta_i,\delta_j\rangle\,\delta_k\big) \\
&\qquad + \nabla\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j + \nabla\delta_i\cdot\left(\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k\right) + \nabla\left(\langle\delta_i,\delta_j\rangle\,\mathrm{Id}_{\mathbb{S}^2}\right)\cdot\delta_k \\
&= \frac{-1}{3}\big(\langle\delta_j,\delta_k\rangle\,\delta_i + \langle\delta_i,\delta_k\rangle\,\delta_j + \langle\delta_i,\delta_j\rangle\,\delta_k\big) \\
&\qquad - \langle\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k,\delta_j\rangle\,\mathrm{Id}_{\mathbb{S}^2} + \mathrm{G}_\Gamma\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j \\
&\qquad - \langle\delta_i,\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k\rangle\,\mathrm{Id}_{\mathbb{S}^2} + \mathrm{G}_\Gamma\,\delta_i\cdot\left(\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k\right) \\
&\qquad + \langle\nabla\delta_i\cdot\delta_k,\delta_j\rangle\,\mathrm{Id}_{\mathbb{S}^2} + \langle\delta_i,\nabla\delta_j\cdot\delta_k\rangle\,\mathrm{Id}_{\mathbb{S}^2} + \langle\delta_i,\delta_j\rangle\,\delta_k \\
&= \frac{-1}{3}\big(\langle\delta_j,\delta_k\rangle\,\delta_i + \langle\delta_i,\delta_k\rangle\,\delta_j - 2\,\langle\delta_i,\delta_j\rangle\,\delta_k\big) + \mathrm{G}_\Gamma\left(\mathrm{G}_\Gamma\,\delta_i\cdot\delta_k\right)\cdot\delta_j + \mathrm{G}_\Gamma\,\delta_i\cdot\left(\mathrm{G}_\Gamma\,\delta_j\cdot\delta_k\right).
\end{aligned}$$

Symmetrizing this expression with respect to j and k, we obtain the result. □

4.2 Stokes solutions on the exterior of a sphere

The results given here are borrowed from [7]. In this section, we use spherical coordinates $(r,\theta,\varphi)\in\mathbb{R}_+\times[0,\pi]\times[0,2\pi)$ which are recalled in Appendix B.

We recall that a spherical harmonic of degree $n\geqslant 0$ is defined by

$$\begin{aligned}
[0,\pi]\times[0,2\pi] &\to \mathbb{R} \\
(\theta,\varphi) &\mapsto \sum_{m=-n}^{n}\gamma_n^m Y_n^m(\theta,\varphi)
\end{aligned} \tag{4.1}$$

and a rigid spherical harmonic of degree $-(n+1)$ by

$$\begin{aligned}
\mathbb{R}_+^*\times[0,\pi]\times[0,2\pi] &\to \mathbb{R} \\
(r,\theta,\varphi) &\mapsto r^{-(n+1)}\sum_{m=-n}^{n}\gamma_n^m Y_n^m(\theta,\varphi)\,,
\end{aligned} \tag{4.2}$$

with $\gamma_n^m\in\mathbb{R}$ and where Y_n^m is defined by

$$\begin{cases}
Y_n^m(\theta,\varphi) = c_n^m\cos(m\varphi)P_n^m(\cos\theta)\,, \\
Y_n^0(\theta,\varphi) = c_n^0 P_n^0(\cos\theta)\,, \\
Y_n^{-m}(\theta,\varphi) = c_n^m\sin(m\varphi)P_n^m(\cos\theta)\,,
\end{cases} \qquad (n\in\mathbb{N},\ 0\leqslant m\leqslant n,\ (\theta,\varphi)\in[0,\pi]\times[0,2\pi])\,, \tag{4.3}$$

with

$$c_n^m = \begin{cases}
\sqrt{\dfrac{(2n+1)}{4\pi}} & \text{if } m=0\,, \\
\sqrt{\dfrac{(2n+1)(n-m)!}{2\pi(n+m)!}} & \text{if } m>0
\end{cases} \qquad (n\in\mathbb{N}\,, 0\leqslant m\leqslant n)$$

and with P_n^m is the associated Legendre polynomial of degree n and order m, that is to say that

$$P_n^m(x) = \frac{1}{2^n n!}\left(1-x^2\right)^{\frac{m}{2}}\frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}}\left((x^2-1)^n\right) \qquad (x\in[-1,1]\,,\ n\in\mathbb{N}\,,\ m\in\{0,\cdots,n\})\,.$$

We recall that the family $\{Y_n^m\}_{\substack{n\in\mathbb{N},\\ m\in\{-n,\cdots,n\}}}$ forms an orthonormal basis of $L^2(\partial\mathcal{S}_0)$, see for instance [16, Chapter VII § 5.3, p. 513]. More precisely, this family is orthonormal for the scalar product

$$\langle\zeta,\Upsilon\rangle = \int_0^{2\pi}\int_0^{\pi}\zeta(\theta,\varphi)\Upsilon(\theta,\varphi)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\,.$$

Remark 4.4. Let us mention that,

$$\begin{cases} P_1^0(x) = \dfrac{1}{2}\dfrac{\mathrm{d}^1}{\mathrm{d}x^1}\left(x^2-1\right) = x\,, \\ P_1^1(x) = \dfrac{1}{2}\left(1-x^2\right)^{\frac{1}{2}}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^2-1\right) = \left(1-x^2\right)^{\frac{1}{2}} \end{cases} \qquad (x\in[-1,1])\,.$$

and hence,

$$\begin{cases} Y_1^1(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\cos\varphi\sin\theta\,, \\ Y_1^0(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\cos\theta\,, \\ Y_1^{-1}(\theta,\varphi) = \sqrt{\dfrac{3}{4\pi}}\sin\varphi\sin\theta\,, \end{cases} \qquad ((\theta,\varphi)\in[0,\pi]\times[0,2\pi])\,.$$

According to Lamb [21], see also Brenner [7, Eq. 2.13], the solution (v,q) of the Stokes equation in an exterior domain can be expressed in spherical coordinates (see Appendix B for the related definition of spherical coordinates and expression of the usual operators ∇, div and rot) as

$$v = \sum_{n=1}^{\infty}\left(\operatorname{rot}\left(\chi_{-(n+1)}r\boldsymbol{e}_r\right) + \nabla\phi_{-(n+1)} - \frac{n-2}{2n(2n-1)}r^2\nabla p_{-(n+1)} + \frac{n+1}{n(2n-1)}p_{-(n+1)}r\boldsymbol{e}_r\right) \tag{4.4a}$$

$$q = \sum_{n=1}^{\infty} p_{-(n+1)} \tag{4.4b}$$

where $\chi_{-(n+1)}, \phi_{-(n+1)}$ and $p_{-(n+1)}$ are rigid spherical harmonics of degree $-(n+1)$ defined as in (4.2). Furthermore, the drag and torque exerted on the immersed domain by the fluid can be expressed as

$$F = -4\pi\nabla\left(r^3 p_{-2}\right), \tag{4.5a}$$

$$T = -8\pi\nabla\left(r^3\chi_{-2}\right). \tag{4.5b}$$

Let us mention that F and T are constant vectors of $\mathbb{R}^3$. In fact, we have,

$$\nabla\left(r^3\left(r^{-2}Y_1^0(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\cos\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla z = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\nabla\left(r^3\left(r^{-2}Y_1^1(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\cos\varphi\sin\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla x = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$\nabla\left(r^3\left(r^{-2}Y_1^{-1}(\theta,\varphi)\right)\right) = \sqrt{\frac{3}{4\pi}}\nabla\left(r\sin\varphi\sin\theta\right) = \sqrt{\frac{3}{4\pi}}\nabla y = \sqrt{\frac{3}{4\pi}}\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

When the exterior domain is the exterior of the unit ball of $\mathbb{R}^3$, $v\cdot\boldsymbol{e}_r$, $\operatorname{div}_\Gamma v$ and $\operatorname{rot}_\Gamma v$ for $r=1$ can be expressed as a sum of spherical harmonics (see (B.1) for the definition of $\operatorname{div}_\Gamma$ and $\operatorname{rot}_\Gamma$),

$$v\cdot\boldsymbol{e}_r = \sum_{n=0}^{\infty} X_n\,, \tag{4.6a}$$

$$\operatorname{div}_\Gamma v = \sum_{n=0}^{\infty} Y_n\,, \tag{4.6b}$$

$$\operatorname{rot}_\Gamma v = \sum_{n=0}^{\infty} Z_n\,, \tag{4.6c}$$

with X_n, Y_n and Z_n spherical harmonics of degree n.

According to [7], $\chi_{-(n+1)}, \phi_{-(n+1)}$ and $p_{-(n+1)}$ are related to X_n, Y_n and Z_n by

$$\chi_{-(n+1)}(r,\theta,\varphi) = \frac{r^{-(n+1)}}{n(n+1)} Z_n(\theta,\varphi)\,, \tag{4.7a}$$

$$\phi_{-(n+1)}(r,\theta,\varphi) = \frac{r^{-(n+1)}}{2(n+1)} (nX_n(\theta,\varphi) + Y_n(\theta,\varphi))\,, \tag{4.7b}$$

$$p_{-(n+1)}(r,\theta,\varphi) = r^{-(n+1)} \frac{2n-1}{n+1} \left((n+2)X_n(\theta,\varphi) + Y_n(\theta,\varphi)\right), \tag{4.7c}$$

for every $n \in \mathbb{N}^*$.

Using the decomposition (4.6) for (3.13) we obtain

$$\partial_s^\alpha N(s)\boldsymbol{e}_j = -\begin{pmatrix} 4\pi\nabla\left(r^3 p_{-2}\right) \\ 8\pi\nabla\left(r^3\chi_{-2}\right)\end{pmatrix}.$$

Since $\partial_s^\alpha D_c^j(s)$ is a tangential field, p_{-2} and χ_{-2} are given by (4.7) with $X = 0$, i.e.

$$p_{-2}(r,\theta,\varphi) = \frac{r^{-2}}{2} Y_1(\theta,\varphi) \qquad \text{and} \qquad \chi_{-2}(r,\theta,\varphi) = \frac{r^{-2}}{2} Z_1(\theta,\varphi)\,,$$

where Y_1 and Z_1 are defined from (4.6) with $v = \partial_s^\alpha D_c^j(s)$. More precisely, we obtain

$$\partial_s^\alpha N(s)\boldsymbol{e}_j = -\sqrt{3\pi} \begin{pmatrix} \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ 2\langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \end{pmatrix}.$$

Let us also recall that for a spherical body, the matrix K_c introduced in (3.3) is (see [20, § 5.2 and 5.3]) $2\pi \begin{pmatrix} 3I_3 & 0 \\ 0 & 4I_3 \end{pmatrix}$ and hence,

$$\partial_s^\alpha V_c^j(s) = -\sqrt{\frac{3}{4\pi}} \begin{pmatrix} \frac{1}{3}I_3 & 0 \\ 0 & \frac{1}{2}I_3 \end{pmatrix} \begin{pmatrix} \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ \langle \mathrm{div}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \\ \langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^1\rangle \\ \langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^{-1}\rangle \\ \langle \mathrm{rot}_\Gamma\, \partial_s^\alpha D_c^j(s), Y_1^0\rangle \end{pmatrix}. \tag{4.8}$$

4.3 Particular choices of δ

In order to fully define the swimmer configuration, $c = (0, \delta)$, we introduce some explicit choices of δ_i's.

The first type of δ_i that we consider is

$$\zeta_n^m(\theta,\varphi) = \partial_\theta Y_n^m(\theta,\varphi)\boldsymbol{e}_\theta + \partial_\varphi Y_n^m(\theta,\varphi)\frac{\boldsymbol{e}_\varphi}{\sin\theta} \tag{4.9}$$

and the second type is

$$\xi_n^m(\theta,\varphi) = \partial_\varphi Y_n^m(\theta,\varphi)\frac{\boldsymbol{e}_\theta}{\sin\theta} - \partial_\theta Y_n^m(\theta,\varphi)\boldsymbol{e}_\varphi\,, \tag{4.10}$$

with $n \in \mathbb{N}$ and $m \in \{-n, \cdots, n\}$.

Let us remind that, according to Proposition 4.1, we have

$$D_c^i(0)(\theta,\varphi) = \delta_i(\theta,\varphi)\,.$$

Let us then compute $V_c^i(s)$ given by (4.8) at $s = 0$ for the two possible choices of δ_i given by (4.9) and (4.10).

• If $\delta_i = \zeta_n^m$. Assume $m \geqslant 0$, the case $m \leqslant 0$ is similar. In order to compute $V_c^i(0)$, one have to compute the solution $v = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi$ of the Stokes equation set on the exterior of the unit ball with the Dirichlet boundary condition $v = \delta_i$, that is to say that at $r = 1$, v_r, v_θ and v_φ shall satisfy

$$v_r(\theta,\varphi) = 0\,, \qquad v_\theta(\theta,\varphi) = \partial_\theta Y_n^m(\theta,\varphi) = -c_n^m \sin\theta (P_n^m)'(\cos\theta)\cos(m\varphi)$$
$$\text{and} \qquad v_\varphi(\theta,\varphi) = \frac{1}{\sin\theta}\partial_\varphi Y_n^m(\theta,\varphi) = \frac{-mc_n^m}{\sin\theta} P_n^m(\cos\theta)\sin(m\varphi)\,.$$

In order to express the solution in a sum of rigid spherical harmonics, we compute the decomposition in spherical harmonics in (4.6),

$$\begin{aligned}
\boldsymbol{e}_r \cdot v &= 0\,,\\
\operatorname{div}_\Gamma v &= \frac{-1}{\sin\theta}\left(\partial_\theta(v_\theta \sin\theta) + \partial_\varphi v_\varphi\right)\\
&= \frac{-c_n^m}{\sin\theta}\left(-\partial_\theta(\sin^2\theta (P_n^m)'(\cos\theta)) - \frac{m^2}{\sin\theta}P_n^m(\cos\theta)\right)\cos(m\varphi)\\
&= -c_n^m\left(\frac{-1}{\sin\theta}\partial_\theta((1-\cos^2\theta)(P_n^m)'(\cos\theta)) - \frac{m^2}{1-\cos^2\theta}P_n^m(\cos\theta)\right)\cos(m\varphi)\\
&= c_n^m n(n+1) P_n^m(\cos\theta)\cos(m\varphi)\\
&= n(n+1) Y_n^m\,,\\
\operatorname{rot}_\Gamma v &= \frac{1}{\sin\theta}\left(\partial_\theta(v_\varphi \sin\theta) - \partial_\varphi v_\theta\right)\\
&= \frac{c_n^m}{\sin\theta}\left(-m\partial_\theta(P_n^m(\cos\theta)) - m\sin\theta (P_n^m)'(\cos\theta)\right)\sin(m\varphi)\\
&= 0\,.
\end{aligned}$$

In the above relations, we have used the property of the associated Legendre polynomials, see for instance [16, Chapter V § 10.3 p. 327]

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left((1-\zeta^2)(P_n^m)'(\zeta)\right) - \frac{m^2}{1-\zeta^2}P_n^m(\zeta) = -n(n+1)P_n^m(\zeta)\,.$$

Consequently, by orthogonality of spherical harmonics, we obtain $V_c^i(0) = 0$, for $n \geqslant 2$.

• If $\delta_i = \xi_n^m$. Assume $m \geqslant 0$, the case $m \leqslant 0$ is similar. Similarly, we have to compute the solution $v = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi$ of the Stokes equation set on the exterior of the unit ball with the Dirichlet boundary condition:

$$v_r = 0\,, \qquad v_\theta = \frac{1}{\sin\theta}\partial_\varphi Y_n^m(\theta,\varphi) = \frac{-mc_n^m}{\sin\theta}P_n^m(\cos\theta)\sin(m\varphi)$$
$$\text{and} \qquad v_\varphi = -\partial_\theta Y_n^m(\theta,\varphi) = c_n^m \sin\theta (P_n^m)'(\cos\theta)\cos(m\varphi)$$

and similarly to the previous case, we obtain

$$\boldsymbol{e}_r \cdot v = 0\,, \quad \operatorname{div}_\Gamma v = 0 \quad \text{and} \quad \operatorname{rot}_\Gamma v = -n(n+1)P_n^m(\cos\theta)e^{im\varphi}\,.$$

Consequently, for $n \geqslant 2$, we have $V_c^i(0) = 0$.

Lie brackets at $s = 0$. Due to the choice of the δ_i's given by (4.9) and (4.10), we obtain (choosing $n \geqslant 2$) $V_c^i(0) = 0$ for every $i \in \{1, \cdots, d\}$. Consequently, at $s = 0$ the expression of the Lie brackets given in (3.9) and (3.11) are

$$f_c^i(0, \mathrm{I}_3, 0) = \mathfrak{p}\,(0, 0, \boldsymbol{e}_i)\,, \tag{4.11a}$$
$$[f_c^j, f_c^i](0, \mathrm{I}_3, 0) = \tilde{\mathfrak{p}}\left(\partial_j V_c^i - \partial_i V_c^j\right), \tag{4.11b}$$
$$[f_c^k, [f_c^j, f_c^i]](0, \mathrm{I}_3, 0) = \tilde{\mathfrak{p}}\left(\partial_k\left(\partial_j V_c^i - \partial_i V_c^j\right)\right). \tag{4.11c}$$

4.4 Explicit computations

In this section we combine (4.8) and Proposition 4.1 in order to compute explicitly (4.11). This computation have been made using the computer algebra system `maxima`.

Case $d = 4$. In this case, we consider $\delta = (\delta_1, \cdots, \delta_4)$, with

$$\delta_1 = \zeta_4^1\,, \qquad \delta_2 = \xi_4^0\,, \qquad \delta_3 = \zeta_3^0 \quad \text{and} \quad \delta_4 = \zeta_3^1\,.$$

Setting $\Delta_{i,j} = \partial_j V_c^i(0) - \partial_i V_c^j(0)$ the 6×6 matrix $(\Delta_{12} \mid \Delta_{1,3} \mid \Delta_{1,4} \mid \Delta_{2,3} \mid \Delta_{2,4} \mid \Delta_{3,4})$ is

$$\begin{pmatrix} 0 & \frac{2^{5/2}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 & 0 & 0 & 0 \\ -\frac{2^{3/2}\,5^{3/2}}{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s\frac{4\,\sqrt{3}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5\,3^{5/2}}{\sqrt{2}\,\sqrt{7}\,\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11\,3^{3/2}}{\sqrt{2}\,\pi} \\ 0 & 0 & 0 & \frac{90}{\sqrt{7}\,\pi} & 0 & 0 \end{pmatrix}.$$

This, together with Lemma 3.4, ensures that the dimension of the Lie algebra generated by $f_c^1, \cdots, f_c^4$ is of maximal dimension (i.e. $6 + 4 = 10$) for $s = 0$.

Case $d = 3$. In this case, we consider $\delta = (\delta_1, \delta_2, \delta_3)$, with

$$\delta_1 = \zeta_4^1\,, \qquad \delta_2 = \xi_4^0 \quad \text{and} \quad \delta_3 = \zeta_3^0\,.$$

Setting $\Delta_{i,j} = \partial_j V_c^i(0) - \partial_i V_c^j(0)$ and $\Delta_{i,j}^k = \partial_k \left(\partial_j V_c^i(s) - \partial_i V_c^j(s)\right)\big|_{s=0}$ the 6×3 matrix whose columns are formed by the $\Delta_{i,j}$ is

$$\begin{pmatrix} 0 & \frac{2^{5/2}\,5^{3/2}}{\sqrt{7}\,\pi} & 0 \\ -\frac{2^{3/2}\,5^{3/2}}{\pi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{90}{\sqrt{7}\,\pi} \end{pmatrix}$$

and the 6×9 matrix whose columns are formed by the $\Delta_{i,j}^k$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8949}{286\,\sqrt{7}\,\pi^{3/2}} & 0 & 0 & 0 & -\frac{2360\,\sqrt{7}}{143\,\pi^{3/2}} & 0 \\ 0 & 0 & \frac{249\;5^{7/2}}{143\;2^{3/2}\,\sqrt{7}\,\pi^{3/2}} & 0 & -\frac{3879\,\sqrt{5}}{13\;2^{3/2}\,\sqrt{7}\,\pi^{3/2}} & 0 & \frac{753\,\sqrt{5}\,7^{3/2}}{143\,\sqrt{2}\,\pi^{3/2}} & 0 & 0 \\ 0 & \frac{65367\;5^{3/2}}{1001\;2^{3/2}\,\pi^{3/2}} & 0 & 0 & 0 & \frac{1623\,\sqrt{5}}{11\;2^{3/2}\,\pi^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This, together with Lemma 3.4, ensures that the dimension of the Lie algebra generated by f_c^1, f_c^2, f_c^3 is of maximal dimension (i.e. $6 + 3 = 9$) for $s = 0$.

In conclusion of the above computations, we have the following results.

Lemma 4.5. *For every $d \geqslant 2$ and every $(c, s) \in \mathcal{A}^2(d)$, we set $\Delta_{i,j}(s) = \partial_j V_c^i(s) - \partial_i V_c^j(s)$ and $\Delta_{i,j}^k(s) = \partial_k \Delta_{i,j}(s)$.*

- *For every $d \geqslant 4$, the analytic maps*

$$\begin{array}{rcl} \mathcal{A}^2(d) & \to & \mathbb{R} \\ (c, s) & \mapsto & \det\left(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{1,4}(s) \mid \Delta_{2,3}(s) \mid \Delta_{2,4}(s) \mid \Delta_{3,4}(s)\right) \end{array}$$

and (at $s = 0$)

$$\begin{array}{rcl} \mathcal{SC}^2(d) & \to & \mathbb{R} \\ c & \mapsto & \det\left(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{1,4}(0) \mid \Delta_{2,3}(0) \mid \Delta_{2,4}(0) \mid \Delta_{3,4}(0)\right) \end{array}$$

are non identically 0.

- *For every $d \geqslant 3$, the analytic maps*

$$\begin{array}{rcl} \mathcal{A}^2(d) & \to & \mathbb{R} \\ (c,s) & \mapsto & \det\left(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{2,3}(s) \mid \Delta^2_{1,2}(s) \mid \Delta^3_{1,2}(s) \mid \Delta^1_{1,3}(s)\right) \end{array}$$

and (at $s = 0$)

$$\begin{array}{rcl} \mathcal{SC}^2(d) & \to & \mathbb{R} \\ c & \mapsto & \det\left(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{2,3}(0) \mid \Delta^2_{1,2}(0) \mid \Delta^3_{1,2}(0) \mid \Delta^1_{1,3}(0)\right) \end{array}$$

are non identically 0.

Remark 4.6. We tried to prove a similar result for $d = 2$ but our numerical simulations seems to indicate that it is not possible. More precisely, we went up to the computation of Lie brackets of fifth order. In all the computations we performed the maximal rank of the Lie algebra evaluated at $s = 0$ was 3. In these computations, we have considered all possible choices of δ given by (4.9) and (4.10) up to spherical harmonics of order 6. We have also taken the parameters m and n in (4.9) and (4.10) randomly, using a Poisson law for n, and again the maximal rank obtained was 3.
However, we believe that the generic result, Theorem 2.5 is still valid for $d = 2$ but probably the spherical swimmers are too symmetric to be controllable with only two elementary deformations.

From Lemma 4.5, we deduce:

Proposition 4.7. *Given $d \geqslant 3$, $\varepsilon > 0$ and $\overline{c} = (\overline{\Psi}_0, \overline{\delta}) \in \mathcal{SC}^2(d)$, there exists $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ such that*

$$\|c - \overline{c}\| < \varepsilon$$

and

$$\dim \mathrm{Lie}_{(h,Q,s)} \left\{ f_c^1, \cdots, f_c^d \right\} = d + 6 \qquad ((h,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3))$$

for almost every $s \in \mathcal{J}(\delta)$ and every $s \in \mathbb{R}^d$ small enough.

Furthermore, c can be chosen such that for almost every $s \in \mathcal{J}(\delta)$ and every $(h,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)$, we have

- *for $d = 4$,*

$$\mathrm{Lie}_{(h,Q,s)} \left\{ f_c^1, \cdots, f_c^d \right\} = \mathrm{Span}\left(\left\{ f_c^1(h,Q,s), \cdots, f_c^d(h,Q,s), \right\} \cup \left\{ [f_c^j, f_c^i](h,Q,s)\,,\ i,j \in \{1,\cdots,d\} \right\} \right);$$

- *for $d = 3$,*

$$\begin{aligned} \mathrm{Lie}_{(h,Q,s)} \left\{ f_c^1, \cdots, f_c^d \right\} = \mathrm{Span}\Big(\left\{ f_c^1(h,Q,s), \cdots, f_c^d(h,Q,s) \right\} \cup \left\{ [f_c^j, f_c^i](h,Q,s)\,,\ i,j \in \{1,\cdots,d\} \right\} \\ \cup \left\{ [f_c^k, [f_c^j, f_c^i]](h,Q,s)\,,\ i,j,k \in \{1,\cdots,d\} \right\} \Big). \end{aligned}$$

Proof. Let us sketch the proof for $d \geqslant 4$. The proof in the case $d = 3$ is similar.

The analyticity of the map

$$F : c \in \mathcal{SC}^2(d) \mapsto \det\left(\Delta_{1,2}(0) \mid \Delta_{1,3}(0) \mid \Delta_{1,4}(0) \mid \Delta_{2,3}(0) \mid \Delta_{2,4}(0) \mid \Delta_{3,4}(0)\right)$$

given in Lemma 4.5 and its non nullity ensure that for every $\overline{c}$ there exists c such that $\|c - \overline{c}\| < \varepsilon$ and $F(c) \neq 0$. This together with the analyticity of

$$(c,s) \in \mathcal{A}(d) \mapsto \det\left(\Delta_{1,2}(s) \mid \Delta_{1,3}(s) \mid \Delta_{1,4}(s) \mid \Delta_{2,3}(s) \mid \Delta_{2,4}(s) \mid \Delta_{3,4}(s)\right)$$

and Lemma 3.4 ensures that $\dim \mathrm{Lie}_{(h,Q,s)} \left\{ f_c^i, \cdots, f_c^d \right\} = d + 6$ for every $h \in \mathbb{R}^3$, $Q \in \mathcal{SO}(3)$ and for almost every $s \in \mathcal{J}(\delta)$ and in particular for every $s \in \mathbb{R}^d$, with $|s|$ small enough. □

4.5 Proof of Theorem 2.5

In this paragraph, we prove Theorem 2.5 using Proposition 4.7 together with the orbit Theorem and Rashevsky-Chow Theorem (see [1, Chapter 5]).

When considering a controllable swimmer, one can show that any path can be tracked. To prove this result, we use Proposition 4.7 together with the orbit Theorem and Rashevsky-Chow Theorem (see [1, Chapter 5]). In order to prove that any path can be tracked, we use the fact that the dimension of the Lie algebra is independent of h and Q and the set of points s where the Lie algebra is not Lie bracket generating is included in an analytic manifold.

Proposition 4.8. *Assume $d \geqslant 3$, $\varepsilon > 0$ and $c = (\Psi_0, \delta) \in \mathcal{SC}^\infty(d)$ such that there exists $s_0 \in \mathcal{J}(\delta)$ so that*

$$\dim \mathrm{Lie}_{(h,Q,s_0)} \left\{ f_c^1, \cdots, f_c^d \right\} = d + 6 \qquad ((h,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)).$$

Then for every time $T > 0$ and every path $(\overline{h}, \overline{Q}, \overline{s}) \in C^0\big([0,T], \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathcal{J}(\delta)\big)$, there exists $s \in C^\infty\big([0,T], \mathcal{J}(\delta)\big)$ satisfying

$$s(0) = \overline{s}(0)\,, \qquad s(T) = \overline{s}(T) \quad \textit{and} \quad \sup_{t\in[0,T]} |s(t) - \overline{s}(t)| < \eta\,,$$

such that the corresponding solution (h,Q) of (2.7) with initial conditions

$$h(0) = \overline{h}(0)\,, \quad Q(0) = \overline{Q}(0)$$

satisfies

$$\sup_{t\in[0,T]} \big(|h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)|\big) < \eta$$

together with

$$h(T) = \overline{h}(T)\,, \quad Q(T) = \overline{Q}(T)\,.$$

Proof. First of all, since $\overline{s}$ is a continuous curve and since $\mathcal{J}(\delta)$ is an open set of $\mathbb{R}^d$, the path $t \mapsto \overline{s}(t)$ can be approximated by a piecewise affine and continuous function in $\mathcal{J}(\delta)$ joining $\overline{s}(0)$ at $t = 0$ to $\overline{s}(T)$ at $t = T$. Consequently, it is enough to prove the result for $\overline{s}$ an affine function.

Since for $s = s_0$, we have $\dim \mathrm{Lie}_{(h,Q,s_0)} \left\{ f_c^1, \cdots, f_c^d \right\} = \dim \mathrm{Lie}_{(0,\mathrm{I}_3,s_0)} \left\{ f_c^1, \cdots, f_c^d \right\} = d + 6$, there exist 6 Lie brackets $b_1(s), \cdots, b_6(s)$, of order greater than 1, such that $F(s_0) \neq 0$, with F defined by

$$F(s) = \det\left(\tilde{\mathfrak{p}}^{-1}(b_1(s)),\, \tilde{\mathfrak{p}}^{-1}(b_2(s)),\, \tilde{\mathfrak{p}}^{-1}(b_3(s)),\, \tilde{\mathfrak{p}}^{-1}(b_4(s)),\, \tilde{\mathfrak{p}}^{-1}(b_5(s)),\, \tilde{\mathfrak{p}}^{-1}(b_6(s))\right) \qquad (s \in \mathcal{J}(\delta)).$$

where $\tilde{\mathfrak{p}}$ is defined by (3.10). Let us then define

$$\mathcal{Z} = \{ s \in \mathcal{J}(\delta) \;\mid\; F(s) = 0 \}\,.$$

Similarly to Lemma 4.5, F is an analytic map on $\mathcal{J}(\delta)$ and hence, $\mathcal{Z}$ is an analytic manifold of $\mathbb{R}^d$. Furthermore, since $s_0 \notin \mathcal{Z}$, $\mathcal{Z}$ is not equal to $\mathbb{R}^d$.

Since $t \mapsto \overline{s}(t)$ is affine, the compact set $\{\overline{s}(t)\,,\ t \in [0,T]\}$ is also an included in an analytic manifold of $\mathbb{R}^d$. Thus, using the property on zeros of analytic functions, three situations can hold,

1. $\overline{s}(t) \notin \mathcal{Z}$ for every $t \in [0,T]$;
2. there exists a finite number of times $t_1, \ldots, t_k$ such that $\overline{s}(t_i) \in \mathcal{Z}$ $(i = 1, \ldots, k)$ and $\overline{s}(t) \notin \mathcal{Z}$ for every $t \in [0,T] \setminus \{t_1, \ldots, t_k\}$.
3. $\overline{s}(t) \in \mathcal{Z}$ for every $t \in [0,T]$;

Let us treat each cases.

1. In this situation, for every $t \in [0,T]$, we have $\dim \mathrm{Lie}_{(h,Q,\overline{s}(t))} \left\{ f_c^1, \cdots, f_c^d \right\} = d + 6$ for every $t \in [0,T]$ and every $(h,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)$. Hence, we conclude using the orbit Theorem and Rashevsky-Chow Theorem (see [1, Chapter 5]).

2. Without loss of generality, we can reduce this situation to the case $k = 1$ and $t_1 = 0$ or $t_1 = T$.
   Consider first the situation, that is to say $\overline{s}(0) \in \mathcal{Z}$ and $\overline{s}(t) \notin \mathcal{Z}$ for every $t \in (0, T]$.
   Using the uniform continuity of $\overline{h}$ and $\overline{Q}$, one can find $\tau > 0$ small enough such that the solution (h, Q) of (3.8) with control $s = \overline{s}|_{[0,\tau]}$ satisfies $|h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)| \leqslant \eta/2$ for every $t \in [0, \tau]$. Now, since $\overline{s}(t) \notin \mathcal{Z}$ for every $t \in [\tau, T]$, we use the first case to conclude that there exists a control s on $[\tau, T]$, with

$$s(\tau) = \overline{s}(\tau)\,, \quad s(T) = \overline{s}(T) \quad \text{and} \quad \sup_{t\in[\tau,T]} |s(t) - \overline{s}(t)| < \eta/2$$

   such that the solution (h, Q) on $[\tau, T]$ of (3.8) satisfies

$$\sup_{t\in[\tau,T]} |h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)| < \eta\,, \qquad h(T) = \overline{h}(T) \quad \text{and} \quad Q(T) = \overline{Q}(T)\,.$$

   The second situation, that is to say $\overline{s}(T) \in \mathcal{Z}$ and $\overline{s}(t) \notin \mathcal{Z}$ for every $t \in [0, T)$, can be treated with the above arguments and reverting the time.

3. In this situation, there exists $\sigma \in \mathbb{R}^d$ which can be chosen arbitrarily small such that $\overline{s}(0) + \sigma \notin \mathcal{Z}$. Consequently, the curve $t \mapsto \overline{s}(t) + \sigma$ is not included in $\mathcal{Z}$. Using the uniform continuity of $\overline{s}$, $\overline{h}$ and $\overline{Q}$, there exists $\tau > 0$ small enough such that the solution (h, Q) of (3.8) with control $s(t) = (\sigma - \overline{s}(0) + \overline{s}(\tau))t/\tau + \overline{s}(0)$ for $t \in [0, \tau]$ satisfies

$$|s(t) - \overline{s}(t)| < \frac{\eta}{2} \quad \text{and} \quad |h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)| < \frac{\eta}{2} \qquad (t \in [0, \tau]).$$

   Using the situations described in the two previous cases, one can find a control s on $[\tau, T/2]$ such that

$$s(\tau) = \overline{s}(\tau) + \sigma\,, \qquad s(T/2) = \overline{s}(T/2) + \sigma \quad \text{and} \quad |s(t) - (\overline{s}(t) + \sigma)| < \frac{\eta}{2} \qquad (t \in [\tau, T/2])$$

   and the solution of (3.8) satisfy

$$h(T/2) = \overline{h}(T/2)\,, \qquad Q(T/2) = \overline{Q}(T/2) \quad \text{and} \quad |h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)| < \eta \qquad (t \in [\tau, T/2]).$$

   Reverting the time, one can similarly build a control s on $[T/2, T]$ such that,

$$s(T/2) = \overline{s}(T/2) + \sigma\,, \qquad s(T) = \overline{s}(T) \quad \text{and} \quad |s(t) - \overline{s}(t)| \leqslant \eta \qquad (t \in [T/2, T])$$

   and the corresponding solution (h, Q) of (3.8) satisfy

$$h(T) = \overline{h}(T)\,, \qquad Q(T) = \overline{Q}(T) \quad \text{and} \quad |h(t) - \overline{h}(t)| + |Q(t) - \overline{Q}(t)| < \eta \qquad (t \in [T/2, T]).$$

The above construction leads to a control s which has a $W^{1,\infty}$-regularity. In order to prove that the control problem can be solved with a control s of arbitrary regularity, we use a classical smoothing procedure. □

Combining Propositions 4.7 and 4.8 together with the fact that the set of C^k-diffeomorphism of $\mathbb{S}^2$ is open and nonempty in $C^k(\mathbb{S}^2)$ (see [9, Proposition 2 p. 287] and [10, p. 1]), we deduce the proof of Theorem 2.5.

A Function spaces

We give here some notation.

- $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^d$ or a norm on $\mathcal{M}_3(\mathbb{R})$.
- Given $k \in \mathbb{N}$, $C_0^k(\mathbb{R}^3)$ is defined by

$$C_0^k(\mathbb{R}^3) = \left\{ f \in C^k(\mathbb{R}^3) \mid \lim_{|x|\to\infty} |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} f(x)| = 0\,,\ \forall\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \text{ s.t. } \alpha_1 + \alpha_2 + \alpha_3 \leqslant k \right\}.$$

  This is a Banach space when endowed with the norm:

$$\|f\|_{C_0^k(\mathbb{R}^3)} = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\in\mathbb{N},\\ \alpha_1+\alpha_2+\alpha_3\leqslant k}} \sup_{x\in\mathbb{R}^3} |\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} f(x)|\,.$$

  We also set $C_0^\infty(\mathbb{R}^3) = \bigcap_{k=0}^{\infty} C_0^k(\mathbb{R}^3)$.

- Given $k \in \mathbb{N}^* \cup \{\infty\}$, $\mathcal{D}_0^k$ is the connected component of

$$\left\{ f \in C_0^k(\mathbb{R}^3)^3 \mid \mathrm{Id}_{\mathbb{R}^3} + f \text{ is a } C^1\text{-diffeomorphism of } \mathbb{R}^3 \right\}$$

containing 0. In particular, we have (see [24]) for $k \in \mathbb{N}^*$,

Lemma A.1. *$\mathcal{D}_0^k$ contains the unit ball of $C_0^k(\mathbb{R}^3)^3$ and is an open set of $C_0^k(\mathbb{R}^3)^3$.*

- For a C^∞-manifold $\mathcal{M}$, $\mathrm{T}\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and $C^k(\mathcal{M}, \mathrm{T}\mathcal{M})$ is the set of k-differentiable tangent vector fields of $\mathcal{M}$.

B Formula in spherical coordinates

These results are borrowed from [20, § A.15] and are recalled here for the sake of completeness.

Consider the spherical coordinates:

$$x = r\sin\theta\cos\varphi\,, \quad y = r\sin\theta\sin\varphi \quad \text{and} \quad z = r\cos\theta\,,$$

with $(r, \theta, \varphi) \in \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]$. We have:

$$\begin{aligned}
\boldsymbol{e}_r &= \sin\theta\cos\varphi\boldsymbol{e}_1 + \sin\theta\sin\varphi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3\,,\\
\boldsymbol{e}_\theta &= \cos\theta\cos\varphi\boldsymbol{e}_1 + \cos\theta\sin\varphi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3\,,\\
\boldsymbol{e}_\varphi &= -\sin\varphi\boldsymbol{e}_1 + \cos\varphi\boldsymbol{e}_2\,,
\end{aligned}$$

Let $f, v_r, v_\theta, v_\varphi$ be scalar functions and set $v = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi$, then we have

$$\begin{aligned}
\nabla f &= \partial_r f\boldsymbol{e}_r + \frac{1}{r}\partial_\theta f\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\partial_\varphi f\boldsymbol{e}_\varphi\,,\\
\operatorname{div} v &= \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi\,,\\
\operatorname{rot} v &= \frac{1}{r\sin\theta}\left(\partial_\theta(v_\varphi\sin\theta) - \partial_\varphi v_\theta\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi v_r - \partial_r(r v_\varphi)\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\partial_r(r v_\theta) - \partial_\theta v_r\right)\boldsymbol{e}_\varphi\,.
\end{aligned}$$

and we define

$$\begin{aligned}
\operatorname{div}_\Gamma v &:= r\boldsymbol{e}_r\cdot\nabla(\boldsymbol{e}_r\cdot v) - r\operatorname{div} v\\
&= r\left(\partial_r v_r - \frac{1}{r^2}\partial_r(r^2 v_r) - \frac{1}{r\sin\theta}\partial_\theta(v_\theta\sin\theta) - \frac{1}{r\sin\theta}\partial_\varphi v_\varphi\right)\\
&= -2v_r - \frac{1}{\sin\theta}\left(\partial_\theta(v_\theta\sin\theta) + \partial_\varphi v_\varphi\right)\,, \qquad \text{(B.1a)}\\
\operatorname{rot}_\Gamma v &:= r\boldsymbol{e}_r\cdot\operatorname{rot} v\\
&= \frac{1}{\sin\theta}\left(\partial_\theta(v_\varphi\sin\theta) - \partial_\varphi v_\theta\right)\,. \qquad \text{(B.1b)}
\end{aligned}$$

Let $u = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_\varphi\boldsymbol{e}_\varphi$ and $v = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi$. Then we have:

$$\begin{aligned}
\nabla u = {}& \partial_r u_r\boldsymbol{e}_r\boldsymbol{e}_r^\top + \partial_r u_\theta\boldsymbol{e}_\theta\boldsymbol{e}_r^\top + \partial_r u_\varphi\boldsymbol{e}_\varphi\boldsymbol{e}_r^\top\\
&+ \frac{1}{r}\left(\partial_\theta u_r - u_\theta\right)\boldsymbol{e}_r\boldsymbol{e}_\theta^\top + \frac{1}{r}\left(\partial_\theta u_\theta + u_r\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta^\top + \frac{1}{r}\partial_\theta u_\varphi\boldsymbol{e}_\varphi\boldsymbol{e}_\theta^\top\\
&+ \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r - u_\varphi\right)\boldsymbol{e}_r\boldsymbol{e}_\varphi^\top + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\boldsymbol{e}_\theta\boldsymbol{e}_\varphi^\top\\
&+ \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + u_r + \operatorname{cotan}\theta u_\theta\right)\boldsymbol{e}_\varphi\boldsymbol{e}_\varphi^\top\,.
\end{aligned}$$

So that we have,

$$\begin{aligned}\nabla u \cdot v = v_r \left(\partial_r u_r \boldsymbol{e}_r + \partial_r u_\theta \boldsymbol{e}_\theta + \partial_r u_\varphi \boldsymbol{e}_\varphi\right) + v_\theta \left(\frac{1}{r}\left(\partial_\theta u_r - u_\theta\right)\boldsymbol{e}_r + \frac{1}{r}\left(\partial_\theta u_\theta + u_r\right)\boldsymbol{e}_\theta + \frac{1}{r}\partial_\theta u_\varphi \boldsymbol{e}_\varphi\right)\\ + v_\varphi \left(\frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r - u_\varphi\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\boldsymbol{e}_\theta\right.\\ \left. + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + u_r + \operatorname{cotan}\theta u_\theta\right)\boldsymbol{e}_\varphi\right)\\ = \left(v_r\partial_r u_r + \frac{v_\theta}{r}\left(\partial_\theta u_r - u_\theta\right) + \frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_r - u_\varphi\right)\right)\boldsymbol{e}_r\\ + \left(v_r\partial_r u_\theta + \frac{v_\theta}{r}\left(\partial_\theta u_\theta + u_r\right) + \frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta\\ + \left(v_r\partial_r u_\varphi + \frac{v_\theta}{r}\partial_\theta u_\varphi + \frac{v_\varphi}{r}\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + u_r + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi\end{aligned}$$

and in particular, for $u = u_\theta(\theta,\varphi)\boldsymbol{e}_\theta + u_\varphi(\theta,\varphi)\boldsymbol{e}_\varphi$ and $v = v_\theta(\theta,\varphi)\boldsymbol{e}_\theta + v_\varphi(\theta,\varphi)\boldsymbol{e}_\varphi$, we have at $r = 1$,

$$\begin{aligned}\nabla u \cdot v &= -\left(v_\theta u_\theta + v_\varphi u_\varphi\right)\boldsymbol{e}_r + \left(v_\theta\partial_\theta u_\theta + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta\\ &\qquad + \left(v_\theta\partial_\theta u_\varphi + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi\\ &= -\langle u, v\rangle\,\boldsymbol{e}_r + \mathrm{G}_\Gamma\, u \cdot v\,,\end{aligned} \tag{B.2}$$

where we have set,

$$\mathrm{G}_\Gamma\, u \cdot v = \left(v_\theta\partial_\theta u_\theta + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\theta - \operatorname{cotan}\theta u_\varphi\right)\right)\boldsymbol{e}_\theta + \left(v_\theta\partial_\theta u_\varphi + v_\varphi\left(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + \operatorname{cotan}\theta u_\theta\right)\right)\boldsymbol{e}_\varphi\,. \tag{B.3}$$

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