

# SEMISMOOTH NEWTON SOLVER FOR PERIODICALLY-FORCED SOLUTIONS TO UNILATERAL CONTACT FORMULATIONS

Yulin Shi (yulin.shi@mail.mcgill.ca) & Mathias Legrand (mathias.legrand@mcgill.ca)

Structural Dynamics and Vibration Laboratory, McGill University, Montreal, Canada

## Introduction

- ▶ Vibratory response of periodically-forced mechanical systems undergoing unilateral contact condition is addressed
- ▶ Complementarity contact condition reformulated as root set of Lipschitz continuous function
- ▶ Galerkin methods implemented to seek periodic solutions
- ▶ *semismooth Newton method* to solve the resulting nonsmooth equations performed
- ▶ Frequency-Energy Plots show nonlinear resonances

## Linear modal analysis

- ▶ Governing equation for linear mass-spring oscillators  $\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{0}, \quad \forall t$  (1) where  $u(t)$  is the displacement

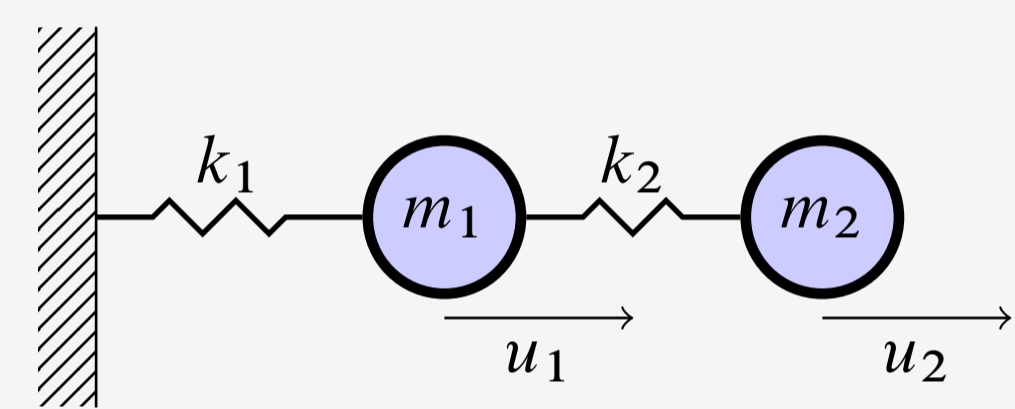


Figure 1: Two-DOF mass-spring oscillator

- ▶ Solutions are harmonic  $\mathbf{u}(t) = \bar{\mathbf{u}}e^{i\omega t}$  (2) where  $\omega$  is the vibratory frequency
- ▶ For the two-dof system, non-trivial solutions with  $k_1 = k_2 = 1 = m_1 = m_2 = 1$  are  $\omega_{n1} = 0.618$  and  $\omega_{n2} = 1.618$  which are called *natural frequencies*. This information can be shown in a Frequency-Energy Plot (FEP) as *backbone curves*, see Fig. 2

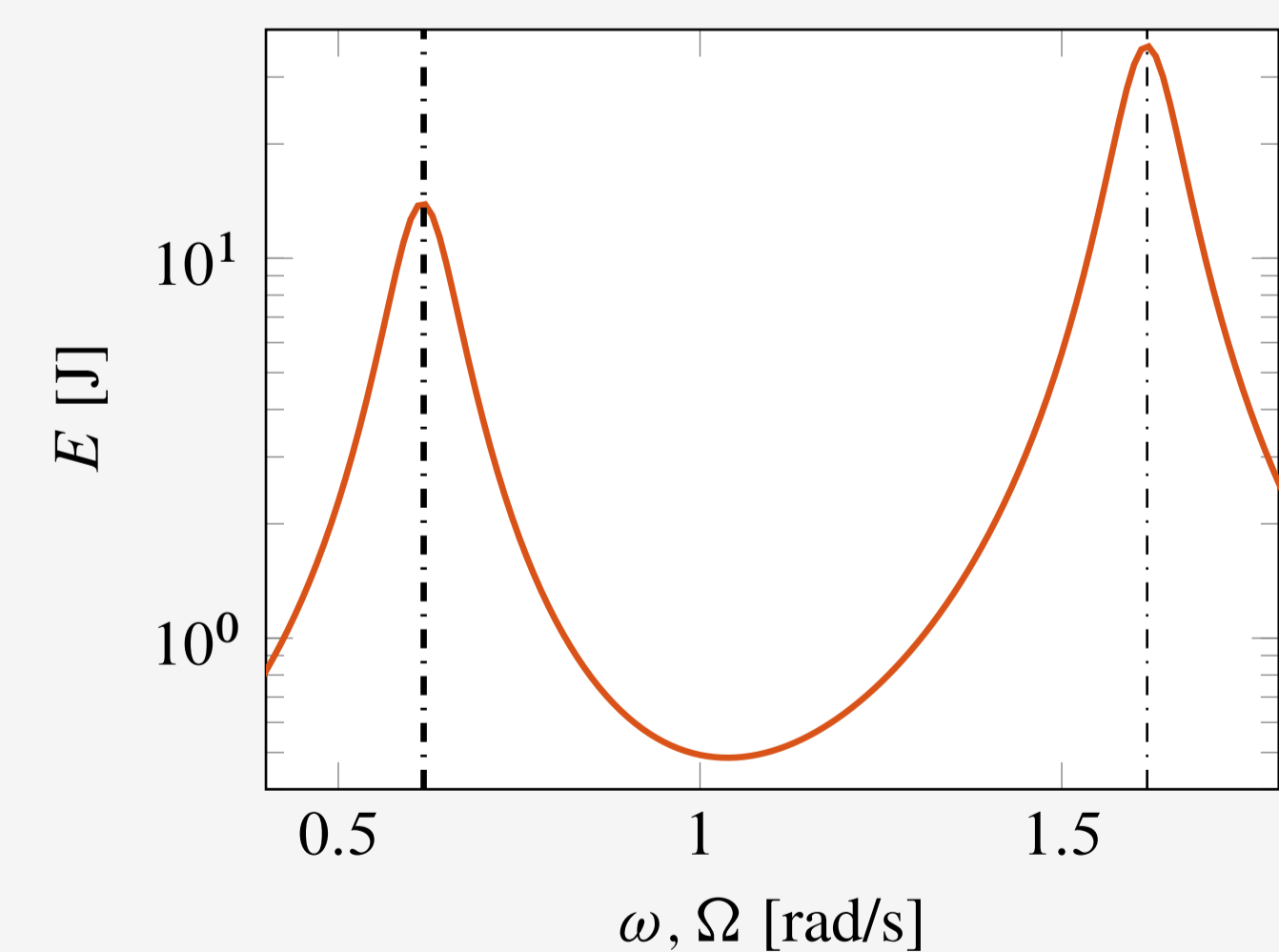


Figure 2: FEP of systems (1) (---) and (3) (—)

- ▶ Backbone curves predict the resonance of periodically forced systems with light damping  $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}\cos(\Omega t)$  (3) The energy of vibration is defined as the average energy over one period

## Unilateral contact: semismooth formulation

- ▶ Nonlinearity arises when non-penetrable wall presents  $\mathbf{r}(\mathbf{u}, \lambda) := \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{G}^T\lambda = \mathbf{0}$  (4) where  $\mathbf{u}$  is the displacement vector and  $\lambda$ , the contact force

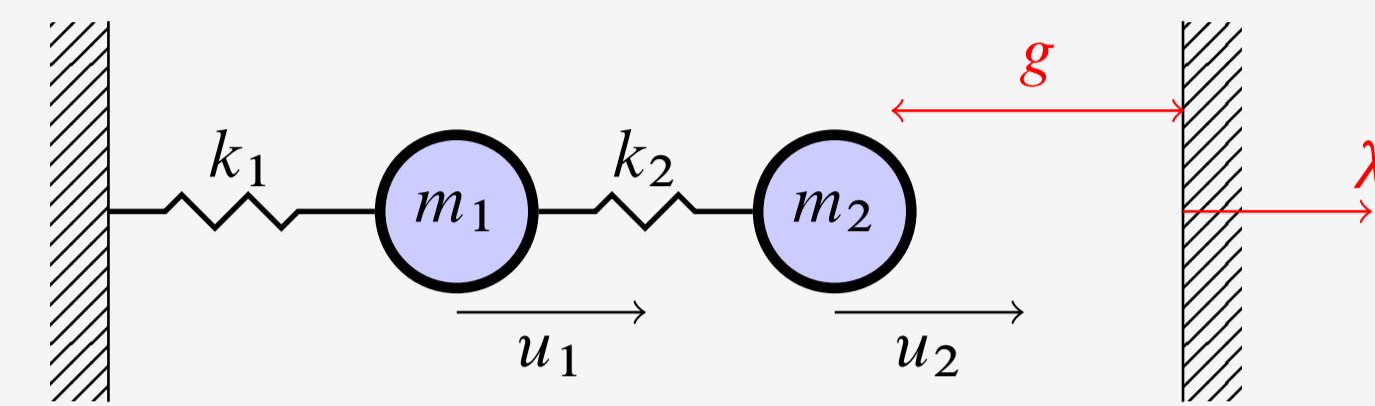


Figure 3: A two-DOF oscillator under contact

- ▶ A complementarity contact condition is added to (4) to completely describe the nonlinear system  $\mathbf{G}\mathbf{u} - g \leq 0, \quad \lambda \leq 0, \quad (\mathbf{G}\mathbf{u} - g)\lambda = 0$  (5)

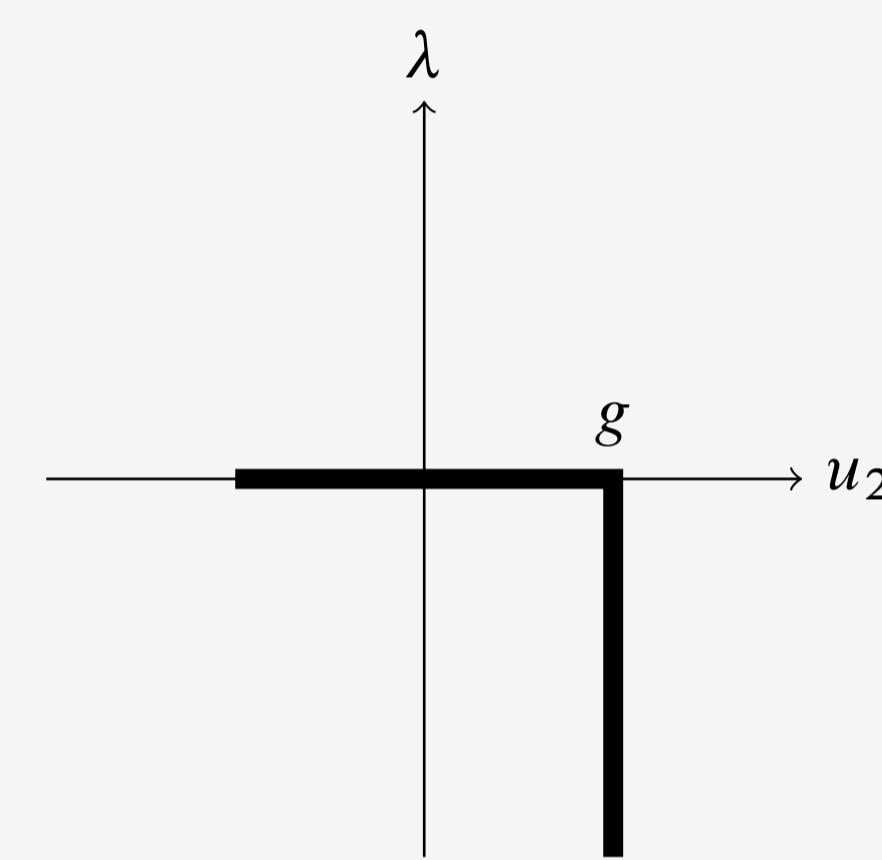


Figure 4: Multi-valued unilateral contact constitutive law

- ▶ Satisfying conditions (5) is equivalent to finding the root-set of the Lipschitz continuous function  $s(\mathbf{u}, \lambda) := \lambda + \max\{0, c(\mathbf{G}\mathbf{u} - g) - \lambda\} = 0, \quad \forall c > 0$  (6) as plotted in Figure 5

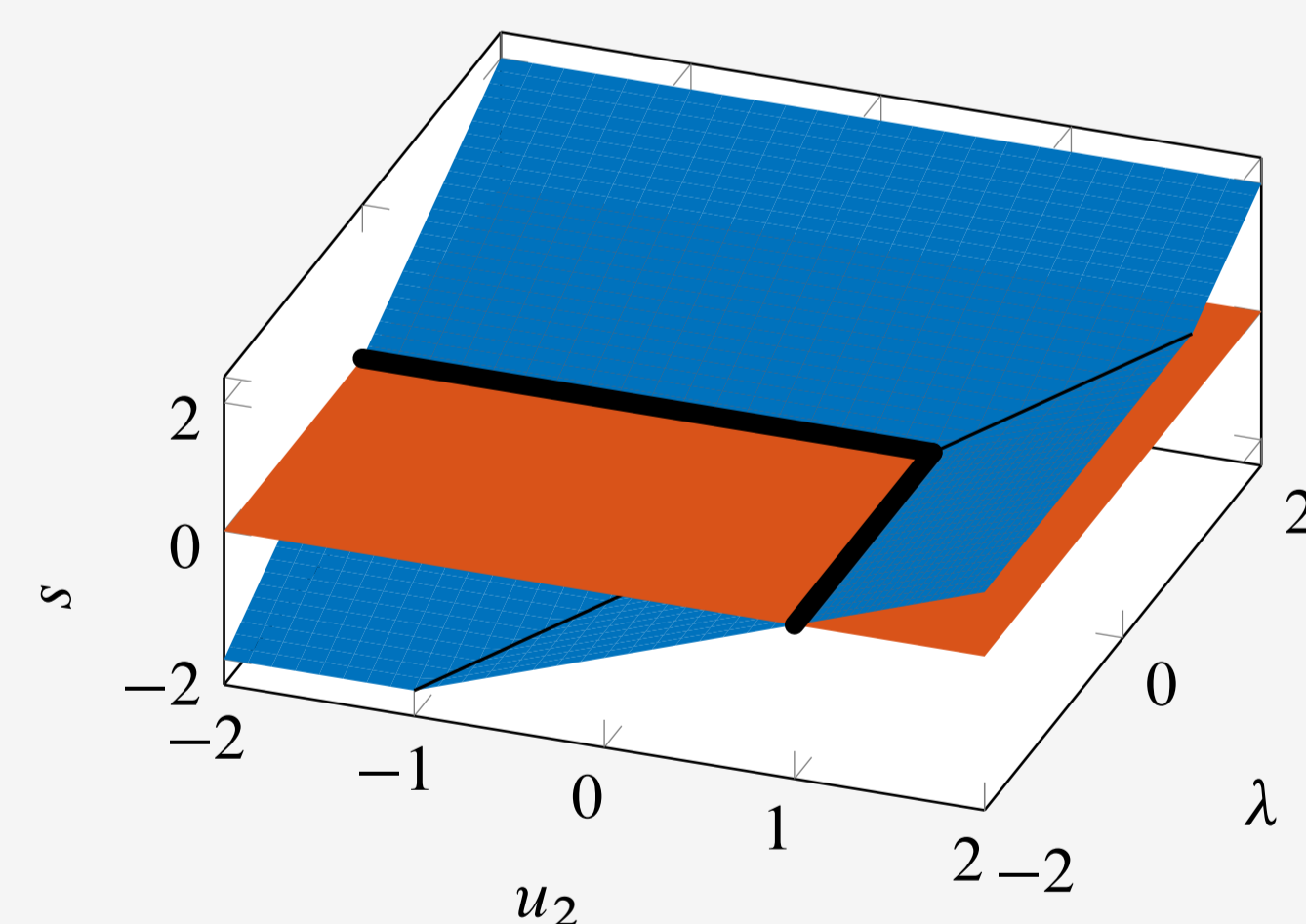


Figure 5: Complementarity conditions as the root-set of a two-dimensional Lipschitz continuous function

## References

- [1] Michael Hintermüller. "Semismooth Newton methods and applications." *Department of Mathematics, Humboldt-University of Berlin*, (2010).
- [2] Pierre Alart, Curnier Alain. "A mixed formulation for frictional contact problems prone to Newton like solution methods." *Computer methods in applied mechanics and engineering* 92.3 (1991): 353-375.
- [3] Vincent Acary, Brogliato Bernard. Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics. *Springer Science & Business Media*, 2008.

## Galerkin method

- ▶ Periodic solutions to (4) and (6) are approximated through a Galerkin method in time. Solution is expanded as a truncated series  $\mathbf{u}(t) \approx \sum_{m=1}^M \phi_m(t)\mathbf{u}_m$  and  $\lambda(t) \approx \sum_{m=1}^M \phi_m(t)\lambda_m$  (7) where  $\phi_m(t)$  are  $T$ -periodic shape functions;  $\mathbf{u}_m$  and  $\lambda_m$  are the unknown coefficients to be found
- ▶ Galerkin projection creates a set of semismooth nonlinear equations in the unknown coefficients  $\mathbf{u}_m$  and  $\lambda_m$   $\int_0^T \phi_k(t) \mathbf{r} \left( \sum_{m=1}^M \phi_m(t)\mathbf{u}_m, \sum_{m=1}^M \phi_m(t)\lambda_m \right) dt = \mathbf{0}$  (8)  $\int_0^T \phi_k(t) s \left( \sum_{m=1}^M \phi_m(t)\mathbf{u}_m, \sum_{m=1}^M \phi_m(t)\lambda_m \right) dt = 0$  for all  $k = 1, \dots, M$

## Semismooth Newton method

- ▶ Lipschitz continuous equation (8) written in vector form  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  where  $\mathbf{x}$  denotes the vector of unknown coefficients can be solved using the *semismooth Newton method*  $\mathbf{x}^{k+1} = \mathbf{x}^k - \mathbf{G}^{-1}(\mathbf{x}^k)\mathbf{f}(\mathbf{x}^k)$  (9) where  $\mathbf{G}(\mathbf{x}) \in \partial\mathbf{f}(\mathbf{x})$ .  $\partial\mathbf{f}(\mathbf{x})$  denotes the sub-differential of  $\mathbf{f}$  in literature [1, 2]

## Nonlinear modal analysis

- ▶ Displacement validates the accuracy of HBM

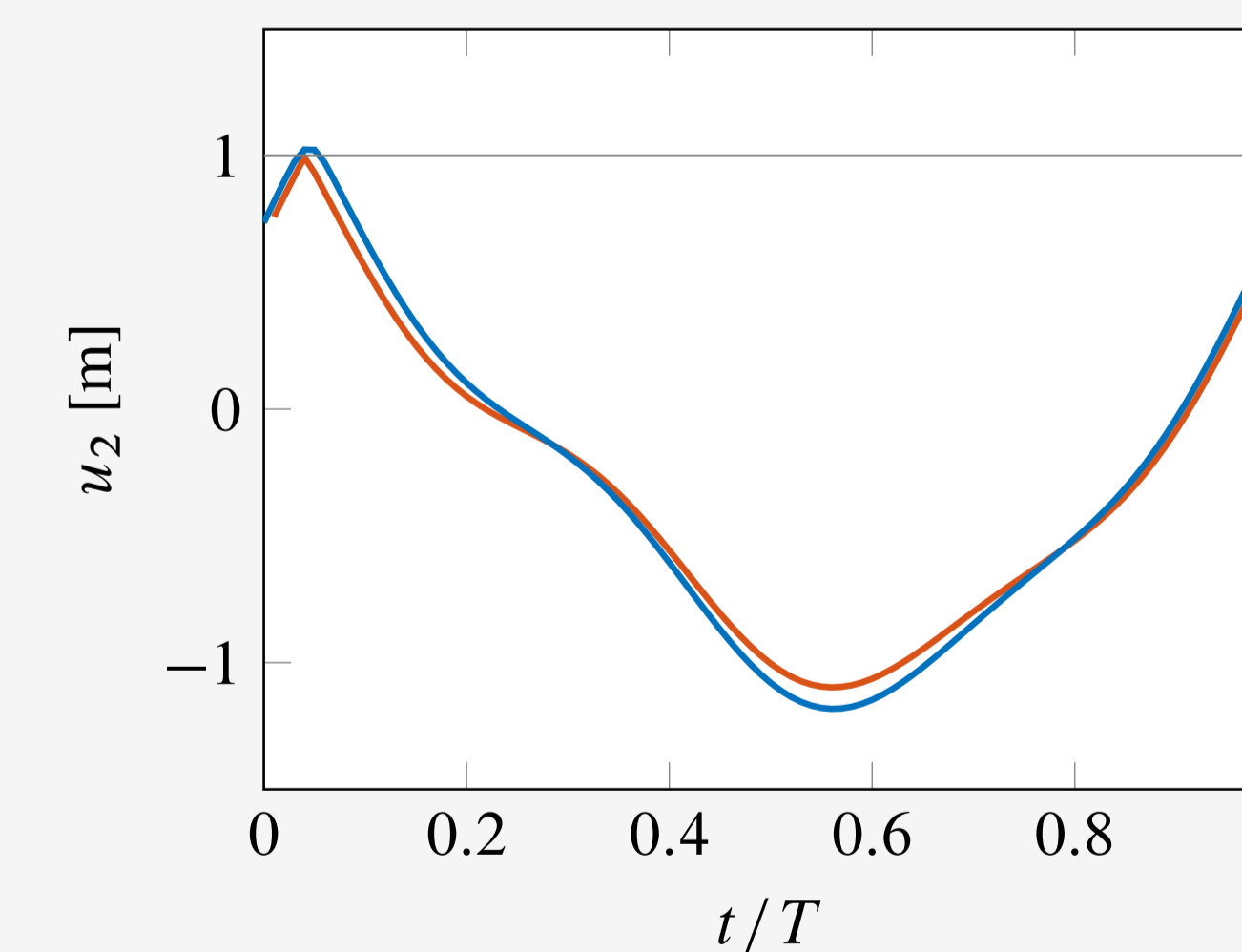


Figure 6: Periodic displacement of the system in Fig. 3 excited at  $\Omega = 0.618$ , using HBM (---) and event-driven (—) [3]

## Nonlinear modal analysis (Cont'd)

- ▶ Main and sub-harmonic resonances are found (see Fig 7)

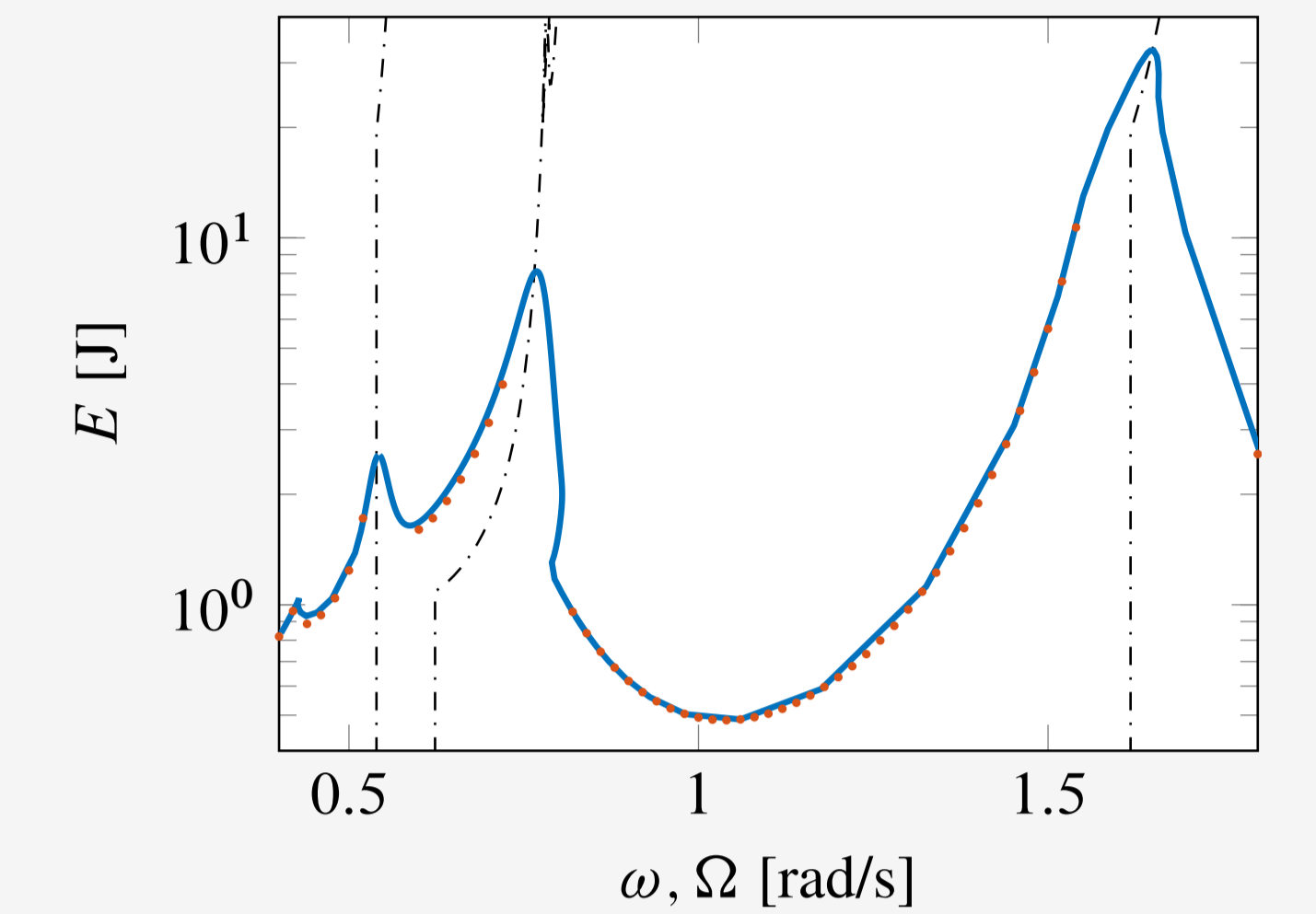


Figure 7: Resonance curves of the system in Fig. 3 solved using HBM (---) and event-driven method (---). Backbone curves (---)

- ▶ Bifurcation diagram exhibits chaos (typical property of periodically-forced impact-systems)

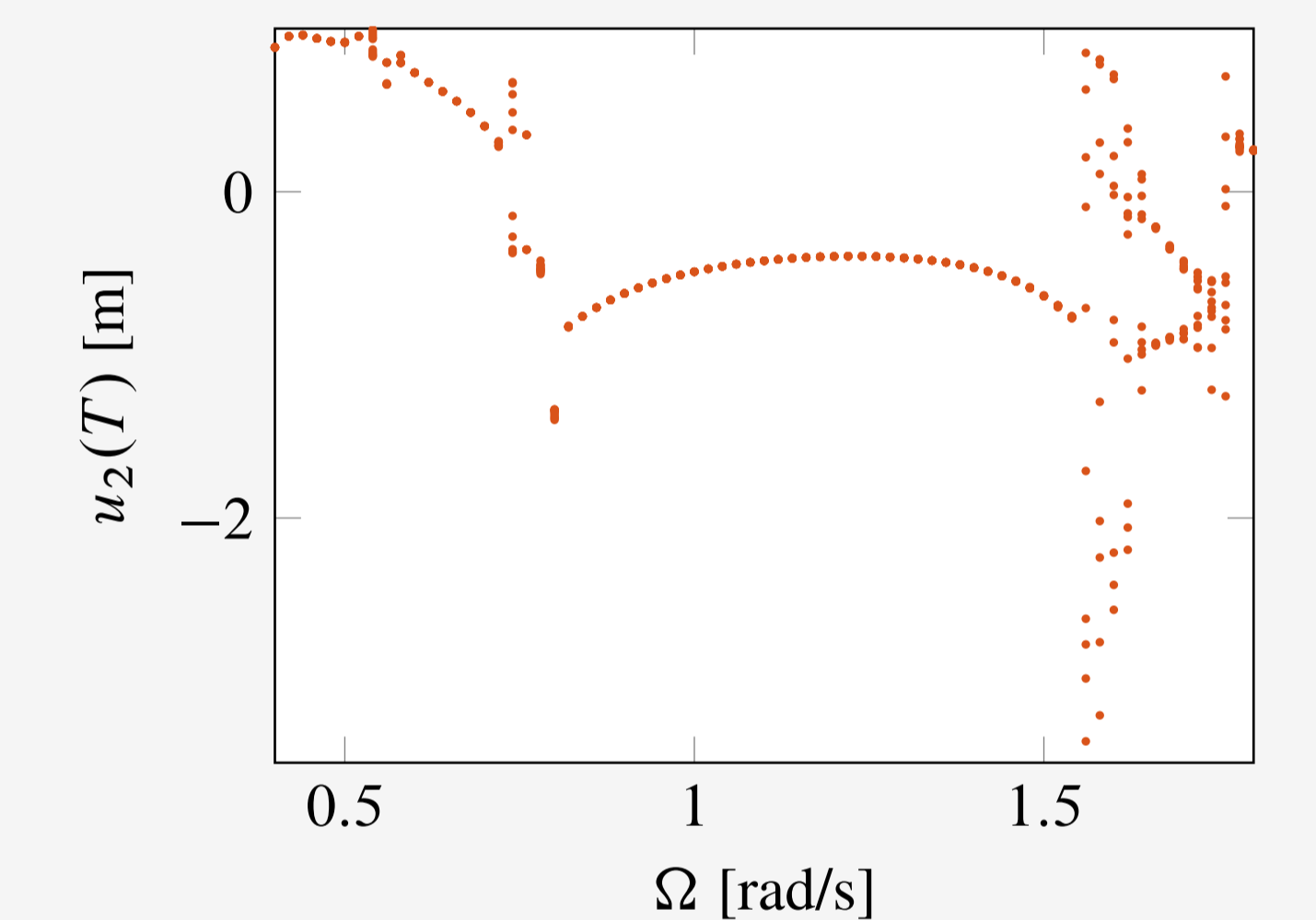


Figure 8: Bifurcation diagram of the system in Fig. 3 solved using event-driven method (---)

## Conclusion

- ▶ Methodology applicable to multi-DOF systems
- ▶ Vibratory resonances and sub-harmonic resonances
- ▶ Galerkin method combined to numerical continuation in order to find solution branches (either autonomous or periodically forced)
- ▶ Reduced-order models by exploiting the local contact nonlinearity
- ▶ Stability analysis can be done based on the approximate periodic solutions
- ▶ Slow convergence but robust numerical algorithm
- ▶ Residual penetration can be controlled by the number of shape functions