Semismooth Newton solver for periodically-forced solutions to a unilateral contact formulation

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SEMISMOOTH NEWTON SOLVER FOR PERIODICALLY-FORCED SOLUTIONS TO A UNILATERAL CONTACT FORMULATION

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Summary The vibratory response of a periodically-forced generic mechanical system undergoing a unilateral contact condition is addressed. The unilateral contact constraint is reformulated as a nonsmooth Lipschitz continuous function. This allows the use of the so-called *semismooth Newton method* capable of solving the equations governing the dynamics and the unilateral contact constraints simultaneously. The assumed periodic solution and the contact force are approximated by truncated Fourier series before being incorporated in the solver after projection of the equations on the Fourier basis. Continuation of the solution harmonics with respect to the forcing frequency is performed. For a medium size system of 20 degrees-of-freedom, it it shown that convergence is achieved by comparing with the "reference" time-marching solution.

INTRODUCTION

Fast and robust numerical algorithms for periodic solutions to unilateral contact problems under harmonic excitations are relevant to the industrial sphere. Applications include rotor-stator contact in turbomachines, milling, crack detection, and fatigue experiment rigs which induce dovetail-disk contact to name a few. Various solution methods such as time marching techniques, harmonic balance strategies, finite element methods are available for finding periodic solutions to smooth polynomial problems. However, they all face numerical difficulties when unilateral contact conditions are incorporated. State-of-the-art algorithms combining space discretization and time-domain Newmark numerical integration algorithms dedicated to unilateral contact problems are reviewed in [1]. In a vast majority, these algorithms exhibit either numerical spurious oscillations, numerical energy dissipation, or residual non-physical penetrations between contacting bodies. Alternating frequency/time domain techniques [2, 3] are used to find periodic solutions to unilateral contact problems. The finite element method in time-domain with periodicity and continuity conditions is also reported [4]. The unilateral contact problem is then reformulated as a linear complementarity problem which can be solved by Lemke's classical pivoting algorithm: it is known that uniqueness of the solution is not guaranteed [5].

Inspired by [6, 7, 8], this work reformulates the unilateral contact problem in the form of a Lipschitz continuous equation which allows the use of semismooth iterative Newton techniques. It is proved that for unilateral contact problems in statics within the small perturbation framework, the semismooth Newton algorithm converges Q-quadratically and is equivalent to the active-set method [9]. Yet, in a dynamic framework, proof of convergence is not available. Still, a Signorini condition seen as a Lipschitz continuous equation can be easily projected onto any finite-dimensional space of functions, as for example the Fourier functions used in this work.

INVESTIGATED SYSTEM AND FORMULATION

The system of interest is a simple serial N-degree-of-freedom oscillator stemming from the finite element discretization of a clamped-free periodically forced rod undergoing longitudinal displacement only. The free end of the rod is subjected to unilateral contact conditions. The governing equation together with the complementarity condition are

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{G}^\top \lambda \tag{1}$$

$$\mathbf{G}\mathbf{u} - g \leq 0, \quad \lambda \leq 0, \quad (\mathbf{G}\mathbf{u} - g)\lambda = 0 \tag{2}$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$ are the mass, damping, and stiffness matrices respectively; $\mathbf{u}$ is the displacement, $\mathbf{f}$, the external periodic force, λ, the scalar contact force (all are time-dependent functions), g, the initial separating gap, and finally $\mathbf{G}$, the constant matrix mapping the contact force to the N degrees-of-freedom. The complementarity condition (2) can be equivalently reformulated as the equality [6]

$$\lambda + \max(0, c(\mathbf{G}\mathbf{u} - g) - \lambda) = 0 \tag{3}$$

where c is an arbitrary positive constant. The displacement $\mathbf{u}$ and contact force λ are expanded as truncated Fourier series of period T (which is the period of the external forcing)

$$\mathbf{u}(t) \approx \sum_{i=1}^{M} \bar{\mathbf{u}}_i \phi_i(t) \quad \text{and} \quad \lambda(t) \approx \sum_{i=1}^{M} \bar{\lambda}_i \phi_i(t) \tag{4}$$

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where quantities $\bar{\mathbf{u}}_i$ and $\bar{\lambda}_i$, $i = 1, \ldots, M$, are the unknown coefficients to be found. Expansions (4) are inserted into Equations (1) and (3) and a Galerkin projection on the $\phi_i(t)$ family is performed yielding:

$$\forall j = 1, \ldots, M, \quad \sum_{i=1}^{M} \int_0^T \phi_j(t) \Big[\big(\ddot{\phi}_i(t)\mathbf{M} + \dot{\phi}_i(t)\mathbf{D} + \phi_i(t)\mathbf{K} \big) \bar{\mathbf{u}}_i - \mathbf{f}(t) - \mathbf{G}^\top \bar{\lambda}_i \phi_i(t) \Big] \mathrm{d}t = \mathbf{0} \tag{5}$$

$$\forall j = 1, \ldots, M, \quad \sum_{i=1}^{M} \int_0^T \phi_j(t) \Big[\bar{\lambda}_i \phi_i(t) + \max(0, c(\mathbf{G}\bar{\mathbf{u}}_i \phi_i(t) - g) - \bar{\lambda}_i \phi_i(t)) \Big] \mathrm{d}t = 0 \tag{6}$$

which is a system of Lipschitz continuous equations in the unknown $\bar{\mathbf{u}}_i$ and $\bar{\lambda}_i$: it can be handled by a semismooth Newton solver.

RESULTS

The tip displacement of a 20-dof system forced in the vicinity of its first natural frequency is displayed in Fig. 1. Light damping is included. The initial guess inserted in the iterative solver is the zero displacement/contact force with no convergence issues even with low damping. The reference solution is calculated with a time-stepping method and only the periodic steady-state is shown.

The semismooth Newton solver was then embedded in a pseudo-arclength continuation algorithm to track the magnitude of the forced response with respect to the excitation frequency. The frequency response is depicted in Fig. 2 for various initial gaps. The well-known stiffening effect induced by unilateral contact conditions is retrieved: this is for example reported by researchers solving similar problems using the penalty method [2]. It is also to be noted that the contact force does not statisfies (2) pointwise, but only (6) which is a weaker requirement possible leading to "strange" loops in the frequency response for small initial gaps.

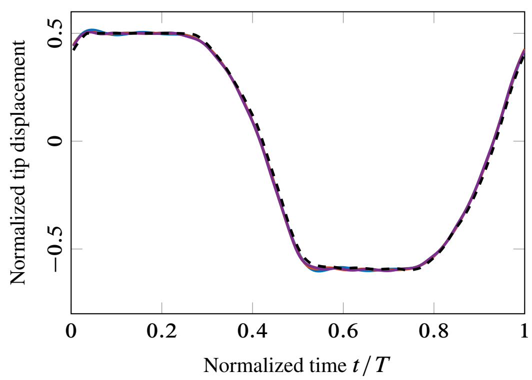

Figure 1: Rod tip displacement with $N = 20$ and $g = 0.5$. First natural frequency as forcing frequency. $M = 10$ (——), $M = 15$ (——), $M = 20$ (——), reference (- - -)

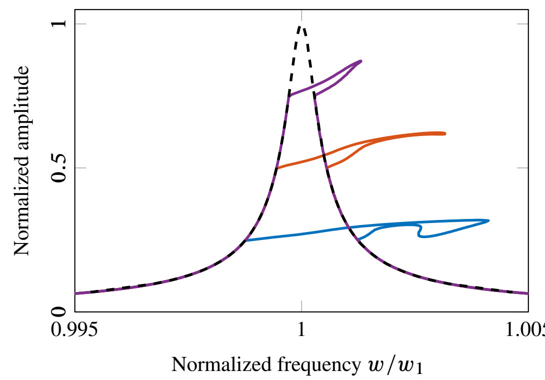

Figure 2: Frequency response curves near natural frequency ω_1 with $N = 20$ and $M = 5$. $g = 0.25$ (——), $g = 0.5$ (——), $g = 0.75$ (——), linear (- - -)

References

[1] David Doyen, Alexandre Ern, and Serge Piperno. Time-integration schemes for the finite element dynamic signorini problem. *SIAM Journal on Scientific Computing*, 33(1):223–249, 2011.

[2] Gotz Von Groll and David Ewins. The harmonic balance method with arc-length continuation in rotor/stator contact problems. *Journal of sound and vibration*, 241(2):223–233, 2001.

[3] Denis Laxalde, Fabrice Thouverez, Jean-Jacques Sinou, and Jean-Pierre Lombard. Qualitative analysis of forced response of blisks with friction ring dampers. *European Journal of Mechanics-A/Solids*, 26(4):676–687, 2007.

[4] Markus Meingast, Mathias Legrand, and Christophe Pierre. A linear complementarity problem formulation for periodic solutions to unilateral contact problems. *International Journal of Non-Linear Mechanics*, 66:18–27, 2014.

[5] Richard Cottle and Richard Stone. On the uniqueness of solutions to linear complementarity problems. *Mathematical programming*, 27(2):191–213, 1983.

[6] Andreas Fischer. A special newton-type optimization method. *Optimization*, 24(3-4):269–284, 1992.

[7] Tecla De Luca, Francisco Facchinei, and Christian Kanzow. A theoretical and numerical comparison of some semismooth algorithms for complementarity problems. *Computational Optimization and Applications*, 16(2):173–205, 2000.

[8] Michael Ulbrich. *Semismooth Newton methods for variational inequalities and constrained optimization problems in function spaces*, volume 11. SIAM, 2011.

[9] Michael Hintermüller. Semismooth newton methods and applications. *Department of Mathematics, Humboldt-University of Berlin*, 2010.