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► **To cite this version:**

Nassim Loukkas, John Jairo Martinez Molina, Nacim Meslem. Set-Membership Observer Design Based on Ellipsoidal Invariant Sets. IFAC WC 2017 - 20th IFAC World Congress, Jul 2017, Toulouse, France. 10.1016/j.ifacol.2017.08.1043 . hal-01569456

**HAL Id: hal-01569456**

**<https://hal.science/hal-01569456>**

Submitted on 26 Jul 2017

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# Set-membership observer design based on ellipsoidal invariant sets

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**Abstract:** This paper presents a new set-membership observer design method for linear discrete-time systems. The real process is assumed to be perturbed by unknown but bounded disturbances. The set-membership observer provides a deterministic state interval that is build as the sum of the estimated system states and its corresponding estimation errors bounds. The proposed approach is based on the solutions of a few number of Linear Matrix Inequalities that are suitable modified to provide both the observer parameters and ellipsoidal Robustly Positive Invariant sets. The latter are used to frame the estimation error in a very simple and accurate way. The enhanced precision on the computation of the estimation error bounds has been possible thanks to the use of the a posteriori calculated covariance matrix that allows, in a second time, to better describe the dissipation equation used in the Bounded-real lemma formulation. A numerical example illustrates the behavior of such observer and discuss its easy implementation.

*Keywords:* Set membership state estimation, state observers, robustly positive invariant sets, bounded-real lemma.

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## 1. INTRODUCTION

State estimation is often a necessary task to solve many control problems. Since in practical applications, sensors are often limited in number, can provide measurements with not enough accuracy and/or present low reliability, the knowledge of the full state of the system can be possible by using state observers. Several of the today control problems are related to decision-making process. Such real-time decisions are mostly based on estimated states. However, the associated thresholds for making decisions are often based on statistical results obtained during experimental tests. Those thresholds are intended to provide an admissible domain for starting or not a given action. For instance, in diagnosis and fault-tolerant control systems, several fault detection mechanisms are based on the detection of abnormal values on the observer-based residuals, see for instance Seron et al. (2008), Yetendje et al. (2011). Those residuals nominally belongs to a given interval of values in absence of faults and leaves those intervals in presence of faults. Thus, in this kind of applications, the main problem is to compute variable intervals or thresholds in a very accurate way for avoiding, for instance, false-alarms.

State estimation of disturbed systems can be performed, for instance, by Luenberger observers or Kalman filters which can provide a measurement of the quality of the estimates (e.g. the covariance matrix of the estimates). However, obtain a simple and accurate characterization of the state bounds in presence of bounded disturbances remains an open problem for high dimensional systems. In particular methods based on (real-time) interval arithmetic, see for instance Kieffer et al. (2002), Jaulin (2002), Kieffer and Walter (2002) and Efimov et al. (2013); methods based on online computation of zonotopes, see Combastel (2015)

and Le et al. (2013); methods based on online computation of ellipsoidal sets, see Ben Chabane et al. (2014b) and Durieu et al. (2001); and/or methods that combine zonotopes and ellipsoidal sets Ben Chabane et al. (2014a); are susceptible to increase the computational cost for high order systems and/or solving Linear Matrix Inequalities (LMI) in real-time. Some of the existent approaches do not guarantee stability of the obtained bounds for unstable open-loop systems, and most of the existent approaches are based on the propagation of the initial computed set which implies propagation, and possible amplification, of the initial state bounding error.

In this paper, we explore the concept of Robustly Positive Invariant (RPI) sets for designing set-membership observers. In particular, we explore the use of RPI sets with ellipsoidal form, see for instance Martinez (2015). There are two main reasons for this choice: i) Invariant sets allow to obtain state bounds in a deterministic and guarantee way, and ii) Geometry of ellipsoidal forms can be exploited to obtain more implementable solutions.

This paper presents a new set-membership observer design method for linear discrete-time disturbed systems. The system is assumed to be detectable. This concerns the only necessary condition for the system during the observer design process. The system disturbances are considered to be unknown inputs but belong to a bounded and known set. The proposed set-membership observer provides a deterministic state interval that is build as the sum of the estimated system states and its corresponding estimation errors bounds. The proposed approach is based on the solutions of a few number of Linear Matrix Inequalities that are suitable modified to provide both the observer parameters and ellipsoidal RPI sets. The latter are used to frame the estimation error in a very simple and accurate

way. The enhanced precision on the computation of the estimation error bounds has been possible thanks to the use of the a posteriori calculated covariance matrix that allows, in a second time, to better describe the dissipation equation used in the Bounded-real lemma formulation. Comparing with respect to standard state observers for  $n$ -order systems, the proposed set-membership observer will be a  $(n + 1)$ -order observer which facilitate its implementation for high order systems.

The paper is organized as follows. Section 2 presents the problem statement. Then, in Section 3 the Bounded-real lemma has been used to design a Luenberger observer and provides ellipsoidal Robustly Positive Invariant (RPI) sets. In Section 4, an strategy based on an, *a posteriori* calculated, covariance matrix is used to obtain smaller RPI sets. This Section also summarizes the proposed observer design process. Sections 5 and 6 are dedicated to infer the evolution of such RPI sets starting for any initial estimation error and illustrates the implementation of the set membership observer. Finally in Section 7 a numerical example illustrates the interest of such approach. Conclusions and future work are presented in Section 8.

## 2. PROBLEM STATEMENT

Consider the time invariant linear discrete-time system:

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{F}\mathbf{d}_k \\ \mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{Z}\mathbf{v}_k \end{cases} \quad (1)$$

where  $\mathbf{x}_k \in \mathbb{R}^n$  is the state vector,  $\mathbf{u}_k \in \mathbb{R}^{n_u}$  is the input vector and  $\mathbf{y}_k \in \mathbb{R}^{n_y}$  is the measured output vector. The vectors  $\mathbf{d}_k$  and  $\mathbf{v}_k$  are unknown state disturbances and unknown measurement noises, respectively. The vector of total disturbances, i.e.  $[\mathbf{d}_k \ \mathbf{v}_k]^T$ , is assumed to belong to a bounded set which includes the zero. The matrices  $\mathbf{A}, \mathbf{B}, \mathbf{F}, \mathbf{C}, \mathbf{Z}$  have appropriated dimensions. We assume that the pair  $(\mathbf{A}, \mathbf{C})$  is detectable.

Suppose we can design the following Luenberger observer:

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k \quad (2)$$

where  $\mathbf{L}$  is the observer gain matrix. One of the objectives of this matrix is to guarantee that the estimation dynamics (2) will be stable.

Defining the estimation error at a given instant  $k$  as  $\mathbf{e}_k := \mathbf{x}_k - \hat{\mathbf{x}}_k$ , the dynamics of the estimation error can be obtained from (1) and (2), as follows:

$$\mathbf{e}_{k+1} = \mathbf{A}_o\mathbf{e}_k + \mathbf{E}\mathbf{w}_k \quad (3)$$

where  $\mathbf{A}_o = (\mathbf{A} - \mathbf{L}\mathbf{C})$ ,  $\mathbf{E} = [\mathbf{F} \ -\mathbf{L}\mathbf{Z}]$  and  $\mathbf{w}_k \in \mathbb{R}^m$ , defined as  $\mathbf{w}_k := \begin{pmatrix} \mathbf{d}_k \\ \mathbf{v}_k \end{pmatrix}$ .

Remark that the matrix  $\mathbf{L}$  guarantees the stability of the error dynamics (3). However, it also intervenes on the amplification or on the attenuation of the disturbances  $\mathbf{w}_k$ . In the sequel we assume that disturbance  $\mathbf{w}_k$  can be bounded as follows  $\mathbf{w}_k^T\mathbf{w}_k \leq \bar{\mathbf{w}}^T\bar{\mathbf{w}}$  for all  $k \geq 0$ , where  $\bar{\mathbf{w}} \in \mathbb{R}^m$ ,  $\bar{\mathbf{w}} \geq 0$  and  $\bar{\mathbf{w}} := \sup\{\mathbf{w}_k\}$ .

Now suppose that we exactly known a bound of the estimation error  $\mathbf{e}_k$ , denoted  $\bar{\mathbf{e}}_k$ . At each time-instant, the real state vector satisfies

$$\hat{\mathbf{x}}_k - \bar{\mathbf{e}}_k \leq \mathbf{x}_k \leq \hat{\mathbf{x}}_k + \bar{\mathbf{e}}_k \quad (4)$$

In other words, at every time-instant we can guarantee that the system state belongs to a set defined by the vectors  $\underline{\mathbf{x}}_k := \hat{\mathbf{x}}_k - \bar{\mathbf{e}}_k$  and  $\bar{\mathbf{x}}_k := \hat{\mathbf{x}}_k + \bar{\mathbf{e}}_k$ .

In this case, a set-membership observer could be implemented as follows:

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k \quad (5)$$

$$\bar{\mathbf{x}}_k = \hat{\mathbf{x}}_k + \bar{\mathbf{e}}_k \quad (6)$$

$$\underline{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \bar{\mathbf{e}}_k \quad (7)$$

Therefore the set-membership observer design problem can be reduced to compute at each time-instant the vectors  $\bar{\mathbf{x}}_k$  and  $\underline{\mathbf{x}}_k$  as the sum of both a state estimation and a bound of the estimation error  $\bar{\mathbf{e}}_k$  and  $-\bar{\mathbf{e}}_k$ , respectively.

Nevertheless, computation of a deterministic, simple and accurate bound  $\bar{\mathbf{e}}_k$  is not an easy task. In this paper we propose a new method to design a set-membership observer in a very accurate and implementable way. The proposed method is based on the computation of ellipsoidal Robustly Positive Invariant (RPI) sets. Such sets are used to compute suitable deterministic bounds of the estimation error  $\mathbf{e}_k$ .

## 3. SET-MEMBERSHIP OBSERVER DESIGN

In this Section, a first observer design based on  $H_\infty$  synthesis is presented. Additionally, the ellipsoidal RPI sets are derived and used for bounding the estimation error at steady-state regime.

### 3.1 $H_\infty$ Observer design using the Bounded-real lemma

Consider the estimation error dynamics (3). An  $H_\infty$  observer is intended for minimizing the impact of the system disturbances  $\mathbf{w}_k$  on the estimation error  $\mathbf{e}_k$ .

Now consider the following candidate Lyapunov function  $\mathbf{V}_k = \mathbf{e}_k^T\mathbf{P}\mathbf{e}_k$  for system (3). Suppose there exist a symmetric positive definite matrix  $\mathbf{P}$  and a scalar  $\gamma > 0$  verifying the following *dissipation* inequality:

$$\mathbf{V}_{k+1} - \mathbf{V}_k \leq -\mathbf{e}_k^T\mathbf{e}_k + \gamma^2\mathbf{w}_k^T\mathbf{w}_k \quad (8)$$

We can use the Bounded-real Lemma for discrete-time systems for designing a state observer, i.e. find the matrix  $\mathbf{L}$ , the matrix  $\mathbf{P}$  and the minimum scalar  $\gamma$  which verify:

$$\begin{pmatrix} \mathbf{A}_o^T\mathbf{P}\mathbf{A}_o - \mathbf{P} + \mathbf{I}_n & \mathbf{A}_o^T\mathbf{P}\mathbf{E} \\ \mathbf{E}^T\mathbf{P}\mathbf{A}_o & \mathbf{E}^T\mathbf{P}\mathbf{E} - \gamma^2\mathbf{I}_m \end{pmatrix} \preceq 0 \quad (9)$$

This condition has to be transformed into a Linear Matrix Inequality (LMI) to be solved. Thus, we start by written condition (9) as follows:

$$\begin{pmatrix} -\mathbf{P} + \mathbf{I}_n & 0 \\ 0 & -\gamma^2\mathbf{I}_m \end{pmatrix} + \begin{pmatrix} \mathbf{A}_o^T\mathbf{P}\mathbf{A}_o & \mathbf{A}_o^T\mathbf{P}\mathbf{E} \\ \mathbf{E}^T\mathbf{P}\mathbf{A}_o & \mathbf{E}^T\mathbf{P}\mathbf{E} \end{pmatrix} \preceq 0 \quad (10)$$

then, by using the Schur complement we obtain:

$$\begin{pmatrix} -\mathbf{P} + \mathbf{I}_n & 0 & \mathbf{A}_o^T\mathbf{P} \\ 0 & -\gamma^2\mathbf{I}_m & \mathbf{E}^T\mathbf{P} \\ \mathbf{P}\mathbf{A}_o & \mathbf{P}\mathbf{E} & -\mathbf{P} \end{pmatrix} \preceq 0 \quad (11)$$

Replacing  $\mathbf{A}_o = (\mathbf{A} - \mathbf{L}\mathbf{C})$  and  $\mathbf{E} = [\mathbf{F} \ -\mathbf{L}\mathbf{Z}]$ , we have:

$$\begin{pmatrix} -\mathbf{P} + \mathbf{I}_n & 0 & \mathbf{A}^T\mathbf{P} - \mathbf{C}^T\mathbf{L}^T\mathbf{P} \\ 0 & -\gamma^2\mathbf{I}_m & [\mathbf{P}\mathbf{F} \ -\mathbf{P}\mathbf{L}\mathbf{Z}]^T \\ \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{L}\mathbf{C} & [\mathbf{P}\mathbf{F} \ -\mathbf{P}\mathbf{L}\mathbf{Z}] & -\mathbf{P} \end{pmatrix} \preceq 0 \quad (12)$$

by performing a change of variable in (12), i.e.  $\mathbf{U} := \mathbf{P}\mathbf{L}$ , we obtain the following LMI:

$$\begin{pmatrix} -\mathbf{P} + \mathbf{I}_n & 0 & \mathbf{A}^T \mathbf{P} - \mathbf{C}^T \mathbf{U}^T \\ 0 & -\gamma^2 \mathbf{I}_m & [\mathbf{P}\mathbf{F} - \mathbf{U}\mathbf{Z}]^T \\ \mathbf{P}\mathbf{A} - \mathbf{U}\mathbf{C} & [\mathbf{P}\mathbf{F} - \mathbf{U}\mathbf{Z}] & -\mathbf{P} \end{pmatrix} \preceq 0 \quad (13)$$

Once the matrices  $\mathbf{P}$ ,  $\mathbf{U}$  and the minimum scalar  $\gamma$  are found, the state observer gain  $\mathbf{L}$  can be computed as  $\mathbf{L} = \mathbf{P}^{-1}\mathbf{U}$ . Then, the first part of the Set-membership state observer, equation (5), can be implemented. The problem now is to compute suitable bounds of the estimation error for implementing (6) and (7).

### 3.2 Computing ellipsoidal RPI sets

Considering that the norm-2 of the disturbances can be bounded as  $\mathbf{w}_k^T \mathbf{w}_k \leq \bar{\mathbf{w}}^T \bar{\mathbf{w}}$ , from (8) we have:

$$\mathbf{V}_{k+1} - \mathbf{V}_k \leq -\mathbf{e}_k^T \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \leq 0 \quad (14)$$

if  $\mathbf{e}_k^T \mathbf{e}_k \geq \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}$ . Then, outside the ball

$$B_w := \{\mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{e} \leq \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}\} \quad (15)$$

the difference of the Lyapunov function along the trajectories, i.e.  $\mathbf{V}_{k+1} - \mathbf{V}_k$ , is negative, which implies that a level set of the Lyapunov function  $\Omega := \{\mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq c\}$  that contains the ball  $B_w$  is an attractive invariant set. A value of  $c$  which guarantees that the ball  $B_w$  is included into the set  $\Omega$  can be calculated as  $c = \lambda_{\max}(\mathbf{P}) \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}$ . Thus, the set  $\Phi$  defined below is an attractive invariant set for the system (3):

$$\Phi := \{\mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \lambda_{\max}(\mathbf{P}) \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}\} \quad (16)$$

The symbol  $\lambda_{\max}(\mathbf{P})$  denotes the maximum eigenvalue of matrix  $\mathbf{P}$ .

Now, using geometrical properties of the ellipsoids, steady-state bounds on  $\mathbf{e}_k$  (i.e. for  $k \rightarrow \infty$ ), denoted  $\bar{\mathbf{e}}_\infty$ , can be obtained as follows:

$$\bar{\mathbf{e}}_\infty = \text{diag} \left( \left( \frac{\mathbf{P}}{\lambda_{\max}(\mathbf{P}) \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}} \right)^{-1/2} \right) \quad (17)$$

The symbol  $\text{diag}(M)$  denotes the diagonal of matrix  $M$ .

The above bounds on  $\mathbf{e}_k$  presents two drawbacks: i) those bounds only characterize the steady-state regime of the estimation error, and ii) the used ellipsoidal RPI set could present an important volume, providing very conservative bounds of the estimation error.

In the next Section, we will propose a method to reduce the size (the volume) of the obtained RPI sets to enhance the precision of the obtained bounds. In addition, in Section 5 we will use the evolution of the Lyapunov level sets, describing the RPI sets, for finding suitable bounds during the whole period of the state observer evolution, i.e. for  $k > 0$ .

## 4. SHRINKING ELLIPSOIDAL RPI SETS

### 4.1 Suitable modification of the dissipation matrix

It is possible to shrink the ellipsoidal RPI set (16) by finding a novel condition of the contraction of the Lyapunov function. That is, assuming the existence of a symmetric

positive definite matrix  $\mathbf{Q}$ , we can modify (14) to obtain the following less restrictive dissipation inequality:

$$\mathbf{V}_{k+1} - \mathbf{V}_k \leq -\mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \leq 0 \quad (18)$$

which holds if  $\mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k \geq \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}$ .

Thus, the LMI (13), used for designing a state observer, can be transformed into a more general LMI which consider any dissipation matrix  $\mathbf{Q}$ , that is:

$$\begin{pmatrix} -\mathbf{P} + \mathbf{Q} & 0 & \mathbf{A}^T \mathbf{P} - \mathbf{C}^T \mathbf{U}^T \\ 0 & -\gamma^2 \mathbf{I}_m & [\mathbf{P}\mathbf{F} - \mathbf{U}\mathbf{Z}]^T \\ \mathbf{P}\mathbf{A} - \mathbf{U}\mathbf{C} & [\mathbf{P}\mathbf{F} - \mathbf{U}\mathbf{Z}] & -\mathbf{P} \end{pmatrix} \preceq 0 \quad (19)$$

Once the matrix  $\mathbf{P}$  and the minimum scalar  $\gamma$  have been computed from (19) for any initial dissipation matrix  $\mathbf{Q}$ . A new and refined matrix  $\mathbf{Q}$  can be obtained, in a second time, by minimizing the volume of the ellipsoid defined by this matrix. That is, find  $\mathbf{Q}$  which *minimizes*  $-\ln \det(\mathbf{Q})$

and verifies the following LMI:

$$\begin{pmatrix} \mathbf{A}_o^T \mathbf{P} \mathbf{A}_o - \mathbf{P} + \mathbf{Q} & \mathbf{A}_o^T \mathbf{P} \mathbf{E} \\ \mathbf{E}^T \mathbf{P} \mathbf{A} & \mathbf{E}^T \mathbf{P} \mathbf{E} - \gamma^2 \mathbf{I}_m \end{pmatrix} \preceq 0 \quad (20)$$

In this way, a smaller ellipsoidal RPI set can be obtained as follows:

$$\Psi := \left\{ \mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \frac{1}{\lambda} \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \right\} \quad (21)$$

where, for a given non-zero vector  $\mathbf{e}$ , the scalar  $\lambda$  satisfy

$$\frac{\mathbf{e}^T \mathbf{Q} \mathbf{e}}{\mathbf{e}^T \mathbf{P} \mathbf{e}} \geq \lambda \quad (22)$$

which is known as the **generalized Rayleigh quotient**. The scalar  $\lambda$  satisfying (22) can be obtained as the minimum generalized eigenvalue of the pair  $(\mathbf{Q}, \mathbf{P})$ .

The ellipsoid (21) represents the smallest level set of the Lyapunov function which includes the set:

$$B_w^* := \{\mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{Q} \mathbf{e} \leq \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}\} \quad (23)$$

### 4.2 Using the a posteriori steady-state covariance matrix

Considering that the expected value of  $\mathbf{e}_k \in \mathbb{R}^n$  in (3) is equal to zero, its steady-state covariance equal to  $V$  and for any real number  $t > 0$ , we can use the multidimensional **Chebyshev's inequality**:

$$\Pr(\mathbf{e}_k^T V^{-1} \mathbf{e}_k > t^2) \leq \frac{n}{t^2} \quad (24)$$

for computing a stochastic ellipsoidal set. Even if this set could have very small volume, this set is not an invariant set because there is a probability that some trajectories of the estimation error  $\mathbf{e}$  go out this set. However, its shape matrix can be used to update the dissipation matrix  $\mathbf{Q}$  during the observer design.

Remember that the Lyapunov function in (18) only decreases if the following condition holds:

$$\mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k > \gamma^2 \bar{\mathbf{w}}_k^T \bar{\mathbf{w}}_k \quad (25)$$

Remark that this deterministic condition has the same shape than the inequality (24), if we get  $\mathbf{Q} = V^{-1}$ .

Hence, the problem now is to calculate the matrix  $V$ . The steady-state covariance of the estimation error, denoted  $V$ , can be obtained by solving the following Lyapunov equation:

$$\mathbf{A}_o V \mathbf{A}_o^T - V = -W \quad (26)$$

where  $W$  represents the co-variance matrix for disturbances  $\mathbf{E}\mathbf{w}_k$  in (3). In practical applications where the co-variance matrix for disturbances is not available, we can assume that every element of the disturbance vector  $\mathbf{w}_k$  is uniform distributed but bounded in a given interval  $[a \ b]$ . In this case, its variance can be computed as

$$W = \text{var}(\mathbf{E}\mathbf{w}_k) = (1/12)(b - a)^2 \mathbf{E}\mathbf{E}^T \quad (27)$$

Remark that the computation of  $V$  is possible once the matrix  $\mathbf{A}_o$  is available, i.e. an *a priori* observer matrix gain  $\mathbf{L}$  has to be calculated using an initial and arbitrary matrix  $\mathbf{Q}$ , for instance  $\mathbf{Q} = \mathbf{I}_n$ . After applying this *heuristic* procedure, a significant reduction of the RPI set volume can be obtained by re-starting the observer design process with the new computed matrix  $\mathbf{Q} = V^{-1}$ .

The complete observer design process is summarizing in Algorithm 1.

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**Algorithm 1** set-membership observer design based on ellipsoidal invariant sets.

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**Require:** Matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}$  and  $\mathbf{Z}$  describing system

- (1). Initialization of  $\mathbf{Q} = \mathbf{I}_n$  and  $i = 0$ .
  - 1: Increment  $i$  by one
  - 2: Find matrices  $\mathbf{P}, \mathbf{U}$  and the minimum  $\gamma$  who satisfy the LMI (19).
  - 3: Compute  $\mathbf{L} = \mathbf{P}^{-1}\mathbf{U}$ .
  - 4: Compute  $\mathbf{A}_o = \mathbf{A} - \mathbf{L}\mathbf{C}$ .
  - 5: Compute  $\mathbf{E} = [\mathbf{F} \quad -\mathbf{L}\mathbf{Z}]$
  - 6: Using  $\mathbf{P}$  and  $\gamma$ , find a new matrix  $\mathbf{Q}$  which satisfies the LMI (20) and minimize  $(-\log(\det(\mathbf{Q})))$
  - 7: Compute the minimum  $\lambda$  which satisfies (22)
  - 8: **if**  $i < 2$  **then**
  - 9:     Compute the disturbance variance  $W$  using (27).
  - 10:    Obtain the covariance matrix  $V$  using (26).
  - 11:    Do  $\mathbf{Q} = V^{-1}$ .
  - 12:    Go to step 1.
  - 13: **end if**
  - 14: **return** The observer parameters  $\mathbf{L}, \mathbf{P}, \gamma$  and  $\lambda$ .
- 

Now, proceeding in a similar way that this proposed in Section 3.2, smaller steady-state bounds on  $\mathbf{e}_k$ , denoted  $\bar{\mathbf{e}}_\infty$ , can be obtained as follows:

$$\bar{\mathbf{e}}_\infty = \text{diag} \left( \left( \frac{\mathbf{P}}{\frac{1}{\lambda} \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}} \right)^{-1/2} \right) \quad (28)$$

In the next Section, we will use the evolution of the Lyapunov level sets, describing the RPI sets, for finding suitable bounds during the whole period of the state observer evolution, i.e. for  $k > 0$ .

## 5. CHARACTERIZING THE EVOLUTION OF THE RPI SETS

Suppose that the initial estimation error, denoted  $\mathbf{e}_0$  is unknown but belongs to an initial bounded set  $\mathcal{E}_0 \subset \mathbb{R}^n$ .

There exist a scalar  $\mu \geq 1$  such that the following condition holds:

$$\mathcal{E}_0 \subset \bar{\Psi} \quad (29)$$

with

$$\bar{\Psi} := \{ \mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \frac{1}{\lambda} \mu \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \} \quad (30)$$

Remark that the set (30) corresponds to an expansion of the invariant set (21), using  $\mu \geq 1$ . For this reason the set (30) is also an invariant set.

Starting by the inequality (18) and using the relation (22):

$$\mathbf{V}_{k+1} \leq \mathbf{V}_k - \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \quad (31)$$

$$\mathbf{e}_{k+1}^T \mathbf{P} \mathbf{e}_{k+1} \leq \mathbf{e}_k^T \mathbf{P} \mathbf{e}_k - \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \quad (32)$$

$$\leq \frac{1}{\lambda} \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k - \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \quad (33)$$

$$\leq \left( \frac{1}{\lambda} - 1 \right) \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \quad (34)$$

Now considering the fact that the set  $\bar{\Psi}$  includes the set:

$$\bar{B}_w^* := \{ \mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{Q} \mathbf{e} \leq \mu \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \} \quad (35)$$

we have

$$\mathbf{e}_{k+1}^T \mathbf{P} \mathbf{e}_{k+1} \leq \left( \frac{1}{\lambda} - 1 \right) \mu \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} + \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \quad (36)$$

we can now explicitly compute the one-step ahead RPI set, denoted  $\bar{\Psi}^+$ , as follows:

$$\bar{\Psi}^+ := \left\{ \mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \left( \frac{1}{\lambda} \mu - \mu + 1 \right) \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}} \right\} \quad (37)$$

By defining  $\bar{c} := \frac{1}{\lambda} \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}$  we have a more compact expression of the expanded invariant set (30) and its one-step ahead invariant set evolution (37), that is

$$\begin{cases} \bar{\Psi} & : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \mu \bar{c} \\ \bar{\Psi}^+ & : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \underbrace{\left( (1 - \lambda) \mu + \lambda \right)}_{\mu^+} \bar{c} \end{cases} \quad (38)$$

These expressions can be used to infer a recursive relationship between  $\mu$  and its one-step ahead value, denoted  $\mu^+$  in (38). Thus, for a given initial condition  $\mu_0 \geq 1$  the dynamics of this scalar, at every time-instant  $k$ , obeys:

$$\mu_{k+1} = (1 - \lambda) \mu_k + \lambda \quad (39)$$

This dynamics is necessarily stable because it characterizes the contraction of the invariant set (30). Remark that  $\mu_{k+1} \leq \mu_k$  because  $\bar{\Psi}^+ \subseteq \bar{\Psi}$ . In addition,  $\mu_k$  asymptotically converges to 1 as long as the time-instant  $k \rightarrow \infty$ .

Hence, at every time instant, the invariant sets obeys the following dynamics:

$$\bar{\Psi}_k := \{ \mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{P} \mathbf{e} \leq \mu_k \bar{c} \} \quad (40)$$

and its ellipsoidal shape matrix can be used to compute a more accuracy bounds for the estimation error  $\mathbf{e}_k$ . That is,

$$\bar{\mathbf{e}}_k = \text{diag} \left( \left( \frac{\mathbf{P}}{\mu_k \bar{c}} \right)^{-1/2} \right) \quad (41)$$

equivalently,

$$\bar{\mathbf{e}}_k = \bar{\mathbf{e}}_\infty \mu_k^{1/2} \quad (42)$$

with  $\bar{\mathbf{e}}_\infty$  a known constant column vector defined in (28).

It only remains to compute an appropriated initial value for the scalar  $\mu$ . To do this, suppose that the initial condition for the estimation error verifies  $e(0) \in \mathcal{E}_0 \subseteq B_0$ ,



where, for a given scalar  $\gamma_\epsilon \geq 0$ , the ball  $B_0$  defined as follows:

$$B_0 := \{\mathbf{e} \in \mathbb{R}^n : \mathbf{e}^T \mathbf{e} \leq \gamma_\epsilon^2\} \quad (43)$$

will be included into the invariant set

$$\bar{\Psi}_0 := \{\mathbf{e} \in \mathbb{R}^n : \{\mathbf{e}^T \mathbf{P} \mathbf{e} \leq \lambda_{max}(\mathbf{P})\gamma_\epsilon^2\}\} \quad (44)$$

Thus, using (40) and (44) a suitable initial value of  $\mu$  verifies:

$$\mu_0 \bar{c} = \lambda_{max}(\mathbf{P})\gamma_\epsilon^2 \quad (45)$$

and then, we can chose  $\mu_0 = \lambda_{max}(\mathbf{P})\gamma_\epsilon^2 / \bar{c}$ .

The next Section summarizes the set-membership observer implementation.

## 6. THE SET-MEMBERSHIP OBSERVER IMPLEMENTATION

Once the Algorithm 1 returns the observer parameters  $\mathbf{L}$ ,  $\mathbf{P}$ ,  $\gamma$  and  $\lambda$ , and after to initialize the scalar  $\mu$  in a suitable way. The dynamical equations of the set-membership observer will be implemented as follows:

$$\hat{\mathbf{x}}_{k+1} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k \quad (46)$$

$$\mu_{k+1} = (1 - \lambda)\mu_k + \lambda \quad (47)$$

$$\bar{\mathbf{x}}_k = \hat{\mathbf{x}}_k + \bar{\mathbf{e}}_\infty \mu_k^{1/2} \quad (48)$$

$$\underline{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \bar{\mathbf{e}}_\infty \mu_k^{1/2} \quad (49)$$

with a constant column vector:

$$\bar{\mathbf{e}}_\infty = \text{diag} \left( \left( \frac{\mathbf{P}}{\bar{c}} \right)^{-1/2} \right) \quad (50)$$

where  $\bar{c} = \frac{1}{\lambda} \gamma^2 \bar{\mathbf{w}}^T \bar{\mathbf{w}}$ .

The initial condition for the scalar  $\mu$  can be obtained as

$$\mu_0 = \frac{\lambda_{max}(\mathbf{P})\gamma_\epsilon^2}{\bar{c}} \quad (51)$$

for any initial estimation error insides a given ball with radius  $\gamma_\epsilon \geq 0$ , i.e.

$$\mathbf{e}_0^T \mathbf{e}_0 \leq \gamma_\epsilon^2 \quad (52)$$

with  $\mathbf{e}_0 := \mathbf{x}_0 - \hat{\mathbf{x}}_0$ .

Recall that we assume that the initial estimation error is bound, and its bound is known, that is  $\gamma_\epsilon$  is known. This value allows to properly initialize the scalar  $\mu$  using (51).

The implementation of the proposed set-membership observer is relatively simple, since it only requires to extend its dynamics by including a scalar dynamical equation. Thus, the order of the set-membership state observer will be only of  $n + 1$ , for any  $n$ -order system, i.e.  $\mathbf{x} \in \mathbb{R}^n$ .

## 7. A NUMERICAL EXAMPLE

Consider a second order linear discrete-time system (1) with matrices:

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.2 \\ 0 & 0.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \mathbf{C} = (1 \ 0)$$

and  $\mathbf{Z} = 0.1$ .

After applying the Algorithm 1, the obtained matrices which describe the set-membership state observer are:

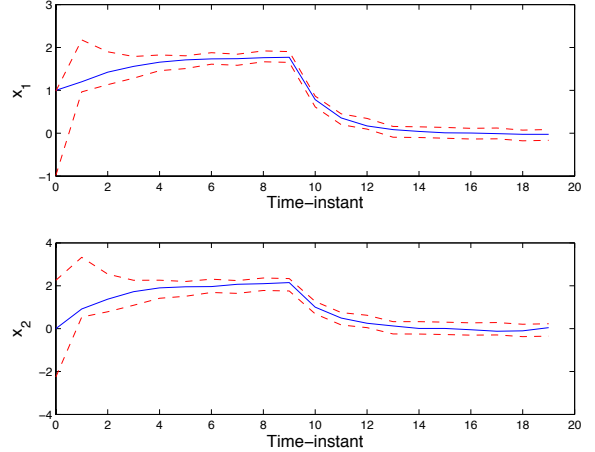


Fig. 1. Set-membership estimation. The dashed lines correspond to the bounds obtained from the set-membership state observer. For comparison, the solid lines corresponds to the real system state.

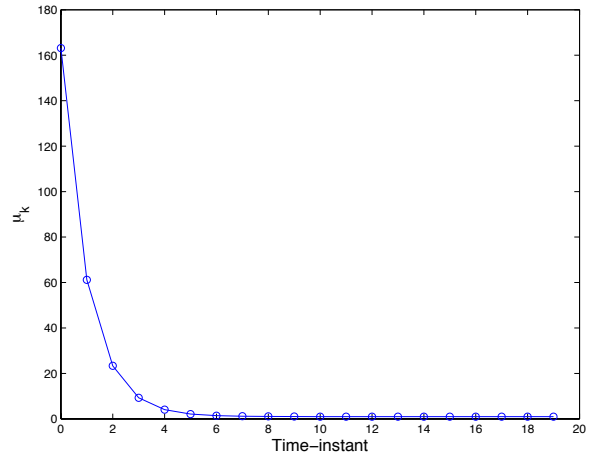


Fig. 2. Evolution of the scalar  $\mu_k$ . This scalar characterizes the contraction of the initial RPI set. The value of  $\mu_k$  converges to 1 as  $k \rightarrow \infty$ .

$$\mathbf{L} = \begin{pmatrix} 0.5453 \\ 0.8919 \end{pmatrix}, \quad \mathbf{P} = 1e3 \begin{pmatrix} 2.5084 & -0.8209 \\ -0.8209 & 0.4749 \end{pmatrix}$$

and the scalars:  $\gamma = 2.3223$  and  $\lambda = 0.6289$ .

We suppose that for all  $k$ , the disturbances  $\mathbf{w}_k$  are random variables with uniform distribution but bounded by the vector  $\bar{\mathbf{w}} = [1 \ 1]^T$ . That is,  $-\bar{\mathbf{w}} \leq \mathbf{w}_k \leq \bar{\mathbf{w}}$ .

In this example, we consider a constant system input  $\mathbf{u}_k = 1$  for time-instants  $k < 9$ . After the time-instant  $k \geq 9$  the system input is  $\mathbf{u}_k = 0$ .

The set-membership state observer has been implemented using equations (46)-(49). The initial conditions for the observer states  $\hat{\mathbf{x}}$  and the scalar  $\mu$  are:  $\hat{\mathbf{x}}_0 = (0 \ 0)^T$  and  $\mu_0 = 163.17$ , respectively. The latter has been computed using (51) by considering  $(-1 \ 0)^T \leq \mathbf{x}_0 \leq (1 \ 0)^T$  and then  $\gamma_\epsilon = 1$ .

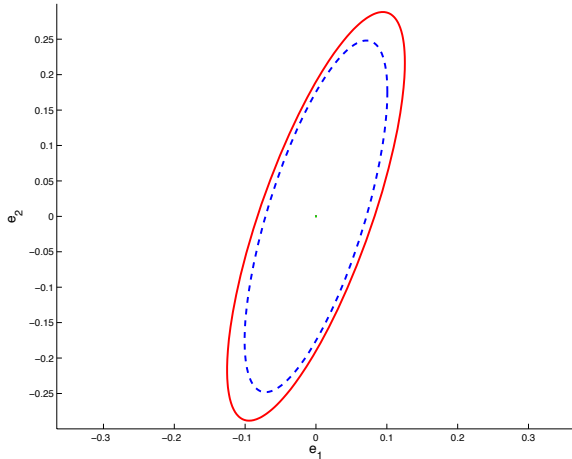


Fig. 3. Invariant set and states covariances. The solid line corresponds to the obtained RPI set used for computing *deterministic* bounds of the estimation error. The dashed line corresponds to the stochastic ellipsoidal defined by (24) with  $t = 3$ .

Figure 1 illustrates the behavior of the observer. The dashed lines correspond to the bounds obtained from the set-membership state observer. For comparison, we have included the solid lines which corresponds to the real system state. Remark that the obtained bounds are very accuracy for both during the transient and during the steady-state periods.

Figure 2 depicts the behavior of the scalar  $\mu_k$  during the whole period of the estimation. Remark that even if its value starts at 163.17, it converges asymptotically to 1 with a behavior compatible with the estimation error dynamics.

In Figure 3, the solid-line ellipsoid corresponds to the obtained RPI set used for computing *deterministic* bounds of the estimation error (for  $\mu = 1$ ). The dashed-line corresponds to the stochastic ellipsoidal defined by (24) with  $t = 3$ . Remark that the obtained RPI set has taken a shape that is very close to that characterizing the states covariances. The latter set represents the set of possible values of the estimation error with a probability greater than 77.78%. Thus, according to the Chebyshev's inequality (24) there is a probability of obtaining estimation errors outside this set with a probability less or equal to 22.22%. However, for initial conditions starting inside the RPI set (solid-line ellipsoid), it is possible to conclude that, for all  $k > 0$ , the estimation error always remains inside this set.

## 8. CONCLUSIONS AND FUTURE WORK

In this paper we has presented a new method for designing state set-membership observers. The proposed observer is able to provide, at every time-instant, accurate bounds of the current system state. The observer design is based on the computation of an  $H_\infty$  observer using LMIs and including a modified Bounded-real lemma formulation. The implementation of the observer is very simple. Comparing with respect to standard state observers for  $n$ -order systems, the proposed set-membership observer will be a

$(n + 1)$ -order observer which facilitate its implementation for high order systems.

As future work, we expect to perform an extension of this work for parameter uncertain systems and its application on fault detection and fault-tolerant control problems.

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