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EQUIVARIANT EXTENSIONS OF \mathbb{G}_a -TORSORS OVER PUNCTURED SURFACES

ADRIEN DUBOULOZ, ISAC HEDÉN, AND TAKASHI KISHIMOTO

ABSTRACT. Motivated by the study of the structure of algebraic actions the additive group on affine threefolds X , we consider a special class of such varieties whose algebraic quotient morphisms $X \rightarrow X//\mathbb{G}_a$ restrict to principal homogeneous bundles over the complement of a smooth point of the quotient. We establish basic general properties of these varieties and construct families of examples illustrating their rich geometry. In particular, we give a complete classification of a natural subclass consisting of threefolds X endowed with proper \mathbb{G}_a -actions, whose algebraic quotient morphisms $\pi : X \rightarrow X//\mathbb{G}_a$ are surjective with only isolated degenerate fibers, all isomorphic to the affine plane \mathbb{A}^2 when equipped with their reduced structures.

INTRODUCTION

Algebraic actions of the complex additive group $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{C}}$ on normal complex affine surfaces S are essentially fully understood: the ring of invariants $\mathcal{O}(S)^{\mathbb{G}_{a,\mathbb{C}}}$ is a finitely generated algebra whose spectrum is a smooth affine curve $C = S//\mathbb{G}_a$, and the inclusion $\mathcal{O}(S)^{\mathbb{G}_a} \subset \mathcal{O}(S)$ defines a surjective morphism $\pi : S \rightarrow C$ whose general fibers coincide with general orbits of the action, hence are isomorphic to the affine line \mathbb{A}^1 on which \mathbb{G}_a acts by translations. The degenerate fibers of such \mathbb{A}^1 -fibrations are known to consist of finite disjoint unions of smooth affine curves isomorphic to \mathbb{A}^1 when equipped with their reduced structure. A complete description of isomorphism classes of germs of invariant open neighborhoods of irreducible components of such fibers was established by Fieseler [8].

In contrast, very little is known so far about the structure of \mathbb{G}_a -actions on complex normal affine threefolds. For such a threefold X , the ring of invariants $\mathcal{O}(X)^{\mathbb{G}_a}$ is again finitely generated [13] and the morphism $\pi : X \rightarrow S$ induced by the inclusion $\mathcal{O}(X)^{\mathbb{G}_a} \subset \mathcal{O}(X)$ is an \mathbb{A}^1 -fibration over a normal affine surface S . But in general, π is neither surjective nor equidimensional. Furthermore, it can have degenerate fibers over closed subsets of pure codimension 1 as well as of codimension 2. All of these possible degeneration are illustrated by the following example:

The restriction of the projection $\text{pr}_{x,y}$ to the smooth threefold $X = \{x^2(x-1)v + yu^2 - x = 0\}$ in \mathbb{A}^4 is an \mathbb{A}^1 -fibration $\pi : X \rightarrow \mathbb{A}^2$ which coincides with the algebraic quotient morphism of the \mathbb{G}_a -action on X associated to the locally nilpotent derivation $\partial = x^2(x-1)\partial_u - 2yu\partial_v$ of its coordinate ring. The restriction of π over the principal open subset $x^2(x-1) \neq 0$ of \mathbb{A}^2 is a trivial principal \mathbb{G}_a -bundle, but the fibers of π over the points $(1,0)$ and $(0,0)$ are respectively empty and isomorphic to \mathbb{A}^2 . Furthermore, for every $y_0 \neq 0$, the inverse images under π of the points $(0, y_0)$ and $(1, y_0)$ are respectively isomorphic to \mathbb{A}^1 but with multiplicity 2, and to the disjoint union of two reduced copies of \mathbb{A}^1 .

Partial results concerning the structure of one-dimensional degenerate fibers of \mathbb{G}_a -quotient \mathbb{A}^1 -fibrations were obtained by Gurjar-Masuda-Miyayoshi [9]. In the present article, as a step towards the understanding of the structure of two-dimensional degenerate fibers, we consider a particular type of non equidimensional surjective \mathbb{G}_a -quotient \mathbb{A}^1 -fibrations $\pi : X \rightarrow S$ which have the property that they restrict to \mathbb{G}_a -torsors¹ over the complement of a finite set of smooth points in S . These are simpler than the general case illustrated in the previous example since they do not admit additional degeneration of their fibers over curves in S passing through the given points. The local and global study of some classes of such fibrations was initiated by the second author [10]. He constructed in particular many examples of \mathbb{G}_a -quotient \mathbb{A}^1 -fibrations on smooth affine threefolds X with image \mathbb{A}^2 whose restrictions over the complement of the

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¹sometimes also referred to as Zariski locally trivial principal \mathbb{G}_a -bundles

origin are isomorphic to the geometric quotient $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2/\mathbb{G}_a$ of SL_2 by the action of unitary upper triangular matrices.

One of the simplest examples of this type is the smooth threefold $X_0 \subset \mathbb{A}_{x,y,p,q,r}^5$ defined by the equations

$$X_0 : \begin{cases} xr - yq & = 0 \\ yp - x(q-1) & = 0 \\ pr - q(q-1) & = 0 \end{cases}$$

and equipped with the \mathbb{G}_a -action associated to the locally nilpotent $\mathbb{C}[x, y]$ -derivation $x^2\partial_p + xy\partial_q + y^2\partial_r$ of its coordinate ring. The equivariant open embedding $\mathrm{SL}_2 = \{xv - yu = 1\} \hookrightarrow X_0$ is given by $(x, y, u, v) \mapsto (x, y, xu, xv, yv)$. The \mathbb{G}_a -quotient morphism coincides with the surjective \mathbb{A}^1 -fibration $\pi_0 : \mathrm{pr}_{x,y} : X_0 \rightarrow \mathbb{A}^2$. Its restriction over $\mathbb{A}^2 \setminus \{(0, 0)\}$ is isomorphic to the quotient morphism $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2/\mathbb{G}_a$, while its fiber over $(0, 0)$ is the smooth quadric $\{pr - q(q-1) = 0\} \subset \mathbb{A}_{p,q,r}^3$, isomorphic to the quotient $\mathrm{SL}_2/\mathbb{G}_m$ of SL_2 by the action of its diagonal torus (see Example 2.1). A noteworthy property of this example is that the \mathbb{G}_a -quotient morphism $\pi : X_0 \rightarrow \mathbb{A}^2$ factors through a locally trivial \mathbb{A}^1 -bundle $\rho : X_0 \rightarrow \tilde{\mathbb{A}}^2$ over the the blow-up $\tau : \tilde{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$ of the origin.

It is a general fact that every irreducible component of a degenerate fiber of pure codimension one of a \mathbb{G}_a -quotient \mathbb{A}^1 -fibration $\pi : X \rightarrow S$ on a smooth affine threefold is an \mathbb{A}^1 -uniruled affine surface (see Proposition 1.3). We do not know whether every \mathbb{A}^1 -uniruled surface can be realized as an irreducible component of the degenerate fiber of a \mathbb{G}_a -extension. But besides the smooth affine quadric $\mathrm{SL}_2/\mathbb{G}_m$ appearing in the previous example, the following one confirms that the affine plane \mathbb{A}^2 can also be realized (see also Examples 1.4 and 1.5 for other types of surfaces that can be realized): Let $X_1 \subset \mathbb{A}_{x,y,z_1,z_2,w}^5$ be the smooth affine threefold defined by the equations

$$X_1 : \begin{cases} xw - y(yz_1 + 1) & = 0 \\ xz_2 - z_1(yz_1 + 1) & = 0 \\ z_1w - yz_2 & = 0, \end{cases}$$

equipped with the \mathbb{G}_a -action associated to the locally nilpotent $\mathbb{C}[x, y]$ -derivation $x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w$ of its coordinate ring. The morphism $\mathrm{SL}_2 \hookrightarrow X_1$ given by $(x, y, u, v) \mapsto (x, y, u, uv, yv)$ is equivariant open embedding. The \mathbb{G}_a -quotient morphism coincides with the surjective \mathbb{A}^1 -fibration $\pi_1 = \mathrm{pr}_{x,y} : X_1 \rightarrow \mathbb{A}^2$, whose fiber over the origin is the affine plane $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[z_2, w])$ and whose restriction over $\mathbb{A}^2 \setminus \{(0, 0)\}$ is again isomorphic to the quotient morphism $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2/\mathbb{G}_a$. A special additional feature is that the \mathbb{G}_a -action on X_1 extending that on SL_2 is not only fixed point free but actually *proper*: its geometric quotient X_1/\mathbb{G}_a is separated. One can indeed check that X_1/\mathbb{G}_a is isomorphic to the complement $\tilde{\mathbb{A}}^2 \setminus \{o_1\}$ of a point o_1 supported on the exceptional divisor E of the blow-up $\tilde{\mathbb{A}}^2$ of \mathbb{A}^2 at the origin (see Example 3.2).

Relaxing the hypothesis that the \mathbb{A}^1 -fibration $\pi : X \rightarrow S$ arises as the quotient of a \mathbb{G}_a -action on an affine threefold X to consider the broader problem of describing the geometry of degeneration of \mathbb{A}^1 -fibrations over irreducible closed subsets of pure codimension two of their base, we are led to the following more general notion:

Definition. Let (S, o) be a pair consisting of a normal separated 2-dimensional scheme S essentially of finite type over a field k of characteristic zero and of a closed point o contained in the smooth locus of S . A \mathbb{G}_a -extension of a \mathbb{G}_a -torsor $\rho : P \rightarrow S \setminus \{o\}$ is a \mathbb{G}_a -equivariant open embedding $j : P \hookrightarrow X$ into an integral scheme X equipped with a surjective morphism $\pi : X \rightarrow S$ of finite type and a $\mathbb{G}_{a,S}$ -action, such that the commutative diagram

$$\begin{array}{ccc} P & \xhookrightarrow{j} & X \\ \rho \downarrow & & \downarrow \pi \\ S \setminus \{o\} & \xhookrightarrow{\quad} & S \end{array}$$

is cartesian.

The examples X_0 and X_1 above provide motivation to study the following natural classes of \mathbb{G}_a -extensions $\pi : X \rightarrow S$ of a \mathbb{G}_a -torsor $\rho : P \rightarrow S \setminus \{o\}$, which are arguably the simplest possible types of \mathbb{G}_a -extensions from the viewpoints of their global geometry and of the properties of their \mathbb{G}_a -actions:

- (Type I) Extensions for which π factors through a locally trivial \mathbb{A}^1 -bundle over the blow-up $\tau : \tilde{S} \rightarrow S$ of the point o , the fiber $\pi^{-1}(o)$ being then the total space of a locally trivial \mathbb{A}^1 -bundle over the exceptional divisor of τ .
- (Type II) Extensions for which $\pi^{-1}(o)_{\text{red}}$ is isomorphic to the affine plane \mathbb{A}_{κ}^2 over the residue field κ of S at o , X is smooth along $\pi^{-1}(o)$ and the $\mathbb{G}_{a,S}$ -action on X is proper.

The first main result of this article, Proposition 2.3 and Theorem 2.5, is a complete description of \mathbb{G}_a -extensions of Type I together with an effective characterization of which among them have the additional property that the morphism $\pi : X \rightarrow S$ is affine. Our second main result, Theorem 3.7, consists of a classification of \mathbb{G}_a -extensions of Type II, under the additional assumption that the morphism $\pi : X \rightarrow S$ is quasi-projective. More precisely, given a \mathbb{G}_a -torsor $\rho : P \rightarrow S \setminus \{o\}$ and a \mathbb{G}_a -extension $\pi : X \rightarrow S$ with proper $\mathbb{G}_{a,S}$ -action and reduced fiber $\pi^{-1}(o)_{\text{red}}$ isomorphic to \mathbb{A}_{κ}^2 , we establish that the possible geometric quotients $S' = X/\mathbb{G}_a$ belong to a very special class of surfaces isomorphic to open subsets of blow-ups of S with centers over o which we fully describe in § 3.1. We show conversely that every such surface is indeed the geometric quotient of a \mathbb{G}_a -extension of $\rho : P \rightarrow S \setminus \{o\}$ with the desired properties.

In a second step, we tackle the question of existence of \mathbb{G}_a -extensions $\pi : X \rightarrow S$ of Type II for which the structure morphism π is not only quasi-projective but affine. Our method to produce extensions with this property is inspired by the observation that the threefolds X_0 and X_1 above are not only birational to each other due to the property that they both contain SL_2 as open subset, but in fact that the birational morphism

$$\eta : X_1 \rightarrow X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, y, p, q, r) = (x, y, xz_1, yz_1 + 1, w)$$

expresses X_1 as a \mathbb{G}_a -equivariant affine modification of X_0 in the sense of Kalimian and Zaidenberg [11]. This suggests that extensions of Type II for which X is affine over S could be obtained as equivariant affine modification in a suitable generalized sense from extensions of Type I with the same property. Using this technique, we are able to show in Theorem 3.8 that for each possible geometric quotient S' above, there exist \mathbb{G}_a -extensions $\pi : X \rightarrow S$ of $\rho : P \rightarrow S \setminus \{o\}$ with geometric quotient $X/\mathbb{G}_a = S'$ such that π is an affine morphism.

As an application towards the initial question of the structure \mathbb{G}_a -quotient \mathbb{A}^1 -fibrations on affine threefolds, we in particular derive from this construction the existence of uncountably many pairwise non isomorphic smooth affine threefolds X endowed with proper \mathbb{G}_a -actions, containing SL_2 as an invariant open subset with complement \mathbb{A}^2 , whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine, and whose algebraic quotients are all isomorphic to \mathbb{A}^2 .

The scheme of the article is the following. The first section begins with a review of general properties of \mathbb{G}_a -extensions. We then set up the basic tools which will be used through all the article: locally trivial \mathbb{A}^1 -bundles with additive group actions and equivariant affine birational morphisms between these. In section two, we study \mathbb{G}_a -extensions of Type I. The last section is devoted to the classification of quasi-projective \mathbb{G}_a -extensions of Type II.

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1. PRELIMINARIES

Notation 1.1. In the rest of the article, the term *surface* refers to a normal separated 2-dimensional scheme essentially of finite type over a field k of characteristic zero. A *punctured surface* $S_* = S \setminus \{o\}$ is the complement of a closed point o contained in the smooth locus of a surface S . We denote by κ the residue field of S at o .

Remark 1.2. We do not require that the residue field κ of S at o is an algebraic extension of k . For instance, S can very well be the spectrum of the local ring $\mathcal{O}_{X,Z}$ of an arbitrary smooth k -variety X at an irreducible closed subvariety Z of codimension two in X and o its unique closed point, in which case the residue field κ is isomorphic to the field of rational functions on Z .

In this section, we first review basic geometric properties of equivariant extensions of \mathbb{G}_a -torsors over punctured surfaces. We then collect various technical results on additive group actions on affine-linear bundles of rank one and their behavior under equivariant affine modifications.

1.1. Equivariant extensions of \mathbb{G}_a -torsors. A \mathbb{G}_a -torsor over punctured surface $S_* = S \setminus \{o\}$ is an S_* -scheme $\rho : P \rightarrow S_*$ equipped with a \mathbb{G}_a -action $\mu : \mathbb{G}_{a,S_*} \times_{S_*} P \rightarrow P$ for which there exists a Zariski open cover $f : Y \rightarrow S_*$ of S_* such that $P \times_{S_*} Y$ is equivariantly isomorphic to $\mathbb{G}_{a,Y}$ acting on itself by translations. In the present article, we primarily focus on \mathbb{G}_a -torsors $\rho : P \rightarrow S_*$ whose restrictions $P \times_{S_*} U \rightarrow U \setminus \{o\}$ over every Zariski open neighborhood U of o in S are nontrivial. Since in this case the total space of P is affine over S (see e.g. [4, Proposition 1.2] whose proof carries over verbatim to our more general situation), it follows that for every \mathbb{G}_a -extension $j : P \hookrightarrow X$ the fiber $\pi^{-1}(o) \subset X$ of the surjective morphism $\pi : X \rightarrow S$ has pure codimension one in X . Two important families of examples of non trivial normal \mathbb{G}_a -extensions $j : \mathrm{SL}_2 \rightarrow X$ of the \mathbb{G}_a -torsor $\rho : \mathrm{SL}_2 \rightarrow \mathrm{SL}_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\}$, where \mathbb{G}_a acts on SL_2 via left multiplication by upper triangular unipotent matrices, were constructed in [10, Section 5 and 6]. Various other extensions were obtained from these by performing suitable equivariant affine modifications. One can observe that for all these extensions, the fiber $\pi^{-1}(\{(0,0)\})$ is an \mathbb{A}^1 -ruled surface, a property which is a consequence of the following more general fact:

Proposition 1.3. *Let $\rho : P \rightarrow S_*$ be a non trivial \mathbb{G}_a -torsor over the punctured spectrum $S \setminus \{o\}$ of a regular local ring of dimension 2 over an algebraically closed field k and with residue field $\kappa(o) = k$, and let $\pi : X \rightarrow S$ be a \mathbb{G}_a -extension of P . If X is smooth along $\pi^{-1}(o)$, then every irreducible component F of $\pi^{-1}(o)_{\mathrm{red}}$ is a uniruled surface. Furthermore, if X is affine then F is \mathbb{A}^1 -uniruled, hence \mathbb{A}^1 -ruled when it is normal.*

Proof. Since $\pi^{-1}(o)$ has pure codimension one in X and X is smooth along $\pi^{-1}(o)$, every irreducible component of $\pi^{-1}(o)$ is a \mathbb{G}_a -invariant Cartier divisor on X . The complement X' in X of all but one irreducible component of $\pi^{-1}(o)$ is thus again a \mathbb{G}_a -extension of P , and we may therefore assume without loss of generality that $F = \pi^{-1}(o)_{\mathrm{red}}$ is irreducible. Let $x \in F$ be a closed point in the regular locus of F . Since F and X are smooth at x and X is connected, there exists a curve $C \subset X$, smooth at x and intersecting F transversally at x . The image $\pi(C)$ of C is a curve on S passing through o , and the closure B of $\pi^{-1}(\pi(C) \cap S_*)$ in X is a surface containing C . Since $\rho : P \rightarrow S_*$ is a \mathbb{G}_a -torsor, the restriction of π to $B \cap P$ is a trivial \mathbb{G}_a -torsor over the affine curve $\pi(C)$. So $\pi|_B : B \rightarrow \pi(C)$ is an \mathbb{A}^1 -fibration. Let $\nu : \tilde{C} \rightarrow \pi(C)$ be the normalization of $\pi(C)$. Then $\pi|_B$ lifts to an \mathbb{A}^1 -fibration $\theta : \tilde{B} \rightarrow \tilde{C}$ on the normalization \tilde{B} of B . The fiber of θ over every point in $\nu^{-1}(o)$ is a union of rational curves. Since the normalization morphism $\mu : \tilde{B} \rightarrow B$ is surjective, one of the irreducible components of $\nu^{-1}(o)$ is mapped by μ onto a rational curve in F passing through x . This shows that for every smooth closed point x of F , there exists a non constant rational map $h : \mathbb{P}^1 \dashrightarrow F$ such that $x \in h(\mathbb{P}^1)$. Thus F is uniruled. If X is in addition affine, then B and \tilde{B} are affine surfaces, and the fibers of the \mathbb{A}^1 -fibration $\theta : \tilde{B} \rightarrow \tilde{C}$ consist of disjoint union of curves isomorphic to \mathbb{A}^1 when equipped with their reduced structure. This implies that F is not only uniruled but actually \mathbb{A}^1 -uniruled. \square

Example 1.4. Let X be the smooth affine threefold in $\mathbb{A}^2 \times \mathbb{A}^4 = \text{Spec}(k[x, y][c, d, e, f])$ defined by the equations

$$\begin{cases} xd - y(c+1) & = 0 \\ xc^2 - y^2e & = 0 \\ yf - c(c+1) & = 0 \\ xf^2 - (c+1)^2e & = 0 \\ de - cf & = 0 \end{cases}$$

equipped with the \mathbb{G}_a -action induced by the locally nilpotent $k[x, y]$ -derivation

$$xy\partial_c + y^2\partial_d + x(2c+1)\partial_f + (2x^2f - 2xye)\partial_e$$

of its coordinate ring. The morphism $j : \text{SL}_2 = \{xv - yu = 1\} \rightarrow X$ defined by $(x, y, u, v) \mapsto (x, y, yu, yv, xu^2, xuv)$ is an open embedding of SL_2 in X as the complement of the fiber over $o = (0, 0)$ of the projection $\pi = \text{pr}_{x, y} : X \rightarrow \mathbb{A}^2$. So $j : \text{SL}_2 \rightarrow X$ is an affine \mathbb{G}_a -extension of the \mathbb{G}_a -torsor $\rho : \text{SL}_2 \rightarrow \text{SL}_2/\mathbb{G}_a = \mathbb{A}^2 \setminus \{o\}$, for which $\pi^{-1}(o)$ consists of the disjoint union of two copies $D_1 = \{x = y = c = 0\} \simeq \text{Spec}(k[d, f])$ and $D_2 = \{x = y = c + 1 = 0\} \simeq \text{Spec}(k[d, e])$ of \mathbb{A}^2 . Note that the induced \mathbb{G}_a -action on each of these is the trivial one.

Example 1.5. Let X be the affine \mathbb{G}_a -extension constructed in the previous example and let $C \subset D_1$ be any smooth affine curve. Let $\tau : \tilde{X} \rightarrow X$ be the blow-up of X along C , let $i : X' \hookrightarrow \tilde{X}$ be the open immersion of the complement of the proper transform of $D_1 \cup D_2$ in \tilde{X} and let $\pi' = \pi \circ \tau \circ i : X' \rightarrow \mathbb{A}^2$. Since C and $D_1 \cup D_2$ are \mathbb{G}_a -invariant, the \mathbb{G}_a -action on X lifts to a \mathbb{G}_a -action on \tilde{X} which restricts in turn to X' . By construction, π' is surjective, with fiber $\pi'^{-1}(o)$ isomorphic to $C \times \mathbb{A}^1$ and $\tau \circ i : X' \rightarrow X$ restricts to an equivariant isomorphism between $X' \setminus \pi'^{-1}(o)$ and $X \setminus \pi^{-1}(o) \simeq \text{SL}_2$. So $\pi' : X' \rightarrow \mathbb{A}^2$ is a \mathbb{G}_a -extension of the \mathbb{G}_a -torsor $\rho : \text{SL}_2 \rightarrow \text{SL}_2/\mathbb{G}_a = \mathbb{A}^2 \setminus \{o\}$.

1.2. Recollection on affine-linear bundles. Affine-linear bundles of rank one over a scheme are natural generalization of \mathbb{G}_a -torsors. To fix the notation, we briefly recall their basic definitions and properties

By a line bundle on a scheme S , we mean the relative spectrum $p : M = \text{Spec}(\text{Sym} \mathcal{M}^\vee) \rightarrow S$ of the symmetric algebra of the dual of an invertible sheaf of \mathcal{O}_S -module \mathcal{M} . Such a line bundle M can be viewed as a locally constant group scheme over S for the group law $m : M \times_S M \rightarrow M$ whose co-morphism

$$m^\# : \text{Sym} \mathcal{M}^\vee \rightarrow \text{Sym} \mathcal{M}^\vee \otimes \text{Sym} \mathcal{M}^\vee \simeq \text{Sym}(\mathcal{M}^\vee \oplus \mathcal{M}^\vee)$$

is induced by the diagonal homomorphism $\mathcal{M}^\vee \rightarrow \mathcal{M}^\vee \oplus \mathcal{M}^\vee$. An M -torsor is then an S -scheme $\theta : W \rightarrow S$ equipped with an action $\mu : M \times_S W \rightarrow W$ which is Zariski locally over S isomorphic to M acting on itself by translations.

This is the case precisely when there exists a Zariski open cover $f : Y \rightarrow S$ and an \mathcal{O}_Y -algebra isomorphism $\psi : f^* \mathcal{A} \rightarrow \text{Sym} f^* \mathcal{M}^\vee$ such that over $Y' = Y \times_S Y$ the automorphism $p_1^* \psi \circ p_2^* \psi^{-1} : \text{Sym} \mathcal{M}_{Y'}^\vee \rightarrow \text{Sym} \mathcal{M}_{Y'}^\vee$, of the symmetric algebra of $\mathcal{M}_{Y'}^\vee = p_2^* f^* \mathcal{M}^\vee = p_1^* f^* \mathcal{M}^\vee$ is *affine-linear*, i.e. induced by an $\mathcal{O}_{Y'}$ -module homomorphism $\mathcal{M}_{Y'}^\vee \rightarrow \text{Sym} \mathcal{M}_{Y'}^\vee$ of the form

$$(1.1) \quad \beta \oplus \text{id} : \mathcal{M}_{Y'}^\vee \rightarrow \mathcal{O}_{Y'} \oplus \mathcal{M}_{Y'}^\vee \hookrightarrow \bigoplus_{n \geq 0} (\mathcal{M}_{Y'}^\vee)^{\otimes n} = \text{Sym} \mathcal{M}_{Y'}^\vee$$

for some $\beta \in \text{Hom}_{Y'}(\mathcal{M}_{Y'}^\vee, \mathcal{O}_{Y'}) \simeq H^0(Y', \mathcal{M}_{Y'})$ which is a Čech 1-cocycle with values in \mathcal{M} for the Zariski open cover $f : Y \rightarrow S$. Standard arguments show that the isomorphism class of $\theta : W \rightarrow S$ depends only on the class of β in the Čech cohomology group $\check{H}^1(S, \mathcal{M})$, and one eventually gets a one-to-one correspondence between isomorphism classes of M -torsors over S and elements of the cohomology group $H^1(S, M) = H^1(S, \mathcal{M}) \simeq \check{H}^1(S, \mathcal{M})$ with the zero element corresponding to the trivial torsor $p : M \rightarrow S$.

It is classical that every locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow S$ over a reduced scheme S can be equipped with the additional structure of a torsor under a uniquely determined line bundle M on S . The existence of this additional structure will be frequently used in the sequel, and we now quickly review its construction (see also e.g. [2, § 2.3 and § 2.4]). Letting $\mathcal{A} = \theta_* \mathcal{O}_W$, there exists by definition a Zariski open cover $f : Y \rightarrow S$ and a quasi-coherent \mathcal{O}_Y -algebra isomorphism $\varphi : f^* \mathcal{A} \rightarrow \mathcal{O}_Y[u]$. Over $Y' = Y \times_S Y$ equipped with the two projections p_1 and p_2 to Y , the $\mathcal{O}_{Y'}$ -algebra isomorphism $\Phi = p_1^* \varphi \circ p_2^* \varphi^{-1}$ has the form

$$(1.2) \quad \Phi : \mathcal{O}_{Y'}[u] \rightarrow \mathcal{O}_{Y'}[u], \quad u \mapsto au + b$$

for some $a \in \Gamma(Y', \mathcal{O}_{Y'}^*)$ and $b \in \Gamma(Y', \mathcal{O}_{Y'})$ whose pull back over $Y'' = Y \times_S Y \times_S Y$ by the three projections $p_{12}, p_{23}, p_{13} : Y'' \rightarrow Y'$ satisfy the cocycle relations $p_{13}^* a = p_{23}^* a \cdot p_{12}^* a$ and $p_{13}^* b = p_{23}^* a \cdot p_{12}^* b + p_{23}^* b$ in $\Gamma(Y'', \mathcal{O}_{Y''}^*)$ and $\Gamma(Y'', \mathcal{O}_{Y''})$ respectively. The first one says that a is a Čech 1-cocycle with values in \mathcal{O}_S^* for the cover $f : Y \rightarrow S$, which thus determines, via the isomorphism $H^1(S, \mathcal{O}_S^*) \simeq \text{Pic}(S)$, a unique invertible sheaf \mathcal{M} on S together with an \mathcal{O}_Y -module isomorphism $\alpha : f^* \mathcal{M}^\vee \rightarrow \mathcal{O}_Y$ such that $p_1^* \alpha \circ p_2^* \alpha^{-1} : \mathcal{O}_{Y'} \rightarrow \mathcal{O}_{Y'}$ is the multiplication by a . The second one can be equivalently reinterpreted as the fact that $\beta = p_2^*(\alpha)(b) \in \Gamma(Y', \mathcal{M}_{Y'})$ is a Čech 1-cocycle with values in \mathcal{M} for the Zariski open cover $f : Y \rightarrow S$. Letting $\text{Sym}(\alpha) : \text{Sym}(f^* \mathcal{M}^\vee) \rightarrow \mathcal{O}_Y[u]$ be the graded \mathcal{O}_Y -algebra isomorphism induced by α , the isomorphism $\psi = \text{Sym}(\alpha^{-1}) \circ \varphi : f^* \mathcal{A} \rightarrow \text{Sym}(f^* \mathcal{M}^\vee)$ has the property that $p_1^* \psi \circ p_2^* \psi^{-1}$ is affine-linear, induced by the homomorphism $\beta \oplus \text{id} : \mathcal{M}_{Y'}^\vee \rightarrow \mathcal{O}_{Y'} \oplus \mathcal{M}_{Y'}^\vee$. So $\theta : W \rightarrow S$ is a torsor under the line bundle $M = \text{Spec}(\text{Sym}(\mathcal{M}^\vee))$, with isomorphism class in $H^1(S, M)$ equal to the cohomology class of the cocycle β . Summing up, we obtain;

Proposition 1.6. *Let $\theta : W \rightarrow S$ be a locally trivial \mathbb{A}^1 -bundle. Then there exists a unique pair (M, g) consisting of a line bundle M on S and a class $g \in H^1(S, M)$ such that $\theta : W \rightarrow S$ is an M -torsor with isomorphism class g .*

1.3. Additive group actions on affine-linear bundles of rank one. Given a locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow S$, which we view as an M -torsor for a line bundle $M = \text{Spec}(\text{Sym}(\mathcal{M}^\vee)) \rightarrow S$ on S , with corresponding action $\mu : M \times_S W \rightarrow W$, every nonzero group scheme homomorphism $\xi : \mathbb{G}_{a,S} \rightarrow M$ induces a nontrivial $\mathbb{G}_{a,S}$ -action $\nu = \mu \circ (\xi \times \text{id}) : \mathbb{G}_{a,S} \times_S W \rightarrow W$ on W . A nonzero group scheme homomorphism $\xi : \mathbb{G}_{a,S} = \text{Spec}(\mathcal{O}_S[t]) \rightarrow M = \text{Spec}(\text{Sym}(\mathcal{M}^\vee))$ is uniquely determined by a nonzero \mathcal{O}_S -module homomorphism $\mathcal{M}^\vee \rightarrow \mathcal{O}_S$, equivalently by a nonzero global section $s \in \Gamma(S, \mathcal{M})$. The following proposition asserts conversely that every nontrivial $\mathbb{G}_{a,S}$ -action on an M -torsor $\theta : W \rightarrow S$ uniquely arises from such a section.

Proposition 1.7. ([1, Chapter 3]) *Let $\theta : W \rightarrow S$ be a torsor under the action $\mu : M \times_S W \rightarrow W$ of a line bundle $M = \text{Spec}(\text{Sym}(\mathcal{M}^\vee)) \rightarrow S$ on S and let $\nu : \mathbb{G}_{a,S} \times_S W \rightarrow W$ be a non trivial $\mathbb{G}_{a,S}$ -action on W . Then there exists a non zero global section $s \in \Gamma(S, \mathcal{M})$ such that $\nu = \mu \circ (\xi \times \text{id})$ where $\xi : \mathbb{G}_{a,S} \rightarrow M$ is the group scheme homomorphism induced by s .*

Proof. Let $\mathcal{A} = \theta_* \mathcal{O}_W$ and let $f : Y \rightarrow S$ be a Zariski open cover such that there exists an \mathcal{O}_Y -algebra isomorphism $\varphi : f^* \mathcal{A} \rightarrow \mathcal{O}_Y[u]$, and let

$$\Phi = p_1^* \varphi \circ p_2^* \varphi^{-1} : \mathcal{O}_{Y'}[u] \rightarrow \mathcal{O}_{Y'}[u], \quad u \mapsto au + b$$

be as in (1.2) above. Since $\theta : W \rightarrow S$ is an M -torsor, φ also determines an \mathcal{O}_Y -module isomorphism $\alpha : f^* \mathcal{M}^\vee \rightarrow \mathcal{O}_Y$ such that $p_1^* \alpha \circ p_2^* \alpha^{-1} : \mathcal{O}_{Y'} \rightarrow \mathcal{O}_{Y'}$ is the multiplication by a . The $\mathbb{G}_{a,S}$ -action ν on W pulls back to a $\mathbb{G}_{a,Y}$ -action $\nu \times \text{id}$ on $W \times_S Y$. The co-morphism $\eta : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$ of the nontrivial $\mathbb{G}_{a,Y}$ -action $\varphi \circ (\nu \times \text{id}) \circ (\text{id} \times \varphi^{-1})$ on $\text{Spec}(\mathcal{O}_Y[u])$ has the form $u \mapsto u \otimes 1 + 1 \otimes \gamma t$ for some nonzero $\gamma \in \Gamma(Y, \mathcal{O}_Y)$. Letting $\mathcal{I} = \gamma \cdot \mathcal{O}_Y$ be the ideal sheaf generated by γ , η factors as

$$\eta = (\text{id} \otimes j) \circ \tilde{\eta} : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \text{Sym}(\mathcal{I}) \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$$

where $\tilde{\eta}$ is the co-morphism of an action of the line bundle $\text{Spec}(\text{Sym}(\mathcal{I})) \rightarrow Y$ on $\mathbb{A}_S^1 \times_S Y \simeq W \times_S Y$ and $j : \text{Sym}(\mathcal{I}) \rightarrow \mathcal{O}_Y[t]$ is the homomorphism induced by the inclusion $\mathcal{I} \subset \mathcal{O}_Y$. Pulling back to Y' , we find that $p_2^* \gamma = a \cdot p_1^* \gamma$, which implies that ${}^t \alpha(\gamma) \in \Gamma(Y, f^* \mathcal{M})$ is the pull-back $f^* s$ to Y of a nonzero global section $s \in \Gamma(S, \mathcal{M})$. Letting $D = \text{div}_0(s)$ be the divisors of zeros of s , we have $\mathcal{M}^\vee \simeq \mathcal{O}_S(-D) \subset \mathcal{O}_S$ and $f^* \mathcal{M}^\vee \simeq \mathcal{O}_Y(-f^* D) \subset \mathcal{O}_Y$ is equal to the ideal $\mathcal{I} = \gamma \cdot \mathcal{O}_Y$. The global section $f^* s$ viewed as a homomorphism $f^* \mathcal{M}^\vee \rightarrow \mathcal{O}_Y$ coincides via these isomorphisms with the inclusion $\gamma \cdot \mathcal{O}_Y \hookrightarrow \mathcal{O}_Y$. We can thus rewrite η in the form

$$\eta = (\text{id} \otimes \text{Sym}(f^* s)) \circ \tilde{\eta} : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \text{Sym}(f^* \mathcal{M}^\vee) \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t].$$

By construction $\tilde{\eta} = (\varphi \otimes \text{id}) \circ f^* \mu^\sharp \circ \varphi^{-1}$ where $f^* \mu^\sharp$ is the pull-back of the co-morphism $\mu^\sharp : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym}(\mathcal{M}^\vee)$ of the action $\mu : M \times_S W \rightarrow W$ of M on W . It follows that the pull-back $f^* \nu^\sharp$ of the co-morphism of the action $\nu : \mathbb{G}_{a,S} \times W \rightarrow W$ factors as

$$f^* \nu^\sharp = (\text{id} \otimes \text{Sym}(f^* s)) \circ f^* \mu^\sharp = f^* \mathcal{A} \rightarrow f^* \mathcal{A} \otimes \text{Sym}(f^* \mathcal{M}^\vee) \rightarrow f^* \mathcal{A} \otimes \mathcal{O}_Y[t]$$

This in turn implies that ν^\sharp factors as $(\text{id} \otimes \text{Sym}(s)) \circ \mu^\sharp : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym}(\mathcal{M}^\vee) \rightarrow \mathcal{A} \otimes \mathcal{O}_Y[t]$ as desired. \square

Remark 1.8. In the setting of Proposition 1.7, letting $U \subset S$ be the complement of the zero locus of s , the morphism ξ restricts to an isomorphism of group schemes $\xi|_U : \mathbb{G}_{a,U} \rightarrow M|_U$ for which $W|_U$ equipped with the $\mathbb{G}_{a,U}$ -action $\nu|_U : \mathbb{G}_{a,U} \times_U W|_U \rightarrow W|_U$ is a $\mathbb{G}_{a,U}$ -torsor. This isomorphism class in $H^1(U, \mathcal{O}_U)$ of this $\mathbb{G}_{a,U}$ -torsor coincides with the image of the isomorphism class $g \in H^1(S, \mathcal{M})$ of W by the composition of the restriction homomorphism $\text{res} : H^1(S, \mathcal{M}) \rightarrow H^1(U, \mathcal{M}|_U)$ with the inverse of the isomorphism $H^1(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{M}|_U)$ induced by $s|_U$.

1.4. \mathbb{G}_a -equivariant affine modifications of affine-linear bundles of rank one. Recall [3] that given an integral scheme X with sheaf of rational functions \mathcal{K}_X , an effective Cartier divisor D on X and a closed subscheme $Z \subset X$ whose ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ contains $\mathcal{O}_X(-D)$, the *affine modification of X with center (\mathcal{I}, D)* is the affine X -scheme $\sigma : X' = \text{Spec}(\mathcal{O}_X[\mathcal{I}/D]) \rightarrow X$ where $\mathcal{O}_X[\mathcal{I}/D]$ denotes the quotient of the Rees algebra

$$\mathcal{O}_X[(\mathcal{I} \otimes \mathcal{O}_X(D))] = \bigoplus_{n \geq 0} (\mathcal{I} \otimes \mathcal{O}_X(D))^n t^n \subset \mathcal{K}_X[t]$$

of the fractional ideal $\mathcal{I} \otimes \mathcal{O}_X(D) \subset \mathcal{K}_X$ by the ideal generated by $1 - t$. In the case where $X = \text{Spec}(A)$ is affine, $D = \text{div}(f)$ is principal and Z is defined by an ideal $I \subset A$ containing f then \tilde{X} is isomorphic to the affine modification $X' = \text{Spec}(A[I/f])$ of X with center (I, f) in the sense of [11].

Now let S be an integral scheme and let $\theta : W \rightarrow S$ be a locally trivial \mathbb{A}^1 -bundle. Let $C \subset S$ be an integral Cartier divisor, let $D = \theta^{-1}(C)$ be its inverse image in W and let $Z \subset D$ be a non empty integral closed subscheme of D on which θ restricts to an open embedding $\theta|_Z : Z \hookrightarrow C$. Equivalently, Z is the closure in D of the image $\alpha(U)$ of a rational section $\alpha : C \rightarrow D$ of the locally trivial \mathbb{A}^1 -bundle $\theta|_D : D \rightarrow C$ defined over a non empty open subset U of C . The complement F of $\theta|_Z(Z)$ in C is a closed subset of C hence of S . Letting $i : S \setminus F \hookrightarrow S$ be the natural open embedding, we have the following result:

Lemma 1.9. *Let $\sigma : W' \rightarrow W$ be the affine modification of W with center (\mathcal{I}_Z, D) . Then the composition $\theta \circ \sigma : W' \rightarrow S$ factors through a locally trivial \mathbb{A}^1 -bundle $\theta' : W' \rightarrow S \setminus F$ in such a way that we have a cartesian diagram*

$$\begin{array}{ccc} W' & \xrightarrow{\sigma} & W \\ \theta' \downarrow & & \downarrow \theta \\ S \setminus F & \xrightarrow{i} & S. \end{array}$$

Proof. The question being local with respect to a Zariski open cover of S over which $\theta : W \rightarrow S$ becomes trivial, we can assume without loss of generality that $S = \text{Spec}(A)$, $W = \text{Spec}(A[x])$, $C = \text{div}(f)$ for some non zero element $f \in A$. The integral closed subscheme $Z \subset D$ is then defined by an ideal I of the form (f, g) where $g(x) \in A[x]$ is an element whose image in $(A/f)[x]$ is a polynomial of degree one in x . So $g(x) = a_0 + a_1x + x^2fR(x)$ where $a_0 \in A$, $a_1 \in A$ has non zero residue class in A/f and $R(x) \in A[x]$. The condition that $\theta|_Z : Z \rightarrow C$ is an open embedding implies further that the residue classes \bar{a}_0 and \bar{a}_1 of a_0 and a_1 in A/f generate the unit ideal. The complement F of the image of $\theta|_Z(Z)$ in C is then equal to the closed subscheme of C with defining ideal $(\bar{a}_1) \subset A/f$, hence to the closed subscheme of S with defining ideal $(f, a_1) \subset A$. The algebra $A[t][I/f]$ is isomorphic to

$$A[x][u]/(g - fu) = A[x][u - x^2R(x)]/(a_0 + a_1x - f(u - x^2R(x))) \simeq A[x][v]/(a_0 + a_1x - fv).$$

One deduces from this presentation that the morphism $\theta \circ \sigma : W' = \text{Spec}(A[I/f]) \rightarrow \text{Spec}(A)$ corresponding to the inclusion $A \rightarrow A[I/f]$ factors through a locally trivial \mathbb{A}^1 -bundle $\theta' : W' \rightarrow S \setminus F$ over the complement of F . Namely, since \bar{a}_0 and \bar{a}_1 generate the unit ideal in A/f , it follows that a_1 and f generate the unit ideal in $A[x][u]/(g - fu)$. So W' is covered by the two principal affine open subsets

$$\begin{aligned} W'_{a_1} &\simeq \text{Spec}(A_{a_1}[x][v]/(a_0 + a_1x - fv)) \simeq \text{Spec}(A_{a_1}[v]) \simeq S_{a_1} \times \mathbb{A}^1 \\ W'_f &\simeq \text{Spec}(A_f[x][v]/(a_0 + a_1x - fv)) \simeq \text{Spec}(A_f[x]) \simeq S_f \times \mathbb{A}^1 \end{aligned}$$

on which θ' restricts to the projection onto the first factor. \square

With the notation above, $\theta : W \rightarrow S$ and $\theta' : W' \rightarrow S \setminus F$ are torsors under the action of line bundles $M = \text{Spec}(\text{Sym} \mathcal{M}^\vee)$ and $M' = \text{Spec}(\text{Sym} \mathcal{M}'^\vee)$ for certain uniquely determined invertible sheaves \mathcal{M} and \mathcal{M}' on S and $S \setminus F$ respectively.

Lemma 1.10. ([1, §4.3]) *Let $\sigma : W' \rightarrow W$ be the affine modification of W with center (\mathcal{I}_Z, D) as is Lemma 1.9. Then $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(-C)|_{S \setminus F}$ and the cartesian diagram of Lemma 1.9 is equivariant for the group scheme homomorphism $\xi : M' \rightarrow M$ induced by the homomorphism $\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(-C) \rightarrow \mathcal{M}$ obtained by tensoring the inclusion $\mathcal{O}_S(-C) \hookrightarrow \mathcal{O}_S$ by \mathcal{M} .*

Proof. Since M and M' are uniquely determined, the question is again local with respect to a Zariski open cover of S over which $\theta : W \rightarrow S$, hence M , becomes trivial. We can thus assume as in the proof of Lemma 1.9 that $S = \text{Spec}(A)$, $W = \text{Spec}(A[x])$, that $C = \text{div}(f)$ for some non zero element $f \in A$ and that $Z \subset D$ is defined by the ideal (f, g) for some $g = a_0 + a_1x + fx^2R(x) \in A[x]$. Furthermore, the action of $M \simeq \mathbb{G}_{a,S} = \text{Spec}(A[t])$ on $W \simeq S \times \mathbb{A}^1$ is the one by translations $x \mapsto x + t$ on the second factor. Let $N = \text{Spec}(\text{Sym}^* \mathcal{O}_S(C)) \simeq \text{Spec}(\text{Sym}^* f^{-1}A)$ where $f^{-1}A$ denotes the free sub- A -module of the field of fractions $\text{Frac}(A)$ of A generated by f^{-1} . As in the proof of Proposition 1.7, the inclusion $\mathcal{O}_S(-C) = f \cdot \mathcal{O}_S \hookrightarrow \mathcal{O}_S$ induces a group-scheme homomorphism $\xi : N \rightarrow M$ whose co-morphism ξ^\sharp coincides with the inclusion $A[t] \subset \text{Sym}^* f^{-1}A = A[(f^{-1}t)]$. The co-morphism of the corresponding action of N on W is given by

$$A[x] \rightarrow A[x] \otimes A[f^{-1}t], \quad x \mapsto x \otimes 1 + 1 \otimes t = x \otimes 1 + f \otimes f^{-1}t.$$

This action lifts on $W' \simeq \text{Spec}(A[x][v]/(a_0 + a_1x - fv))$ to an action $\nu : N \times_S W' \rightarrow W'$ whose co-morphism

$$A[x][v]/(a_0 + a_1x - fv) \rightarrow A[x][v]/(a_0 + a_1x - fv) \otimes A[f^{-1}t]$$

is given by $x \mapsto x \otimes 1 + 1 \otimes t$ and $v \mapsto v \otimes 1 + a_1 \otimes f^{-1}t$. By construction, the principal open subsets $W'_{a_1} \simeq \text{Spec}(A_{a_1}[v]) \simeq \text{Spec}(A_{a_1}[v/a_1])$ and $W'_f \simeq \text{Spec}(A_f[x]) \simeq \text{Spec}(A_f[x/f])$ of W' equipped with the induced actions of $N|_{S_{a_1}}$ and $N|_{S_f}$ respectively are equivariantly isomorphic to $N|_{S_{a_1}}$ and $N|_{S_f}$ acting on themselves by translations. So $\theta' : W' \rightarrow S \setminus F$ is an $N|_{S \setminus F}$ -torsor, showing that $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(-C)|_{S \setminus F}$ as desired. \square

2. EXTENSIONS OF \mathbb{G}_a -TORSORS OF TYPE I: LOCALLY TRIVIAL BUNDLES OVER THE BLOW-UP OF A POINT

Given a surface S and a locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow \tilde{S}$ over the blow-up $\tau : \tilde{S} \rightarrow S$ of a closed point o in the smooth locus of S , the restriction of W over the complement $\tilde{S} \setminus E$ of the exceptional divisor E of τ is a locally trivial \mathbb{A}^1 -bundle $\tau \circ \theta : W|_{\tilde{S} \setminus E} \rightarrow \tilde{S} \setminus E \xrightarrow{\cong} S \setminus \{o\}$. This observation combined with the following re-interpretation of an example constructed in [10] suggests that locally trivial \mathbb{A}^1 -bundles over the blow-up of closed point o in the smooth locus of a surface S form a natural class of schemes in which to search for nontrivial \mathbb{G}_a -extension of \mathbb{G}_a -bundles over punctured surfaces.

Example 2.1. Let $o = V(x, y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface S . Let $\rho : P \rightarrow S \setminus \{o\}$ and $\pi_0 : X_0 \rightarrow S$ be the affine S -schemes with defining sheaves of ideals $(xv - yu - 1)$ and $(xr - yq, yp - x(q - 1), pr - q(q - 1))$ in $\mathcal{O}_S[u, v]$ and $\mathcal{O}_S[p, q, r]$ respectively. The morphism of S -schemes $j_0 : P \rightarrow X_0$ defined by $(x, y, u, v) \mapsto (x, y, xu, xv, yv)$ is an open embedding, equivariant for the $\mathbb{G}_{a,S}$ -actions on P and X_0 associated with the locally nilpotent \mathcal{O}_S -derivations $x\partial_u + y\partial_v$ and $x^2\partial_p + xy\partial_q + y^2\partial_r$ of $\rho_*\mathcal{O}_P$ and $(\pi_0)_*\mathcal{O}_{X_0}$ respectively. It is straightforward to check that $\rho : P \rightarrow S \setminus \{o\}$ is a \mathbb{G}_{a,S^*} -torsor and that $\pi_0 : X_0 \rightarrow S$ is a \mathbb{G}_a -extension of P whose fiber over o is isomorphic to the smooth affine quadric $\{pr - q(q - 1) = 0\} \subset \mathbb{A}_\kappa^3$. Viewing the blow-up \tilde{S} of o as the closed subscheme of $S \times_k \text{Proj}(k[u_0, u_1])$ with equation $xu_1 - yu_0 = 0$, the morphism of S -schemes $\theta : X_0 \rightarrow \tilde{S}$ defined by

$$(x, y, p, q, r) \mapsto ((x, y), [x : y]) = ((x, y), [q : r]) = ((x, y), [p : q - 1])$$

is a locally trivial \mathbb{A}^1 -bundle, actually a torsor under the line bundle corresponding to the invertible sheaf $\mathcal{O}_{\tilde{S}}(-2E)$, where $E \simeq \mathbb{P}_\kappa^1$ denotes the exceptional divisor of the blow-up.

Notation 2.2. Given a surface S and a closed point o in the smooth locus of S , with residue field κ , we denote by $\tau : \tilde{S} \rightarrow S$ be the blow-up of o , with exceptional divisor $E \simeq \mathbb{P}_\kappa^1$. We identify $\tilde{S} \setminus E$ and $S_* = S \setminus \{o\}$ by the isomorphism induced by τ . For every $\ell \in \mathbb{Z}$, we denote by $M(\ell) = \text{Spec}(\text{Sym}^* \mathcal{O}_{\tilde{S}}(-\ell E))$ the line bundle on \tilde{S} corresponding to the invertible sheaf $\mathcal{O}_{\tilde{S}}(\ell E)$.

The aim of this section is to give a classification of all possible \mathbb{G}_a -equivariant extensions of Type I of a given \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$, that is \mathbb{G}_a -extensions $\pi : W \rightarrow S$ that factor through locally trivial \mathbb{A}^1 -bundles $\theta : W \rightarrow \tilde{S}$.

2.1. Existence of \mathbb{G}_a -extensions of Type I. By virtue of Propositions 1.6 and 1.7, there exists a one-to-one correspondence between \mathbb{G}_a -equivariant extensions of a \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ that factor through a locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow \tilde{S}$ and pairs (M, ξ) consisting of an M -torsor $\theta : W \rightarrow \tilde{S}$ for some line bundle M on \tilde{S} and a group scheme homomorphism $\xi : \mathbb{G}_{a, \tilde{S}} \rightarrow M$ restricting to an isomorphism over $\tilde{S} \setminus E$, such that W equipped with the $\mathbb{G}_{a, \tilde{S}}$ -action deduced by composition with ξ restricts on $S_* = \tilde{S} \setminus E$ to a \mathbb{G}_{a, S_*} -torsor $\theta|_{S_*} : W|_{S_*} \rightarrow S_*$ isomorphic to $\rho : P \rightarrow S_*$. The condition that $\xi : \mathbb{G}_{a, \tilde{S}} \rightarrow M$ restricts to an isomorphism outside E implies that $M \simeq M(\ell)$ for some ℓ , which is necessarily non negative, and that ξ is induced by the canonical global section of $\mathcal{O}_{\tilde{S}}(\ell E)$ with divisor ℓE .

Proposition 2.3. *Let $\rho : P \rightarrow S_*$ be a \mathbb{G}_{a, S_*} -torsor. Then there exists an integer $\ell_0 \geq 0$ depending on P only such that for every $\ell \geq \ell_0$, P admits a \mathbb{G}_a -extension to a uniquely determined $M(\ell)$ -torsor $\theta_\ell : W(P, \ell) \rightarrow \tilde{S}$ equipped with the $\mathbb{G}_{a, \tilde{S}}$ -action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ with divisor ℓE .*

Proof. The invertible sheaves $\mathcal{O}_{\tilde{S}}(nE)$, $n \geq 0$, form an inductive system of sub- $\mathcal{O}_{\tilde{S}}$ -modules of the sheaf $\mathcal{K}_{\tilde{S}}$ of rational function on \tilde{S} , where for each n , the injective transition homomorphism $j_{n, n+1} : \mathcal{O}_{\tilde{S}}(nE) \hookrightarrow \mathcal{O}_{\tilde{S}}((n+1)E)$ is obtained by tensoring the canonical section $\mathcal{O}_{\tilde{S}} \rightarrow \mathcal{O}_{\tilde{S}}(E)$ with divisor E with $\mathcal{O}_{\tilde{S}}(nE)$. Let $i : S_* = \tilde{S} \setminus E \hookrightarrow \tilde{S}$ be the open inclusion. Since E is a Cartier divisor, it follows from [6, Théorème 9.3.1] that $i_*\mathcal{O}_{S_*} \simeq \text{colim}_{n \geq 0} \mathcal{O}_{\tilde{S}}(nE)$. Furthermore, since $E \simeq \mathbb{P}_\kappa^1$ is the exceptional divisor of $\tau : \tilde{S} \rightarrow S$, we have $\mathcal{O}_{\tilde{S}}(E)|_E \simeq \mathcal{O}_{\mathbb{P}_\kappa^1}(-1)$, and the long exact sequence of cohomology for the short exact sequence

$$(2.1) \quad 0 \rightarrow \mathcal{O}_{\tilde{S}}(nE) \rightarrow \mathcal{O}_{\tilde{S}}((n+1)E) \rightarrow \mathcal{O}_{\tilde{S}}((n+1)E)|_E \rightarrow 0, \quad n \geq 0,$$

combined with the vanishing of $H^0(\mathbb{P}_\kappa^1, \mathcal{O}_{\mathbb{P}_\kappa^1}(-n-1))$ for every $n \geq 0$ implies that the transition homomorphisms

$$H^1(j_{n, n+1}) : H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \rightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}((n+1)E)), \quad n \geq 0,$$

are all injective. By assumption, S whence \tilde{S} is noetherian, and $i : S_* \rightarrow \tilde{S}$ is an affine morphism as E is a Cartier divisor on \tilde{S} . We thus deduce from [12, Theorem 8] and [7, Corollaire 1.3.3] that the canonical homomorphism

$$(2.2) \quad \psi : \text{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \rightarrow H^1(S_*, \mathcal{O}_{S_*})$$

obtained as the composition of the canonical homomorphisms

$$\text{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)) \rightarrow H^1(\tilde{S}, \text{colim}_{n \geq 0} \mathcal{O}_{\tilde{S}}(nE)) = H^1(\tilde{S}, i_*\mathcal{O}_{S_*})$$

and $H^1(\tilde{S}, i_*\mathcal{O}_{S_*}) \rightarrow H^1(S_*, \mathcal{O}_{S_*})$ is an isomorphism.

Let $g \in H^1(S_*, \mathcal{O}_{S_*})$ be the isomorphism class of the \mathbb{G}_{a, S_*} -torsor $\rho : P \rightarrow S_*$. If $g = 0$, then since ψ is an isomorphism, we have $\psi^{-1}(g) = 0$ and, since the homomorphisms $H^1(j_{n, n+1})$ are injective, it follows that $\psi^{-1}(g)$ is represented by the zero sequence $(0)_n \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$, $n \geq 0$. Consequently, the only \mathbb{G}_a -extensions of P are the line bundles $W(P, \ell) = M(\ell)$, $\ell \geq 0$, each equipped with the $\mathbb{G}_{a, \tilde{S}}$ -action induced by its canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$.

Otherwise, if $g \neq 0$, then $h = \psi^{-1}(g) \neq 0$, and since the homomorphisms $H^1(j_{n, n+1})$, $n \geq 0$ are injective, it follows that there exists a unique minimal integer ℓ_0 such that h is represented by the sequence

$$(2.3) \quad h_n = H^1(j_{n-1, n}) \circ \cdots \circ H^1(j_{\ell_0, \ell_0+1})(h_{\ell_0}) \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)), \quad n \geq \ell_0$$

for some non zero $h_{\ell_0} \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell_0 E))$. It then follows from Proposition 1.7 that for every $\ell \geq \ell_0$, the $M(\ell)$ -torsor $\theta_\ell : W(P, \ell) \rightarrow \tilde{S}$ with isomorphism class h_ℓ equipped with the $\mathbb{G}_{a, \tilde{S}}$ -action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ is a \mathbb{G}_a -extension of P .

Conversely, for every \mathbb{G}_a -extension of P into an $M(\ell)$ -torsor $\theta : W \rightarrow \tilde{S}$ equipped with the $\mathbb{G}_{a, \tilde{S}}$ -action induced by the canonical global section $s_\ell \in \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$, it follows from Proposition 1.7 again that the

image of the isomorphism class $h_\ell \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell E))$ of W in $H^1(\tilde{S} \setminus E, \mathcal{O}_{\tilde{S}}(\ell E)|_{\tilde{S} \setminus E}) \simeq H^1(S_*, \mathcal{O}_{S_*})$ is equal to g . Letting $h \in \text{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ be the element represented by the sequence

$$h_n = (H^1(j_{n-1,n} \circ \cdots \circ j_{\ell,\ell+1})(h_\ell))_{n \geq \ell} \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE)), \quad n \geq \ell$$

we have $\psi(h) = g$ and since ψ is an isomorphism, we conclude that $W \simeq W(P, \ell)$. \square

2.2. \mathbb{G}_a -extensions with affine total spaces. The extensions $\theta : W \rightarrow \tilde{S}$ we get from Proposition 2.3 are not necessarily affine over S . In this subsection we establish a criterion for affineness which we then use to characterize all extensions $\theta : W \rightarrow \tilde{S}$ of a \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ whose total spaces W are affine over S .

Lemma 2.4. *Let $S = \text{Spec}(A)$ be an affine surface and let $o = V(x, y)$ be a global scheme-theoretic complete intersection point in the smooth locus of S . Let $\tau : \tilde{S} \rightarrow S$ be the blow-up of o with exceptional divisor E and let $\theta : W \rightarrow \tilde{S}$ be an $M(\ell)$ -torsor for some $\ell \geq 0$. Then the following hold:*

a) $H^1(W, \mathcal{O}_W) = 0$.

b) If $H^1(W, \theta^* \mathcal{O}_{\tilde{S}}(\ell E)) = 0$ for some $\ell \geq 2$ then W is an affine scheme.

Proof. Since o is a scheme-theoretic complete intersection, we can identify \tilde{S} with the closed subvariety of $S \times_k \mathbb{P}_k^1 = S \times_k \text{Proj}(k[t_0, t_1])$ defined by the equation $xt_1 - yt_0 = 0$. The restriction $p : \tilde{S} \rightarrow \mathbb{P}_k^1$ of the projection to the second factor is an affine morphism. More precisely, letting $U_0 = \mathbb{P}_k^1 \setminus \{[1 : 0]\} \simeq \text{Spec}(k[z])$ and $U_\infty = \mathbb{P}_k^1 \setminus \{[0 : 1]\} \simeq \text{Spec}(k[z'])$ be the standard affine open cover of \mathbb{P}_k^1 , we have $p^{-1}(U_0) \simeq \text{Spec}(A[z]/(x - yz))$ and $p^{-1}(U_\infty) \simeq \text{Spec}(A[z']/(y - xz'))$. The exceptional divisor $E \simeq \mathbb{P}_k^1$ of $\tau : \tilde{S} \rightarrow S$ is a flat quasi-section of p with local equations $y = 0$ and $x = 0$ in the affine charts $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$ respectively. Every $M(\ell)$ -torsor $\theta : W \rightarrow \tilde{S}$ for some $\ell \geq 0$ is isomorphic to the scheme obtained by gluing $W_0 = p^{-1}(U_0) \times \text{Spec}(k[u])$ with $W_\infty = p^{-1}(U_\infty) \times \text{Spec}(k[u'])$ over $U_0 \cap U_\infty$ by an isomorphism induced by a k -algebra isomorphism of the form

$$A[(z')^{\pm 1}]/(y - xz')[u'] \ni (z', u') \mapsto (z^{-1}, z^\ell u + p) \in A[z^{\pm 1}]/(x - yz)[u]$$

for some $p \in A[z^{\pm 1}]/(x - yz)$. Since $H^1(W, \mathcal{O}_W) \simeq \check{H}^1(W, \mathcal{O}_W) \simeq \check{H}^1(\{W_0, W_\infty\}, \mathcal{O}_W)$, it is enough in order to prove a) to check that every Čech 1-cocycle g with value in \mathcal{O}_W for the covering of W by the affine open subsets W_0 and W_∞ is a coboundary. Viewing g as an element $g = g(z^{\pm 1}, u) \in A[z^{\pm 1}]/(x - yz)[u]$, it is enough to show that every monomial $g_s = hz^r u^s$ where $h \in A$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ is a coboundary, which is the case if and only if there exist $a(z, u) \in A[z]/(f - gz)[u]$ and $b(z', u') \in A[z']/(y - xz')[u']$ such that $g = b(z^{-1}, z^\ell u + p) - b(z, u)$. If $r \geq 0$ then $g \in A[z]/(x - yz)[u]$ is a coboundary. We thus assume from now on that $r < 0$. Suppose that $s > 0$. Then we can write $u^s = z^{-\ell s}(z^\ell u + p)^s - R(u)$ where $R \in A[z^{\pm 1}]/(x - yz)[u]$ is polynomial whose degree in u is strictly less than s . Then since $r < 0$,

$$\begin{aligned} hz^r u^s &= hz^{r-\ell s}(z^\ell u + p)^s - hz^r R(u) \\ &= b(z^{-1}, z^\ell u + p) - hz^r R(u) \end{aligned}$$

where $b(z', u') = h(z')^{-r+\ell s}(u')^s \in A[z']/(y - xz')[u']$. So g_s is a coboundary if and only if $-hz^r R(u)$ is. By induction, we only need to check that every monomial $g_0 = hz^r \in A[z^{\pm 1}]/(x - yz)[u]$ of degree 0 in u is a coboundary. But such a cocycle is simply the pull-back to W of a Čech 1-cocycle h_0 with value in $\mathcal{O}_{\tilde{S}}$ for the covering of \tilde{S} by the affine open subsets $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$. Since the canonical homomorphism

$$H^1(S, \mathcal{O}_S) = H^1(S, \tau_* \mathcal{O}_{\tilde{S}}) \rightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) \simeq \check{H}^1(\{p^{-1}(U_0), p^{-1}(U_\infty)\}, \mathcal{O}_{\tilde{S}})$$

is an isomorphism and $H^1(S, \mathcal{O}_S) = 0$ as S is affine, we conclude that h_0 is a coboundary, hence that g_0 is a coboundary too. This proves a).

Now suppose that $H^1(W, \theta^* \mathcal{O}_{\tilde{S}}(\ell E)) = 0$ for some $\ell \geq 2$. Let $\eta : V \rightarrow \mathbb{P}_k^1$ be a non trivial $\mathcal{O}_{\mathbb{P}_k^1}(-\ell)$ -torsor and consider the fiber product $W \times_{p \circ \theta, \mathbb{P}_k^1, \eta} V$:

$$\begin{array}{ccc} & W \times_{p \circ \theta, \mathbb{P}_k^1, \eta} V & \\ & \swarrow \quad \searrow & \\ W & & V \\ & \searrow \quad \swarrow & \\ & \mathbb{P}_k^1 & \end{array}$$

$p \circ \theta$ η

By virtue of [5, Proposition 3.1], V is an affine surface. Since $p \circ \theta : W \rightarrow \mathbb{P}_k^1$ is an affine morphism, so is $\text{pr}_V : W \times_{\mathbb{P}_k^1} V \rightarrow V$ and hence, $W \times_{\mathbb{P}_k^1} V$ is an affine scheme. On the other hand, since $p^* \mathcal{O}_{\mathbb{P}_k^1}(-1) \simeq \mathcal{O}_{\tilde{S}}(E)$, the projection $\text{pr}_W : W \times_{\mathbb{P}_k^1} V \rightarrow W$ is a $\theta^* M(\ell)$ -torsor, hence is isomorphic to the trivial one $q : \theta^* M(\ell) \rightarrow W$ by hypothesis. So W is isomorphic to the zero section of $\theta^* M(\ell)$, which is a closed subscheme of the affine scheme $W \times_{\mathbb{P}_k^1} V$, hence an affine scheme. \square

We are now ready to prove the following characterization:

Theorem 2.5. *A \mathbb{G}_{a, S_*} -torsor $\rho : P \rightarrow S_*$ admits a \mathbb{G}_a -extension to a locally trivial \mathbb{A}^1 -bundle whose total space is affine over S if and only if for every Zariski open neighborhood U of o , $P \times_{S_*} U \rightarrow U_* = U \setminus \{o\}$ is a non trivial \mathbb{G}_{a, U_*} -torsor.*

When it exists, the corresponding locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow \tilde{S}$ is unique and is an $M(\ell_0)$ -torsor for some $\ell_0 \geq 2$, whose restriction to $E \simeq \mathbb{P}_k^1$ is a non trivial $\mathcal{O}_{\mathbb{P}_k^1}(-\ell_0)$ -torsor.

Proof. The scheme W is affine over S if and only if its restriction $W|_E$ over $E \subset \tilde{S}$ is a nontrivial torsor. Indeed, if $W|_E$ is a trivial torsor then it is a line bundle over $E \simeq \mathbb{P}_k^1$. Its zero section is then a proper curve contained in the fiber of $\pi = \tau \circ \theta : W \rightarrow S$, which prevents π from being an affine morphism. Conversely, if $W|_E$ is nontrivial, then it is a torsor under a uniquely determined line bundle $\mathcal{O}_{\mathbb{P}_k^1}(-m)$ for some $m \geq 2$ necessarily. Since by construction π restricts over S_* to $\rho : P \rightarrow S_*$ which is an affine morphism, π is affine if and only if there exists an open neighborhood U of o in S such that $\pi^{-1}(U)$ is affine. Replacing S by a suitable affine open neighborhood of o , we can therefore assume without loss of generality that $S = \text{Spec}(A)$ is affine and that o is a scheme-theoretic complete intersection $o = V(x, y)$ for some elements $x, y \in A$. By virtue of [5, Proposition 3.1] every nontrivial $\mathcal{O}_{\mathbb{P}_k^1}(-m)$ -torsor, $m \geq 2$, has affine total space. The Cartier divisor $D = W|_E$ in W is thus an affine surface, and so $H^1(D, \mathcal{O}_W((n+1)D)|_D) = 0$ for every $n \in \mathbb{Z}$. By a) in Lemma 2.4, $H^1(W, \mathcal{O}_W) = 0$, and we deduce successively from the long exact sequence of cohomology for the short exact sequence

$$0 \rightarrow \mathcal{O}_W(nD) \rightarrow \mathcal{O}_W((n+1)D) \rightarrow \mathcal{O}_W((n+1)D)|_D \rightarrow 0$$

in the case $n = 0$ and then $n = 1$ that $H^1(W, \mathcal{O}_W(D)) = H^1(W, \mathcal{O}_W(2D)) = 0$. Since $\mathcal{O}_W(2D) \simeq \theta^* \mathcal{O}_{\tilde{S}}(2E)$, we conclude from b) in the same lemma that W is affine.

The condition that $P \times_{S_*} U \rightarrow U_*$ is nontrivial for every open neighborhood U of o is necessary for the existence of an extension $\theta : W \rightarrow \tilde{S}$ of P for which $W|_E$ is a nontrivial torsor. Indeed, if there exists a Zariski open neighborhood U of o such that the restriction of P over U_* is the trivial \mathbb{G}_{a, U_*} -torsor, then the image in $H^1(U_*, \mathcal{O}_{U_*})$ of the isomorphism class g of P is zero and so, arguing as in the proof of Proposition 2.3, every \mathbb{G}_a -extension $\theta : W \rightarrow \tilde{S}$ restricts on $\tau^{-1}(U)$ to the trivial $M(\ell)|_{\tau^{-1}(U)}$ -torsor $M(\ell)|_{\tau^{-1}(U)} \rightarrow \tau^{-1}(U)$, hence to a trivial torsor on $E \subset \tau^{-1}(U)$.

Now suppose that $\rho : P \rightarrow S_*$ is a \mathbb{G}_{a, S_*} -torsor with isomorphism class $g \in H^1(S_*, \mathcal{O}_{S_*})$ such that $P \times_{S_*} U \rightarrow U_*$ is non trivial for every open neighborhood U of o . The inverse image $h = \psi^{-1}(g) \in \text{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ of g by the isomorphism (2.2) is represented by a sequence of nonzero elements $h_n \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(nE))$ as in (2.3) above. By the long exact sequence of cohomology of the short exact sequence (2.1), the image \bar{h}_n of h_n in $H^1(E, \mathcal{O}_{\tilde{S}}(nE)|_E) \simeq H^1(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(-n))$ is nonzero if and only if h_n is not in the image of the injective homomorphism $H^1(j_{n, n-1})$. Since \bar{h}_n coincides with the isomorphism class of the restriction $W_n|_E$ of an $M(n)$ -torsor $\theta_n : W_n \rightarrow \tilde{S}$ with isomorphism class h_n , we conclude

that there exists a unique $\ell_0 \geq 2$ such that the restriction to E of an $M(\ell_0)$ -torsor $\theta_{\ell_0} : W_{\ell_0} \rightarrow \tilde{S}$ with isomorphism class $h_{\ell_0} \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\ell_0 E))$ is a nontrivial $\mathcal{O}_{\mathbb{P}^1}(-\ell_0)$ -torsor. \square

2.3. Examples. In this subsection, we consider \mathbb{G}_a -torsors of the punctured affine plane. So $S = \mathbb{A}^2 = \text{Spec}(k[x, y])$, $o = (0, 0)$ and $\mathbb{A}_*^2 = \mathbb{A}^2 \setminus \{o\}$. We let $\tau : \tilde{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$ be the blow-up of o , with exceptional divisor $E \simeq \mathbb{P}^1$ and we let $i : \mathbb{A}_*^2 \hookrightarrow \tilde{\mathbb{A}}^2$ be the immersion of \mathbb{A}_*^2 as the open subset $\tilde{\mathbb{A}}^2 \setminus E$. We further identify $\tilde{\mathbb{A}}^2$ with the total space $f : \tilde{\mathbb{A}}^2 \rightarrow \mathbb{P}^1$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ in such a way that E corresponds to the zero section of this line bundle.

2.3.1. A simple case: homogeneous \mathbb{G}_a -torsors. Following [4, §1.3], we say that a non trivial $\mathbb{G}_{a, \mathbb{A}_*^2}$ -torsor $\rho : P \rightarrow \mathbb{A}_*^2$ is homogeneous if it admits a lift of the \mathbb{G}_m -action $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ on \mathbb{A}_*^2 which is locally linear on the fibers of ρ . By [4, Proposition 1.6], this is the case if and only if the isomorphism class g of P in $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ can be represented on the open covering of \mathbb{A}_*^2 by the principal open subsets \mathbb{A}_x^2 and \mathbb{A}_y^2 by a Čech 1-cocycle of the form $x^{-m}y^{-n}p(x, y)$ where $m, n \geq 0$ and $p(x, y) \in k[x, y]$ is a homogeneous polynomial of degree $r \leq m + n - 2$. Equivalently, P is isomorphic the $\mathbb{G}_{a, \mathbb{A}_*^2}$ -torsor

$$\rho = \text{pr}_{x,y} : P_{m,n,p} = \{x^m v - y^n u = p(x, y)\} \setminus \{x = y = 0\} \rightarrow \mathbb{A}_*^2,$$

which admits an obvious lift $\lambda \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{m-d}u, \lambda^{n-d}v)$, where $d = m + n - r$, of the \mathbb{G}_m -action on \mathbb{A}_*^2 . Let $q : \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2/\mathbb{G}_m = \mathbb{P}^1$ be the quotient morphism of the aforementioned \mathbb{G}_m -action on \mathbb{A}_*^2 . Then it follows from [4, Example 1.8] that the inverse image by the canonical isomorphism

$$\bigoplus_{k \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \simeq H^1(\mathbb{P}^1, q_* \mathcal{O}_{\mathbb{A}_*^2}) \rightarrow H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$$

of the isomorphism class g of such an homogeneous torsor is an element h of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$. Furthermore, the \mathbb{G}_m -equivariant morphism $\rho : P \rightarrow \mathbb{A}_*^2$ descends to a locally trivial \mathbb{A}^1 -bundle $\bar{\rho} : P/\mathbb{G}_m \rightarrow \mathbb{P}^1 = \mathbb{A}_*^2/\mathbb{G}_m$ which is an $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor with isomorphism class $h \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$.

Since $f^* \mathcal{O}_{\mathbb{P}^1}(-d) \simeq \mathcal{O}_{\tilde{\mathbb{A}}^2}(dE)$, the fiber product $W(P, d) = \tilde{\mathbb{A}}^2 \times_{\mathbb{P}^1} P/\mathbb{G}_m$ is equipped via the restriction of the first projection with the structure of an $M(d)$ -torsor $\theta : W(P, d) \rightarrow \tilde{\mathbb{A}}^2$ with isomorphism class $f^* h \in H^1(\tilde{\mathbb{A}}^2, \mathcal{O}_{\tilde{\mathbb{A}}^2}(dE))$. On the other other hand, $W(P, d)$ is a line bundle over P/\mathbb{G}_m via the second projection, hence is an affine threefold as P/\mathbb{G}_m is affine. By construction, we have a commutative diagram

$$\begin{array}{ccccc} & & W(P, d) & & \\ & j \nearrow & \downarrow \theta & \searrow & \\ P & \xrightarrow{\quad} & P/\mathbb{G}_m & & \\ \rho \downarrow & & \downarrow & & \downarrow \bar{\rho} \\ \mathbb{A}_*^2 & \xrightarrow{i} & \tilde{\mathbb{A}}^2 & \xrightarrow{f} & \mathbb{P}^1 \\ & \searrow q & & \nearrow & \\ & & \mathbb{A}_*^2 & & \end{array}$$

in which each square is cartesian. In other words, $W(P, d)$ is obtained from the \mathbb{G}_m -torsor $P \rightarrow P/\mathbb{G}_m$ by “adding the zero section”. The open embedding $j : P \hookrightarrow W(P, d)$ is equivariant for the \mathbb{G}_a -action on $W(P, d)$ induced by the canonical global section of $\mathcal{O}_{\tilde{\mathbb{A}}^2}(dE)$ with divisor dE (see Proposition 1.7). By Theorem 2.5, $\theta : W(P, d) \rightarrow \tilde{\mathbb{A}}^2$ is the unique \mathbb{G}_a -extension of $\rho : P \rightarrow \mathbb{A}_*^2$ with affine total space.

In the simplest case $d = 2$, the unique homogeneous $\mathbb{G}_{a, \mathbb{A}_*^2}$ -torsor is the geometric quotient $\text{SL}_2 \rightarrow \text{SL}_2/\mathbb{G}_a$ of the group SL_2 by the action of its subgroup of upper triangular unipotent matrices equipped with the diagonal \mathbb{G}_m -action, and we recover Example 2.1.

2.3.2. General case. Here, given an arbitrary non trivial \mathbb{G}_a -torsor $\rho : P \rightarrow \mathbb{A}_*^2$, we describe a procedure to explicitly determine the unique \mathbb{G}_a -extension $\theta : W \rightarrow \tilde{\mathbb{A}}^2$ of P with affine total space W from a Čech 1-cocycle $x^{-m}y^{-n}p(x, y)$, where $m, n \geq 0$ and $p(x, y) \in k[x, y]$ is a non zero polynomial of degree $r \leq m + n - 2$, representing the isomorphism class $g \in H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ of P on the open covering of \mathbb{A}_*^2 by the principal open subsets \mathbb{A}_x^2 and \mathbb{A}_y^2 .

Write $p(x, y) = p_d + p_{d+1} + \cdots + p_r$ where the $p_i \in k[x, y]$ are the homogeneous components of p , and $p_d \neq 0$. In the decomposition

$$H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq H^1(\mathbb{P}^1, q_* \mathcal{O}_{\mathbb{A}_*^2}) \simeq \bigoplus_{s \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$$

a non zero homogeneous component $x^{-m}y^{-n}p_i$ of $x^{-m}y^{-n}p(x, y)$ corresponds to a non zero element of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m-n+i))$. On the other hand, since for every $\ell \in \mathbb{Z}$, $\mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E) = f^* \mathcal{O}_{\mathbb{P}^1}(-\ell)$ and $f : \tilde{\mathbb{A}}^2 \rightarrow \mathbb{P}^1$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, it follows from the projection formula that

$$H^1(\tilde{\mathbb{A}}^2, \mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E)) \simeq H^1(\mathbb{P}^1, f_* \mathcal{O}_{\tilde{\mathbb{A}}^2} \otimes \mathcal{O}_{\mathbb{P}^1}(-\ell)) \simeq \bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t - \ell)).$$

The image of $x^{-m}y^{-n}p(x, y)$ in $\bigoplus_{s \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$ belongs to $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t - \ell))$ if and only if $\ell \geq \ell_0 = m + n - d \geq 2$. Given such an ℓ , the image $(h_t)_{t \geq 0} \in \bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t - \ell))$ of $x^{-m}y^{-n}p(x, y)$ then defines a unique $M(\ell)$ -torsor $\theta_\ell : W(P, \ell) \rightarrow \tilde{\mathbb{A}}^2$ whose restriction over the complement of E is isomorphic to $\rho : P \rightarrow \mathbb{A}_*^2$ when equipped with the action \mathbb{G}_a -action induced by the canonical section of $\mathcal{O}_{\tilde{\mathbb{A}}^2}(\ell E)$ with divisor ℓE . On the other hand, the restriction of $W|_E \rightarrow E$ over E is an $\mathcal{O}_{\mathbb{P}^1}(-\ell)$ -torsor with isomorphism class $h_0 \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-\ell))$. By definition, h_0 is non zero if and only if $\ell = \ell_0$, and we conclude from Theorem 2.5 that $\theta_{\ell_0} : W(P, \ell_0) \rightarrow \tilde{\mathbb{A}}^2$ is the unique \mathbb{G}_a -extension of $\rho : P \rightarrow \tilde{\mathbb{A}}^2$ with affine total space.

3. QUASI-PROJECTIVE \mathbb{G}_a -EXTENSIONS OF TYPE II

In this section we consider the following subclass of extensions of Type II of a \mathbb{G}_a -torsor over a punctured surface.

Definition 3.1. A \mathbb{G}_a -extension $\pi : X \rightarrow S$ of a \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ over a punctured surface $S_* = S \setminus \{o\}$ is said to be a *quasi-projective* extension of Type II if it satisfies the following properties

- i) X is quasi-projective over S and the $\mathbb{G}_{a,S}$ -action on X is proper,
- ii) X is smooth along $\pi^{-1}(o)$ and $\pi^{-1}(o)_{\text{red}} \simeq \mathbb{A}_\kappa^2$.

Example 3.2. Let $o = V(x, y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface S and let $\rho : P \rightarrow S \setminus \{o\}$ be the \mathbb{G}_a -torsor with defining sheaf of ideals $(xv - yu - 1) \subset \mathcal{O}_S[u, v]$ as in Example 2.1. Let $\pi_1 : X_1 \rightarrow S$ be the affine S -scheme with defining sheaf of ideals $(xw - y(yz_1 + 1), xz_2 - z_1(yz_1 + 1), z_1w - yz_2) \subset \mathcal{O}_S[z_1, z_2, w]$. The morphism of S -schemes $j_1 : P \rightarrow X_1$ defined by $(x, y, u, v) \mapsto (x, y, u, uv, yu)$ is an open embedding, equivariant for the $\mathbb{G}_{a,S}$ -action on X_1 associated with the locally nilpotent \mathcal{O}_S -derivation $x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w$ of $\pi_* \mathcal{O}_{X_1}$. The fiber $\pi_1^{-1}(o)$ is isomorphic to $\mathbb{A}_\kappa^2 = \text{Spec}(\kappa[z_2, w])$ on which the $\mathbb{G}_{a,S}$ -action restricts to $\mathbb{G}_{a,\kappa}$ -action by translations associated to the derivation ∂_{z_2} of $\kappa[z_2, w]$. It is straightforward to check that X_1 is smooth along $\pi_1^{-1}(o)$. We claim that the geometric quotient of the $\mathbb{G}_{a,S}$ -action on X_1 is isomorphic to the complement of a κ -rational point o_1 in the blow-up $\tau : \tilde{S} \rightarrow S$ of o . Such a surface being in particular separated, the $\mathbb{G}_{a,S}$ -action on X_1 is proper, implying that $j_1 : P \hookrightarrow X_1$ is a quasi-projective extension of P of Type II.

Indeed, let us identify \tilde{S} with the closed subvariety of $S \times_k \text{Proj}(k[u_0, u_1])$ with equation $xu_1 - yu_0 = 0$ in such a way that τ coincides with the restriction of the first projection. The morphism $f : X_1 \rightarrow \tilde{S}$ defined by

$$(x, y, z, u, v) \mapsto ((x, y), [x : y]) = ((x, y), [yz_1 + 1 : w])$$

is \mathbb{G}_a -invariant and maps $\pi_1^{-1}(o)$ dominantly onto the exceptional divisor $E \simeq \text{pr}_S^{-1}(o) \simeq \text{Proj}(\kappa[u_0, u_1])$ of τ . The induced morphism

$$f|_{\pi^{-1}(o)} : \pi^{-1}(o) = \text{Spec}(\kappa[z_2, w]) \rightarrow E, \quad (z_2, w) \mapsto [1 : w]$$

factors as the composition of the geometric quotient $\pi_1^{-1}(o) \rightarrow \pi_1^{-1}(o)/\mathbb{G}_{a,\kappa} \simeq \text{Spec}(\kappa[w])$ with the open immersion $\pi_1^{-1}(o)/\mathbb{G}_{a,\kappa} \hookrightarrow E$ of $\pi_1^{-1}(o)/\mathbb{G}_{a,\kappa}$ as the complement of the κ -rational point $o_1 = ((0, 0), [0 : 1]) \in E$. On the other hand, the composition

$$\tau \circ f \circ j_1 : P \xrightarrow{\simeq} X_1 \setminus \pi_1^{-1}(o) \rightarrow \tilde{S} \setminus E \xrightarrow{\simeq} S \setminus \{o\}$$

coincides with the geometric quotient morphism $\rho : P \rightarrow S \setminus \{o\}$. So $f : X_1 \rightarrow \tilde{S}$ factors through a surjective morphism $q : X_1 \rightarrow \tilde{S} \setminus \{o_1\}$ whose fibers all consist of precisely one \mathbb{G}_a -orbit. Since q is a smooth morphism, q is a \mathbb{G}_a -torsor which implies that $X_1/\mathbb{G}_a \simeq \tilde{S} \setminus \{o_1\}$.

The scheme of the classification of quasi-projective extensions of Type II of a given \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ which we give below is as follows: we first construct in §3.1 families of such extensions, in the form of \mathbb{G}_a -torsors $q : X \rightarrow S'$ over quasi-projective S -schemes $\tau : S' \rightarrow S$ such that $\tau^{-1}(o)_{\text{red}}$ is isomorphic to \mathbb{A}_κ^1 , S' is smooth along $\tau^{-1}(o)$, and $\tau : S' \setminus \tau^{-1}(o) \rightarrow S_*$ is an isomorphism. We then show in § 3.2 that for quasi-projective \mathbb{G}_a -extension $\pi : X \rightarrow S$ of Type II of a given \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$, the structure morphism $\pi : X \rightarrow S$ factors through a \mathbb{G}_a -torsor $q : X \rightarrow S'$ over one of these S -schemes S' . In the last subsection, we focus on the special case where $\pi : X \rightarrow S$ has the stronger property of being an affine morphism.

3.1. A family of \mathbb{G}_a -extensions over quasi-projective S -schemes.

Let again (S, o) be a pair consisting of a surface and a closed point o contained in the smooth locus of S , with residue field κ . We let $\bar{\tau}_1 : \bar{S}_1 \rightarrow S$ be the blow-up of o , with exceptional divisor $\bar{E}_1 \simeq \mathbb{P}_\kappa^1$. Then for every $n \geq 2$, we let $\bar{\tau}_{n,1} : \bar{S}_n = \bar{S}_n(o_1, \dots, o_{n-1}) \rightarrow \bar{S}_1$ be the scheme obtained from \bar{S}_1 by performing the following sequence of blow-ups of κ -rational points:

- The first step $\bar{\tau}_{2,1} : \bar{S}_2(o_1) \rightarrow \bar{S}_1$ is the blow-up of a κ -rational point $o_1 \in \bar{E}_1$ with exceptional divisor $\bar{E}_2 \simeq \mathbb{P}_\kappa^1$,
 - Then for every $2 \leq i \leq n-2$, we let $\bar{\tau}_{i+1,i} : \bar{S}_{i+1}(o_1, \dots, o_i) \rightarrow \bar{S}_i(o_1, \dots, o_{i-1})$ be the blow-up of a κ -rational point $o_i \in \bar{E}_i$, with exceptional divisor $\bar{E}_{i+1} \simeq \mathbb{P}_\kappa^1$.
 - Finally, we let $\bar{\tau}_{n,n-1} : \bar{S}_n(o_1, \dots, o_{n-1}) \rightarrow \bar{S}_{n-1}(o_1, \dots, o_{n-2})$ be the blow-up of a κ -rational point $o_{n-1} \in \bar{E}_{n-1}$ which is a smooth point of the reduced total transform of \bar{E}_1 by $\bar{\tau}_1 \circ \dots \circ \bar{\tau}_{n-1,n-2}$.
- We let $\bar{E}_n \simeq \mathbb{P}_\kappa^1$ be the exceptional divisor of $\bar{\tau}_{n,n-1}$ and we let

$$\bar{\tau}_{n,1} = \bar{\tau}_{2,1} \circ \dots \circ \bar{\tau}_{n,n-1} : \bar{S}_n(o_1, \dots, o_{n-1}) \rightarrow \bar{S}_1.$$

The inverse image of o in $\bar{S}_n(o_1, \dots, o_{n-1})$ by $\bar{\tau}_1 \circ \bar{\tau}_{n,1}$ is a tree of κ -rational curves in which \bar{E}_n intersects the reduced proper transform of $\bar{E}_1 \cup \dots \cup \bar{E}_{n-1}$ in $\bar{S}_n(o_1, \dots, o_{n-1})$ transversally in a unique κ -rational point.

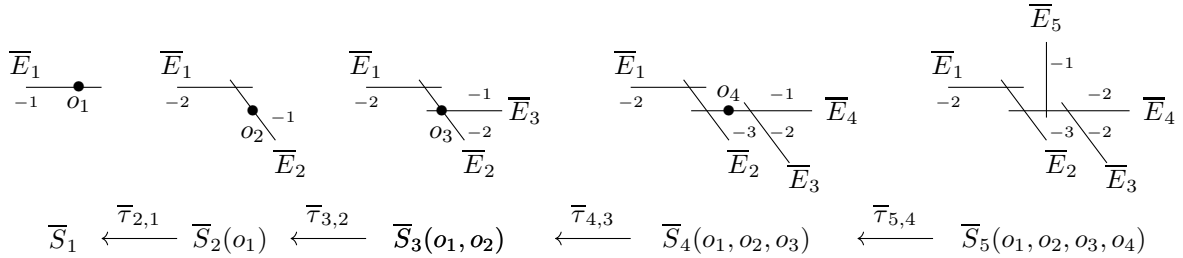


FIGURE 3.1. The successive total transforms of \bar{E}_1 in a possible construction of a surface of the form $\bar{S}_5(o_1, \dots, o_4)$ over a k -rational point o . The integers indicate the self-intersections of the corresponding curves.

Notation 3.3. For every κ -rational point $o_1 \in \bar{E}_1$, we let $S_1(o_1) = \bar{S}_1 \setminus \{o_1\}$, $E_1 = \bar{E}_1 \cap S_1 \simeq \mathbb{A}_\kappa^1$ and we let $\tau_1 : S_1(o_1) \rightarrow S$ be the restriction of $\bar{\tau}_1$.

For $n \geq 2$, we let $S_n(o_1, \dots, o_{n-1}) = \bar{S}_n(o_1, \dots, o_{n-1}) \setminus \bar{E}_1 \cup \dots \cup \bar{E}_{n-1}$ and $E_n = S_n(o_1, \dots, o_{n-1}) \cap \bar{E}_n \simeq \mathbb{A}_\kappa^1$. We denote by $\tau_{n,1} : S_n(o_1, \dots, o_{n-1}) \rightarrow \bar{S}_1$ the birational morphism induced by $\bar{\tau}_{n,1}$ and we let $\tau_n = \bar{\tau}_1 \circ \tau_{n,1} : S_n(o_1, \dots, o_{n-1}) \rightarrow S$.

The following lemma summarizes some basic properties of the so-constructed S -schemes:

Lemma 3.4. *For every $n \geq 1$, the following hold for $S_n = S_n(o_1, \dots, o_{n-1})$:*

- $\tau_n : S_n \rightarrow S$ is quasi-projective and restricts to an isomorphism over S_* while $\tau_n^{-1}(o)_{\text{red}} = E_n$,
- S_n is smooth along $\tau_n^{-1}(o)$

c) $\tau_n^* : \Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(S_n, \mathcal{O}_{S_n})$ is an isomorphism.

Moreover for $n \geq 2$, the morphism $\tau_{n,1} : S_n \rightarrow \overline{S}_1$ is affine.

Proof. The first three properties are straightforward consequences of the construction. For the last one, let $D = \overline{E}_1 + \sum_{i=2}^{n-1} a_i \overline{E}_i$ where a_i is a sequence of positive rational numbers and let $m \geq 1$ be so that mD is a Cartier divisor on \overline{S}_n . Then a direct computation shows that the restriction of $\mathcal{O}_{\overline{S}_n}(mD)$ to $\overline{\tau}_{n,1}^{-1}(o_1)_{\text{red}} = \bigcup_{i=2}^n \overline{E}_i$ is an ample invertible sheaf provided that the sequence $(a_i)_{i=2, \dots, n-1}$ decreases rapidly enough with respect to the distance of \overline{E}_i to \overline{E}_1 in the dual graph of $\overline{E}_1 \cup \dots \cup \overline{E}_{n-1}$. Since $\overline{\tau}_{n,1}$ restricts to an isomorphism over $\overline{S}_1 \setminus \{o_1\}$, it follows from [7, Théorème 4.7.1] that $\mathcal{O}_{\overline{S}_n}(mD)$ is $\overline{\tau}_{n,1}$ -ample on \overline{S}_n . Since by definition $\tau_{n,1}$ is the restriction of the projective morphism $\overline{\tau}_{n,1} : \overline{S}_n \rightarrow \overline{S}_1$ to $S_n = \overline{S}_n \setminus \overline{E}_1 \cup \dots \cup \overline{E}_{n-1} = \overline{S}_n \setminus \text{Supp}(D)$, we conclude that $\tau_{n,1}$ is an affine morphism. \square

Remark 3.5. By construction, $\tau_1^{-1}(o) = E_1$ in $S_1(o_1)$, but for $n \geq 2$, we have $\tau_n^{-1}(o) = mE_n$ for some integer $m \geq 1$ which depends on the sequence of κ -rational points o_1, \dots, o_{n-1} blown-up to construct $S_n(o_1, \dots, o_{n-1})$. For instance, it is straightforward to check that $m = 1$ if and only if for every $i \geq 1$, $o_i \in \overline{E}_i$ is a smooth point of the reduced total transform of \overline{E}_1 in $\overline{S}_i(o_1, \dots, o_{i-1})$.

The structure morphism of a \mathbb{G}_a -torsor being affine, hence quasi-projective, the total space of any \mathbb{G}_a -torsor $q : X \rightarrow S_n$ over an S -scheme $\tau_n : S_n = S_n(o_1, \dots, o_n) \rightarrow S$ is a quasi-projective S -scheme $\pi = \tau_n \circ q : X \rightarrow S$ equipped with a proper $\mathbb{G}_{a,S}$ -action. Furthermore $\pi^{-1}(o)_{\text{red}} = q^{-1}(E_n) \simeq E_n \times \mathbb{A}_\kappa^1 \simeq \mathbb{A}_\kappa^2$ and X is smooth along $\pi^{-1}(o)$ as S_n is smooth along E_n . On the other hand, $\pi : X \rightarrow S$ is by construction a \mathbb{G}_a -extension of its restriction $\rho : P \rightarrow S_n \setminus E_n \simeq S_*$ over $S_n \setminus E_n$, hence is a quasi-projective \mathbb{G}_a -extension of P of Type II. The following proposition shows conversely that every \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ admits a quasi-projective \mathbb{G}_a -extension of Type II into a \mathbb{G}_a -torsor $q : X \rightarrow S_n$.

Proposition 3.6. *Let $\rho : P \rightarrow S_*$ be a \mathbb{G}_a -torsor. Then for every $n \geq 1$ and every S -scheme $\tau_n : S_n(o_1, \dots, o_{n-1}) \rightarrow S$ as in Notation 3.3 there exist a \mathbb{G}_a -torsor $q : X \rightarrow S_n(o_1, \dots, o_{n-1})$ and an equivariant open embedding $j : P \hookrightarrow X$ such that in the following diagram*

$$\begin{array}{ccc} P & \xrightarrow{j} & X \\ \rho \downarrow & & \downarrow q \\ S_n(o_1, \dots, o_{n-1}) \setminus E_n & \hookrightarrow & S_n(o_1, \dots, o_{n-1}) \\ \tau_n \downarrow \wr & & \downarrow \tau_n \\ S_* & \xrightarrow{\quad} & S \end{array}$$

all squares are cartesian. In particular, $j : P \hookrightarrow X$ is a quasi-projective \mathbb{G}_a -extension of P of Type II.

Proof. Letting $S_n = S_n(o_1, \dots, o_n)$, we have to prove that every \mathbb{G}_a -torsor $\rho : P \rightarrow S_n \setminus E_n \simeq S_*$ is the restriction of a \mathbb{G}_a -torsor $q : X \rightarrow S_n$, or equivalently that the restriction homomorphism $H^1(S_n, \mathcal{O}_{S_n}) \rightarrow H^1(S_n \setminus E_n, \mathcal{O}_{S_n \setminus E_n})$ is surjective. It is enough to show that there exists a Zariski open neighborhood U of E_n in S_n and a \mathbb{G}_a -torsor $q : Y \rightarrow U$ such that $Y|_{U \setminus E_n} \simeq P|_{U \setminus E_n}$. Indeed, if so then a \mathbb{G}_a -torsor $q : X \rightarrow S_n$ with the desired property is obtained by gluing P and Y over $U \setminus E_n$ by the isomorphism $Y|_{U \setminus E_n} \simeq P|_{U \setminus E_n}$. In particular, we can replace S_n by the inverse image by $\tau_n : S_n \rightarrow S$ of any Zariski open neighborhood of o in S . We can thus assume from the very beginning that $S = \text{Spec}(A)$ is affine and that $o = V(f, g)$ is a scheme-theoretic intersection for some $f, g \in A$. Up to replacing f and g by other generators of the maximal ideal of o in A , we can assume that the proper transform L_1 in $\overline{\tau}_1 : \overline{S}_1 \rightarrow S$ of the curve $L = V(f) \subset S$ intersects \overline{E}_1 in o_1 . We denote by $M_1 \subset \overline{S}_1$ the proper transform of the curve $M = V(g) \subset S$.

We first treat the case $n = 1$. The open subset $U_1 = \overline{S}_1 \setminus L$ of \overline{S}_1 is then affine and contained in S_1 . Furthermore $U_1 \setminus E_1 = \overline{S}_1 \setminus \overline{\tau}_1^{-1}(L) \simeq S \setminus L$ is also affine. The Mayer-Vietoris long exact sequence of cohomology of \mathcal{O}_{S_1} for the open covering of S_1 by $S_1 \setminus E_1$ and U_1 then reads

$$\begin{aligned} 0 \rightarrow H^0(S_1, \mathcal{O}_{S_1}) &\rightarrow H^0(U_1, \mathcal{O}_{S_1}) \oplus H^0(S_1 \setminus E_1, \mathcal{O}_{S_1}) \rightarrow H^0(U_1 \setminus E_1, \mathcal{O}_{S_n}) \rightarrow \dots \\ \dots \rightarrow H^1(S_1, \mathcal{O}_{S_1}) &\rightarrow H^1(U_1, \mathcal{O}_{U_1}) \oplus H^1(S_1 \setminus E_1, \mathcal{O}_{S_1}) \rightarrow H^1(U_1 \setminus E_1, \mathcal{O}_{U_1}) \rightarrow \dots \end{aligned}$$

Since $U_1 \setminus E_1$ is affine, $H^1(U_1 \setminus E_1, \mathcal{O}_{U_1}) = 0$ and so, the homomorphism $H^1(S_1, \mathcal{O}_{S_1}) \rightarrow H^1(S_1 \setminus E_1, \mathcal{O}_{S_1})$ is surjective as desired.

In the case where $n \geq 2$, the open subset $V_1 = \overline{S}_1 \setminus M_1$ of \overline{S}_1 is affine and it contains o_1 since M_1 intersects \overline{E}_1 in a point distinct from o_1 . Since $\tau_{n,1} : S_n \rightarrow \overline{S}_1$ is an affine morphism by Lemma 3.4, $U_n = \tau_{n,1}^{-1}(U_1)$ is an affine open neighborhood of E_n in S_n . By construction, S_n is then covered by the two open subset U_n and $S_n \setminus E_n$ which intersect along the affine open subset $U_n \cap S_n \setminus E_n = U_n \setminus E_n = \tau_{n,1}^{-1}(\overline{S}_1 \setminus \overline{E}_1)$ of S_n . The conclusion then follows from the Mayer-Vietoris long exact sequence of cohomology of \mathcal{O}_{S_n} for the open covering of S_n by $S_n \setminus E_n$ and U_n . \square

3.2. Classification.

The following theorem shows that every quasi-projective \mathbb{G}_a -extension of Type II of a given \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ is isomorphic to one of the schemes $q : X \rightarrow S_n$ constructed in § 3.1.

Theorem 3.7. *Let $\rho : P \rightarrow S_*$ be a \mathbb{G}_a -torsor and let*

$$\begin{array}{ccc} P & \xleftarrow{j} & X \\ \rho \downarrow & & \downarrow \pi \\ S_* & \xleftarrow{} & S \end{array}$$

be a quasi-projective \mathbb{G}_a -extension of P of Type II. Then there exists an integer $n \geq 1$ and a scheme $\tau_n : S_n(o_1, \dots, o_{n-1}) \rightarrow S$ such that X is a \mathbb{G}_a -torsor $q : X \rightarrow S_n(o_1, \dots, o_{n-1}) \simeq X/\mathbb{G}_a$ and $\rho : P \rightarrow S_*$ coincides with the restriction of q to $S_n(o_1, \dots, o_{n-1}) \setminus E_n \simeq S_*$.

Proof. Since the $\mathbb{G}_{a,S}$ -action on X is proper, the geometric quotient $X/\mathbb{G}_{a,S}$ exists in the form of a separated algebraic S -space $\delta : X/\mathbb{G}_{a,S} \rightarrow S$. Furthermore, since by definition of an extension $\pi^{-1}(S_*) \simeq P$, we have $\pi^{-1}(S_*)/\mathbb{G}_{a,S} \simeq P/\mathbb{G}_{a,S} \simeq S_*$ and so δ restricts to an isomorphism over S_* . On the other hand, $\pi^{-1}(o) \simeq \mathbb{A}_\kappa^2$ is equipped with the induced proper $\mathbb{G}_{a,\kappa}$ -action, whose geometric quotient $\mathbb{A}_\kappa^2/\mathbb{G}_{a,\kappa}$ is isomorphic to \mathbb{A}_κ^1 . It follows from the universal property of geometric quotient that $\delta^{-1}(o) = \mathbb{A}_\kappa^2/\mathbb{G}_{a,\kappa} = \mathbb{A}_\kappa^1$.

Since X is smooth in a neighborhood of $\pi^{-1}(o)$, $X/\mathbb{G}_{a,S}$ is smooth in neighborhood of $\delta^{-1}(o)$. In particular, $\pi^{-1}(o)$ and $\delta^{-1}(o)$ are Cartier divisors on X and $X/\mathbb{G}_{a,S}$ respectively. Let $\overline{\tau}_1 : \overline{S}_1 \rightarrow S$ be the blow-up of o . Then by the universal property of blow-ups [14, Tag 085P], the morphisms $\pi : X \rightarrow S$ and $\delta : X/\mathbb{G}_{a,S} \rightarrow S$ lift to morphisms $\pi_1 : X \rightarrow \overline{S}_1$ and $\delta_1 : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_1$ respectively, and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_1} & \overline{S}_1 \\ \downarrow & \nearrow \delta_1 & \downarrow \overline{\tau}_1 \\ X/\mathbb{G}_a & \xrightarrow{\delta} & S \end{array}$$

Furthermore, since $\delta : X/\mathbb{G}_{a,S} \rightarrow S$ and $\overline{\tau}_1 : \overline{S}_1 \rightarrow S$ are separated, it follows that $\delta_1 : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_1$ is separated. By construction, the image of $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ by δ_1 is contained in \overline{E}_1 .

If δ_1 is not constant on $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ then δ_1 is a separated quasi-finite birational morphism. Since \overline{S}_1 is normal, δ_1 is thus an open immersion by virtue of Zariski Main Theorem for algebraic spaces [14, Tag 05W7]. Since $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}_\kappa^1$, the only possibility is that $\overline{S}_1 \setminus \delta_1(X/\mathbb{G}_{a,S})$ consists of a unique κ -rational point $o_1 \in \overline{E}_1$ and $\delta_1 : X/\mathbb{G}_{a,S} \rightarrow S_1(o_1) = \overline{S}_1 \setminus \{o_1\}$ is an isomorphism. So $\pi_1 : X \rightarrow S_1(o_1)$ is \mathbb{G}_a -torsor whose restriction to $S_1(o_1) \setminus E_1 \simeq S_*$ coincides with $\rho : P \rightarrow S_*$.

Otherwise, if δ_1 is constant on $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$, then its image consists of a unique κ -rational point $o_1 \in \overline{E}_1$. The same argument as above implies that $\pi_1 : X \rightarrow \overline{S}_1$ and $\delta_1 : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_1$ lift to a $\mathbb{G}_{a,S}$ -invariant morphism $\pi_2 : X \rightarrow \overline{S}_2(o_1)$ and a separated morphism $\delta_2 : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_2(o_1)$ to the blow-up $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \rightarrow \overline{S}_1$ of \overline{S}_1 at o_1 , with exceptional divisor \overline{E}_2 . If the restriction of δ_2 to $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ is not constant then δ_2 is an open immersion and the image of $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ is an open subset of \overline{E}_2 isomorphic to \mathbb{A}_κ^1 . The only possibility is that $\delta_2(\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}) = \overline{E}_2 \setminus \overline{E}_1$. Indeed, otherwise $\overline{S}_2 \setminus \delta_2(X/\mathbb{G}_{a,S})$ would consist of the disjoint union of a point in $\overline{E}_2 \setminus (\overline{E}_1 \cap \overline{E}_2)$ and of the curve $\overline{E}_1 \setminus (\overline{E}_1 \cap \overline{E}_2)$ which is not closed in \overline{S}_2 , in contradiction to the fact that δ_2 is an open immersion. Summing up, $\delta_2 : X/\mathbb{G}_{a,S} \rightarrow S_2(o_1) = \overline{S}_2(o_1) \setminus \overline{E}_1$ is an isomorphism mapping $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ isomorphically onto E_2 . So $\pi_2 : X \rightarrow S_2(o_1)$ is \mathbb{G}_a -torsor whose restriction to $S_2(o_1) \setminus E_2 \simeq S_*$ coincides with $\rho : P \rightarrow S_*$.

Otherwise, if δ_2 is constant on $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$, then $\delta_2(\pi^{-1}(o)/\mathbb{G}_{a,\kappa})$ is a κ -rational point $o_2 \in \overline{E}_2$, and there exists a unique minimal sequence of blow-ups $\overline{\tau}_{k+1,k} : \overline{S}_{k+1}(o_1, \dots, o_k) \rightarrow \overline{S}_k(o_1, \dots, o_{k-1})$, $k = 2, \dots, m-1$ of successive κ -rational points $o_k \in \overline{E}_k \subset \overline{S}_k(o_1, \dots, o_{k-1})$, with exceptional divisor $\overline{E}_{k+1} \subset \overline{S}_{k+1}(o_1, \dots, o_k)$ such that $\pi_2 : X \rightarrow \overline{S}_2(o_1)$ and $\delta_2 : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_2(o_1)$ lift respectively to a $\mathbb{G}_{a,S}$ -invariant morphism $\pi_m : X \rightarrow \overline{S}_m(o_1, \dots, o_{m-1})$ and a separated morphism $\delta_m : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_m(o_1, \dots, o_{m-1})$ with the property that the restriction of δ_m to $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ is non constant. By Zariski Main Theorem [14, Tag 05W7] again, we conclude that δ_m is an open immersion, mapping $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}_\kappa^1$ isomorphically onto an open subset of $\overline{E}_m \simeq \mathbb{P}_\kappa^1$. As in the previous case, the image of $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ in \overline{E}_m must be equal to the complement of the intersection of \overline{E}_m with the proper transform of $\overline{E}_1 \cup \dots \cup \overline{E}_{m-1}$ in $\overline{S}_m(o_1, \dots, o_{m-1})$ since otherwise $\overline{S}_m(o_1, \dots, o_{m-1}) \setminus \delta_m(X/\mathbb{G}_{a,S})$ would not be closed in $\overline{S}_m(o_1, \dots, o_{m-1})$. Since $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}_\kappa^1$, it follows that \overline{E}_m intersects the proper transform of $\overline{E}_1 \cup \dots \cup \overline{E}_{m-1}$ in a unique κ -rational point, implying in turn that $o_{m-1} \in \overline{E}_{m-1}$ is a smooth κ -rational point of the reduced total transform $\overline{E}_1 \cup \dots \cup \overline{E}_{m-1}$ of \overline{E}_1 in $\overline{S}_{m-1}(o_1, \dots, o_{m-2})$. Summing up,

$$\delta_m : X/\mathbb{G}_{a,S} \rightarrow \overline{S}_m(o_1, \dots, o_{m-1}) \setminus \overline{E}_1 \cup \dots \cup \overline{E}_{m-1}$$

is an isomorphism with an S -scheme of the form $S_m(o_1, \dots, o_{m-1})$ as constructed in §3.1, mapping $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ isomorphically onto $E_m = S_m(o_1, \dots, o_{m-1}) \cap \overline{E}_m$. It follows in turn that $\pi_m : X \rightarrow S_m(o_1, \dots, o_{m-1})$ is a \mathbb{G}_a -torsor whose restriction to $S_m(o_1, \dots, o_{m-1}) \setminus E_m \simeq S_*$ coincides with $\rho : P \rightarrow S_*$. This completes the proof. \square

3.3. Affine \mathbb{G}_a -extensions of Type II. In this subsection, given a \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ we consider the existence of quasi-projective \mathbb{G}_a -extensions of Type II

$$\begin{array}{ccc} P & \xleftarrow{j} & X \\ \rho \downarrow & & \downarrow \pi \\ S_* & \xleftarrow{\quad} & S \end{array}$$

with the additional for which X is affine over S . As in the case of extension to \mathbb{A}^1 -bundles over the blow-up of o treated in § 2.2, a necessary condition for the existence of such extensions is that the restriction of P over every open neighborhood of the closed point o in S is nontrivial. Indeed, if there exists an affine open neighborhood U of o over which P is trivial, then $P \simeq U \setminus \{o\} \times \mathbb{A}_\kappa^1$ is strictly quasi-affine, hence cannot be the complement of a Cartier divisor $\pi^{-1}(o)$ is any affine U -scheme $X|_U$. The next theorem shows that this condition is actually sufficient:

Theorem 3.8. *Let $\rho : P \rightarrow S_*$ be a \mathbb{G}_a -torsor such that for every open neighborhood U of o in S , the restriction $P \times_{S_*} U \rightarrow U \setminus \{o\}$ is non trivial. Then for every $n \geq 1$ and every S -scheme $\tau_n : S_n(o_1, \dots, o_{n-1}) \rightarrow S$ as in Notation 3.3 there exists a quasi-projective \mathbb{G}_a -extension of P of Type II into the total space of a \mathbb{G}_a -torsor $q : X \rightarrow S_n(o_1, \dots, o_{n-1})$ for which $\pi = \tau_n \circ q : X \rightarrow S$ is an affine morphism.*

The following example illustrates the strategy of the proof given below, which consists in constructing such affine extensions $\pi : X \rightarrow S$ by performing a well-chosen equivariant affine modification of extensions of $\rho : P \rightarrow S_*$ into locally trivial \mathbb{A}^1 -bundles $\theta : W(P) \rightarrow \tilde{S}$ over the blow-up $\tau : \tilde{S} \rightarrow S$ of the point o .

Example 3.9. Let again X_0 and X_1 be the \mathbb{G}_a -extensions of $\rho : P = \{xv - yu = 1\} \rightarrow S \setminus \{o\}$ considered in Example 2.1 and 3.2. Recall that X_0 and X_1 are the affine S -schemes in \mathbb{A}_S^3 defined respectively by the equations

$$X_0 : \begin{cases} xr - yq & = 0 \\ yp - x(q-1) & = 0 \\ pr - q(q-1) & = 0 \end{cases} \quad \text{and} \quad X_1 : \begin{cases} xw - y(yz_1 + 1) & = 0 \\ xz_2 - z_1(yz_1 + 1) & = 0 \\ z_1w - yz_2 & = 0 \end{cases}$$

equipped with the $\mathbb{G}_{a,S}$ -actions associated with the locally nilpotent \mathcal{O}_S -derivations $\partial_0 = x^2\partial_p + xy\partial_q + y^2\partial_r$ and $\partial_1 = x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w$ respectively.

The morphism $\pi_0 : X_0 \rightarrow S$ factors through the structure morphism $\theta : X_0 \rightarrow \tilde{S}$ of a torsor under a line bundle on the blow-up $\tau : \tilde{S} \rightarrow S$ of the origin, with the property that the restriction of X_0 to exceptional divisor $E = \mathbb{P}_\kappa^1$ of τ is a nontrivial torsor under the total space of the line bundle $\mathcal{O}_{\mathbb{P}_\kappa^1}(-2)$.

The $\mathbb{G}_{a,S}$ -action on X_0 restricts to the trivial one on $X_0|_E = \pi_0^{-1}(o)$. More precisely, ∂_0 is a global section of the sheaf $\mathcal{T}_{X_0} \otimes \mathcal{O}_{X_0}(-2X_0|_E)$ of vector fields on X_0 that vanish at order 2 along $X_0|_E$. One way to obtain from X_0 a \mathbb{G}_a -extension $\pi : X \rightarrow S$ of $\rho : P \rightarrow S \setminus \{o\}$ with fiber $\pi^{-1}(o)_{\text{red}}$ isomorphic to \mathbb{A}_κ^2 and a fixed point free action is thus to perform an equivariant affine modification which simultaneously replaces $X_0|_E$ by a copy of \mathbb{A}_κ^2 and decreases the “fixed point order of ∂_0 along $X_0|_E$ ”, typically a modification with divisor D equal to $X_0|_E$ and whose center $Z \subset X_0|_E$ is supported by a curve isomorphic to \mathbb{A}_κ^1 which is mapped isomorphically onto its image by the restriction of θ . The birational S -morphism

$$\eta : X_1 \rightarrow X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, y, xz_1, yz_1 + 1, w)$$

is equivariant for the $\mathbb{G}_{a,S}$ -actions on X_0 and X_1 and corresponds to an equivariant affine modification of this type: it restricts to an isomorphism outside the fibers of π_0 and π_1 over o , and it contracts $\pi_1^{-1}(o) = \text{Spec}(\kappa[z_2, w])$ onto the curve $\{p = q - 1 = 0\} \subset \pi_0^{-1}(o) = \{pr - q(q - 1) = 0\}$. This curve is isomorphic to $\mathbb{A}_\kappa^1 = \text{Spec}(\kappa[r])$ and it is mapped by the restriction

$$\theta|_{\pi_0^{-1}(o)} : \pi_0^{-1}(o) \simeq \{pr - q(q - 1) = 0\} \rightarrow E = \mathbb{P}_\kappa^1, \quad (p, q, r) \mapsto [p : q - 1] = [q : r]$$

of θ isomorphically onto the complement of the κ -rational point $[0 : 1] \in \mathbb{P}_\kappa^1$.

Proof of Theorem 3.8. By virtue of Theorem 2.5, there exists a unique integer $\ell_0 \geq 2$ such that $\rho : P \rightarrow S_*$ is the restriction of a torsor $\theta_1 : W_1 \rightarrow \overline{S}_1$ under the line bundle $M_1(\ell_0) = \text{Spec}(\text{Sym } \mathcal{O}_{\overline{S}_1}(-\ell_0 \overline{E}_1)) \rightarrow \overline{S}_1$ whose total space W_1 is affine over \overline{S}_1 . We now treat the case of $S_1(o_1)$ and $S_n(o_1, \dots, o_{n-1})$, $n \geq 2$ separately.

Given a κ -rational point $o_1 \in \overline{E}_1$, the restriction of W_1 over $E_1 = \overline{E}_1 \setminus \{o_1\} \simeq \mathbb{A}_\kappa^1$ is the trivial \mathbb{A}_κ^1 -bundle $E_1 \times \mathbb{A}_\kappa^1$. Since on the other hand the restriction $\theta_1|_{\overline{E}_1} : W_1|_{\overline{E}_1} \rightarrow \overline{E}_1$ is a non trivial $\mathcal{O}_{\mathbb{P}^1}(-\ell_0)$ -torsor (see Theorem 2.5), it follows that for every section $s : E_1 \rightarrow W_1|_{E_1}$ the image Z_1 of E_1 in $W_1|_{\overline{E}_1}$ is a closed curve isomorphic to E_1 . Indeed, otherwise if Z_1 is not closed in $W_1|_{\overline{E}_1}$ then its closure \overline{Z}_1 would be a section of $\theta_1|_{\overline{E}_1}$ in contradiction with the fact that $\theta_1|_{\overline{E}_1} : W_1|_{\overline{E}_1} \rightarrow \overline{E}_1$ is a non trivial $\mathcal{O}_{\mathbb{P}^1}(-\ell_0)$ -torsor. Let $D_1 = \theta_1^{-1}(\overline{E}_1)$ and let $\sigma_1 : W'_1 \rightarrow W_1$ be the affine modification of W_1 with center (\mathcal{I}_{Z_1}, D_1) . By virtue of Lemmas 1.9 and 1.10, $\theta_1 \circ \sigma_1 : W'_1 \rightarrow \overline{S}_1$ factors through a torsor $\theta'_1 : W'_1 \rightarrow \overline{S}_1 \setminus \{o_1\} = S_1(o_1)$ under the line bundle

$$M'_1(\ell_0 - 1) = \text{Spec}(\text{Sym } \mathcal{O}_{S_1(o_1)}((-\ell_0 + 1)E_1)) \rightarrow S_1(o_1).$$

Now since $E_1 \simeq \mathbb{A}_\kappa^1$ is affine, the restriction of θ'_1 over $E_1 \subset S_1(o_1)$ is the trivial $M'_1(\ell_0 - 1)|_{E_1}$ -torsor. Letting $D_2 = \theta'_1{}^{-1}(E_1)$ and $Z_2 \subset D_2$ be any section of $\theta'_1|_{D_2} : D_2 \rightarrow E_1$, the affine modification $\sigma_2 : W'_2 \rightarrow W'_1$ with center (\mathcal{I}_{Z_2}, D_2) is then an $M'_1(\ell_0 - 2)$ -torsor $\theta'_2 : W'_2 \rightarrow S_1(o_1)$. Iterating this construction $\ell_0 - 1$ times, we reach a $\mathbb{G}_{a,S_1(o_1)}$ -torsor $q = \theta'_{\ell_0+1} : X = W'_{\ell_0+1} \rightarrow S_1(o_1)$. Since $\sigma_1 : W'_1 \rightarrow W_1$ and each $\sigma_i : W'_i \rightarrow W'_{i-1}$, $i \geq 2$, restricts to an isomorphism over the complement of E_1 , the restriction of $q : X \rightarrow S_1(o_1)$ over $S_1(o_1) \setminus E_1 \simeq S_*$ is isomorphic to $\rho : P \rightarrow S_*$. Furthermore, since the morphisms σ_i , $i = 1, \dots, \ell_0 + 1$ are affine and $\overline{\tau}_1 \circ \theta_1 : W_1 \rightarrow S$ is an affine morphism, it follows that

$$\tau_1 \circ q = \overline{\tau}_1 \circ \theta_1 \circ \sigma_1 \circ \dots \circ \sigma_{\ell_0+1} : X \rightarrow S$$

is an affine morphism. So $q : X \rightarrow S_1(o_1)$ is a \mathbb{G}_a -extension of $\rho : P \rightarrow S_*$ with the desired property.

Now suppose that $n \geq 2$. It follows from the construction of the morphism $\tau_{n,1} : S_n = S_n(o_1, \dots, o_{n-1}) \rightarrow \overline{S}_1$ given in subsection 3.1 that $\tau_{n,1}^* \mathcal{O}_{\overline{S}_1}(\ell_0 \overline{E}_1) \simeq \mathcal{O}_{S_n}(mE_n)$ for some $m \geq 2$. The fiber product $W_n = W_1 \times_{\overline{S}_1} S_n$ is thus a torsor $\theta_n : W_n \rightarrow S_n$ under the line bundle

$$M_n(m) = \text{Spec}(\text{Sym } \mathcal{O}_{S_n}(-mE_n)) \rightarrow S_n$$

whose restriction to $S_n \setminus E_n \simeq S_*$ is isomorphic to $\rho : P \rightarrow S_*$. Furthermore, since $\tau_{n,1}$ is an affine morphism by virtue of Lemma 3.4, so is the projection $\text{pr}_{W_1} : W_n \rightarrow W_1$. Since $\overline{\tau}_1 \circ \theta_1 : W_1 \rightarrow S$ is an affine morphism, we conclude that $\tau_n \circ \theta_n = \overline{\tau}_1 \circ \tau_{n,1} \circ \theta_n = \overline{\tau}_1 \circ \theta \circ \text{pr}_{W_1} : W_n \rightarrow S$ is an affine morphism as well. Since $E_n \simeq \mathbb{A}_\kappa^1$, the restriction of θ_n over E_n is the trivial $M_n(m)|_{E_n}$ -torsor. The desired \mathbb{G}_{a,S_n} -torsor $q : X \rightarrow S_n$ extending $\rho : P \rightarrow S_*$ is then obtained from $\theta_n : W_n \rightarrow S_n$ by performing a sequence of m successive affine modifications similar to those applied in the previous case. \square

Remark 3.10. In the case where S is affine, the total spaces X of the varieties $q : X \rightarrow S_n(o_1, \dots, o_{n-1})$ of Theorem 3.8 are all affine. To our knowledge, these are the first instances of smooth affine threefolds

equipped with proper \mathbb{G}_a -actions whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine.

We do not know in general if under the conditions of Theorem 3.8 every quasi-projective \mathbb{G}_a -extensions of P of Type II into the total space of a \mathbb{G}_a -torsor $q : X \rightarrow S_n(o_1, \dots, o_{n-1})$ has the property that $\pi = \tau_n \circ q : X \rightarrow S$ is an affine morphism. In particular, we ask the following:

Question 3.11. *Is the total space X of a quasi-projective \mathbb{G}_a -extension $\pi : X \rightarrow \mathbb{A}^2$ of $\rho = \text{pr}_{x,y} : \text{SL}_2 = \{xv - yu = 1\} \rightarrow \mathbb{A}_*^2$ of Type II always an affine variety ?*

3.4. Examples. In the next paragraphs, we construct two countable families of quasi-projective \mathbb{G}_a -extensions of the \mathbb{G}_a -torsor $\text{SL}_2 \rightarrow \text{SL}_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\}$ of Type II with affine total spaces. As a consequence of [10, Section 3], for any nontrivial \mathbb{G}_a -torsor $\rho : P \rightarrow S_*$ over a local punctured surface S_* , these provide, by suitable base changes, families of examples of \mathbb{G}_a -extensions of P whose total spaces are all affine over S .

3.4.1. A family of \mathbb{G}_a -extensions of SL_2 of “Type II-A”. Let $S = \mathbb{A}^2 = \text{Spec}(k[x, y_0])$ and let $X_n \subset \mathbb{A}_S^{n+2} = \text{Spec}(k[x, y_0][z_1, z_2, y_1, \dots, y_n])$, $n \geq 1$, be the smooth threefold defined by the system of equations

$$\begin{cases} y_i y_j - y_k y_\ell & = 0 & i, j, k, \ell = 0, \dots, n, \quad i + j = k + \ell \\ z_2 y_i - z_1 y_{i+1} & = 0 & i = 0, \dots, n-1 \\ x y_{i+1} - y_i (y_0 z_1 + 1) & = 0 & i = 0, \dots, n-1 \\ x z_2 - z_1 (y_0 z_1 + 1) & = 0. \end{cases}$$

The threefold X_n can be endowed with a fixed point free $\mathbb{G}_{a,S}$ -action induced by the locally nilpotent $k[x, y_0]$ -derivation

$$x \partial_{z_1} + (2y_0 z_1 + 1) \partial_{z_2} + \sum_{i=1}^n i y_0 y_{i-1} \partial_{y_i}$$

of its coordinate ring. The scheme-theoretic fiber over $o = \{(0,0)\}$ of the \mathbb{G}_a -invariant morphism $\pi_n = \text{pr}_{x,y} : X_n \rightarrow S$ is isomorphic $\mathbb{A}^2 = \text{Spec}(k[z_2, y_n])$, on which the induced \mathbb{G}_a -action is a translation induced by the derivation ∂_{z_2} of $k[z_2, y_n]$. On the other hand, the morphism $j : \text{SL}_2 = \{xv - y_0 u = 1\} \rightarrow X_n$ defined by

$$(x, y, u, v) \mapsto (x, u, uv, y, yv, yv^2, \dots, yv^n)$$

is an equivariant open embedding of SL_2 equipped with the \mathbb{G}_a -action induced by the locally nilpotent derivation $x \partial_u + y_0 \partial_v$ of its coordinate ring into X_n with image equal to $\pi^{-1}(\mathbb{A}^2 \setminus \{o\})$. So $j : \text{SL}_2 \hookrightarrow X_n$ is a quasi-projective \mathbb{G}_a -extension of SL_2 into the affine variety X_n , with $\pi_n^{-1}(o) \simeq \mathbb{A}_k^2$.

The restrictions of the projection $\mathbb{A}_S^{n+3} \rightarrow \mathbb{A}_S^{n+2}$ onto the first $n+2$ variables induce a sequence of \mathbb{G}_a -equivariant birational morphisms $\sigma_{n+1,n} : X_{n+1} \rightarrow X_n$. The threefolds X_n thus form a countable tower of \mathbb{G}_a -equivariant affine modifications of X_1 . It follows from Example 3.2 that X_1 is a quasi-projective extension of SL_2 of Type II with geometric quotient isomorphic to a quasi-projective surface of the form $S_1(o_1)$. More generally, we have the following result.

Proposition 3.12. *For every $n \geq 2$, the morphism $j : \text{SL}_2 \hookrightarrow X_n$ is a quasi-projective \mathbb{G}_a -extension of Type II. The geometric quotient X_n/\mathbb{G}_a is isomorphic to a quasi-projective surface $S_n = S_n(o_1, \dots, o_n)$ as in § 3.1 for which $\overline{S}_n(o_1, \dots, o_{n-1}) \setminus S_n$ consists of a chain of $n-1$ smooth rational curves with self-intersection -2 , i.e. the exceptional set of the minimal resolution of a surface singularity of type A_{n-1} .*

Proof. To see this, we consider the following sequence of blow-ups: the first one $\overline{\tau}_1 : \overline{S}_1 \rightarrow U_0 = \mathbb{A}^2$ is the blow-up of the origin, with exceptional divisor \overline{E}_1 , and we let $U_1 \simeq \mathbb{A}^2 = \text{Spec}(k[x, w_1])$ be the affine chart of \overline{S}_1 on which $\overline{\tau}_1 : \overline{S}_1 \rightarrow \mathbb{A}^2$ is given by $(x, w_1) \mapsto (x, xw_1)$. Then we let $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \rightarrow \overline{S}_1$ be the blow-up of the point $o_1 = (0,0) \in U_1 \subset \overline{S}_1$ with exceptional divisor \overline{E}_2 , and we let $U_2 \simeq \mathbb{A}^2 = \text{Spec}(k[x, w_2])$ be the affine chart of $\overline{S}_2(o_1)$ on which the restriction of $\overline{\tau}_{2,1} : \overline{S}_2(o_1) \rightarrow \overline{S}_1$ coincides with the morphism $U_2 \rightarrow U_1$, $(x, w_2) \mapsto (x, xw_2)$. For every $2 < m \leq n$, we define by induction the blow-up

$$\overline{\tau}_{m,m-1} : \overline{S}_m(o_1, \dots, o_{m-1}) \rightarrow \overline{S}_{m-1}(o_1, \dots, o_{m-2})$$

of the point $o_{m-1} = (0, 0) \in U_{m-1} \subset \overline{S}_{m-1}(o_1, \dots, o_{m-2})$ with exceptional divisor \overline{E}_m and we let $U_m \simeq \mathbb{A}^2 = \text{Spec}(k[x, w_m])$ be the affine chart of $\overline{S}_m(o_1, \dots, o_{m-1})$ on which the restriction of $\overline{\tau}_{m,m-1}$ coincides with the morphism $U_m \rightarrow U_{m-1}, (x, w_m) \mapsto (x, xw_m)$. By construction, we have a commutative diagram

$$\begin{array}{ccccccc} \overline{S}_n(o_1, \dots, o_{n-1}) & \xrightarrow{\overline{\tau}_{n,n-1}} & \overline{S}_{n-1}(o_1, \dots, o_{n-2}) & \xrightarrow{\overline{\tau}_{n-1,n-2}} & \dots & \xrightarrow{\overline{\tau}_{2,1}} & \overline{S}_1 & \xrightarrow{\overline{\tau}_1} & \mathbb{A}^2 \\ \uparrow & & \uparrow & & & & \uparrow & & \parallel \\ U_n & \longrightarrow & U_{n-1} & \longrightarrow & \dots & \longrightarrow & U_1 & \longrightarrow & \mathbb{A}^2 = U_0. \end{array}$$

The total transform of \overline{E}_1 in $\overline{S}_n(o_1, \dots, o_{n-1})$ is a chain $\overline{E}_1 \cup \overline{E}_2 \cup \dots \cup \overline{E}_{n-1} \cup \overline{E}_n$ is a chain formed of $n-1$ curves with self-intersection -2 and the curve \overline{E}_n which has self-intersection -1 .



FIGURE 3.2. Dual graph of the total transform of \overline{E}_1 in $\overline{S}_n(o_1, \dots, o_n)$.

The morphism $\pi : X_n \rightarrow S$ lifts to a morphism $\pi_1 : X_n \rightarrow \overline{S}_1$ defined by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto ((x, y_0), [x : y_0]) = ((x, y), [y_0 z_1 + 1 : y_1]).$$

This morphism contracts $\pi^{-1}(o)$ onto the point $o_1 = ((0, 0), [1 : 0])$ of the exceptional divisor \overline{E}_1 of $\overline{\tau}_1$. The induced rational map $\pi_1 : X_n \dashrightarrow U_1$ is given by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto \left(x, \frac{y_1}{y_0 z_1 + 1}\right)$$

and it contracts $\pi^{-1}(o)$ onto the origin $o_1 = (0, 0)$. So π_1 lifts to a morphism $\pi_2 : X_n \rightarrow \overline{S}_2(o_1)$, and with our choice of charts, the induced rational map $\pi_2 : X_n \dashrightarrow U_2$ is given by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto \left(x, \frac{y_2}{(y_0 z_1 + 1)^2}\right)$$

If $n = 2$ then the image of $\pi^{-1}(o) = \text{Spec}(k[z_2, y_2])$ by π_2 is equal to $\overline{E}_2 \cap U_2$ and $\pi_2^{-1}(\overline{E}_2 \cap U_2)$ is equivariantly isomorphic to $(\overline{E}_2 \cap U_2) \times \text{Spec}(k[z_2])$ on which \mathbb{G}_a acts by translations on the second factor. So $\pi_2 : X_n \rightarrow \overline{S}_2(o_1)$ factors through a \mathbb{G}_a -bundle $q_2 : X_2 \rightarrow S_2(o_1) = \overline{S}_2(o_1) \setminus \overline{E}_1$ and $X_2/\mathbb{G}_a \simeq S_2(o_1)$. Otherwise, if $n > 2$ then π_2 contracts $\pi^{-1}(o)$ onto the point $o_2 = (0, 0) \in \overline{E}_2 \cap U_2 \subset \overline{S}_2(o_1)$. So $\pi_2 : X_n \rightarrow \overline{S}_2(o_1)$ lifts to a morphism $\pi_3 : X_n \rightarrow \overline{S}_3(o_1, o_2)$. With our choice of charts, for each $2 < m < n$, the induced rational map $\pi_m : X_n \dashrightarrow U_m$ is given by

$$(x, z_1, z_2, y_0, y_1, \dots, y_n) \mapsto \left(x, \frac{y_m}{(y_0 z_1 + 1)^m}\right)$$

hence contracts $\pi^{-1}(o)$ onto the point $o_m = (0, 0) \in U_m \subset \overline{S}_m(o_1, \dots, o_{m-1})$. It thus lifts to a morphism $\pi_m : X_n \rightarrow \overline{S}_m(o_1, \dots, o_{m-1})$. At the last step, the image of $\pi^{-1}(o) = \text{Spec}(k[z_2, y_n])$ by the rational map $\pi_n : X_n \dashrightarrow U_n$ induced by $\pi_n : X_n \rightarrow \overline{S}_n(o_1, \dots, o_{n-1})$ is equal to $\overline{E}_n \cap U_n$, and we conclude as above that $\pi_n : X_n \rightarrow \overline{S}_n(o_1, \dots, o_{n-1})$ factors through a \mathbb{G}_a -bundle

$$q_n : X_n \rightarrow S_n(o_1, \dots, o_{n-1}) = \overline{S}_n(o_1, \dots, o_{n-1}) \setminus (\overline{E}_1 \cup \dots \cup \overline{E}_{n-1}),$$

hence that X_n/\mathbb{G}_a is isomorphic to the quasi-projective surface $S_n(o_1, \dots, o_{n-1})$. \square

3.4.2. *A family of \mathbb{G}_a -extensions of SL_2 of "Type II-D".* To conclude this section, we present as an illustration of the proof of Theorem 3.8 another countable family of quasi-projective \mathbb{G}_a -extensions of SL_2 of Type II with affine total spaces.

Let again $\overline{\tau}_1 : \overline{S}_1 \rightarrow S = \mathbb{A}^2$ be the blow-up of the origin $o = \{(0, 0)\}$ in $\mathbb{A}^2 = \text{Spec}(k[x, y])$ with exceptional divisor $\overline{E}_1 \simeq \mathbb{P}^1$, identified with closed subvariety of $\mathbb{A}^2 \times \mathbb{P}^1_{[w_0:w_1]}$ with equation $xw_1 - yw_0 = 0$ in such a way that τ coincides with the restriction of the first projection. The second projection identifies \overline{S}_1 with the total space $p : \overline{S}_1 \rightarrow \mathbb{P}^1$ of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. We fix trivializations $p^{-1}(U_\infty) = \text{Spec}(k[z_\infty][u_\infty])$ and $p^{-1}(U_0) = \text{Spec}(k[z_0][u_0])$ over the open subsets $U_\infty = \mathbb{P}^1 \setminus \{[0 : 1]\} = \text{Spec}(k[z_\infty])$

and $U_0 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \text{Spec}(k[z_0])$ in such a way that the gluing of $p^{-1}(U_\infty)$ and $p^{-1}(U_0)$ over $U_0 \cap U_\infty$ is given by the isomorphism $(z_0, u_0) \mapsto (z_\infty, u_\infty) = (z_0^{-1}, z_0 u_0)$.

For every $n \geq 1$, we let $S_{2n+3,0} = \text{Spec}(k[z_0, u_0^{\pm 1}])$,

$$S_{2n+3,\infty} = \text{Spec}(k[z_\infty, u_\infty, v_\infty]/(u_\infty^n v_\infty - z_\infty^2 - u_\infty)),$$

and we let S_{2n+3} be the surface obtained by gluing $S_{2n+3,0}$ and $S_{2n+3,\infty}$ along the open subsets $S_{2n+3,0} \setminus \{z_0 = 0\}$ and $S_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the isomorphism

$$(z_0, u_0) \mapsto (z_\infty, u_\infty, v_\infty) = (z_0^{-1}, z_0 u_0, (z_0 u_0)^{-n}(z_0^{-2} + z_0 u_0)).$$

The canonical open immersion $S_{2n+3,0} \hookrightarrow p^{-1}(U_0)$ and the projection $\text{pr}_{z_\infty, u_\infty} : S_{2n+3,\infty} \rightarrow p^{-1}(U_\infty)$ glue to a global birational affine morphism $\tau_{2n+3,1} : S_{2n+3} \rightarrow \overline{S}_1$ restricting to an isomorphism $S_{2n+3} \setminus \{z_\infty = u_\infty = 0\} \rightarrow \overline{S}_1 \setminus \overline{E}_1$ where we identified the closed subset $E_{2n+3} = \{z_\infty = u_\infty = 0\} \simeq \text{Spec}(k[v_\infty])$ of $S_{2n+3,\infty}$ with its image in S_{2n+3} . We leave to the reader to check that with the notation of § 3.1, $S_{2n+3} = S_{2n+3}(o_1, \dots, o_{2n+2})$ for a surface $\overline{\tau}_{2n+3,1} : \overline{S}_{2n+3,1}(o_1, \dots, o_{2n+2}) \rightarrow \overline{S}_1$ obtained by first blowing-up the point $o_1 = (0, 0) \in p^{-1}(U_\infty)$ with exceptional divisor \overline{E}_2 , then the point $o_2 = \overline{E}_1 \cap \overline{E}_2$ with exceptional divisor \overline{E}_3 , then a point $o_3 \in \overline{E}_3 \setminus (\overline{E}_1 \cup \overline{E}_2)$ with exceptional divisor \overline{E}_4 and then a sequence of point $o_i \in \overline{E}_i \setminus \overline{E}_{i-1}$ with exceptional divisor \overline{E}_{i+1} , $i = 5, \dots, 2n+2$ in such a way that the total transform of \overline{E}_1 in $\overline{S}_{2n+3,1}$ is a tree depicted in Figure 3.3. Letting $\tau_{2n+3} = \overline{\tau}_1 \circ \tau_{2n+3,1} : S_{2n+3} \rightarrow \mathbb{A}^2$, we have $\tau_{2n+3}^{-1}(o)_{\text{red}} = E_{2n+3} \simeq \mathbb{A}^1$ and $\tau_{2n+3}^*(o) = 2E_{2n+3}$.

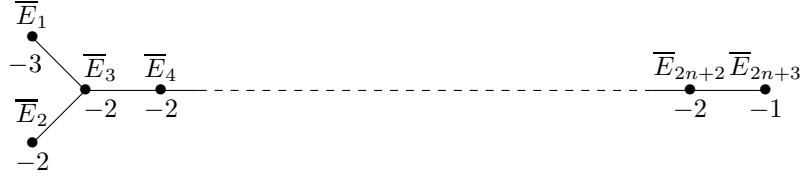


FIGURE 3.3. Dual graph of the total transform of \overline{E}_1 in $\overline{S}_{2n+3}(o_1, \dots, o_{2n+2})$.

Now we let $q : X_{2n+3} \rightarrow S_{2n+3}$ be the \mathbb{G}_a -bundle defined as the gluing of the trivial \mathbb{G}_a -bundles $X_{2n+3,0} = S_{2n+3,0} \times \text{Spec}(k[t_0])$ and $X_{2n+3,\infty} = S_{2n+3,\infty} \times \text{Spec}(k[t_\infty])$ over $S_{2n+3,0}$ and $S_{2n+3,\infty}$ respectively along the open subsets $X_{2n+3,0} \setminus \{z_0 = 0\}$ and $X_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the \mathbb{G}_a -equivariant isomorphism

$$(z_0, u_0, t_0) \mapsto (z_\infty, u_\infty, v_\infty, t_\infty) = (z_0^{-1}, z_0 u_0, (z_0 u_0)^{-n}(z_0^{-2} + z_0 u_0), t_0 + z_0^{-1} u_0^{-2}).$$

Let $\pi_{2n+3} = \overline{\tau}_1 \circ \tau_{2n+3,1} \circ q : X_{2n+3} \rightarrow \mathbb{A}^2$.

Proposition 3.13. *For every $n \geq 1$, the variety X_{2n+3} is affine and there exists a \mathbb{G}_a -equivariant open embedding $j : \text{SL}_2 \hookrightarrow X_{2n+3}$ which makes $\pi_{2n+3} : X_{2n+3} \rightarrow \mathbb{A}^2$ a quasi-projective \mathbb{G}_a -extension of SL_2 of Type II, with fiber $\pi_{2n+3}^{-1}(o)$ isomorphic to \mathbb{A}^2 of multiplicity two, and geometric quotient $X_{2n+3}/\mathbb{G}_a \simeq S_{2n+3}$.*

Proof. Let $j_1 : \text{SL}_2 \hookrightarrow W = W(\text{SL}_2, 2)$ be the \mathbb{G}_a -extension of SL_2 into a locally trivial \mathbb{A}^1 -bundle $\theta : W \rightarrow \overline{S}_1$ with affine total space constructed in Example 2.1. Recall that the image of j_1 coincides with the restriction of θ to $\overline{S}_1 \setminus \overline{E}_1 = \mathbb{A}^2 \setminus \{o\}$. With our choice of coordinates, the open subsets $W_0 = \theta^{-1}(q^{-1}(U_0))$ and $W_\infty = \theta^{-1}(q^{-1}(U_\infty))$ of W are respectively isomorphic to $p^{-1}(U_0) \times \text{Spec}(k[w_0])$ and $p^{-1}(U_\infty) \times \text{Spec}(k[w_\infty])$ glued over $U_0 \cap U_\infty$ by the isomorphism

$$(z_0, u_0, w_0) \mapsto (z_\infty, u_\infty, w_\infty) = (z_0^{-1}, z_0 u_0, z_0^2 w_0 + z_0).$$

The \mathbb{G}_a -action on W_0 and W_∞ are given respectively by $\alpha \cdot (z_0, u_0, w_0) = (z_0, u_0, w_0 + \alpha u_0^2)$ and $\alpha \cdot (z_\infty, u_\infty, w_\infty) = (z_\infty, u_\infty, w_\infty + \alpha u_\infty^2)$.

Let $W' = W \times_{\overline{S}_1} S_{2n+3}$, equipped with the natural lift of the \mathbb{G}_a -action on W . Since $\tau_{2n+3,1} : S_{2n+3} \rightarrow \overline{S}_1$ restricts to an isomorphism over $\overline{S}_1 \setminus \overline{E}_1$, the composition $j' = \tau_{2n+3,1}^{-1} \circ j_1 : \text{SL}_2 \rightarrow W'$ is a \mathbb{G}_a -equivariant open embedding. Furthermore, since W is affine and $\tau_{2n+3,1}$ is an affine morphism, it follows

that W' is affine. By construction, W' is covered by the two open subsets

$$\begin{cases} W'_0 = W \times_{p^{-1}(U_0)} S_{2n+3,0} & \simeq S_{2n+3,0} \times \text{Spec}(k[w_0]) \\ W'_\infty = W \times_{p^{-1}(U_\infty)} S_{2n+3,\infty} & \simeq S_{2n+3,\infty} \times \text{Spec}(k[w_\infty]). \end{cases}$$

The local \mathbb{G}_a -equivariant morphisms

$$\begin{cases} \beta_0 : X_{2n+3,0} = S_{2n+3,0} \times \text{Spec}(k[t_0]) \rightarrow W'_0 \\ \beta_\infty : X_{2n+3,\infty} = S_{2n+3,\infty} \times \text{Spec}(k[t_\infty]) \rightarrow W'_\infty \end{cases}$$

of schemes over $S_{2n+1,0}$ and $S_{2n+3,\infty}$ respectively defined by $t_0 \mapsto w_0 = u_0^2 t_0$ and $t_\infty \mapsto w_\infty = u_\infty^2 t_\infty$ glue to a global \mathbb{G}_a -equivariant birational affine morphism $\beta : X_{2n+3} \rightarrow W'$, restricting to an isomorphism over $S_{2n+3} \setminus E_{2n+3} \simeq \mathbb{A}^2 \setminus \{o\}$. Summing up, X_{2n+3} is affine over W' hence affine, and the composition $\beta^{-1} \circ j' : \text{SL}_2 \hookrightarrow X_{2n+3}$ is a \mathbb{G}_a -equivariant open embedding which realizes $\pi : X_{2n+3} \rightarrow \mathbb{A}^2$ as a \mathbb{G}_a -extension of SL_2 of Type II with affine total space. By construction, $\pi_{2n+3}^{-1}(o) = q^{-1}(2E_{2n+3})$ is isomorphic to \mathbb{A}^2 , with multiplicity two, while the geometric quotient X_{2n+3}/\mathbb{G}_a is isomorphic to S_{2n+3} . \square

Remark 3.14. For every $n \geq 1$, the birational morphism $S_{2(n+1)+3,\infty} \rightarrow S_{2n+3,\infty}$, $(z_\infty, u_\infty, v_\infty) \mapsto (z_\infty, u_\infty, u_\infty v_\infty)$ extends to a birational morphism $S_{2(n+1)+3} \rightarrow S_{2n+3}$ which lifts in turn in a unique way to a \mathbb{G}_a -equivariant birational morphism $\gamma_{n+1,n} : X_{2(n+1)+3} \rightarrow X_{2n+3}$. So in a similar way as for the family constructed in § 3.4.1, the family of threefolds X_{2n+3} , $n \geq 1$, form a tower of \mathbb{G}_a -equivariant affine modifications of the initial one X_5 .

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