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► **To cite this version:**

Mirko Fiacchini, Gilles Millérioux. Dead-beat stabilizability of autonomous switched linear discrete-time systems. 20th IFAC World Congress, IFAC WC 2017, Jul 2017, Toulouse, France. 10.1016/j.ifacol.2017.08.734 . hal-01569247

HAL Id: hal-01569247

<https://hal.science/hal-01569247>

Submitted on 19 Nov 2020

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Dead-beat stabilizability of autonomous switched linear discrete-time systems ^{*}

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Abstract: The paper deals with dead-beat stabilizability of autonomous switched linear discrete-time systems. More precisely, it is investigated with the problem of finding a condition on the sequences of switches which guarantee that the state of the system reaches the origin in finite steps. A literature overview highlights the fact that such a problem is an open problem in the general case. First, we give necessary and sufficient conditions under which such an autonomous switched linear discrete-time system is dead-beat stabilizable. Then, an algorithm is proposed for deciding when such a sequence exists and for providing the sequence. It is shown that the complexity of the test can be much lower than the exhaustive search.

Keywords: Switched linear systems, dead-beat stabilizability

1. INTRODUCTION

Switching systems are dynamical systems for which the state dynamics vary between different operating modes according to a switching sequence Liberzon (2003). Such systems are found in many practical and theoretical domains. For example they appear in the study of networked control systems Alur et al. (2011); Jungers et al. (2012), in congestion control for computer networks Shorten et al. (2006), in viral mitigation Hernandez-Vargas et al. (2011), as abstractions of more complex hybrid systems Liberzon and Morse (1999), and other fields (see e.g. Jungers (2009); Lin and Antsaklis (2009); Shorten et al. (2007) and references therein).

The problem considered here is to find a condition on the sequences under which a switched system reaches the origin in finite steps. As somehow related problems, the following literature deserves a special attention.

In the survey Sun and Ge (2005), conditions for controllability and observability of switched linear systems based on the concept of A -invariant are provided. Special results concerning the discrete-time case given by Conner and Stanford (1984, 1987) are recalled. These papers provide bounds on the minimal length of a controlling sequence for completely controllable systems. In particular, a bound on the length of the switching sequence guaranteeing that a state is reached from the origin is given. However, the results are given for switched systems with nonsingular transition matrices (defined as reversible in Sun and Ge (2005)) and with control inputs.

The other, and more recent, survey on the topic is Lin and Antsaklis (2009), where the autonomous cases in the

continuous and discrete contexts are examined. There, after dealing with the conditions for stabilizability of continuous-time autonomous switched linear systems, it is claimed that the extension of the synthesis method to discrete-time counterpart is not obvious. A sufficient condition, provided by the authors themselves, is presented. Another criterion, probably less conservative, is given in Geromel and Colaneri (2006) but is also sufficient only. Necessary and sufficient conditions are provided in Fiacchini and Jungers (2014) but they are computationally complex. The difficulty in obtaining some relevant results stems from the fact that the question of deciding the stabilizability of a switching system is known to be hard in general (see Jungers (2009), Section 2.2, for hardness results). Finite-time stabilizability, that is the problem of checking whether there exists a finite sequence of switches that gives the null matrix, is even more challenging.

Two papers deal specifically on the dead-beat stability issue for autonomous switched linear discrete-time systems, namely on the problem of checking if every sufficiently long sequence of switches leads to a null matrix. In Parriaux and Millérioux (2013), dead-beat stability has been used to characterize flatness of controlled switched linear discrete-time systems. Actually, it has been shown that a switched linear system is flat if a so-called auxiliary system, taking the form of an autonomous switched linear system, is dead-beat stable. In that typical case, the dynamical matrices of the auxiliary system may be not invertible. The issue has been tackled with the notion of nilpotent semigroups. Dead-beat stability has also been examined in Philippe et al. (2016). The specificity is that the switching sequence is constrained. The dead-beat stability is expressed in terms of joint spectral radius. A polynomial time algorithm to decide on the dead-beat stability is provided. Concerning the problem of dead-beat stabilizability, we must quote

^{*} This work was supported by Research Grants ANR-13-INSE-0005-01 from the Agence Nationale de la Recherche

the papers Paterson (1970); Blondel and Tsitsiklis (1997); Bournez and Branicky (2002) which deal with the problem of mortality of a set of matrices. A set of matrices is mortal if the zero matrix can be expressed as the product of finite length of matrices. Clearly, this issue is closely related to the concern under investigation in the present paper. The decidability on the mortality of a set of matrices is discussed in the cited papers, which prove that, except some particular cases whose mortality problem results NP-complete, e.g. pairs of integer matrices, the problem is unsolvable. For instance Paterson (1970) proves that the problem is unsolvable for finite sets of 3×3 matrices over the integers. Thus, in general, the problem of checking mortality of a set of matrices is unsolvable.

From this literature overview, it can be concluded that dead-beat stabilizability of switched linear discrete-time systems is still an open problem. This paper is an attempt to go further regarding this issue. In particular we are interested in providing a constructive method to check if a finite sequence of switches gives the null matrix and to compute it, if it exists. To show the benefit of the proposed method, it will be compared with the brute-force approach based on the exhaustive search over all the possible sequences of bounded length.

The paper is organized as follows. In Section 2, a necessary and sufficient condition ensuring the existence of a sequence of finite length such that the state reaches the origin from any arbitrary initial condition, is provided. Then, in Section 3, an algorithm for testing dead-beat stabilizability is provided. The complexity issue is discussed. In Section 4, numeric simulations highlight the efficiency of the approach.

Notation Given $n \in \mathbb{N}$ define $\mathbb{N}_n = \{j \in \mathbb{N} : 1 \leq j \leq n\}$, the set of integers ranging between 1 and n . The set of q switching modes is $\mathcal{I} = \mathbb{N}_q$ and the related matrices forms a finite collection $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$, whose i -th element is denoted with A_i , i.e. $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$, with $A_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathcal{I}$. All the possible sequences of modes of length N is $\mathcal{I}^N = \prod_{j=1}^N \mathcal{I}$ and $|\sigma| = N$ if $\sigma \in \mathcal{I}^N$. Given $\sigma \in \mathcal{I}^N$ and $i, j \in \mathbb{N}_N$ with $i \leq j$, σ_i is the i -th element of σ and $\sigma_{[i:j]}$ the subsequence of σ given by the elements from the i -th to the j -th, with $\sigma_{[i:j]}$ the empty sequence if $i > j$. Given σ, δ sequences of modes, (σ, δ) is their concatenation. Given $\sigma \in \mathcal{I}^N$, define:

$$\mathbb{A}_\sigma = \prod_{j=1}^N A_{\sigma_j} = A_{\sigma_N} \cdots A_{\sigma_1},$$

and $\prod_{j=m}^n A_{\sigma_j} = I$ if $m > n$ where I stands for the identity matrix. Given $a \in \mathbb{R}$, $\lceil a \rceil$ is the smallest integer greater than or equal to a .

2. NECESSARY AND SUFFICIENT CONDITION FOR DEAD-BEAT STABILIZABILITY

2.1 Problem statement

Consider the switched linear discrete-time system

$$x_{k+1} = A_{\sigma(k)} x_k, \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \rightarrow \mathcal{I}$ is the switching law.

We aim at giving a necessary and sufficient condition for the existence of an integer $K \in \mathbb{N}$ and a sequence $\sigma \in \mathcal{I}^K$ such that for any initial condition $x_0 \in \mathbb{R}^n$, the state at time K reaches zero. That is

$$\exists K \in \mathbb{N}, \exists \sigma \in \mathcal{I}^K \quad \text{s.t.} \quad x_K = 0, \quad \forall x_0 \in \mathbb{R}^n,$$

that is equivalent to

$$\exists K \in \mathbb{N}, \exists \sigma \in \mathcal{I}^K \quad \text{s.t.} \quad A_{\sigma(K)} \cdots A_{\sigma(0)} = 0 \quad (2)$$

We also aim at using the necessary and sufficient condition for computing such a sequence, when numerically possible.

Remark 1. According to the considerations made in the introduction on the notion of mortality, the existence of such a K cannot, in general, be performed in finite time, resulting in an unsolvable problem. In this paper we are interested in conditions to determine K and σ , whenever they exist.

2.2 Main result

Denote with

$$\mathcal{I}_s = \{i \in \mathcal{I} : \det A_i = 0\}, \quad \mathcal{I}_{ns} = \{i \in \mathcal{I} : \det A_i \neq 0\}, \quad (3)$$

which means that \mathcal{I}_s and \mathcal{I}_{ns} are the sets of singular and nonsingular matrices in \mathcal{I} , respectively. The following lemma will be instrumental for the sequel.

Lemma 1. (Meyer (2000)) For every pair of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, we have that

$$\dim \ker(AB) = \dim \ker(B) + \dim(\text{im}(B) \cap \ker(A)).$$

The following proposition holds.

Proposition 1. The system is dead-beat stabilizable if and only if there exist $m \in \mathbb{N}_n$ finite mode sequences $\sigma^j \in \mathcal{I}^{n^j}$ with $n^j \in \mathbb{N}$, where $j \in \mathbb{N}_m$, such that

$$\sum_{k=1}^m \dim \left(\text{im} \left(\prod_{j=1}^{k-1} \mathbb{A}_{\sigma^j} \right) \cap \ker(\mathbb{A}_{\sigma^k}) \right) = n, \quad (4)$$

and with $\sigma_{n^j}^j \in \mathcal{I}_s$ for every $j \in \mathbb{N}_m$.

Proof 1. The system is dead-beat stabilizable if and only if there exists a finite sequence $\gamma \in \mathcal{I}^n$ such that $\mathbb{A}_\gamma = 0$, which is equivalent to $\ker(\mathbb{A}_\gamma) = \mathbb{R}^n$ and also to $\dim \ker(\mathbb{A}_\gamma) = n$. Hence, from Lemma 1, the system is dead-beat stabilizable if and only if there exist $m \in \mathbb{N}_n$ sequences $\delta^j \in \mathcal{I}^{r^j}$ with $r^j \in \mathbb{N}$ for all $j \in \mathbb{N}_s$ such that

$$\begin{aligned} \mathbb{A}_\gamma &= \prod_{j=1}^m \mathbb{A}_{\delta^j} \quad \text{and} \\ n &= \dim \ker(\mathbb{A}_\gamma) = \dim \ker(\mathbb{A}_{\delta^{m-1}} \mathbb{A}_{\delta^{m-2}} \cdots \mathbb{A}_{\delta^1}) \\ &\quad + \dim(\text{im}(\mathbb{A}_{\delta^{m-1}} \mathbb{A}_{\delta^{m-2}} \cdots \mathbb{A}_{\delta^1}) \cap \ker(\mathbb{A}_{\delta^m})) \\ &= \dim \ker(\mathbb{A}_{\delta^{m-2}} \mathbb{A}_{\delta^{m-3}} \cdots \mathbb{A}_{\delta^1}) \\ &\quad + \dim(\text{im}(\mathbb{A}_{\delta^{m-2}} \mathbb{A}_{\delta^{m-3}} \cdots \mathbb{A}_{\delta^1}) \cap \ker(\mathbb{A}_{\delta^{m-1}})) \\ &\quad + \dim(\text{im}(\mathbb{A}_{\delta^{m-1}} \mathbb{A}_{\delta^{m-2}} \cdots \mathbb{A}_{\delta^1}) \cap \ker(\mathbb{A}_{\delta^m})) \\ &= \sum_{k=1}^m \dim \left(\text{im} \left(\prod_{j=1}^{k-1} \mathbb{A}_{\delta^j} \right) \cap \ker(\mathbb{A}_{\delta^k}) \right). \end{aligned} \quad (5)$$

Clearly, the number $\dim \ker(\mathbb{A}_{\delta^k})$ can be assumed to be greater than 0 for all $k \in \mathbb{N}_m$. Since $\dim \ker(AB) = \dim \ker(A)$ if $\dim \ker(B) = 0$, then, for every $j \in \mathbb{N}_m$, there

must exist $i \in \mathbb{N}_{r^j}$ such that $\delta_i^j \in \mathcal{I}_s$, i.e. $\det(A_{\delta_i^j}) = 0$ and then $\det(\mathbb{A}_{\delta^j}) = 0$.

Now, what remains to be proved is that, if there exists $\delta^j \in \mathcal{I}^{n^j}$ such that (5) holds, then the condition can be satisfied also for $\sigma^j \in \mathcal{I}^{n^j}$ whose last element is related to a singular matrix, i.e. such that $\sigma_{n^j}^j \in \mathcal{I}_s$. For every $j \in \mathbb{N}_m$, define

$$i^j = \max_{i \in \mathbb{N}_{r^j}} \{i : \delta_i^j \in \mathcal{I}_s\}, \quad (6)$$

that is the higher index value related to a singular matrix among those involved in \mathbb{A}_{δ^j} . Such an index exists since $\det(\mathbb{A}_{\delta^j}) = 0$. For every $j \in \mathbb{N}_m$, define

$$\underline{\delta}^j = \delta_{[1:i^j]}^j, \quad \bar{\delta}^j = \delta_{[i^j+1:n^j]}^j, \quad (7)$$

which means that the sequence $\underline{\delta}^j$ is the prefix of δ^j of length i^j and $\bar{\delta}^j$ is the sequence of length $|\delta^j| - i^j$ such that $\delta^j = (\underline{\delta}^j, \bar{\delta}^j)$. Notice that $\bar{\delta}^j$ is empty if $i^j = |\delta^j|$. Define also

$$\sigma^j = \begin{cases} \underline{\delta}^j, & \text{if } j = 1, \\ (\bar{\delta}^{j-1}, \underline{\delta}^j), & \text{if } 2 \leq j \leq m, \end{cases}$$

and $n^j = |\sigma^j|$. By construction, $\sigma_{n^j}^j \in \mathcal{I}_s$ and $\bar{\delta}^j$ exclusively involves modes in \mathcal{I}_{n_s} , which implies that $\mathbb{A}_{\bar{\delta}^j}$ is nonsingular. We have to prove that

$$\begin{aligned} d_k &= \dim \left(\text{im} \left(\prod_{j=1}^{k-1} \mathbb{A}_{\delta^j} \right) \cap \ker(\mathbb{A}_{\delta^k}) \right) \\ &= \dim \left(\text{im} \left(\prod_{j=1}^{k-1} \mathbb{A}_{\sigma^j} \right) \cap \ker(\mathbb{A}_{\sigma^k}) \right) \end{aligned} \quad (8)$$

for all $k \in \mathbb{N}_m$. Consider first $k \geq 2$. From the first equality in (8), there exist d_k linearly independent vectors $v_l \in \mathbb{R}^n$, with $l \in \mathbb{N}_{d_k}$, that are both in the image of $\prod_{j=1}^{k-1} \mathbb{A}_{\delta^j}$ and in the kernel of \mathbb{A}_{σ^k} , which means that there exists $x_l \in \mathbb{R}^n$ such that

$$\begin{aligned} v_l &= \prod_{j=1}^{k-1} \mathbb{A}_{\delta^j} x_l = \mathbb{A}_{\delta^{k-1}} \dots \mathbb{A}_{\delta^1} x_l = \mathbb{A}_{\bar{\delta}^{k-1}} \mathbb{A}_{\underline{\delta}^{k-1}} \mathbb{A}_{\bar{\delta}^{k-2}} \\ &\quad \dots \mathbb{A}_{\bar{\delta}^2} \mathbb{A}_{\bar{\delta}^1} \mathbb{A}_{\underline{\delta}^1} x_l = \mathbb{A}_{\bar{\delta}^{k-1}} \mathbb{A}_{\sigma^{k-1}} \dots \mathbb{A}_{\sigma^2} \mathbb{A}_{\sigma^1} x_l, \\ 0 &= \mathbb{A}_{\delta^k} v_l = \mathbb{A}_{\bar{\delta}^k} \mathbb{A}_{\underline{\delta}^k} v_l. \end{aligned}$$

Recalling that, by construction, $\mathbb{A}_{\bar{\delta}^j}$ is nonsingular, (see (6)), then d_k linearly independent vectors can be defined as $u_l = \mathbb{A}_{\bar{\delta}^{k-1}}^{-1} v_l$, that are such that

$$\begin{aligned} u_l &= \mathbb{A}_{\bar{\delta}^{k-1}}^{-1} v_l = \mathbb{A}_{\sigma^{k-1}} \dots \mathbb{A}_{\sigma^2} \mathbb{A}_{\sigma^1} x_l, \\ 0 &= \mathbb{A}_{\bar{\delta}^k} \mathbb{A}_{\underline{\delta}^k} \mathbb{A}_{\bar{\delta}^{k-1}}^{-1} u_l = \mathbb{A}_{\bar{\delta}^k} \mathbb{A}_{\sigma^k} u_l. \end{aligned} \quad (9)$$

Then, u_l belongs to the image of $\prod_{j=1}^{k-1} \mathbb{A}_{\sigma^j}$ and also to the kernel of \mathbb{A}_{σ^k} , since $\ker(AB) = \ker(B)$ if A is nonsingular. Then, for $k \geq 2$, condition (8) is satisfied. For $k = 1$, condition (8) reduces to

$$d_1 = \dim \ker(\mathbb{A}_{\delta^1}) = \dim \ker(\mathbb{A}_{\sigma^1})$$

that holds since $\ker(\mathbb{A}_{\delta^1}) = \ker(\mathbb{A}_{\bar{\delta}^1} \mathbb{A}_{\underline{\delta}^1}) = \ker(\mathbb{A}_{\bar{\delta}^1} \mathbb{A}_{\sigma^1})$, being $\mathbb{A}_{\bar{\delta}^1}$ nonsingular. This completes the proof.

In other words, Proposition 1 asserts that the system (1) is dead-beat stabilizable if and only if there exists a set of m switching sequences σ^j of finite length n^j , with $j \in \mathbb{N}_m$

and $1 \leq m \leq n$, whose last element is related to a singular matrix, i.e. $\sigma_{n^j}^j \in \mathcal{I}_s$, and such that the intersection of the kernel of \mathbb{A}_{σ^j} and the image of the matrices preceding it has positive dimension. Moreover, the dimension of those intersections is equal to that of the state space \mathbb{R}^n . This result provides a necessary and sufficient condition for dead-beat stabilizability that would lead to an approach for searching one switching sequence that leads to (2). This approach acts as an alternative to an exhaustive search, that is the brute-force test of all the possible switching sequences that leads to (2).

3. ALGORITHM

3.1 Preliminary considerations

Before detailing a method for testing dead-beat stabilizability, we provide some preliminary considerations. First, one may wonder if, assuming that condition (5) holds, the subspaces given by the intersections provide a basis of the \mathbb{R}^n . The answer is no, as illustrated by the following simple counterexample.

Example 1. Consider the system with $q = 1$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $\ker(A_1) = \text{im}(A_1) = \text{span}\{[1, 0]^T\}$ and then

$\dim \ker(A_1 A_1) = \dim \ker(A_1) + \dim(\text{im}(A_1) \cap \ker(A_1)) = 2$, but the one-dimensional subspaces $\ker(A_1)$ and $\text{im}(A_1) \cap \ker(A_1)$ have the same basis, i.e. $[1, 0]^T$.

From Example 1, another useful consideration, detailed in Remark 2 below, can be done.

Remark 2. It seems reasonable to consider, instead of the singular matrices A_i with $i \in \mathcal{I}_s$, their power of order k_i , that is the smallest non-negative integer such that $\ker(A_i^{k_i}) = \ker(A_i^{k_i+1})$. For the trivial case in Example 1, indeed $k_1 = 2$ and $\dim \ker(A_1^2) = 2$ and dead-beat stabilizability results directly, no image and intersection computation were required.

Finally, it can be noticed that the first element of the set of sequences σ^j can be simplified, as specified in the following corollary.

Corollary 2. If the set of sequences $\sigma^j \in \mathbb{N}_{n^j}$ with $j \in \mathbb{N}_m$ satisfies Proposition 1, then the set of sequence ν^j with $j \in \mathbb{N}_m$ given by

$$\nu^j = \begin{cases} \sigma_{[r:n^1]}^1, & \text{if } j = 1, \\ \sigma^j, & \text{if otherwise,} \end{cases} \quad (10)$$

with $r = \min_{i \in \mathbb{N}_{n^1}} \{i : \sigma_i^1 \in \mathcal{I}_s\}$, also fulfills Proposition 1.

Proof 2. Since $\text{im}(AB) = \text{im}(A)$ if B is nonsingular, then all the images in (4) are unchanged when replacing \mathbb{A}_{σ^1} by \mathbb{A}_{ν^1} . In fact, we have that $\mathbb{A}_{\sigma^1} = \mathbb{A}_{\sigma_{[r:n^1]}^1} \mathbb{A}_{\sigma_{[1:r-1]}^1} = \mathbb{A}_{\nu^1} \mathbb{A}_{\sigma_{[1:r-1]}^1}$ and $\mathbb{A}_{\sigma_{[1:r-1]}^1}$ is nonsingular by definition of r . Moreover, since $\dim \ker(AB) = \dim \ker(A)$ if B is nonsingular, for $k = 1$ in (4) one has

$$\begin{aligned} \dim \left(\text{im} \left(\prod_{j=1}^{k-1} \mathbb{A}_{\sigma^j} \right) \cap \ker(\mathbb{A}_{\sigma^k}) \right) &= \dim \ker(\mathbb{A}_{\sigma^1}) \\ &= \dim \ker(\mathbb{A}_{\nu^1}), \end{aligned}$$

being $\mathbb{A}_{\sigma^1_{[1:r-1]}}$ nonsingular.

Corollary 2 substantially claims that if the dead-beat stabilizability condition is satisfied by the sequences σ^j , it also holds if the prefix of σ^1 involving nonsingular matrices is removed. Alternatively speaking, one can focus only on stabilizing sequence candidates starting and terminating with a singular matrix.

3.2 Computation and complexity

In this subsection, we provide Algorithm 1 to find a dead-beat stabilizing sequence. From Proposition 1 and Corollary 2, every sequence $\gamma \in \mathcal{I}^N$ such that $A_\gamma = 0$ is composed by $m \in \mathbb{N}$ subsequences σ^j terminating with an element of \mathcal{I}_s . Clearly, there might be an infinite number of such γ . The objective is to find such a γ , whose length is minimal.

First, let us assume for simplicity that the sequence of elements terminating the subsequences σ^j , with $j \in \mathbb{N}_m$, is known (this assumption can be removed as explained in Section 3.3 below). Denote such a sequence by $s \in \mathcal{I}_s^m$. Algorithm 1, based on Lemma 3 given below, allows to determine σ^j , with $j \in \mathbb{N}_m$, and the related $\gamma_s = (\sigma^1, \dots, \sigma^m)$. It is worth stressing that it is not really an algorithm but a semi-algorithm since it might not terminate in finite time, an additional termination condition should be added in practice to stop its execution.

Algorithm 1 Define the dead-beat stabilizing sequence γ_s of the system (1), given the sequence $s \in \mathcal{I}_s^m$.

Input: matrices $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$, indexes \mathcal{I} and \mathcal{I}_s and sequence $s \in \mathcal{I}_s^m$.

```

1:  $p \leftarrow 1$ ;            $\triangleright$  Initialize the sequence counter
2:  $\sigma^1 \leftarrow s_1$ ;    $\triangleright$  Define the first sequence  $\sigma^1$ 
3:  $n^1 \leftarrow 1$ ;        $\triangleright$  Length of sequence  $\sigma^1$ 
4:  $D \leftarrow \dim \ker(A_{s_1})$ ;  $\triangleright$  Initialize dimension counter
5: while  $D < n$  do
6:    $p \leftarrow p + 1$ ;    $\triangleright$  Update the sequence counter
7:    $i \leftarrow s_p$ ;      $\triangleright$  Singular matrix closing  $\sigma^p$ 
8:   compute a basis matrix  $X$  of  $\text{im} \left( \prod_{j=1}^{p-1} \mathbb{A}_{\sigma^j} \right)$ ;
9:   find  $h \in \mathbb{N}$  and  $\delta \in \mathcal{I}^h$  such that
      $\dim \ker(A_i \mathbb{A}_\delta X) = d_p > 0$ ;  $\triangleright$  Equivalent to (11)
10:   $\sigma^p \leftarrow (\delta, i)$ ;  $\triangleright$  Define the  $p$ -th sequence  $\sigma^p$ 
11:   $n^p \leftarrow h + 1$ ;    $\triangleright$  Length of sequence  $\sigma^p$ 
12:   $D \leftarrow D + d_p$ ;    $\triangleright$  Update the dimension counter
13: end while
Output:  $\gamma_s \leftarrow (\sigma^1, \dots, \sigma^m)$ .
```

Steps 8 and 9 are based on the following lemma.

Lemma 3. (Meyer (2000)) Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ the equality

$$\dim \ker(A X) = \dim(\text{im}(B) \cap \ker(A))$$

holds, where X is a basis matrix of $\text{im}(B)$.

Lemma 3 permits, given $i \in \mathcal{I}$, $h \in \mathbb{N}$ and $\delta \in \mathcal{I}^h$, to check

$$\dim \left(\text{im} \left(\prod_{j=1}^{p-1} \mathbb{A}_{\sigma^j} \right) \cap \ker(A_i \mathbb{A}_\delta) \right) = d_p > 0; \quad (11)$$

by testing $\dim \ker(A_i \mathbb{A}_\delta X) = d_p > 0$, where X is a basis of $\text{im} \left(\prod_{j=1}^{p-1} \mathbb{A}_{\sigma^j} \right)$.

In practice, Algorithm 1 consists in building the sequences σ^j , each terminating with s_j by assumption, such that

$$\dim \text{im} \left(\prod_{j=1}^p \mathbb{A}_{\sigma^j} \right) < \dim \text{im} \left(\prod_{j=1}^{p-1} \mathbb{A}_{\sigma^j} \right).$$

When the image dimension dropping D reaches n , the overall image is the origin and then $\mathbb{A}_{\gamma_s} = 0$. Algorithm 1 can be modified to take into account the considerations made in Remark 2. Indeed, if the index k_i of a matrix $A_i \in \mathcal{A}$ is larger than one, considering $A_i^{k_i}$ instead of A_i would reduce the number of iterations.

3.3 Considerations on complexity

Algorithm 1 applies once the sequence $s \in \mathcal{I}_s^m$ is given. Then, the sequence s that corresponds to the last elements of the subsequences σ^j composing the sequence γ_s such that $\mathbb{A}_{\gamma_s} = 0$ should be known *a priori*. This is not the case in general, unless $|\mathcal{I}_s| = 1$. Moreover, even if s were known, Algorithm 1 does not necessarily find such a γ_s . In fact, it could happen that a sequence δ , found at Step 9, satisfies $\dim \ker(A_i \mathbb{A}_\delta X) = d_p > 0$ but is not the right subsequence of γ_s . Both problems can be overcome by modifying Algorithm 1 such that all the $i \in \mathcal{I}_s$ and the $\delta \in \mathcal{I}^h$ for $h \in \mathbb{N}_{\bar{h}}$, with $\bar{h} \in \mathbb{N}$ design parameter, are evaluated at Step 9 and all those satisfying $\dim \ker(A_i \mathbb{A}_\delta X) = d_p > 0$ are stored and used to compute the basis at Step 8. Notice that an exhaustive search is necessary to generate the sequences δ in Step 9, a search that seems unavoidable due to the nature of the problem. In practice, the resulting algorithm is able to compute all the mortal sequences whose subsequences are not longer than $\bar{h} + 1$. We preferred not to give here the modified version of Algorithm 1, to avoid unnecessary further complexity in its representation, although it is considered in the following complexity considerations and is used in the numerical section below.

To compare our algorithm with the brute-force approach, we should reasonably assume that an exhaustive search over the sequences of, at most, length p , that are $\sum_{s=1}^p q^s$, has computational complexity $\mathcal{O}(q^{p+1})$. Denote q_s the number of singular matrices.

Suppose first, for instance, that the subsequences σ^j have average length of r , which implies that γ_s has length $(m-1)r + 1$. Then our method requires to perform $m-1$ searches over $q_s \sum_{s=1}^{r-1} q^s$ sequences (and one over the q_s modes for the first subsequence), leading to $\mathcal{O}((m-1)q_s q^r + q_s)$, while the brute-force approach needs to enumerate the $\sum_{s=1}^{(m-1)r+1} q^s$ sequences, yielding to $\mathcal{O}(q^{(m-1)r+2})$. Another way to analyse the benefit is denoting with $l \in \mathbb{N}$ the length of the shortest mortal sequence and $r \in \mathbb{N}$ the length of its longer subsequence. Then r is contained between $l - m + 1$ and $\lceil (l-1)/(m-1) \rceil$ and hence our method complexity is bounded between $\mathcal{O}((m-1)q_s q^{l-m+1} + q_s)$ and $\mathcal{O}((m-1)q_s q^{\lceil (l-1)/(m-1) \rceil} + q_s)$, whereas the exhaustive search has a complexity of $\mathcal{O}(q^{l+1})$. Of course both the modified algorithm and the

brute force one should be applied for increasing values of \bar{h} until a mortal sequence is found or a maximal value of \bar{h} is attained.

Summarizing, with our approach the exhaustive search is on the subsequences σ^j instead of the whole sequence $\gamma_s = (\sigma^1, \dots, \sigma^m)$, where $m \leq n$. This means that the exhaustive search is performed in many shorter subsequences instead of a single longer one and the benefit grows with the state dimension.

4. NUMERIC SIMULATIONS

In this section, we compare the results of the proposed algorithm with the brute-force approach consisting in the enumeration of every bounded length sequence of modes. The test consists in searching for a finite length sequence γ_s such that $\mathbb{A}_{\gamma_s} = 0$ by means of both methods: Algorithm 1 and the exhaustive search and to compare the respective execution times.

4.1 Example 1

As a first test, we consider 5 matrices of dimension 3×3 , with $|\mathcal{I}_s| = 1$ and $|\mathcal{I}_{ns}| = 4$. First, we generate the set of 5 matrices corresponding to the sequence γ_s of length l such that $\mathbb{A}_{\gamma_s} = 0$. The singular matrix is generated randomly and its kernel dimension is 1. Some of the nonsingular matrices are generated randomly, while others are computed to obtain $\mathbb{A}_{\gamma_s} = 0$. The length of σ^2 and σ^3 (recall that $|\sigma^1| = 1$) is the parameter $r \in \mathbb{N}$. For different values of r , the sequence γ_s is generated, then $|\gamma_s| = l = 2r + 1$. Algorithm 1 and the exhaustive search methods are applied to compute γ_s . Both methods are coded as MEX functions of Matlab. The time spent by the exhaustive search method and Algorithm 1 are respectively denoted with t_{enu} and t_{alg} . They are reported in Table 1. The exhaustive search method has not been applied for $r > 5$, since too much time would be necessary.

r	$l = 2r + 1$	$Q = q^l$	t_{enu}	t_{alg}
4	9	$1.95 \cdot 10^6$	2.25s	0.014s
5	11	$4.48 \cdot 10^7$	83s	0.04s
6	13	$1.22 \cdot 10^9$		0.08s
7	15	$3.05 \cdot 10^{10}$		0.24s
8	17	$7.63 \cdot 10^{11}$		0.97s
9	19	$1.9 \cdot 10^{13}$		4.6s

Table 1.

Notice that, even for relatively short sequences γ_s , Algorithm 1 gets the solution more than 100 time faster, for $r = 4$, and more than 2000 faster, for $r = 5$. Notice also that only 4.6s are sufficient to get the solution for $r = 9$, among $1.9 \cdot 10^{13}$ possible sequences.

4.2 Example 2

As a second test, we consider the state dimension $n = 4$ and the modes $q = 6$, with one singular matrix whose kernel is of dimension one. Some nonsingular matrices are generated randomly, others are chosen to get $\mathbb{A}_{\gamma_s} = 0$. The length of the three subsequences σ^i with $i = 2, 3, 4$

are randomly chosen. The total length of the sequence γ_s is imposed to be 9. Thus σ^i with $i = 2, 3, 4$ can be of length 2, 3 or 4 (the length of σ^1 is 1). Different sets of matrices are generated and both methods are applied to retrieve γ_s . The test is repeated for 100 different sets of matrices. The average, the minimal and the maximal computation time is reported in Table 2. Notice that Algorithm 1 gives the solution around 1000 times faster than the exhaustive search approach.

	minimal	average	maximal
t_{alg}	0.0101s	0.0125s	0.0294s
t_{enu}	9.86s	15.62s	17.6s

Table 2.

5. CONCLUSION

In this paper, we have considered the problem of characterizing the dead-beat stabilizability for discrete-time switched systems. We have provided a constructive necessary and sufficient condition for dead-beat stabilizability. Then we have derived from the condition a semi-algorithm for computing stabilizing switching sequences, whenever they exist, with substantially lower complexity than the brute-force exhaustive search. One of the most likely interest of this result is dead-beat control of switched linear systems. Another research line, which should deserve a special attention, is on the use of the condition for constructing dynamics admitting mortal sequences. The resulting dynamical systems would take the form of a new class of automata with finite memory which are typically involved in symmetric cryptography (statistical self-synchronizing stream ciphers as initiated by Daemen et al. (1992)).

REFERENCES

- Alur, R., D’Innocenzo, A., Johansson, K.H., Pappas, G.J., and Weiss, G. (2011). Compositional modeling and analysis of multi-hop control networks. *IEEE Transactions on Automatic control*, 56(10), 2345–2357.
- Blondel, V.D. and Tsitsiklis, J.N. (1997). When is a pair of matrices mortal? *Information Processing Letters*, 63, 283–286.
- Bournez, O. and Branicky, M. (2002). The mortality problem for matrices of low dimensions. *Theory of Computing Systems*, 35(4), 433–448.
- Conner, L.T.J. and Stanford, D.P. (1984). State dead-beat response and observability in multi-modal systems. *SIAM Journal of Control and Optimization*, 22(4), 630–644.
- Conner, L.T.J. and Stanford, D.P. (1987). The structure of the controllable set for multimodal systems. *Linear Algebra and its Applications*, 95, 171–180.
- Daemen, J., Govaerts, R., and Vandewalle, J. (1992). On the design of high speed self-synchronizing stream ciphers. In *Proc. of the ICCS/ISITA’92 conference*, volume 1, 279–283. Singapore.
- Fiacchini, M. and Jungers, M. (2014). Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach. *Automatica*, 50(1), 75 – 83.
- Geromel, J.C. and Colaneri, P. (2006). Stability and stabilization of discrete-time switched systems. *International Journal of Control*, 79(7), 719–728.

- Hernandez-Vargas, E.A., Middleton, R.H., and Colaneri, P. (2011). Optimal and mpc switching strategies for mitigating viral mutation and escape. In *Proc. of the 18th IFAC World Congress Milano (Italy) August*.
- Jungers, R. (2009). The joint spectral radius. *Lecture Notes in Control and Information Sciences*, 385.
- Jungers, R.M., D’Innocenzo, A., and Di Benedetto, M.D. (2012). Feedback stabilization of dynamical systems with switched delays. In *Proc. of the 51st IEEE Conference on Decision and Control*, 1325–1330.
- Liberzon, D. (2003). *Switching in Systems and Control*. Systems and Control: Foundations and Applications. Birkhäuser, Boston, MA.
- Liberzon, D. and Morse, A.S. (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5), 59–70.
- Lin, H. and Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transaction on Automatic Control*, 54(2), 308–322.
- Meyer, C.D. (2000). *Matrix analysis and applied linear algebra*. SIAM.
- Parriaux, J. and Millérioux, G. (2013). Nilpotent semigroups for the characterization of flat outputs of switched linear and LPV discrete-time systems. *Systems and Control Letters*, 62(8), 679–685.
- Paterson, M.S. (1970). Unsolvability in 3×3 matrices. *Studies in Applied Mathematics*, 49(1), 105–107.
- Philippe, M., Millerioux, G., and Jungers, R. (2016). Deciding the boundedness and dead-beat stability of constrained switching systems. *Nonlinear Analysis: Hybrid Systems*.
- Shorten, R., Wirth, F., and Leith, D. (2006). A positive systems model of tcp-like congestion control: asymptotic results. *IEEE/ACM Transactions on Networking*, 14(3), 616–629.
- Shorten, R., Wirth, F., Mason, O., Wulff, K., and King, C. (2007). Stability criteria for switched and hybrid systems. *SIAM review*, 49(4), 545–592.
- Sun, Z. and Ge, S.S. (2005). Analysis and synthesis of switched linear control systems. *Automatica*, 41, 181–195.