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We are interested in mechanical systems with a finite number of degrees of freedom submitted to frictionless unilateral constraints. We consider the case of a convex, non-smooth set of admissible positions given by $K=\{q \in \mathbb{R}^d;\ \varphi_\alpha(q) \geqslant 0,\ 1 \leqslant \alpha \leqslant \nu\}$, $\nu \geqslant 1$, and we assume inelastic shocks at impacts. We propose a time-discretization of the measure differential inclusion which describes the dynamics and we prove the convergence of the approximate solutions to a limit motion which satisfies the constraints. Moreover, if the geometric properties ensuring continuity on data hold at the limit, we show that the transmission of velocities at impacts follows the inelastic shocks rule.

Keywords: Vibro-impact; Non-smooth convex constraints; Measure differential inclusion; Inelastic shocks; Time-discretization scheme; Convergence; Existence

1. Introduction and statement of the result

We consider a mechanical system with d degrees of freedom which unconstrained motion is described by the following ODE

$$\ddot{u} = f(t, u, \dot{u}),$$

where $u \in \mathbb{R}^d$ is the representative point of the system.

We assume that the trajectory must remain in a given closed subset K of $\mathbb{R}^d$ i.e.

$$u(t) \in K \quad \text{for all } t \in [0, \tau].$$

This unilateral constraint may lead to some discontinuities for the velocity. Indeed let us assume for instance that $u(t) \in \mathrm{Int}(K)$ for all $t \in (t_0, t_1) \cup (t_1, t_2) \subset [0, \tau]$ and $u(t_1) \in \partial K$. Then the constraint implies that

$$\begin{aligned} \dot{u}(t_1 - 0) &\in -T_K\big(u(t_1)\big), \\ \dot{u}(t_1 + 0) &\in T_K\big(u(t_1)\big), \end{aligned}$$

where $T_K(q)$ denotes the tangent cone to K at q given by

$$T_K(q) = \overline{\cup_{\lambda>0} \lambda(K - q)}.$$

Hence, if $\dot{u}(t_1 - 0) \notin T_K\big(u(t_1)\big)$, it is clear that $\dot{u}$ is discontinuous at $t = t_1$. It follows that the equation of motion has to be modified by adding a measure μ to the right-hand side i.e.

$$\ddot{u} = f(t, u, \dot{u}) + \mu.$$

This measure μ describes the reaction force due to the unilateral constraint and

$$\mathrm{Supp}(\mu) \subset \big\{ t \in [0, \tau];\, u(t) \in \partial K \big\}.$$

Let us assume moreover that the constraint is perfect i.e. frictionless. We infer (see [7,8]) that

$$-\mu \in T_K(u)^{\perp} = N_K(u)$$

and the motion is described by the following measure differential inclusion (MDI)

$$\mu = \ddot{u} - f(t, u, \dot{u}) \in -N_K(u). \tag{1.1}$$

The discontinuities of the velocity at impacts are now characterized by

$$\dot{u}(t+0) \in T_K\big(u(t)\big),\ \ \dot{u}(t-0) \in -T_K\big(u(t)\big),$$
$$\dot{u}(t-0) - \dot{u}(t+0) = -\mu\big(\{t\}\big) \in N_K\big(u(t)\big)$$

but these equations do not define uniquely $\dot{u}(t+0)$ and we have to complete the description of the motion. Following Moreau [7,8] (see also [15,17] for a mathematical justification of this impact law by a penalty method) we assume inelastic impacts i.e.

$$\dot{u}(t+0) = \mathrm{Proj}\big(T_K\big(u(t)\big), \dot{u}(t-0)\big) \tag{1.2}$$

for all $t \in (0,\tau)$.

Remark. We may observe that

$$N_K\big(u(t)\big) = \{0\}, \quad T_K\big(u(t)\big) = \mathbb{R}^d$$

if $t \in (0,\tau)$ and $u(t) \in \mathrm{Int}(K)$: in this case the impact law (1.2) implies simply that $\dot{u}$ is continuous at t.

Let $(u_0, v_0) \in K \times T_K(u_0)$ be admissible initial data. We consider the following Cauchy problem (P):

Problem (P). Find $u : [0,\tau] \to \mathbb{R}^d$ $(\tau > 0)$ such that
(P1) u is continuous with values in K,
(P2) $\dot{u}$ belongs to $BV(0,\tau;\mathbb{R}^d)$,
(P3) the measure $\mu = \ddot{u} - f(t,u,\dot{u})$ is such that

$$\mathrm{Supp}(\mu) \subset \big\{t \in [0,\tau];\, u(t) \in \partial K\big\}$$

and the MDI (1.1) is satisfied in the following sense (see [18]):

$$\langle \mu, v-u\rangle \geqslant 0 \quad \forall v \in C^0\big([0,\tau];K\big),$$

(P4) the initial data are satisfied in the following sense:

$$u(0) = u_0, \quad \dot{u}(0+0) = v_0,$$

(P5) for all $t \in (0,\tau)$

$$\dot{u}(t+0) = \mathrm{Proj}\big(T_K\big(u(t)\big), \dot{u}(t-0)\big).$$

The existence of a solution for this Cauchy problem is still an open problem in the general case. When the boundary of K is smooth enough, the set K can be described at least locally with a single inequality

$$u \in K \Longleftrightarrow \varphi(u) \geqslant 0.$$

In this case (single-constraint case), several existence results have been obtained. The corresponding proofs rely on the study of a sequence of approximate solutions which are built either by means of a time-discretization of the MDI (see [2–4,9,13,14,16]) nor by means of a penalization (see [12,19]). The convergence of the sequence of approximate solutions gives both a theoretical result of existence and a numerical method to compute approximate solutions of (P).

In a more general case, when K is described by several inequalities (multi-constraint case) i.e.

$$K = \left\{q \in \mathbb{R}^d;\ \ \varphi_\alpha(q) \geqslant 0 \quad \forall \alpha \in \{1, \ldots, \nu\}\right\}, \quad \nu \geqslant 1$$

the existence of a solution has been established by Ballard in [1] if all the data are analytical. His proof uses a very different technic based on existence results for ODE and variational inequalities. Unfortunately, this very nice proof does not yield directly a numerical method. Observing that the time-discretization schemes proposed by M. Monteiro-Marques or L. Paoli and M. Schatzman in the single-constraint case can be also defined in the multi-constraint case, it is natural to try to extend their convergence proofs in order to complete P.Ballard's result when the data are not analytical and to obtain well-suited numerical methods. For this last point another question arises immediately: what do we know about continuity on data? In the analytical case, Ballard has proved (see [1]) that continuity on initial data holds if the active constraints along the limit motion remain orthogonal. Moreover, the study of the model problem of a free material point in an angular domain K of $\mathbb{R}^2$ shows that continuity on data does not hold in general if the active constraints create an obtuse angle and leads to the opposite conjecture in case of acute angles (see [10]). The proof of this last result is given in a very recent paper (see [11]).

In this framework, we will extend in this paper the convergence result of the time-discretization scheme proposed in [9,13] to the multi-constrained case. More precisely, we assume that

(H1) f is a continuous function from $[0, \tau] \times \mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d$ ($\tau > 0$) and is Lipschitz continuous in its last two arguments;

(H2) K is a closed convex subset of $\mathbb{R}^d$ with a non-empty interior, given by

$$K = \left\{q \in \mathbb{R}^d;\ \ \varphi_\alpha(q) \geqslant 0 \quad \forall \alpha \in \{1, \ldots, \nu\}\right\}, \quad \nu \geqslant 1 \tag{1.3}$$

with $\varphi_\alpha \in C^1(\mathbb{R}^d; \mathbb{R})$ such that $\nabla \varphi_\alpha$ does not vanish in a neighborhood of $\{q \in \mathbb{R}^d; \varphi_\alpha(q) = 0\}$.

For all $q \in \mathbb{R}^d$ we define the set of active constraints at q by

$$J(q) = \{\alpha \in \{1, \dots, \nu\};\ \varphi_\alpha(q) \leqslant 0\}$$

and we assume that

(H3) for all $q \in K$, $\big(\nabla\varphi_\alpha(q)\big)_{\alpha\in J(q)}$ is linearly independent.

Let F be a function such that

(H4) F is continuous from $[0,\tau] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times [0, h^*]$ to $\mathbb{R}^d$ ($h^* > 0$), F is Lipschitz continuous in its second, third and fourth arguments and is consistent with respect to f i.e.

$$F(t, q, q, v, 0) = f(t, q, v) \quad \forall (t, q, v) \in [0,\tau] \times \mathbb{R}^d \times \mathbb{R}^d.$$

We define a time-discretization of the Cauchy problem (P) with initial data $(u_0, v_0) \in K \times T_K(u_0)$ as follows:

$$U^0 = u_0, \quad U^1 = \mathrm{Proj}\big(K, u_0 + hv_0 + hz(h)\big) \quad \text{with } \lim_{h\to 0} z(h) = 0, \tag{1.4}$$

and, for all $n \in \{1, \dots, \lfloor \tau/h \rfloor\}$

$$U^{n+1} = \mathrm{Proj}\big(K, 2U^n - U^{n-1} + h^2 F^n\big) \tag{1.5}$$

with

$$F^n = F\left(nh, U^n, U^{n-1}, \frac{U^{n+1} - U^{n-1}}{2h}, h\right). \tag{1.6}$$

Let us denote by L the Lipschitz constant of F. Then, by applying Banach's fixed point theorem, we can prove easily that, for all $h \in (0, h^*] \cap (0, 2/L)$ and for all $n \in \{1, \dots, \lfloor \tau/h \rfloor\}$ the system (1.5)–(1.6) admits an unique solution. Possibly decreasing h^* we will assume from now on that $h^* \in (0, 2/L)$ and hence the scheme is correctly defined for all $h \in (0, h^*]$.

We may observe that the projection on K ensures that all the approximate positions satisfy the constraints and, if $2U^n - U^{n-1} + h^2 F^n$ belongs to $\mathrm{Int}(K)$, then Eq. (1.5) reduces to

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{h^2} = F^n,$$

which is simply a centered scheme for the unconstrained motion.

We define now the sequence of approximate solutions $(u_h)_{h\in(0,h^*]}$ by

$$u_h(t) = U^n + (t-nh)\frac{U^{n+1}-U^n}{h} \quad \text{if } t \in \big[nh, (n+1)h\big) \cap [0,\tau] \tag{1.7}$$

for all $h \in (0, h^*]$.

We prove the following result:

Theorem 1.1. *Let us assume that* (H1)–(H2)–(H3)–(H4) *hold. Let* $(u_0, v_0) \in K \times T_K(u_0)$ *be admissible initial data. Then the sequence* $(u_h)_{h\in(0,h^*]}$ *defined by* (1.4)–(1.5)–(1.6)–(1.7) *admits a converging subsequence in* $C^0\big([0,\tau];\mathbb{R}^d\big)$ *and the limit u satisfies the properties* (P1)–(P2)–(P3)–(P4).

If we assume moreover that

$$\text{(H5)} \qquad \big(\nabla\varphi_\alpha\big(u(t)\big), \nabla\varphi_\beta\big(u(t)\big)\big) \leqslant 0 \quad \forall(\alpha,\beta) \in J\big(u(t)\big)^2, \ \alpha \neq \beta \quad \forall t \in (0,\tau),$$

where (v, w) *denotes the euclidean scalar product of the vectors* v *and* w *in* $\mathbb{R}^d$, *then the function u satisfies also the impact law* (P5) *and is a solution of the Cauchy problem* (P).

Remark. Assumption (H5) is the condition which ensures continuity on data (see [10,11]).

2. Proof of the convergence of the scheme

Let us outline the main steps of the proof of Theorem 1.1. First, in Section 2.1, we establish a priori estimates for the discrete velocities and accelerations. Then, in Section 2.2, we pass to the limit as h tends to zero and applying Ascoli's and Helly's theorem, we infer that there exists a subsequence of approximate solutions, denoted $(u_{h_i})_{h_i>0}$, such that

$$\begin{cases} u_{h_i} \to u & \text{strongly in } C^0\big([0,\tau];\mathbb{R}^d\big), \\ \dot{u}_{h_i} \to \dot{u} & \text{weakly* in } L^\infty(0,\tau;\mathbb{R}^d) \text{ and a.e in } (0,\tau), \\ \ddot{u}_{h_i} \to \ddot{u} & \text{weakly* in } M^1(0,\tau;\mathbb{R}^d). \end{cases}$$

Moreover, we prove that u satisfies the properties (P1)–(P2)–(P3)–(P4).

Finally, in Section 2.3, we study the reflexion of $\dot{u}$ at impacts: we show that the right velocities are given by Moreau's rule for inelastic shocks when assumption (H5) is satisfied.

Thoughout this section we will meet some technicalities which cannot be avoided. In order to make the essential ideas as clear as possible, the proof of some lemmas is given in the Appendix A.

2.1. A priori estimates

For all $h \in (0, h^*]$ we define the discrete velocities by

$$V^n = \frac{U^{n+1} - U^n}{h} \quad \forall n \in \{0 \dots, N\}, \;\; N = \left\lfloor \frac{\tau}{h} \right\rfloor.$$

We prove first a uniform estimate for the velocities $(V^n)_{0 \leqslant n \leqslant N}$.

Proposition 2.1. *There exists $h_1 \in (0, h^*]$ and $C > 0$ such that*

$$\|V^n\| \leqslant C \quad \forall n \in \{0, \dots, N\} \quad \forall h \in (0, h_1]. \tag{2.1}$$

Proof. Let us define M by

$$M = \max\left\{ \left\| F(t, u_0, u_0, 0, h) \right\|; \;\; t \in [0, \tau], \;\; h \in [0, h^*] \right\} \tag{2.2}$$

and recall that L is the Lipschitz constant of F with respect to its second, third and fourth arguments.

As a first step we prove the following estimate:

Lemma 2.2. *Let $h \in (0, h^*]$. For all $n \in \{1, \dots, N\}$, we have*

$$\|V^n\| \leqslant \|V^0\| + M\tau + Lh \sum_{k=0}^{n} \|V^k\| + 2Lh^2 \sum_{k=1}^{n} \sum_{p=0}^{k-1} \|V^p\|. \tag{2.3}$$

Proof. Let $h \in (0, h^*]$ and $n \in \{1, \dots, N\}$. By definition of the scheme we have

$$U^{n+1} = \mathrm{Proj}\big(K, 2U^n - U^{n-1} + h^2 F^n\big)$$

which implies that

$$\big(2U^n - U^{n-1} + h^2 F^n - U^{n+1}, z - U^{n+1}\big) \leqslant 0 \quad \forall z \in K$$

i.e.

$$\big(V^{n-1} - V^n + hF^n, z - U^{n+1}\big) \leqslant 0 \quad \forall z \in K. \tag{2.4}$$

Furthermore, we have also $U^n \in K$, thus

$$\big(V^{n-1} - V^n + hF^n, U^n - U^{n+1}\big) = -h\big(V^{n-1} - V^n + hF^n, V^n\big) \leqslant 0. \tag{2.5}$$

We infer immediately that

$$\|V^n\| \leqslant \|V^{n-1}\| + h\|F^n\|$$

and

$$\|V^n\| \leqslant \|V^0\| + h\sum_{k=1}^{n} \|F^k\|. \tag{2.6}$$

Since F is L-lipschitzian with respect to its second, third and fourth arguments, for all $k \in \{1, \ldots, N\}$, we have

$$\begin{aligned}\|F^k\| &\leqslant \left\| F\left(kh, U^k, U^{k-1}, \frac{V^k + V^{k-1}}{2}, h\right)\right\| \\ &\leqslant \left\| F(kh, u_0, u_0, 0, h)\right\| + L\left(\|U^k - U^0\| + \|U^{k-1} - U^0\| + \frac{\|V^k + V^{k-1}\|}{2}\right).\end{aligned}$$

The first term of the right-hand side can be estimated with the constant M defined by (2.2), thus we get

$$\begin{aligned}\|F^k\| &\leqslant M + L\left\|\sum_{p=0}^{k-1} hV^p\right\| + L\left\|\sum_{p=0}^{k-2} hV^p\right\| + L\frac{\|V^k\| + \|V^{k-1}\|}{2} \\ &\leqslant M + 2Lh\sum_{p=0}^{k-1} \|V^p\| + \frac{L}{2}\|V^k\| + \frac{L}{2}\|V^{k-1}\|\end{aligned}$$

and relation (2.6) yields

$$\begin{aligned}\|V^n\| &\leqslant \|V^0\| + nhM + 2Lh^2\sum_{k=1}^{n}\sum_{p=0}^{k-1} \|V^p\| + \frac{Lh}{2}\sum_{k=1}^{n}\left(\|V^k\| + \|V^{k-1}\|\right) \\ &\leqslant \|V^0\| + M\tau + Lh\sum_{k=0}^{n} \|V^k\| + 2Lh^2\sum_{k=1}^{n}\sum_{p=0}^{k-1} \|V^p\|. \qquad \square\end{aligned}$$

Since $\lim_{h\to 0} \|z(h)\| = 0$, there exists $h_1 \in (0, h^*]$ such that

$$\left\|z(h)\right\| \leqslant 1 \quad \forall h \in (0, h_1]$$

and, recalling that $u_0 = U^0 \in K$, for all $h \in (0, h^*]$, we obtain

$$\|V^0\| = \frac{\big\|\mathrm{Proj}(K, U^0 + hv_0 + hz(h)\big) - U^0\big\|}{h} \leqslant \big\|v_0 + z(h)\big\| \leqslant \|v_0\| + 1.$$

Moreover, (2.3) implies that, for all $n \in \{1, \dots, N\}$ and for all $h \in (0, h_1]$

$$\|V^n\|(1 - Lh) \leqslant \|v_0\| + M\tau + 1 + Lh \sum_{k=0}^{n-1} \|V^k\| + 2Lh^2 \sum_{k=1}^{n} \sum_{p=0}^{k-1} \|V^p\|.$$

Possibly decreasing h_1, we may assume without loss of generality that $h_1 \in (0, 1/L)$. Then, for all $h \in (0, h_1]$, we define

$$\varphi^0 = \frac{\|v_0\| + M\tau + 1}{1 - Lh},$$

$$\varphi^n = \frac{1}{1 - Lh} \left(\|v_0\| + M\tau + 1 + Lh \sum_{k=0}^{n-1} \varphi^k + 2Lh^2 \sum_{k=1}^{n} \sum_{p=0}^{k-1} \varphi^p \right) \quad \forall n \geqslant 1.$$

A trivial induction shows that $\|V^n\| \leqslant \varphi^n$ for all $n \in \{0, \dots, N\}$. Moreover, we have the following result:

Lemma 2.3. *There exist $C_1 > 0$ and $\kappa > 0$ such that*

$$0 \leqslant \varphi^n \leqslant C_1 e^{\kappa n h} \quad \forall n \geqslant 0 \quad \forall h \in (0, h_1].$$

Proof. See Lemma A.1 in Appendix A.

It follows that

$$\|V^n\| \leqslant \varphi^n \leqslant C_1 e^{\kappa n h} \leqslant C = C_1 e^{\kappa \tau} \quad \forall n \in \{1, \dots, N\}, \quad \forall h \in (0, h_1]$$

which concludes the proof. □

Let us establish now an estimate for the discrete accelerations.

Proposition 2.4. *There exist $h_1^* \in (0, h_1]$ and $C' > 0$ such that*

$$\sum_{n=1}^{N} \|V^n - V^{n-1}\| \leqslant C' \quad \forall h \in (0, h_1^*]. \tag{2.7}$$

Proof. Let h_1 and C be defined as in Proposition 2.1 and K_1 and M_1 be defined by

$$K_1 = K \cap \overline{B}(u_0, C\tau)$$

and

$$M_1 = \max\{\|F(t,u,u',v,h)\|,\ t \in [0,\tau],\ (u,u') \in K_1^2,\ \|v\| \leqslant C,\ h \in [0,h^*]\}. \quad (2.8)$$

By definition of scheme, we have $U^n \in K$ for all $n \in \{0,\dots,N+1\}$ and, using Proposition 2.1

$$\|U^n - u_0\| = \|U^n - U^0\| = h\left\|\sum_{k=0}^{n-1} V^k\right\| \leqslant Cnh \leqslant C\tau.$$

Thus $U^n \in K_1$ for all $n \in \{0,\dots,N+1\}$ and $\|F^n\| \leqslant M_1$ for all $n \in \{1,\dots,N\}$.

By Lemma A.2, we infer that, for all $q \in K_1$, there exist $a_q \in \mathbb{R}^d$ and two strictly positive numbers δ_q and r_q such that, for all $q' \in \overline{B}(q, 2\delta_q)$

$$\overline{B}(a_q, r_q) \subset T_K(q') \quad (2.9)$$

and

$$\|z - \mathrm{Proj}(T_K(q'), z)\| \leqslant \frac{1}{2r}\Big(\|z-a_q\|^2 - \|\mathrm{Proj}(T_K(q'),z) - a_q\|^2\Big) \quad \forall z \in \mathbb{R}^d. \quad (2.10)$$

It is obvious that $K_1 \subset \bigcup_{q\in K_1} B(q,\delta_q)$, and a compactness argument implies that there exists $(q_i)_{1\leqslant i\leqslant \ell}$ such that

$$K_1 \subset \bigcup_{i=1}^{\ell} B(q_i, \delta_{q_i}).$$

In the remainder of the proof we will simply write δ_i, a_i and r_i instead of δ_{q_i}, a_{q_i} and r_{q_i}. We define

$$r = \min_{1\leqslant i\leqslant \ell} r_i, \quad \delta = \min_{1\leqslant i\leqslant \ell} \delta_i, \quad \tau_1 = \frac{\delta}{C}.$$

Let $h_1^* \in \big(0, \min(h_1, \tau_1)\big)$, $h \in (0, h_1^*]$ and $n \in \{0,\dots,N\}$. Let $i \in \{1,\dots,\ell\}$ be such that $U^{n+1} \in B(q_i,\delta_i)$. Then, for all $m \in \{n,\dots,p\}$ with $p = \min\big(N, n + \lfloor \tau_1/h \rfloor\big)$,

we have

$$\begin{aligned}\|U^{m+1}-q_i\| &\leqslant \|U^{m+1}-U^{n+1}\|+\|U^{n+1}-q_i\| \\ &\leqslant \left\|\sum_{k=n+1}^{m} hV^k\right\|+\delta_i \leqslant hC(m-n)+\delta_i \leqslant \delta+\delta_i \leqslant 2\delta_i.\end{aligned}$$

By applying (2.9)–(2.10), we obtain that, for all $m \in \{n,\dots,p\}$, we have

$$\overline{B}(a_i,r_i) \subset T_K(U^{m+1})$$

and

$$\big\|z-\mathrm{Proj}(T_K(U^{m+1}),z)\big\| \leqslant \frac{1}{2r_i}\big(\|z-a_i\|^2-\big\|\mathrm{Proj}(T_K(U^{m+1}),z)-a_i\big\|^2\big) \quad \forall z\in\mathbb{R}^d.$$

But, relation (2.4) implies that

$$V^{m-1}-V^m+hF^m \in N_K(U^{m+1}).$$

Since $N_K(U^{m+1})$ and $T_K(U^{m+1})$ are two closed convex polar cones, we infer that (see [5])

$$\mathrm{Proj}\big(T_K(U^{m+1}),V^{m-1}-V^m+hF^m\big)=0.$$

Consequently, we obtain

$$\begin{aligned}\|V^{m-1}-V^m+hF^m\| &\leqslant \frac{1}{2r_i}\big(\big\|(V^{m-1}-V^m+hF^m)-a_i\big\|^2-\|a_i\|^2\big) \\ &\leqslant \frac{1}{2r_i}\Big(\big\|V^{m-1}-V^m+hF^m\big\|^2-2\big(a_i,V^{m-1}-V^m+hF^m\big)\Big)\end{aligned}$$

and thus

$$\begin{aligned}\|V^{m-1}-V^m\| \leqslant\ & h\|F^m\|+\frac{1}{2r_i}\Big(\|V^{m-1}-V^m\|^2+2h\big(F^m,V^{m-1}-V^m\big) \\ & +h^2\|F^m\|^2-2\big(a_i,V^{m-1}-V^m+hF^m\big)\Big).\end{aligned}$$

Moreover, relation (2.5) implies that

$$-\big(V^{m-1},V^m\big) \leqslant -\|V^m\|^2+h\big(F^m,V^m\big)$$

which yields

$$\begin{aligned}\|V^{m-1}-V^m\|^2 &= \|V^{m-1}\|^2 - 2\big(V^{m-1},V^m\big)+\|V^m\|^2\\ &\leqslant \|V^{m-1}\|^2-\|V^m\|^2+2h\big(F^m,V^m\big).\end{aligned}$$

It follows that

$$\begin{aligned}\|V^{m-1}-V^m\| \leqslant h\|F^m\| + \frac{1}{2r_i}\Big(&\|V^{m-1}\|^2-\|V^m\|^2-2\big(a_i,V^{m-1}-V^m\big)\\ &+2h\big(F^m,V^{m-1}-a_i\big)+h^2\|F^m\|^2\Big).\end{aligned}$$

Thus, for all $m\in\{n,\dots,p\}$, we have

$$\|V^{m-1}-V^m\|\leqslant hC_2+\frac{1}{2r}\Big(\|V^{m-1}\|^2-\|V^m\|^2-2\big(a_i,V^{m-1}-V^m\big)\Big)$$

with

$$C_2=M_1\left(1+\frac{C+a}{r}\right)+M_1^2\frac{h^*}{2r},\quad a=\max_{1\leqslant i\leqslant \ell}\|a_i\|.$$

By summation we obtain

$$\begin{aligned}\sum_{m=n+1}^{p}\|V^{m-1}-V^m\| &\leqslant (p-n)hC_2+\frac{1}{2r}\Big(\|V^n\|^2-\|V^p\|^2-2\big(a_i,V^n-V^p\big)\Big)\\ &\leqslant (p-n)hC_2+\frac{1}{2r}\big(\|V^n\|^2-\|V^p\|^2+4Ca\big).\end{aligned}$$

Recalling that $p=\min\big(N,n+\lfloor\tau_1/h\rfloor\big)$, we infer that

$$\sum_{m=1}^{N}\|V^{m-1}-V^m\|\leqslant NhC_2+\frac{1}{2r}\big(\|V^0\|^2-\|V^N\|^2\big)+(k_1+1)4Ca,$$

where $k_1\in\mathbb{N}$ is such that

$$k_1\left\lfloor\frac{\tau_1}{h}\right\rfloor\leqslant N<(k_1+1)\left\lfloor\frac{\tau_1}{h}\right\rfloor.$$

Observing that $k_1\leqslant\tau/(\tau_1-h)$, we can conclude the proof with

$$C'=\tau C_2+\frac{C^2}{r}+4Ca\left(\frac{\tau}{\tau_1-h_1^*}+1\right).\qquad\square$$

2.2. Passage to the limit as h tends to zero

Thanks to Proposition 2.1, we know that the functions u_h, $0 < h \leqslant h_1^*$, are C-Lipschitz continuous on $[0, \tau]$. Hence, $(u_h)_{0<h\leqslant h_1^*}$ is a bounded and equicontinuous family of functions of $C^0([0,\tau];\mathbb{R}^d)$. Applying Ascoli's theorem we may extract a subsequence, denoted $(u_{h_i})_{0<h_i\leqslant h_1^*}$, such that:

$$\begin{aligned} u_{h_i} &\to u \quad \text{strongly in } C^0([0,\tau];\mathbb{R}^d), \\ \dot{u}_{h_i} &\to \dot{u} \quad \text{in } L^\infty(0,\tau;\mathbb{R}^d) \text{ weak*}. \end{aligned}$$

Since $U^n \in K$ for all n belonging to $\{0,\dots,N+1\}$, we infer that, for all $h_i \in (0, h_1^*]$ and for all $t \in [0,\tau]$:

$$\operatorname{dist}(u_{h_i}(t), K) \leqslant h_i \max_{0\leqslant j\leqslant N} \|V^j\| \leqslant h_i C.$$

Passing to the limit when h_i tends to 0, we obtain that $u(t) \in K$ for all $t \in [0,\tau]$ and u satisfies the property (P1).

The measure $\ddot{u}_h$ is a sum of Dirac's measures on $(0,\tau)$, more precisely, we have

$$\ddot{u}_h(t) = \sum_{n=1}^{N} (V^n - V^{n-1})\delta(t - nh) \quad \forall h \in (0, h_1^*].$$

Consequently, the total variation of $\dot{u}_h$ on $(0,\tau)$ is equal to

$$TV(\dot{u}_h) = \sum_{n=1}^{N} \|V^n - V^{n-1}\|$$

and estimate (2.7) implies that $(\dot{u}_h)_{0<h\leqslant h_1^*}$ is a bounded family of $BV(0,\tau;\mathbb{R}^d)$. Using Helly's theorem and possibly extracting another subsequence, we may conclude that $(\dot{u}_{h_i})_{0<h_i\leqslant h_1^*}$ converge, except perhaps on a countable set of points, to a function of bounded variation. Hence

$$\dot{u} \in BV(0,\tau;\mathbb{R}^d), \quad \dot{u}_{h_i} \to \dot{u} \quad \text{except perhaps on a countable set of points } \mathcal{D}$$

and

$$\ddot{u}_{h_i} \to \ddot{u} \quad \text{weakly* in } M^1(0,\tau;\mathbb{R}^d).$$

It follows that (P2) is also satisfied. Moreover, let us define the sets $(D_i)_{i\geqslant 0}$ and D_∞ by

$$D_i = (0,\tau)\cap h_i\mathbb{N}, \quad D_\infty = \big\{t\in(0,\tau);\ \dot u(t-0)\neq \dot u(t+0)\big\}$$

and let

$$D = \bigcup_{i\geqslant 0} D_i \cup D_\infty \cup \mathcal{D}. \tag{2.11}$$

Since $\dot u \in BV(0,\tau;\mathbb{R}^d)$, D_∞ and D are denumerable and, for all $t\in(0,\tau)\setminus D$ we have

$$\dot u(t-0)=\dot u(t+0)=\dot u(t), \quad \dot u_{h_i}(t-0)=\dot u_{h_i}(t+0)=\dot u_{h_i}(t) \quad \forall i\geqslant 0$$

and

$$\dot u(t)=\lim_{h_i\to 0}\dot u_{h_i}(t). \tag{2.12}$$

Let F_h be the measure defined on $(0,\tau)$ by

$$F_h(t)=\sum_{n=1}^{N} hF^n\delta(t-nh) \quad \forall h\in(0,h_1^*].$$

Lemma 2.5. *The sequence $(F_{h_i})_{0<h_i\leqslant h_1^*}$ converges weakly* in $M^1(0,\tau;\mathbb{R}^d)$ to $f(t,u,\dot u)$.*

Proof. We know that $(\dot u_{h_i})_{0<h_i\leqslant h_1^*}$ converges to $\dot u$ in $BV\big(0,\tau;\mathbb{R}^d\big)$. In particular Lebesgue's theorem implies that $(\dot u_{h_i})_{0<h_i\leqslant h_1^*}$ converges to $\dot u$ in $L^1\big(0,\tau;\mathbb{R}^d\big)$. We extend $\dot u_{h_i}$ and $\dot u$ to $\mathbb{R}$ by 0 outside of $[0,\tau]$ and still denote the respective extensions $\dot u_{h_i}$ and $\dot u$. The set $\{\dot u_{h_i} : h_i\in(0,h_1^*]\}\cup\{\dot u\}$ is a compact subset of $L^1(\mathbb{R};\mathbb{R}^d)$. The classical characterization of compact subsets of $L^1(\mathbb{R};\mathbb{R}^d)$ implies that

$$\lim_{\theta\to 0}\sup_{0\leqslant h_i\leqslant h_1^*}\int_{\mathbb{R}}\big\|\dot u_{h_i}(t-\theta)-\dot u_{h_i}(t)\big\|\,dt=0.$$

Letting $\theta=h_i$, we can see that $\dot u_{h_i}(.-h_i)$ converges to $\dot u$ in $L^1\big(\mathbb{R};\mathbb{R}^d\big)$.

Let us define an approximate velocity v_{h_i} on $\mathbb{R}$ by

$$v_{h_i}(t) = \frac{\dot{u}_{h_i}(t - h_i + 0) + \dot{u}_{h_i}(t + 0)}{2} \quad \forall t \in \mathbb{R}.$$

The sequence $(v_{h_i})_{0<h_i \leqslant h_1^*}$ converges to $\dot{u}$ in $L^1\big(\mathbb{R}; \mathbb{R}^d\big)$. Moreover, estimate (2.1) implies that

$$\big\|v_{h_i}(t)\big\| \leqslant C \quad \forall t \in \mathbb{R} \quad \forall h_i \in (0, h_1^*].$$

Let $\phi \in C^0\big([0, \tau]; \mathbb{R}^d\big)$. By definition of F_{h_i} we have

$$\begin{aligned}\langle F_{h_i}, \phi\rangle &= \sum_{n=1}^{N} h_i\big(F^n, \phi(nh_i)\big) = \sum_{n=1}^{N-1} \int_{nh_i}^{(n+1)h_i} \big(F^n, \phi(t)\big)\, dt + \int_{Nh_i}^{\tau} \big(F^N, \phi(t)\big)\, dt \\ &\quad + \sum_{n=1}^{N-1} \int_{nh_i}^{(n+1)h_i} \big(F^n, \phi(nh_i) - \phi(t)\big)\, dt + h_i\big(F^N, \phi(Nh_i)\big) \\ &\quad - \int_{Nh_i}^{\tau} \big(F^N, \phi(t)\big)\, dt. \end{aligned} \tag{2.13}$$

Recalling that $\|F^n\| \leqslant M_1$ for all $n \in \{1, \dots, N\}$ we can easily estimate the last two terms:

$$\big\|h_i\big(F^N, \phi(Nh_i)\big)\big\| \leqslant h_i M_1 \|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}$$

and

$$\left\|\int_{Nh_i}^{\tau} \big(F^N, \phi(t)\big)\, dt\right\| \leqslant (\tau - Nh_i) M_1 \|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} \leqslant h_i M_1 \|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}.$$

Moreover, we denote ω_ϕ the modulus of continuity of ϕ on $[0, \tau]$. We get

$$\sum_{n=1}^{N-1} \left\|\int_{nh_i}^{(n+1)h_i} \big(F^n, \phi(nh_i) - \phi(t)\big)\, dt\right\| \leqslant M_1 (N-1) h_i \omega_\phi(h_i) \leqslant M_1 \tau \omega_\phi(h_i).$$

Let us compare now the two first terms of the right-hand side of (2.13) with

$$\int_0^{\tau} \big(f\big(t, u(t), \dot{u}(t)\big), \phi(t)\big)\, dt.$$

For all $n \in \{1, \dots, N\}$ and $t \in \big[nh_i, (n+1)h_i\big) \cap [0, \tau]$ we have

$$\begin{aligned} F^n &= F\left(nh_i, U^n, U^{n-1}, \frac{V^n + V^{n-1}}{2}, h_i\right) \\ &= F\big(nh_i, u_{h_i}(nh_i), u_{h_i}(nh_i - h_i), v_{h_i}(t), h_i\big) \end{aligned}$$

thus

$$\begin{aligned} &\big\| F^n - f\big(t, u_{h_i}(t), v_{h_i}(t)\big)\big\| \\ &\quad \leqslant \big\| F\big(nh_i, u_{h_i}(nh_i), u_{h_i}\big((n-1)h_i\big), v_{h_i}(t), h_i\big) - F\big(nh_i, u_{h_i}(t), u_{h_i}(t), v_{h_i}(t), h_i\big)\big\| \\ &\qquad + \big\| F\big(nh_i, u_{h_i}(t), u_{h_i}(t), v_{h_i}(t), h_i\big) - f\big(nh_i, u_{h_i}(t), v_{h_i}(t)\big)\big\| \\ &\qquad + \big\| f\big(nh_i, u_{h_i}(t), v_{h_i}(t)\big) - f\big(t, u_{h_i}(t), v_{h_i}(t)\big)\big\|. \end{aligned}$$

The first term on the right-hand side is estimated by

$$L\big\|u_{h_i}(nh_i) - u_{h_i}(t)\big\| + L\big\|u_{h_i}\big((n-1)h_i\big) - u_{h_i}(t)\big| \leqslant 3LCh_i.$$

Let us denote by ω_F the modulus of continuity of F on the compact set $[0, \tau] \times \overline{B}(u_0, C\tau)^2 \times \overline{B}(0, C) \times [0, h^*]$. The second term is equal to

$$\big\| F\big(nh_i, u_{h_i}(t), u_{h_i}(t), v_{h_i}(t), h_i\big) - F\big(nh_i, u_{h_i}(t), u_{h_i}(t), v_{h_i}(t), 0\big)\big\|$$

and can be estimated by $\omega_F(h_i)$. Then, by denoting ω_f the modulus of continuity of f on the compact set $[0, \tau] \times \overline{B}(u_0, C\tau) \times \overline{B}(0, C)$, the third term can be estimated by $\omega_f(h_i)$.

Therefore, using the Lipschitz continuity of f with respect to its last two arguments we get

$$\begin{aligned} &\left\| \langle F_{h_i}, \phi\rangle - \int_0^\tau \big(f\big(t, u(t), \dot{u}(t)\big), \phi(t)\big)\, dt \right\| \\ &\quad \leqslant \int_0^\tau \big\| f\big(t, u_{h_i}(t), v_{h_i}(t)\big) - f\big(t, u(t), \dot{u}(t)\big)\big\| \big\|\phi(t)\big\|\, dt \\ &\qquad + 2h_i M_1 \|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} + M_1 \tau \omega_\phi(h_i) \\ &\qquad + \big(3LCh_i + \omega_F(h_i) + \omega_f(h_i)\big) \int_0^\tau \|\phi(t)\|\, dt \\ &\quad \leqslant L\|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} \int_0^\tau \big(\big\|u_{h_i}(t) - u(t)\big\| + \big\|v_{h_i}(t) - \dot{u}(t)\|\big)\, dt \\ &\qquad + 2h_i M_1 \|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} + M_1 \tau \omega_\phi(h_i) \\ &\qquad + \big(3LCh_i + \omega_F(h_i) + \omega_f(h_i)\big) \int_0^\tau \|\phi(t)\|\, dt. \end{aligned} \tag{2.14}$$

With the previous results we know that $(u_{h_i})_{h_1^* \geqslant h_i > 0}$ converges to u in $C^0\big([0,\tau];\mathbb{R}^d\big)$ and $(v_{h_i})_{h_1^* \geqslant h_i > 0}$ converges to $\dot{u}$ in $L^1(\mathbb{R};\mathbb{R}^d)$. Thus, the first integral term on the right-hand side of (2.14) tends to 0 as h_i tends to 0. The convergence to zero of the other terms is clear. $\square$

Let us define $\mu_h = \ddot{u}_h - F_h$ i.e.

$$\mu_h = \sum_{n=1}^{N} (V^n - V^{n-1} - hF^n)\delta(t - nh) \quad \forall h \in (0, h_1^*].$$

With all the previous results, we know that $(\mu_{h_i})_{0<h_i \leqslant h_1^*}$ converges to $\mu = \ddot{u} - f(t,u,\dot{u})$ weakly* in $M^1(0,\tau;\mathbb{R}^d)$. At the limit, we obtain the equality

$$\ddot{u} = f(t,u,\dot{u}) + \mu \quad \text{in } M^1(0,\tau;\mathbb{R}^d).$$

Let us prove now that the measure μ satisfies property (P3).

Proposition 2.6. *The measure μ satisfies property* (P3) *i.e.*

$$\mathrm{Supp}(\mu) \subset \big\{t \in [0,\tau];\, u(t) \in \partial K\big\}$$

and

$$\langle \mu, v - u\rangle \geqslant 0 \quad \forall v \in C^0\big([0,\tau];K\big).$$

Proof. Let us prove first that

$$\langle \mu, v - u\rangle \geqslant 0 \quad \forall v \in C^0\big([0,\tau];K\big).$$

Let v be continuous from $[0,\tau]$ to K. By definition of μ and μ_{h_i} we have

$$\langle \mu, v - u\rangle = \lim_{h_i \to 0} \langle \mu_{h_i}, v - u\rangle = \lim_{h_i \to 0} \sum_{n=1}^{N} \big(V^n - V^{n-1} - h_i F^n, v(nh_i) - u(nh_i)\big).$$

Let $h_i \in (0, h_1^*]$. Using (2.4), we have

$$\big(h_i F^n - V^n + V^{n-1}, z - U^{n+1}\big) \leqslant 0 \quad \forall z \in K \quad \forall n \in \{1,\dots,N\}.$$

Since $v(nh_i) \in K$ for all $n \in \{1, \dots, N\}$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{N} \big(V^n - V^{n-1} - h_i F^n, v(nh_i) - u(nh_i)\big) \\
&\quad \geqslant \sum_{n=1}^{N} \big(V^n - V^{n-1} - h_i F^n, U^{n+1} - u(nh_i)\big) \\
&\quad = \sum_{n=1}^{N} \big(V^n - V^{n-1} - h_i F^n, hV^n + u_{h_i}(nh_i) - u(nh_i)\big)
\end{aligned}$$

and estimates (2.1) and (2.7) yield

$$\begin{aligned}
\langle \mu_{h_i}, v - u \rangle &= \sum_{n=1}^{N} \big(V^n - V^{n-1} - h_i F^n, v(nh_i) - u(nh_i)\big) \\
&\geqslant -\sum_{n=1}^{N} \Big(h_i M_1 + \|V^n - V^{n-1}\|\Big) \Big(h_i C + \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)}\Big) \\
&\geqslant -(\tau M_1 + C') \Big(h_i C + \|u - u_{h_i}\|_{C^0([0,\tau];\mathbb{R}^d)}\Big),
\end{aligned}$$

where M_1 is defined by (2.8). Passing to the limit when h_i tends to zero, we may conclude the first part of the proof.

Let us prove now that

$$\mathrm{Supp}(\mu) \subset \big\{t \in [0, \tau]; u(t) \in \partial K\big\}.$$

Let $\phi \in C^0\big([0, \tau]; K\big)$ be such that $\phi \not\equiv 0$ and

$$\mathrm{Supp}(\phi) \subset [0, \tau] \setminus \big\{t \in [0, \tau]; u(t) \in \partial K\big\} = \big\{t \in [0, \tau]; u(t) \in \mathrm{Int}(K)\big\}.$$

Then, for all $t \in \mathrm{Supp}(\phi)$, there exists $r_t > 0$ such that $\overline{B}\big(u(t), r_t\big) \subset K$. Observing that

$$\mathrm{Supp}(\phi) \subset \bigcup_{t \in \mathrm{Supp}(\phi)} \Big(t - \frac{r_t}{2C}, t + \frac{r_t}{2C}\Big)$$

and that $\mathrm{Supp}(\phi)$ is a compact subset of $\mathbb{R}$, we infer that there exists $\{t_1, \ldots, t_p\} \subset \mathrm{Supp}(\phi)$ such that

$$\mathrm{Supp}(\phi) \subset \bigcup_{k=1}^{p} \left(t_k - \frac{r_{t_k}}{2C}, t_k + \frac{r_{t_k}}{2C}\right).$$

Let $r = \min_{1 \leqslant k \leqslant p} \frac{r_{t_k}}{2}$. Then, for all $t \in \mathrm{Supp}(\phi)$, $\overline{B}\big(u(t), r\big) \subset K$. Indeed, let $t \in \mathrm{Supp}(\phi)$ and $z \in \overline{B}\big(u(t), r\big)$. There exists $k \in \{1, \ldots, p\}$ such that $t \in \left(t_k - \frac{r_{t_k}}{2C}, t_k + \frac{r_{t_k}}{2C}\right)$ and, recalling that u is C-lipschitzian, we get

$$\big\|z - u(t_k)\big\| \leqslant \big\|z - u(t)\big\| + \big\|u(t) - u(t_k)\big\| \leqslant r + C|t - t_k| \leqslant r_{t_k}.$$

Hence $z \in \overline{B}\big(u(t_k), r_{t_k}\big) \subset K$.

Let us define now $v_\pm = u \pm \frac{r}{\|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}}\phi$. It is clear that $v_\pm \in C^0\big([0, \tau]; K\big)$ and, with the first part of the proof,

$$\langle \mu, v_\pm - u \rangle = \pm \frac{r}{\|\phi\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}} \langle \mu, \phi \rangle \geqslant 0.$$

Thus

$$\langle \mu, \phi \rangle = 0$$

which enables us to conclude. □

Let us conclude this subsection with the proof of property (P4).

Proposition 2.7. *The initial conditions (u_0, v_0) are satisfied i.e.*

$$u(0) = u_0, \quad \dot{u}(0 + 0) = v_0.$$

Proof. Since $u_{h_i}(0) = U^0 = u_0$ for all $h_i \in (0, h_1^*]$, the first equality is an immediate consequence of the uniform convergence of $(u_{h_i})_{h_1^* \geqslant h_i > 0}$ to u on $[0, \tau]$. In order to prove the second equality, we begin with the following lemma.

Lemma 2.8. *Under the previous assumptions we have*

$$\big(v_0 - \dot{u}(0 + 0), z - u_0\big) \leqslant 0 \quad \forall z \in K.$$

Proof. Let $z \in K$ and$\rho \in (0, \tau) \setminus D$, where D is defined by (2.11). We will prove that

$$\big(v_0 - \dot{u}_{h_i}(\rho), z - u_0\big) \leqslant \mathcal{O}(\rho) + \mathcal{O}\big(\|z(h_i)\|\big) + \mathcal{O}(h_i).$$

Passing to the limit as h_i tends to zero first, then as ρ tends to zero, we will obtain the announced result.

Let $0 < h_i < \min(h_1^*, \rho)$. By definition of u_{h_i} we have

$$\big(v_0 - \dot{u}_{h_i}(\rho), z - u_0\big) = \big(v_0 - V^p, z - u_0\big) \quad \text{with } p = \left\lfloor \frac{\rho}{h_i} \right\rfloor$$

which we rewrite as

$$\begin{aligned}\big(v_0 - V^p, z - u_0\big) \;=\; & \big(v_0 - V^0 + z(h_i), z - u_0\big) - \big(z(h_i), z - u_0\big) \\ & + \sum_{n=1}^{p} \big(V^{n-1} - V^n + h_i F^n, z - u_0\big) \\ & - \sum_{n=1}^{p} h_i \big(F^n, z - u_0\big).\end{aligned}$$

Using relation (2.4), we know that for all $n \in \{1, \ldots, N\}$

$$\big(V^{n-1} - V^n + h_i F^n, z - U^{n+1}\big) \leqslant 0$$

and, since $U^1 = \mathrm{Proj}\big(u_0 + h_i v_0 + h_i z(h_i)\big)$, we have also

$$h\big(v_0 - V^0 + z(h_i), z - U^1\big) = \big(u_0 + h_i v_0 + h_i z(h_i) - U^1, z - U^1\big) \leqslant 0.$$

Thus we get

$$\begin{aligned}\big(v_0 - \dot{u}_{h_i}(\rho), z - u_0\big) \;\leqslant\; & \big(v_0 - V^0 + z(h_i), U^1 - u_0\big) - \big(z(h_i), z - u_0\big) \\ & + \sum_{n=1}^{p} \big(V^{n-1} - V^n + h_i F^n, U^{n+1} - u_0\big) - \sum_{n=1}^{p} h_i \big(F^n, z - u_0\big).\end{aligned}$$

Let us estimate each term of the right-hand side of this inequality. Using the estimates (2.1) and (2.7), we obtain

$$\big\|U^{n+1} - u_0\big\| \leqslant \sum_{k=0}^{n} h_i \|V^k\| \leqslant C(n+1)h_i \leqslant C(\rho + h_i) \quad \forall n \in \{1, \ldots, p\}$$

and

$$\Big\|\sum_{n=1}^{p}\big(V^{n-1}-V^{n},U^{n+1}-u_0\big)\Big\|\leqslant\sum_{n=1}^{p}\big\|V^{n-1}-V^{n}\big\|\big\|U^{n+1}-u_0\big\|\leqslant CC'(\rho+h_i).$$

Moreover, we have

$$\|F^n\|\leqslant M_1\quad\forall n\in\{1,\dots,N\}$$

with M_1 defined by (2.8). Thus

$$\begin{aligned}&\Big\|\sum_{n=1}^{p}h_i\big(F^n,U^{n+1}-u_0\big)-\sum_{n=1}^{p}h_i\big(F^n,z-u_0\big)\Big\|\\&\quad\leqslant\sum_{n=1}^{p}h_i\|F^n\|\big(\|U^{n+1}-u_0\|+\|z-u_0\|\big)\\&\quad\leqslant ph_iM_1\big(C(\rho+h_i)+\|z-u_0\|\big)\\&\quad\leqslant\rho M_1\big(C(\rho+h_i)+\|z-u_0\|\big).\end{aligned}$$

Finally,

$$\|V^0\|=\left\|\frac{U^1-u_0}{h_i}\right\|=\frac{1}{h_i}\big\|\mathrm{Proj}(K,u_0,+h_iv_0+h_iz(h_i))-u_0\big\|\leqslant\|v_0\|+\big\|z(h_i)\big\|$$

and

$$\big\|\big(v_0-V^0+z(h_i),U^1-u_0\big)\big\|=h_i\big\|\big(v_0-V^0+z(h_i),V^0\big)\big\|\leqslant2h_i\big(\|v_0\|+\big\|z(h_i)\big\|\big)^2.$$

Thus, we get

$$\begin{aligned}\big(v_0-\dot u_{h_i}(\rho),z-u_0\big)&\leqslant CC'(\rho+h_i)+\rho M_1\big(C(\rho+h_i)+\|z-u_0\|\big)\\&\quad+\big\|z(h_i)\big\|\|z-u_0\|+2h_i\big(\|v_0\|+\big\|z(h_i)\big\|\big)^2\end{aligned}$$

which concludes the proof. □

The previous lemma implies that

$$\big(v_0-\dot u(0+0),w\big)\leqslant0\quad\forall w\in T_K(u_0).$$

If $u_0 \in \mathrm{Int}(K)$ then $T_K(u_0) = \mathbb{R}^d$ and the conclusion follows immediately. Otherwise, by choosing successively $w = \dot{u}(0+0)$ and $w = v_0$, we obtain

$$\|v_0\|^2 \leqslant \big(\dot{u}(0+0), v_0\big) \leqslant \big\|\dot{u}(0+0)\big\|^2.$$

With (2.6) we have also

$$\big\|\dot{u}_{h_i}(\rho)\big\| = \|V^p\| \leqslant \|V^0\| + h_i \sum_{k=1}^{p} \|F^k\| \leqslant \|v_0\| + \big\|z(h_i)\big\| + \rho M_1 \quad \text{with } p = \left\lfloor \frac{\rho}{h_i} \right\rfloor$$

for all $\rho \in (0,\tau) \setminus D$ and for all $h_i \in \big(0, \min(h_1^*, \rho)\big)$. By passing to the limit as h_i tends to zero, then as ρ tends to zero, we get

$$\big\|\dot{u}(0+0)\big\| \leqslant \|v_0\|.$$

Hence

$$\|v_0\|^2 = \big(\dot{u}(0+0), v_0\big) = \big\|\dot{u}(0+0)\big\|^2,$$

and $v_0 = \dot{u}(0+0)$. $\square$

2.3. Transmission of the velocities at impacts

In this subsection, we study the behavior of the limit of the scheme at impacts. We will assume from now on that the limit motion satisfies property (H5), i.e.

$$\text{(H5)} \qquad \big(\nabla\varphi_\alpha\big(u(t)\big), \nabla\varphi_\beta\big(u(t)\big)\big) \leqslant 0 \quad \forall(\alpha,\beta) \in J\big(u(t)\big)^2, \alpha \neq \beta \quad \forall t \in (0,\tau)$$

and we will prove that u satisfies Moreau's rule for inelastic shocks, i.e.

$$\dot{u}(t+0) = \mathrm{Proj}\big(T_K\big(u(t)\big), \dot{u}(t-0)\big) \quad \forall t \in (0,\tau). \tag{2.15}$$

More precisely, let $\bar{t} \in (0,\tau)$ and denote $\bar{u} = u(\bar{t})$, $\dot{u}^+ = \dot{u}(\bar{t}+0)$, $\dot{u}^- = \dot{u}(\bar{t}-0)$. Since $u(t) \in K$ for all $t \in [0,\tau]$, we have

$$\dot{u}^+ \in T_K(\bar{u}), \quad \dot{u}^- \in -T_K(\bar{u}).$$

Thus, if $\dot{u}^+ = \dot{u}^-$, we get $\dot{u}^- \in T_K(\bar{u})$ and the impact law is satisfied. Otherwise, we have $\mu\big(\{\bar{t}\}\big) = \dot{u}^+ - \dot{u}^- \neq 0$ and the measure μ has a Dirac mass at $\bar{t}$.

Let us decompose μ with respect to Lebesgue's measure: there exists $g \in L^1(0,\tau;\mathbb{R}^d)$ such that

$$d\mu = g\,dt + d\mu_s,$$

where μ_s is a singular measure with respect to Lebesgue's measure. Using Radon–Nicodym's theorem we infer that there exists a $|\mu_s|$-integrable function h_s such that

$$d\mu_s = h_s d|\mu_s|.$$

Then, property (P3) implies that (see [18])

$$\begin{aligned} &g(t) \in -N_K\big(u(t)\big) \quad dt \text{ a-e on } (0,\tau),\\ &h_s(t) \in -N_K\big(u(t)\big) \quad |\mu_s| \text{ a-e on } (0,\tau). \end{aligned}$$

It follows that

$$\dot{u}^+ - \dot{u}^- = \mu\big(\{\bar{t}\}\big) \in -N_K(\bar{u}).$$

Thus $\bar{u} \in \partial K$ and $J(\bar{u}) \neq \emptyset$.

In order to prove that (2.15) holds also in this case, we will perform a precise study of the discrete velocities V^n in a neighbourhood of $\bar{t}$. Let us introduce some new notations.

From assumption (H3) we know that $\big(\nabla\varphi_\alpha(\bar{u})\big)_{\alpha\in J(\bar{u})}$ is linearly independent. Hence there exists $(e_\beta)_{\beta\in\{1,\dots,d\}\setminus J(\bar{u})}$ such that the family $\big\{\nabla\varphi_\alpha(\bar{u}), \alpha \in J(\bar{u})\big\} \cup \big\{e_\beta, \beta \in \{1,\dots,d\}\setminus J(\bar{u})\big\}$ is a basis of $\mathbb{R}^d$.

For all $\alpha \in \{1,\dots,d\}$ and for all $q \in \mathbb{R}^d$ we define $e_\alpha(q)$ by

$$e_\alpha(q) = \begin{cases} \nabla\varphi_\alpha(q) & \text{if } \alpha \in J(\bar{u}),\\ e_\alpha & \text{if } \alpha \notin J(\bar{u}). \end{cases}$$

Since the functions φ_α, $1\leqslant\alpha\leqslant v$, belong to $C^1(\mathbb{R}^d;\mathbb{R})$, we infer that there exists $r > 0$ such that $\big(e_\alpha(q)\big)_{\alpha=1,\dots,d}$ is a basis of $\mathbb{R}^d$ for all $q \in \overline{B}(\bar{u},r)$. We define the dual basis $\big(\varepsilon_\alpha(q)\big)_{\alpha=1,\dots,d}$ for all $q \in \overline{B}(\bar{u},r)$. It is clear that the mappings ε_α, $1\leqslant\alpha\leqslant d$, are continuous on $\overline{B}(\bar{u},r)$. Moreover, we recall that $\varphi_\alpha(\bar{u}) > 0$ for all $\alpha \notin J(\bar{u})$. Since the functions φ_α, $1\leqslant\alpha\leqslant v$, are continuous, possibly decreasing r, we may assume without loss of generality that

$$\varphi_\alpha(q) > 0 \quad \forall\alpha \notin J(\bar{u}) \quad \forall q \in \overline{B}(\bar{u},r)$$

i.e.

$$J(q) \subset J(\bar{u}) \quad \forall q \in \overline{B}(\bar{u},r). \tag{2.16}$$

Then, using the continuity of u and the convergence of $(u_i)_{h_1^* \geqslant h_i > 0}$ to u in $C^0\big([0,\tau];\mathbb{R}^d\big)$, we can define $\bar\rho > 0$ and $h_2 \in (0, h_1^*]$ such that $[\bar t - \bar\rho, \bar t + \bar\rho] \subset (0,\tau)$ and

$$u(t) \in \overline{B}(\bar u, r) \quad \forall t \in [\bar t - \bar\rho, \bar t + \bar\rho] \tag{2.17}$$

$$U^{n+1} \in \overline{B}(\bar u, r) \quad \forall n h_i \in [\bar t - \bar\rho, \bar t + \bar\rho] \quad \forall h_i \in (0, h_2]. \tag{2.18}$$

Finally we define

$$M_2 = \sup\big\{\|e_\alpha(q)\|, \|\varepsilon_\alpha(q)\|, \quad q \in \overline{B}(\bar u, r), \quad 1 \leqslant \alpha \leqslant \nu\big\}. \tag{2.19}$$

We prove the following result.

Proposition 2.9. *Let us assume that* $\dot u^+ \neq \dot u^-$ *and*

$$\big(\nabla\varphi_\alpha(\bar u), \nabla\varphi_\beta(\bar u)\big) \leqslant 0 \quad \forall(\alpha,\beta) \in J(\bar u)^2, \alpha \neq \beta.$$

Then the impact law is satisfied at $t = \bar t$, *i.e.*

$$\dot u^+ = \mathrm{Proj}\big(T_K(\bar u), \dot u^-\big). \tag{2.20}$$

Proof. With the definition (1.3) of K we can describe $T_K(\bar u)$ and $N_K(\bar u)$ as follows:

$$\begin{aligned} T_K(\bar u) &= \big\{v \in \mathbb{R}^d;\ \big(\nabla\varphi_\alpha(\bar u), v\big) \geqslant 0 \quad \forall \alpha \in J(\bar u)\big\}, \\ N_K(\bar u) &= \big\{w \in \mathbb{R}^d;\ w = \sum_{\alpha \in J(\bar u)} \lambda_\alpha \nabla\varphi_\alpha(\bar u),\ \lambda_\alpha \leqslant 0 \quad \forall \alpha \in J(\bar u)\big\}. \end{aligned}$$

Thus, there exist non-positive numbers μ_α, $\alpha \in J(\bar u)$, such that

$$\dot u^+ - \dot u^- = \sum_{\alpha \in J(\bar u)} -\mu_\alpha \nabla\varphi_\alpha(\bar u).$$

Recalling that $\dot u^+ \in T_K(\bar u)$, $\dot u^- - \dot u^+ \in N_K(\bar u)$ and $T_K(\bar u)$ and $N_K(\bar u)$ are two polar cones, we infer that (2.20) is equivalent to

$$(\dot u^- - \dot u^+, \dot u^+) = 0 = \sum_{\alpha \in J(\bar u)} \mu_\alpha\big(\nabla\varphi_\alpha(\bar u), \dot u^+\big)$$

i.e.

$$\mu_\alpha\big(\nabla\varphi_\alpha(\bar u), \dot u^+\big) = 0 \quad \forall \alpha \in J(\bar u).$$

Let us prove the following lemma.

Lemma 2.10. *Let $\alpha \in J(\bar{u})$ such that $\mu_\alpha \neq 0$. Then, for all $\rho_1 \in (0, \bar{\rho}]$ there exists $h_{\rho_1} \in (0, h_2]$ such that for all $h_i \in (0, h_{\rho_1}]$, there exists $nh_i \in [\bar{t} - \rho_1, \bar{t} + \rho_1]$ such that $\varphi_\alpha(U^{n+1}) \leqslant 0$.*

Proof. Let us assume that the announced result does not hold, i.e. assume that there exists $\rho_1 \in (0, \bar{\rho}]$ such that, for all $h_{\rho_1} \in (0, h_2]$ there exists $h_i \in (0, h_{\rho_1}]$ such that $\varphi_\alpha(U^{n+1}) > 0$ for all $nh_i \in [\bar{t} - \rho_1, \bar{t} + \rho_1]$.

Hence, we can extract from $(h_i)_{i \geqslant 0}$ a subsequence denoted $(h_{\varphi(i)})_{i \geqslant 0}$ such that

$$\varphi_\alpha(U^{n+1}) > 0 \quad \forall nh_{\varphi(i)} \in [\bar{t} - \rho_1, \bar{t} + \rho_1] \quad \forall i \geqslant 0. \tag{2.21}$$

For all $\rho \in (0, \rho_1]$ such that $\bar{t} \pm \rho \in (0, \tau) \setminus D$, let us establish the following estimate:

$$\left|\left(\dot{u}_{h_{\varphi(i)}}(\bar{t} - \rho) - \dot{u}_{h_{\varphi(i)}}(\bar{t} + \rho), \varepsilon_\alpha(\bar{u})\right)\right| \leqslant \mathcal{O}(\rho) + \mathcal{O}\left(h_{\varphi(i)}\right) + \mathcal{O}\left(\|u - u_{h_{\varphi(i)}}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right).$$

Then, by passing to the limit when i tends to $+\infty$, we will infer with (2.12) that

$$\left|\left(\dot{u}(\bar{t} - \rho) - \dot{u}(\bar{t} + \rho), \varepsilon_\alpha(\bar{u})\right)\right| \leqslant \mathcal{O}(\rho)$$

and, when ρ tends to zero, we will obtain

$$\left|\left(\dot{u}(\bar{t} - 0) - \dot{u}(\bar{t} + 0), \varepsilon_\alpha(\bar{u})\right)\right| = |\mu_\alpha| \leqslant 0$$

which gives a contradiction.

Let $\rho \in (0, \rho_1]$ such that $\bar{t} \pm \rho \in (0, \tau) \setminus D$. For all $i \geqslant 0$ we define

$$n_i = \left\lfloor \frac{\bar{t} - \rho}{h_{\varphi(i)}} \right\rfloor + 1, \quad p_i = \left\lfloor \frac{\bar{t} + \rho}{h_{\varphi(i)}} \right\rfloor.$$

Then, for all $nh_{\varphi(i)} \in [\bar{t} - \rho, \bar{t} + \rho]$, we have $n_i \leqslant n \leqslant p_i$ and we infer from (2.4) that

$$V^{n-1} - V^n + h_{\varphi(i)} F^n \in N_K(U^{n+1}).$$

Hence there exist non-positive numbers $(\mu_\beta^n)_{\beta \in J(U^{n+1})}$ such that

$$V^{n-1} - V^n + h_{\varphi(i)} F^n = \sum_{\beta \in J(U^{n+1})} \mu_\beta^n \nabla \varphi_\beta(U^{n+1}).$$

With (2.21) we obtain that $\alpha \notin J(U^{n+1})$ and thus

$$\Big(V^{n-1} - V^n + h_{\varphi(i)}F^n, \varepsilon_\alpha(U^{n+1})\Big) = \left(\sum_{\beta \in J(U^{n+1})} \mu_\beta^n e_\beta(U^{n+1}), \varepsilon_\alpha(U^{n+1})\right) = 0.$$

It follows that

$$\begin{aligned}
\Big(V^{n_i-1} - V^{p_i}, \varepsilon_\alpha(\bar{u})\Big) &= \sum_{n=n_i}^{p_i} \Big(V^{n-1} - V^n, \varepsilon_\alpha(\bar{u})\Big) \\
&= \sum_{n=n_i}^{p_i} \Big(V^{n-1} - V^n, \varepsilon_\alpha(U^{n+1})\Big) + \sum_{n=n_i}^{p_i} \Big(V^{n-1} - V^n, \varepsilon_\alpha(\bar{u}) - \varepsilon_\alpha(U^{n+1})\Big) \\
&= -\sum_{n=n_i}^{p_i} h_{\varphi(i)}\Big(F^n, \varepsilon_\alpha(U^{n+1})\Big) + \sum_{n=n_i}^{p_i} \Big(V^{n-1} - V^n, \varepsilon_\alpha(\bar{u}) - \varepsilon_\alpha(U^{n+1})\Big).
\end{aligned}$$

Let us observe now that $V^{n_i-1} = \dot{u}_{h_{\varphi(i)}}(\bar{t} - \rho)$ and $V^{p_i} = \dot{u}_{h_{\varphi(i)}}(\bar{t} + \rho)$. We obtain

$$\begin{aligned}
&\Big|\big(\dot{u}_{h_{\varphi(i)}}(\bar{t} - \rho) - \dot{u}_{h_{\varphi(i)}}(\bar{t} + \rho), \varepsilon_\alpha(\bar{u})\big)\Big| \\
&\quad \leqslant \sum_{n=n_i}^{p_i} h_{\varphi(i)} M_1 M_2 + \sum_{n=n_i}^{p_i} \|V^{n-1} - V^n\| \big\|\varepsilon_\alpha(\bar{u}) - \varepsilon_\alpha(U^{n+1})\big\|,
\end{aligned}$$

where M_1 and M_2 are defined by (2.8) and (2.19).

Moreover

$$\begin{aligned}
\|\bar{u} - U^{n+1}\| &= \big\|u(\bar{t}) - u_{h_{\varphi(i)}}\big((n+1)h_{\varphi(i)}\big)\big\| \\
&\leqslant \big\|u(\bar{t}) - u_{h_{\varphi(i)}}(\bar{t})\big\| + \big\|u_{h_{\varphi(i)}}(\bar{t}) - u_{h_{\varphi(i)}}\big((n+1)h_{\varphi(i)}\big)\big\| \\
&\leqslant \|u - u_{h_{\varphi(i)}}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} + C\big|\bar{t} - (n+1)h_{\varphi(i)}\big| \\
&\leqslant \|u - u_{h_{\varphi(i)}}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)} + C\big(\rho + h_{\varphi(i)}\big),
\end{aligned}$$

where C is the constant obtained at Proposition 2.1. Hence,

$$\begin{aligned}
&\Big|\big(\dot{u}_{h_{\varphi(i)}}(\bar{t} - \rho) - \dot{u}_{h_{\varphi(i)}}(\bar{t} + \rho), \varepsilon_\alpha(\bar{u})\big)\Big| \\
&\quad \leqslant M_1 M_2 (p_i - n_i + 1) h_{\varphi(i)}
\end{aligned}$$

$$+\omega_\alpha\big(\|u-u_{h_{\varphi(i)}}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}+C(\rho+h_{\varphi(i)})\big)\sum_{n=n_i}^{p_i}\|V^{n-1}-V^n\|$$

$$\leqslant M_1M_2(2\rho+h_{\varphi(i)})+C'\omega_{\varepsilon_\alpha}\big(\|u-u_{h_{\varphi(i)}}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}+C(\rho+h_{\varphi(i)})\big),$$

where C' is the constant defined at Proposition 2.4 and $\omega_{\varepsilon_\alpha}$ is the modulus of continuity of ε_α on $\overline{B}(\bar{u},r)$, which achieves the proof. □

We come now to the last step of the proof of Proposition 2.9.

Lemma 2.11. *Let $\alpha\in J(\bar{u})$ be such that $\mu_\alpha\neq 0$. Then*

$$\big(\nabla\varphi_\alpha(\bar{u}),\dot{u}^+\big)=0.$$

Proof. Let $\alpha\in J(\bar{u})$ such that $\mu_\alpha\neq 0$. Since $\dot{u}^+\in T_K(\bar{u})$ we have $\big(\nabla\varphi_\alpha(\bar{u}),\dot{u}^+\big)\geqslant 0$ and it remains to prove that $\big(\dot{u}^+,\nabla\varphi_\alpha(\bar{u})\big)\leqslant 0$. The main idea of the proof is to obtain an estimate of $\big(\dot{u}(\bar{t}+\rho),\nabla\varphi_\alpha\big(u(\bar{t}+\rho)\big)\big)$ and to pass to the limit when ρ tends to zero.

More precisely, let $\rho\in(0,\bar{\rho}]$ such that $\bar{t}+\rho\in(0,\tau)\setminus D$. We have

$$\dot{u}(\bar{t}+\rho)=\lim_{h_i\to 0}\dot{u}_{h_i}(t+\rho)=\lim_{h_i\to 0}V^{p_i}\quad\text{with}\quad p_i=\left\lfloor\frac{\bar{t}+\rho}{h_i}\right\rfloor\quad\forall i\geqslant 0.$$

Observing that

$$\begin{aligned}\big\|u(\bar{t}+\rho)-U^{p_i+1}\big\|&\leqslant\big\|u(\bar{t}+\rho)-u_{h_i}(\bar{t}+\rho)\big\|+\big\|u_{h_i}(\bar{t}+\rho)-u_{h_i}\big((p_i+1)h_i\big)\big\|\\&\leqslant\|u-u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}+Ch_i\end{aligned}$$

the continuity of $\nabla\varphi_\alpha$ on $\overline{B}(\bar{u},r)$ implies that

$$\begin{aligned}\big(\dot{u}(\bar{t}+\rho),\nabla\varphi_\alpha\big(u(\bar{t}+\rho)\big)\big)&=\lim_{h_i\to 0}\big(\dot{u}_{h_i}(\bar{t}+\rho),\nabla\varphi_\alpha(U^{p_i+1})\big)\\&=\lim_{h_i\to 0}\big(V^{p_i},\nabla\varphi_\alpha(U^{p_i+1})\big)\end{aligned}$$

and we will prove that

$$\big(V^{p_i},\nabla\varphi_\alpha(U^{p_i+1})\big)\leqslant\mathcal{O}(\rho)+\mathcal{O}(h_i)+\mathcal{O}\big(\|u-u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\big).\tag{2.22}$$

Let us apply the previous lemma: there exists $h_\rho\in(0,h_2]$ such that, for all $h_i\in(0,h_\rho]$, there exists $nh_i\in[\bar{t}-\rho,\bar{t}+\rho]$ such that $\varphi_\alpha(U^{n+1})\leqslant 0$ and we define N_i as

the last time step in $[\bar{t}-\rho, \bar{t}+\rho]$ at which the constraint φ_α is active. More precisely, let i be such that $h_i \in (0, h_\rho]$ and define N_i by

$$N_i = \max\big\{n \in \mathbb{N};\, nh_i \in [\bar{t}-\rho, \bar{t}+\rho] \text{ and } \varphi_\alpha(U^{n+1}) \leqslant 0\big\}.$$

Since $V^{N_i} \in -T_K(U^{N_i+1})$ and $\alpha \in J(U^{N_i+1})$, we infer that

$$\big(V^{N_i}, \nabla\varphi_\alpha(U^{N_i+1})\big) \leqslant 0.$$

By definition of the scheme, for all $n \in \{1, \dots, N\}$, we have

$$V^{n-1} - V^n + h_i F^n \in N_K(U^{n+1})$$

and there exist non-positive numbers $(\mu_\beta^n)_{\beta \in J(U^{n+1})}$ such that

$$V^{n-1} - V^n + h_i F^n = \sum_{\beta \in J(U^{n+1})} \mu_\beta^n \nabla\varphi_\beta(U^{n+1}).$$

Thus, for all $h_i \in (0, h_\rho]$ we get

$$\begin{aligned}
\Big(V^{p_i}, \nabla\varphi_\alpha(U^{p_i+1})\Big) &= \Big(V^{N_i}, \nabla\varphi_\alpha(U^{p_i+1})\Big) + \sum_{n=N_i+1}^{p_i} \Big(V^n - V^{n-1}, \nabla\varphi_\alpha(U^{p_i+1})\Big) \\
&\leqslant \Big(V^{N_i}, \nabla\varphi_\alpha(U^{p_i+1}) - \nabla\varphi_\alpha(U^{N_i+1})\Big) \\
&\quad + \sum_{n=N_i+1}^{p_i} \Big(h_i F^n, \nabla\varphi_\alpha(U^{p_i+1})\Big) \\
&\quad + \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} \Big(-\mu_\beta^n \nabla\varphi_\beta(U^{n+1}), \nabla\varphi_\alpha(U^{p_i+1})\Big) \\
&\leqslant C\big\|\nabla\varphi_\alpha(U^{p_i+1}) - \nabla\varphi_\alpha(U^{N_i+1})\big\| + (p_i - N_i)h_i M_1 M_2 \\
&\quad + \sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} \Big(-\mu_\beta^n \nabla\varphi_\beta(U^{n+1}), \nabla\varphi_\alpha(U^{p_i+1})\Big).
\end{aligned}$$

Let us estimate the last term. We observe first that (2.16) and (2.18) imply that $J(U^{n+1}) \subset J(\bar{u})$ for all $nh_i \in [\bar{t}-\bar{\rho}, \bar{t}+\bar{\rho}]$ and by definition of N_i, we have also

$\alpha \notin J(U^{n+1})$ for all $n \in \{N_i + 1, \dots, p_i\}$. Moreover, assumption (H5) implies that

$$\big(\nabla\varphi_\beta(\bar{u}), \nabla\varphi_\alpha(\bar{u})\big) = \big(e_\beta(\bar{u}), e_\alpha(\bar{u})\big) \leqslant 0 \quad \forall \beta \in J(\bar{u}) \setminus \{\alpha\}$$

and thus

$$\sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} \Big(-\mu_\beta^n \nabla\varphi_\beta(U^{n+1}), \nabla\varphi_\alpha(U^{p_i+1})\Big)$$

$$\leqslant \sum_{\beta \in J(U^{n+1})} \sum_{n=N_i+1}^{p_i} (-\mu_\beta^n)\Big\{\big(e_\beta(U^{n+1}), e_\alpha(U^{p_i+1})\big) - \big(e_\beta(\bar{u}), e_\alpha(\bar{u})\big)\Big\}.$$

Let us denote by ω_{e_β} the modulus of continuity of e_β on $\overline{B}(\bar{u}, r)$ for all $\beta \in J(\bar{u})$ and let $\omega = \max_{\beta \in J(\bar{u})} \omega_{e_\beta}$. Arguing as in the previous lemma we obtain

$$\big\|e_\alpha(U^{p_i+1}) - e_\alpha(\bar{u})\big\| \leqslant \omega_{e_\alpha}\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right)$$

and

$$\big\|e_\beta(U^{n+1}) - e_\beta(\bar{u})\big\| \leqslant \omega_{e_\beta}\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right)$$

for all $\beta \in J(U^{n+1})$ and for all $n \in \{N_i + 1, \dots, p_i\}$. Hence

$$\Big\|\big(e_\beta(U^{n+1}), e_\alpha(U^{p_i+1})\big) - \big(e_\beta(\bar{u}), e_\alpha(\bar{u})\big)\Big\| \leqslant 2M_2\omega\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right)$$

for all $\beta \in J(U^{n+1})$ and for all $n \in \{N_i + 1, \dots, p_i\}$. Moreover, by definition of μ_β^n we have

$$\big|\mu_\beta^n\big| = \Big|\big(V^{n-1} - V^n + h_i F^n, \varepsilon_\beta(U^{n+1})\big)\Big| \leqslant M_2\|V^n - V^{n-1}\| + h_i M_1 M_2$$

for all $\beta \in J(U^{n+1})$ and for all $n \in \{N_i + 1, \dots, p_i\}$. It follows that

$$\sum_{n=N_i+1}^{p_i} \sum_{\beta \in J(U^{n+1})} \Big(-\mu_\beta^n \nabla\varphi_\beta(U^{n+1}), \nabla\varphi_\alpha(U^{p_i+1})\Big)$$

$$\leqslant 2dM_2^2\omega\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right)$$

$$\times \left\{ \sum_{n=N_i+1}^{p_i} \|V^n - V^{n-1}\| + (p_i - N_i)h_i M_1 \right\}$$

$$\leqslant 2dM_2^2(C' + 2\rho M_1)\omega\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right),$$

where C' is the constant defined at Proposition 2.6. Finally, for all $h_i \in (0, h_\rho]$ we have

$$\begin{aligned}
&\left\|\nabla\varphi_\alpha(U^{p_i+1}) - \nabla\varphi_\alpha(U^{N_i+1})\right\| \\
&\quad = \left\|e_\alpha(U^{p_i+1}) - e_\alpha(U^{N_i+1})\right\| \\
&\quad \leqslant \omega_{e_\alpha}\left(\|U^{p_i+1} - U^{N_i+1}\|\right) \leqslant \omega_{e_\alpha}(2C\rho)
\end{aligned}$$

and thus

$$\begin{aligned}
&\left(V^{p_i}, \nabla\varphi_\alpha(U^{p_i+1})\right) \\
&\quad \leqslant C\omega_{e_\alpha}(2C\rho) + 2\rho h_i M_1 M_2 \\
&\qquad + 2dM_2^2(C' + 2\rho M_1)\omega\left(C\rho + Ch_i + \|u - u_{h_i}\|_{C^0\left([0,\tau];\mathbb{R}^d\right)}\right)
\end{aligned}$$

which proves (2.22). Passing to the limit as h_i tends to zero, we obtain

$$\left(\dot{u}(\bar{t}+\rho), \nabla\varphi_\alpha\big(u(\bar{t}+\rho)\big)\right) \leqslant C\omega_{e_\alpha}(2C\rho) + 2dM_2^2(C' + 2\rho M_1)\omega(C\rho).$$

Then, passing to the limit as ρ tends to zero, we conclude the proof. □

Appendix A

Lemma A.1. *Let $h_1 \in (0, 1/L)$ and $(\varphi^n)_{n\geqslant 0}$ be defined by*

$$\varphi^0 = \frac{\|v_0\| + M\tau + 1}{1 - Lh},$$

$$\varphi^n = \frac{1}{1-Lh}\left(\|v_0\| + M\tau + 1 + Lh\sum_{k=0}^{n-1}\varphi^k + 2Lh^2\sum_{k=1}^{n}\sum_{p=0}^{k-1}\varphi^p\right) \quad \forall n \geqslant 1.$$

There exists $C_1 > 0$ and $\kappa > 0$ such that

$$0 \leqslant \varphi^n \leqslant C_1 e^{\kappa nh} \quad \forall n \geqslant 0 \quad \forall h \in (0, h_1].$$

Proof. Let $h \in (0, h_1]$. By definition of the sequence $(\varphi^n)_{n\geqslant 0}$, we have

$$(1 - Lh)\varphi^{n+1} + (Lh - 2 - 2Lh^2)\varphi^n + \varphi^{n-1} = 0 \quad \forall n \geqslant 1.$$

Let us denote by ξ and η the roots of the characteristic equation

$$(1 - Lh)X^2 + (Lh - 2 - 2Lh^2)X + 1 = 0$$

i.e.

$$\xi = 1 + h\frac{L + 2Lh + \sqrt{8L + L^2(1-2h)^2}}{2(1 - Lh)} = 1 + hx_1(h)$$

and

$$\eta = 1 + h\frac{L + 2Lh - \sqrt{8L + L^2(1-2h)^2}}{2(1 - Lh)} = 1 + hx_2(h).$$

Then we have

$$\varphi^n = a\xi^n + b\eta^n \quad \forall n \geqslant 0$$

with (a, b) given by the relations

$$\begin{cases} \varphi^0 = a + b = \dfrac{C_0}{1 - Lh}, \\ \varphi^1 = a\xi + b\eta = \dfrac{C_0}{1 - Lh}\left(1 + h\dfrac{L + 2Lh}{1 - Lh}\right) \end{cases}$$

and $C_0 = \|v_0\| + M\tau + 1$. We infer that

$$a = \frac{C_0}{\sqrt{8L + L^2(1-2h)^2}}\left(\frac{L + 2Lh + \sqrt{8L + L^2(1-2h)^2}}{2(1 - Lh)}\right)$$

and

$$b = \frac{C_0}{\sqrt{8L + L^2(1-2h)^2}}\left(-\frac{L + 2Lh - \sqrt{8L + L^2(1-2h)^2}}{2(1 - Lh)}\right).$$

It follows that a and b are two continuous functions of h, which remain non-negative and bounded on $[0, h_1]$. Thus there exists $C_1 > 0$ such that

$$0 \leqslant a \leqslant \frac{C_1}{2}, \quad 0 \leqslant b \leqslant \frac{C_1}{2} \quad \forall h \in [0, h_1].$$

For the same reason, the functions x_1 and x_2 are bounded on $[0, h_1]$ and there exists $\kappa > 0$ such that

$$0 \leqslant \eta \leqslant \xi \leqslant e^{\kappa h}, \quad \forall h \in [0, h_1].$$

Finally we get

$$0 \leqslant \varphi^n \leqslant a\xi^n + b\eta^n \leqslant (a+b)e^{\kappa n h} \leqslant C_1 e^{\kappa n h} \quad \forall n \geqslant 0 \quad \forall h \in (0, h_1]. \qquad \square$$

Lemma A.2. *Let K be defined by*

$$K = \big\{q \in \mathbb{R}^d;\, \varphi_\alpha(q) \geqslant 0,\ \alpha = 1, \ldots, v\big\}$$

with functions φ_α, $\alpha = 1, \ldots, v$, belonging to $C^1(\mathbb{R}^d, \mathbb{R})$ and satisfying (H4). *Then for all $q_0 \in K$, there exist $\delta > 0$, $r > 0$ and $a \in \mathbb{R}^d$ such that, for all $q \in \overline{B}(q_0, 2\delta)$:*

$$\overline{B}(a, r) \subset T_K(q) \tag{A.1}$$

and

$$\big\|z - \mathrm{Proj}\big(T_K(q), z\big)\big\| \leqslant \frac{1}{2r}\big(\|z - a\|^2 - \big\|\mathrm{Proj}\big(T_K(q), z\big) - a\big\|^2\big) \quad \forall z \in \mathbb{R}^d. \tag{A.2}$$

Proof. First, let us prove (A.1).

Let q_0 be in K and recall that, for all $q \in \mathbb{R}^d$,

$$J(q) = \big\{\alpha \in \{1, \ldots, v\};\, \varphi_\alpha(q) \leqslant 0\big\}.$$

Since the functions $(\varphi_\alpha)_{\alpha=1,\ldots v}$ are continuous, we infer that there exists $\delta_1 > 0$ such that, for all $\alpha \notin J(q_0)$, we have

$$\varphi_\alpha(q) > 0 \quad \text{if } \|q - q_0\| \leqslant \delta_1.$$

It follows that $J(q) \subset J(q_0)$ for all $q \in \overline{B}(q_0, \delta_1)$.

Consequently, if $J(q_0) = \emptyset$, we have $J(q) = \emptyset$ for all $q \in \overline{B}(q_0, \delta_1)$ and (A.1) is satisfied for $\delta = \delta_1/2$ and for all $a \in \mathbb{R}^d$ and $r > 0$.

Let us assume now that $J(q_0) \neq \emptyset$. For all $\alpha \in J(q_0)$ we define $\phi_\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\phi_\alpha(q, y) = \big(\nabla\varphi_\alpha(q), y\big) \quad \forall (q, y) \in \mathbb{R}^d \times \mathbb{R}^d$$

and $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\phi(q, y) = \min_{\alpha \in J(q_0)} \phi_\alpha(q, y) \quad \forall (q, y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Since $\varphi_\alpha \in C^1(\mathbb{R}^d)$ for all $\alpha \in \{1, \ldots, v\}$, we obtain that the mappings are continuous. Moreover, since $\big(\nabla\varphi_\alpha(q_0)\big)_{\alpha \in J(q_0)}$ is linearly independent, we can define a basis $(e_i)_{1 \leqslant i \leqslant d}$ of $\mathbb{R}^d$ such that

$$e_\alpha = \nabla\varphi_\alpha(q_0) \quad \forall \alpha \in J(q_0).$$

Let us denote by $(\varepsilon_i)_{1 \leqslant i \leqslant d}$ the dual basis of $(e_i)_{1 \leqslant i \leqslant d}$ and let

$$a = \sum_{\alpha \in J(q_0)} \varepsilon_\alpha.$$

Then, for all $\alpha \in J(q_0)$, we have

$$\phi_\alpha(q_0, a) = \big(\nabla\varphi_\alpha(q_0), a\big) = \left(e_\alpha, \sum_{\beta \in J(q_0)} \varepsilon_\beta\right) = 1$$

and $\phi(q_0, a) = 1$. By continuity, it follows that there exist $r > 0$ and $\delta_2 > 0$ such that

$$\phi(q, y) > 0 \quad \forall (q, y) \in \overline{B}(q_0, \delta_2) \times \overline{B}(a, r).$$

Let $\delta = \frac{1}{2}\min(\delta_1, \delta_2)$. For all $q \in \overline{B}(q_0, 2\delta)$ we have

$$J(q) \subset J(q_0), \quad \phi(q, y) = \min_{\alpha \in J(q_0)} \big(\nabla\varphi_\alpha(q), y\big) > 0 \quad \forall y \in \overline{B}(a, r)$$

which implies that

$$\overline{B}(a, r) \subset T_K(q) = \big\{y \in \mathbb{R}^d; \big(\nabla\varphi_\alpha(q), y\big) \geqslant 0 \quad \forall \alpha \in J(q)\big\}$$

and (A.1) is satisfied.

Then we observe that (A.2) is a direct consequence (with the choice $H = D = \mathbb{R}^d$, $C = T_K(q)$ and $S = \mathrm{Proj}(C, \cdot)$) of the following result due to J.J.Moreau:

Lemma A.3 (*Moreau [6]*). *Let D be a subset of a real Hilbert space H and let $S : D \to D$ be such that*

$$\|Sz - Sz'\| \leqslant \|z - z'\| \quad \forall (z, z') \in D^2.$$

Let $a \in H$ and $r > 0$ such that $\overline{B}(a, r) \subset \{z \in D : Sz = z\}$. Then

$$\|z - Sz\| \leqslant \frac{1}{2r}\left(\|z - a\|^2 - \|Sz - a\|^2\right) \quad \forall z \in D. \qquad \square$$

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